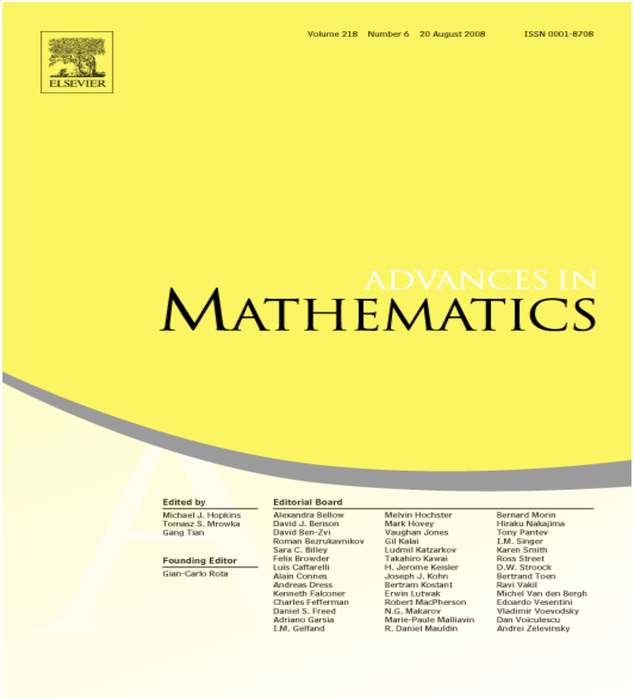
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Homaloidal hypersurfaces and hypersurfaces with vanishing Hessian

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Abstract

We introduce various families of irreducible homaloidal hypersurfaces in projective space $\mathbb{P}^r$, for all $r \geqslant 3$. Some of these are families of homaloidal hypersurfaces whose degrees are arbitrarily large as compared to the dimension of the ambient projective space. The existence of such a family solves a question that has naturally arisen from the consideration of the classes of homaloidal hypersurfaces known so far. The result relies on a fine analysis of hypersurfaces that are dual to certain scroll surfaces. We also introduce an infinite family of determinantal homaloidal hypersurfaces based on a certain degeneration of a generic Hankel matrix. The latter family fit non-classical versions of de Jonquières transformations. As a natural counterpoint, we broaden up aspects of the theory of Gordan–Noether hypersurfaces with vanishing Hessian determinant, bringing over some more precision into the present knowledge.

Keywords: Hypersurfaces; Hessian; Cremona transformations

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0. Introduction

The study of Cremona transformations of $\mathbb{P}^r$ is a classical and fascinating subject(s) in algebraic geometry. The Cremona group of $\mathbb{P}^r$ is well understood only for $r \leqslant 2$. By contrast, in dimension $r \geqslant 3$ it is even problematic to produce non-trivial examples of birational transformations of $\mathbb{P}^r$. Therefore, any relevant addition to the universe of these transformations is very welcome, especially if it bridges up with other interesting concepts in the field.

In this perspective, a good example is that of a *homaloidal hypersurface*. This is a projective hypersurface $X \subset \mathbb{P}^r$, not necessarily reduced or irreducible, defined by a homogeneous polynomial $f = f(x_0, \ldots, x_r)$ of degree $d \geqslant 2$ whose partial derivatives define a Cremona transformation of $\mathbb{P}^r$. Quite generally, the rational map $\phi_f : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ defined by the partial derivatives of f is called the *polar map* of the hypersurface X, so that, if X is reduced, the indeterminacy locus of ϕ_f is precisely the singular locus of X. For instance, a smooth quadric is homaloidal, inasmuch as its polar map is the usual polarity, which is a projective transformation. However, if X is smooth of degree $d \geqslant 3$, then X is never homaloidal, since its polar map has no indeterminacy locus and it is defined by forms of degree $d - 1 > 1$. Indeed, a relevant role in the understanding of homaloidal hypersurfaces is played by the analysis of their singular locus.

On the other hand, an obvious necessary condition in order that X be homaloidal, is the non-vanishing of the Hessian determinant $h(f)$ of f. Note that, if one measures the *complexity* of a hypersurface by the degree of its polar map, the hypersurfaces with vanishing Hessian have to be considered as the *simplest* ones, and the homaloidal hypersurfaces are the simplest among those for which the Hessian is not identically zero. Thus, a couple of natural questions arise: what can we say about hypersurfaces with identically vanishing Hessian? What are the relations, if any, between these and homaloidal hypersurfaces?

As is generally known, both problems—the classification of homaloidal hypersurfaces and of hypersurfaces with vanishing Hessian—play a classical role in the history of algebraic geometry, perhaps with homaloidal running first, as subsumed into Cremona theory, while vanishing Hessian winning in drama ever since Gordan and Noether (see [20]) showed that Hesse (see [22,23]) had previously misapprehended the question.

Although fairly understood, even the theory of plane Cremona transformations is already quite involved. The early results of Noether (see, e.g., [37, Remark 2.3 and ff.]), inspired on Cremona's original work, showed how much more complicated is the theory in $\mathbb{P}^3$. However, it has perhaps been common thought that, notwithstanding the difficulties of the general Cremona theory, homaloidal hypersurfaces would be easier to understand and eventually be subject to classification. For instance, the classification of reduced homaloidal curves in $\mathbb{P}^2$ by Dolgachev (see [10])—which shows that there are only three types up to projective transformations, and, more generally, the examples coming from the theory of pre-homogeneous vector spaces (see [14]), whose degree is bounded in terms of the embedding dimension—could have generated the expectation that the degree of an irreducible, or perhaps even only reduced, homaloidal hypersurface in $\mathbb{P}^r$ is at most $r + 1$. If this were proved to be the case, one would perhaps be half-way from the classification goal.

Alas, nature had the upper hand. Indeed, one of the main objectives of this paper is to show that, as a counterpart to the planar case, in which a full classification is fairly easy to state, the situation is much more complicated in higher dimension. In fact, one of our main results here is to show the existence of families of irreducible hypersurfaces in $\mathbb{P}^r$, for $r \geqslant 3$, with arbitrarily large degree with respect to r (see Section 3.1). We think that this uncovers some complex phenomenology which makes the classification of irreducible homaloidal hypersurfaces quite intricate and therefore deserves a deeper scrutiny, beyond our presently inadequate understanding of the matter. A special role is of course played by the complicated nature of the scheme structure of the base locus of a homaloidal Cremona transformation. In particular, for a homaloidal hypersurface X, this is due to the existence of embedded components originating some infinitely near base points for the linear system of polars of X, which are somehow unexpected inasmuch as they are not singular points of X or do not even belong to X (see, e.g., [1] and Section 3.2). Incidentally, this phenomenon is already present in one of the plane cases appearing in Dolgachev's classification.

As for the second question envisaged here, the problem is after all to find the homogeneous polynomial solutions f of the classical *Monge–Ampère* differential equation $h(f)=0$. It is therefore not surprising to see how far an outpost this question has reached in subsequent geometric developments and how strong a role it has played in various other areas, such as differential geometry and approximation theory (see, for example, [41,16,35]).

In their celebrated work [20], Gordan and Noether constructed counterexamples to Hesse's original claim to the effect that X has vanishing Hessian if and only if it is a cone. The examples have been later revisited and partly extended by several authors (in chronological order, [44,17, 18,32–34,27]). In spite of the difficulty of their original paper, the examples themselves are not all that difficult to understand and can actually be easily described in explicit algebraic terms (see also [31]).

A second goal of this paper is to give a modern overview of the known methods to deal with the problem of vanishing Hessian and to generalize results of Permutti and Perazzo quoted above. One of the challenges is to determine the structure of the dual variety to Gordan–Noether or Permutti hypersurfaces, for which we add a tiny contribution that may help improving our understanding of these defective dual varieties. As it turns, there is a strong relationship between the families of homaloidal hypersurfaces described here and some hypersurfaces with vanishing Hessian. We hope to pursue work along this line in the near future.

We now describe the sections of the paper in somewhat more detail.

The first section contains a recap of known concepts and is primarily meant as a collection of properties of scroll surfaces and their dual varieties that are either spread out or difficult to find in the current literature. The main results are contained in a series of propositions (see Proposition 1.4 through Proposition 1.6). We also describe the behavior of more general rational scroll surfaces containing a so-called *line directrix*, and their dual hypersurfaces (see Propositions 1.8 and 1.9). This section prepares the ground for the more thorough considerations of the third section, for which the present material is essential in the construction of the announced examples.

The second section starts with an overview of the aforementioned polar map ϕ_f associated to a nonzero homogeneous polynomial f. After a brief introduction about the polars and the Hessian of f, we switch to the problem of the vanishing Hessian. Just enough of the Gordan–Noether construction is reviewed in order to state a geometric description of its structure (see Proposition 2.11), based on a notion of *core* of such a hypersurface. We next discuss the work of Permutti extending the previous construction in a special situation, and following the same ideas we also give some features of Permutti's generalized hypersurface (see Proposition 2.13). We proceed to establishing both the structure of the dual variety to a Permutti hypersurface and of its polar image (see Propositions 2.14 and 2.15). The section ends with a generalization of a result of Perazzo (see Proposition 2.18) establishing a bound for the dimension of the image of ϕ_f for a so-called H-hypersurface $X \subset \mathbb{P}^r$ with equation $f=0$, i.e. a reduced hypersurface which contains a subspace of dimension t such that the general subspace of dimension $t+1$ through it cuts out on X a cone with a vertex of dimension at least $r - t - 1$. The dual hypersurfaces to scrolls with a line directrix are special cases of H-hypersurfaces and come up in our examples.

In the third section we introduce families of irreducible homaloidal hypersurfaces, including the case in which they have arbitrarily large degree as compared to the ambient dimension. As a preliminary, we state a general principle for a Cremona transformation saying that such a map always *contracts* its Jacobian, and ask whether, in the case of a polar map ϕ_f, contraction is also sufficient for birationality, provided f, or the corresponding hypersurface X with equation $f = 0$, is *totally Hessian* in the sense that $h(f) = cf^{\frac{(d-2)(r+1)}{d}}$ with $c \in k \setminus \{0\}$. Here a good deal of examples of such forms arises from the theory of *pre-homogeneous vector spaces*, a notion introduced by Kimura and Sato (see [24], also [11,10,14] and Remark 3.5). In this setup f is the so-called *relative invariant* of the pre-homogeneous space, uniquely defined up to a nonzero factor from $\mathbb{C}$. If, moreover, its Hessian is nonzero then it is in fact totally Hessian and f is a homaloidal polynomial such that ϕ_f coincides with its inverse up to a projective transformation (see [11, Theorem 2.8]).

As mentioned, the singularities of a hypersurface $X \subset \mathbb{P}^r$ which is either homaloidal or has vanishing Hessian are not arbitrary. For example, in the second case, if $r \geqslant 3$ then X cannot have isolated singularities. The same result regarding homaloidal hypersurfaces is a conjecture of Dimca–Papadima (see [9]). We give a slight evidence for this conjecture in terms of a resolution of the indeterminacies of the polar map of X by successive blowups along smooth centers, to wit, if $X \subset \mathbb{P}^r$ is homaloidal and its degree exceeds $r + 1$ then, for some blowing-up step, the multiplicity of the proper transform of the general first polar of X is at least the dimension of the center of the blowup (see Proposition 3.6). In other words, the polar linear system of X cannot be *log-canonical* (see [26, p. 56]). This gives a measure of the complexity of the singular locus of X. In particular it shows that a homaloidal hypersurface in $\mathbb{P}^3$, of degree at least 5, cannot have ordinary singularities.

After these preliminaries, we produce, for every $r \geqslant 3$, the promised infinite series of irreducible homaloidal hypersurfaces in $\mathbb{P}^r$ of arbitrarily large degree $d \geqslant 2r - 3$. They are the dual hypersurfaces to certain scroll surfaces with a line directrix. It is relevant to observe that the present examples are not related to the ones based on pre-homogeneous vector spaces as mentioned above. Also they show, perhaps against the ongoing folklore, that there are plenty of homaloidal polynomials around. They even seem to be in majority as compared to polynomials with vanishing Hessian, though a complete classification does not seem to be presently at hand.

The full results are a bit too technical to be narrated here—we refer to the main theorem of the section Theorem 3.13, in which one shows that the dual hypersurfaces to certain rational scroll surfaces $Y(r-2, d-r+2) \subset \mathbb{P}^r$ are homaloidal and have degree $d \geqslant 2r-3$. These examples are obtained via a rather intricate geometric construction linking in an unexpected way hypersurfaces with vanishing Hessian and homaloidal hypersurfaces. A central piece is Theorem 3.12, whose proof is fairly technical but keeps a strong geometric flavor. We then dwell quite a bit into the structure of these scroll surfaces, looking at their construction from various different angles in order to fully apprehend their properties. Finally, in Theorem 3.18 we produce different infinite families of homaloidal examples in $\mathbb{P}^r$, $r \geqslant 4$. These, though still related to some scroll surfaces, do not seem in general to relate to hypersurfaces with vanishing Hessian, which adds to the feeling that the classification of homaloidal hypersurfaces has still a long way to go.

In addition we give a refined analysis of the nature of the singularities of the homaloidal examples in $\mathbb{P}^3$ along with an insight into the degree of the inverse map. That is, here we deal with the scroll $Y(1, d-1)$ which, for $d = 3$ turns out to be a particular case of a series of degenerate determinantal Hankel hypersurfaces considered in the following and last Section 4.

This latter construction, which has a more algebraic flavor, is based on a certain specialization of the generic *Hankel matrix*. The interest of these examples lies in that, besides being irreducible and of degree r, they fit a recent construct generalizing the classical de Jonquières transformations (see [29]) and boil down in particular cases to projections of certain scroll surfaces. The full development of the nature of these homaloidal hypersurfaces relates to several typical concepts of commutative algebra. It also relates to the method devised in [37]. These examples do not come (either) from the theory of pre-homogeneous vector spaces either since, for example, they are not totally Hessian. A marked feature of these homaloidal hypersurfaces is that the corresponding degree is the dimension of the ambient space, while in most examples coming from pre-homogeneous vector spaces the degree of the invariant polynomial is small with respect to the number of variables.

Though somewhat exceptional, all these examples share in common the property of having large degree with respect to the number of variables. Additional inquiry could be made as to whether there are families of totally Hessian polynomials, not necessarily homaloidal, of arbitrary large degree for any $r \geqslant 3$. Or even be wondered if there exists a characterization of all homaloidal polynomials whose Hessian is a nonzero multiple of a linear form such as is the case for the Hankel degeneration examples constructed in the last section.

1. Dual varieties of scroll surfaces

In this section we recall, with no proofs, some general and perhaps mostly well-known facts about projective duality and dual varieties of scroll surfaces. Standing reference for this part are [25,51,36].

1.1. Generalities

Throughout this paper k denotes an algebraically closed field of characteristic zero—though many contentions herein will hold more generally.

Let $\mathbb{P}^r = \mathbb{P}(V)$ be a projective space over k, where V is a k-vector space of dimension $r+1$. The *dual* projective space of $\mathbb{P}^r$ is $\mathbb{P}^{r*} = \mathbb{P}(V^*)$, where $V^* = \mathrm{Hom}_k(V,k)$. If $\Pi = \mathbb{P}(W) \subseteq \mathbb{P}^r$, with $W \subset V$ a vector subspace of dimension $m+1$, then the *orthogonal* projective subspace $\Pi^\perp \subseteq \mathbb{P}^{r*}$ to Π is defined to be $\mathbb{P}((V/W)^*) = \mathbb{P}(\mathrm{Ann}(W)) \subset \mathbb{P}(V^*)$, where $\mathrm{Ann}(W) = \{f \in V^* \mid f(w) = 0,\ \forall w \in W\}$. Note that, geometrically, if one identifies $\mathbb{P}^{r*}$ with the linear system of all hyperplanes in $\mathbb{P}^r$, then $\Pi^\perp$ is identified with the linear system of all hyperplanes in $\mathbb{P}^r$ containing Π and has dimension $r-m-1$.

Let $X \subset \mathbb{P}^r$ be an irreducible projective variety of dimension n. For a smooth point $x \in X$, $T_{X,x}$ will denote the *embedded tangent space* to X at x, a subspace of dimension n.

The *conormal variety* $N(X)$ of $X \subset \mathbb{P}^r$ is the incidence variety defined as the closure of the set of all pairs $(x,\pi) \in \mathbb{P}^r \times \mathbb{P}^{r*}$, such that x is a smooth point of X and $\pi \in T_{X,x}^\perp$—each such a hyperplane π is said to be *tangent* to X at x. Since the fiber of the first projection $N(X) \to X$ over a smooth point $x \in X$ is the projective subspace $T_{X,x}^\perp \simeq \mathbb{P}^{r-n-1}$ of hyperplanes containing $T_{X,x}$, then $N(X)$ is irreducible and $\dim(N(X)) = r-1$.

The image of the projection of $N(X)$ to the second factor is, by definition, the *dual variety* X^* of X. Since k has characteristic zero, one has $N(X) = N(X^*)$ via the natural identification $\mathbb{P}^r = (\mathbb{P}^{r*})^*$—a property known as *reflexivity* (see, e.g., [25]). It follows that $(X^*)^* = X$.

The *dual defect* of $X \subset \mathbb{P}^r$ is the non-negative integer $d(X) := r - 1 - \dim(X^*)$ and $X \subset \mathbb{P}^r$ is said to be (*dual*) *defective* if $d(X) > 0$, i.e. if $X^* \subset \mathbb{P}^{r*}$ is not a hypersurface. Note that $d(X)$ is the dimension of $(T_{X^*,\xi})^{\perp} \subset \mathbb{P}^r$ for smooth $\xi \in X^*$; thus, if ξ corresponds to the general hyperplane π tangent to a point $x \in X$ then π is tangent at all points of $(T_{X^*,\xi})^{\perp} \subset \mathbb{P}^r$.

Also recall that $X \subset \mathbb{P}^r$ is said to be *degenerate* if its linear span $\Pi = \langle X \rangle$ is a proper subspace of $\mathbb{P}^r$, i.e., if its homogeneous defining ideal contains some nonzero linear form.

Let now $\Pi \subset \mathbb{P}^r$ be a subspace of dimension m, and let

$$\sigma_{\Pi} : \mathbb{P}^r \dashrightarrow \left(\Pi^{\perp}\right)^* \simeq \mathbb{P}^{r-m-1}$$

be the projection from Π, defined as $\sigma_{\Pi}(p) = (\ell_1(p) : \ldots : \ell_{r-m}(p))$, where $\ell_1, \ldots, \ell_{r-m}$ are linear forms cutting Π as a linear subspace of $\mathbb{P}^r$. If $X \subset \mathbb{P}^r$ is not contained in Π, the closure X_{Π} of the image of X via σ_{Π} is called the *projection of X from Π*. If $\Pi \cap X = \emptyset$, then σ_{Π}, or X_{Π}, is said to be an *external* projection of X. If $\dim(X) < r - m - 1$ then X_{Π} is a proper subvariety of $(\Pi^{\perp})^* \simeq \mathbb{P}^{r-m-1}$ and one has the following:

Proposition 1.1. *With the previous notation, suppose that $X \subset \mathbb{P}^r$ is non-degenerate and that* $\dim(X) < r - \dim(\Pi) - 1$. *Then:*

(i) $(X_{\Pi})^* \subseteq \Pi^{\perp} \cap X^*$ *and* $(X_{\Pi})^*$ *is an irreducible component of* $\Pi^{\perp} \cap X^*$*;*
(ii) *if* $\Pi^{\perp} \cap X^*$ *is irreducible and reduced, then* $(X_{\Pi})^* = \Pi^{\perp} \cap X^*$ *as a scheme.*

Proof. A general tangent hyperplane to X_{Π} pulls back, via σ_{Π}, to a hyperplane containing Π and tangent to X at a general point. This proves the first assertion in (i).

Let Z be an irreducible component of $\Pi^{\perp} \cap X^*$ containing $(X_{\Pi})^*$, and let ξ be a general point in Z. Then ξ corresponds to a hyperplane containing Π and tangent to X at a general point. Hence its projection via σ_{Π} is a general tangent hyperplane to X_{Π}. This proves (i). Part (ii) follows from (i). □

Proposition 1.2. *Let $\Pi = \mathbb{P}(W) \subset \mathbb{P}^r = \mathbb{P}(V)$ stand for the linear span of the variety $X \subset \mathbb{P}^r$ and let $\widetilde{X}$ denote the variety X as re-embedded into Π. Then $X^* \subset \mathbb{P}^{r*} = \mathbb{P}(V^*)$ is the cone over $\widetilde{X}^* \subset \mathbb{P}(W^*)$ with vertex $\Pi^{\perp} = \mathbb{P}((V/W)^*)$. Conversely the dual of a cone is degenerate, lying on the orthogonal of the vertex of the cone.*

The proof follows immediately from the aforementioned interpretation of $\Pi^{\perp}$ as the set of hyperplanes in $\mathbb{P}^r$ containing Π.

Therefore, a subvariety $X \subset \mathbb{P}^r$ is a cone if and only if its dual $X^* \subset \mathbb{P}^{r*}$ is degenerate. Thus, the study of dual varieties may safely be restricted to non-degenerate varieties.

Finally recall that the *Gauss map* of an embedding $X \subset \mathbb{P}^r$ is the map

$$\gamma_X : x \in X \setminus \mathrm{Sing}(X) \dashrightarrow T_{X,x} \in \mathbb{G}(n, r).$$

The *image of the Gauss map* is the closure of $\gamma_X(X \setminus \mathrm{Sing}(X))$; γ_X is said to be *degenerate* if the fiber of γ_X over a general point of its image has positive dimension, i.e., if the Gauss image of X in $\mathbb{G}(n, r)$ has dimension at most $n - 1$. If $X \subset \mathbb{P}^r$ is a smooth variety, then γ_X is well known to be finite and birational onto its image, see [51, Theorem I.2.3]. More generally, the closure of the general fiber of the Gauss image is a projective subspace (see [21, 2.10] or [51]).

1.2. Scrolls and their dual varieties

As mentioned in the Introduction, scrolls will play a substantial role in the construction of the homaloidal hypersurfaces. Thus, we next proceed to define them.

Definition 1.3. An irreducible variety $X \subset \mathbb{P}^r$ of dimension n is said to be a *scroll* if it is swept out by an irreducible 1-dimensional family $\mathcal{F}(X)$ of linear subspaces of $\mathbb{P}^r$ of dimension $n-1$, called *rulings*, in such a way that through a general point of X there passes a unique member of $\mathcal{F}(X)$.

Equivalently, let C be the normalization of the defining 1-dimensional parameter space $\mathcal{F}(X) \subset \mathbb{G}(1, n-1)$ and let $\pi : Y \to C$ denote the pull-back of the universal family on $\mathbb{G}(1, n-1)$ restricted to $\mathcal{F}(X)$. Then $\pi : Y \to C$ is a $\mathbb{P}^{n-1}$-bundle over C and there exists a birational morphism $\phi : Y \to X \subset \mathbb{P}^r$, induced by the tautological morphism on $\mathbb{G}(1, n-1)$, such that the fibers of π are embedded as linear subspaces of $\mathbb{P}^r$.

With this terminology the scroll $X \subset \mathbb{P}^r$ is said to be *rational* if $C \simeq \mathbb{P}^1$ and *elliptic* if C has genus one. More generally we can define the *genus of* X to be the geometric genus of C.

A scroll $X \subset \mathbb{P}^r$ is said to be *a smooth scroll* if $\phi : Y \to X$ is an isomorphism. As in the classical literature, a (smooth) scroll $X \subset \mathbb{P}^r$ is said to be *normal* if $X \subset \mathbb{P}^r$ is a linearly normal projective variety, i.e. if $X \subset \mathbb{P}^r$ is not a isomorphic linear external projection of a variety $\widetilde{X} \subset \mathbb{P}^{r+1}$.

It is well known that $\pi : Y \to C$ can be naturally identified with $\pi : \mathbb{P}(\mathcal{E}) \to C$, where $\mathcal{E}$ is rank n locally free sheaf over C. Moreover, up to twisting by the pull back of a line bundle on C, we can assume that ϕ is given by a base point free linear system contained in $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. This linear system is complete if and only if $X \subset \mathbb{P}^r$ is a normal scroll. Thus we can also assume that $\mathcal{E}$ is generated by global sections and, if $C \simeq \mathbb{P}^1$, that $\mathcal{E} \simeq \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$ for suitable integers $0 \leqslant a_1 \leqslant \cdots \leqslant a_n$. In this case, if $d = a_1 + \cdots + a_n$, then $S(a_1, \ldots, a_n) \subset \mathbb{P}^{d+n}$ will denote the rational scroll obtained as the image of the birational morphism $\phi : \mathbb{P}(\bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)) \to \mathbb{P}^{d+n}$ given by the complete linear system $|\mathcal{O}(1)|$. In this situation, d is the degree of $S(a_1, \ldots, a_n) \subset \mathbb{P}^{d+n}$.

In the above setting, a smooth non-normal scroll $X \subset \mathbb{P}^r$ is an external projection of a normal smooth scroll. From the point of view of the theory of dual varieties these examples are particularly interesting since every smooth scroll $X \subset \mathbb{P}^r$ has $d(X) = n-2$ (see, e.g., [25]). The simplest of these examples is perhaps the Segre embedding $X = \mathrm{Seg}(1, n-1) = S(1, \ldots, 1) \subset \mathbb{P}^{2n-1}$ of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$—here $\dim(X^*) = n$ and $X^* \subset \mathbb{P}^{2n-1*}$ is projectively equivalent to the original X, i.e. these Segre varieties are *self-dual*.

In dimension 2 the picture turns out to be the following. Consider a non-degenerate surface $X \subset \mathbb{P}^r$, $r \geqslant 3$. Here $n = 2$, and $d(X) = 1$ if and only if X is *developable*. This condition is equivalent to γ_X being degenerate which in turn happens to be the case if and only if X is either a cone with vertex a point $p \in \mathbb{P}^r$ or the tangent developable to a curve C, i.e., its *tangential surface*

$$X = \overline{\bigcup_{x \in C \setminus \mathrm{Sing}(C)} T_{C,x}}$$

(see [21, 3.19]). By Proposition 1.2, the first alternative takes place if and only if X^* is degenerate, contained in the hyperplane $p^\perp \subset \mathbb{P}^{r*}$.

We collect further remarks in the form of a proposition for ready reference.

Proposition 1.4. *Let $X \subset \mathbb{P}^r$ be a non-degenerate scroll surface, $r \geqslant 3$. Let d denote the degree of X, which we assume to be at least* 3.

(i) *If X is not developable then X^* is a hypersurface of degree d which is swept out by the $(r-2)$-dimensional subspaces $F^\perp$, where F varies in the algebraic family $\mathcal{F}(X)$ determined by the rulings of X.*
(ii) *Conversely, if $Y \subset \mathbb{P}^{r*}$ is a hypersurface which is swept out by a one-dimensional family $\mathcal{F}(Y)$ of subspaces of dimension $r-2$, then $Y^* \subset \mathbb{P}^r$ is either a* 2*-dimensional scroll or else a curve. Moreover, Y^* is a curve if and only if one of the following equivalent conditions holds:*
   (a) *Y is* developable*, that is to say, the general fiber of the Gauss map γ_Y coincides with the general element of $\mathcal{F}(Y)$;*
   (b) *$\mathcal{F}(Y)$ is the family of the $(r-2)$-dimensional subspaces $(r-1)$-osculating a curve.*

Proof. Part (i) follows from the fact that a hyperplane ξ is tangent to X if and only if it contains a ruling so that a general pencil of hyperplanes cuts X^* exactly in d points. As for (ii), see [21, Section 2]. □

1.2.1. Smooth rational normal scroll surfaces

We now go deeper into the structure of rational scroll surfaces.

Let $X = S(a,b) \subset \mathbb{P}^{a+b+1}$, $0 < a \leqslant b$, be a smooth rational normal scroll surface of degree $d = a + b$, in its standard embedding. Recall that $S(a,b)$ is swept out by all lines joining corresponding points on rational normal curves of degree a and b spanning $\mathbb{P}^{a+b+1}$. This makes sense even if $a = 0$, in which case $S(0,b)$ is the cone over a rational normal curve of degree b. The homogeneous defining ideal is generated by the 2-minors of the piecewise $2 \times (a+b)$ catalecticant matrix

$$\left(\begin{array}{cccc|cccc} x_0 & x_1 & \dots & x_{a-1} & x_{a+1} & x_{a+2} & \dots & x_{a+b} \\ x_1 & x_2 & \dots & x_a & x_{a+2} & x_{a+3} & \dots & x_{a+b+1} \end{array}\right)$$

(see [13]).

We collect the main features of these scrolls.

Proposition 1.5. *Let $S(a,b) \subset \mathbb{P}^{a+b+1}$, $0 < a \leqslant b$, be as above.*

(i) *$S(a,b)$ is a linear section of the Segre embedding* $\mathrm{Seg}(1, a+b-1)$ *of $\mathbb{P}^1 \times \mathbb{P}^{a+b-1}$ into $\mathbb{P}^{2(a+b)-1}$ by a subspace Π of dimension $a+b+1$.*
(ii) *$S(a,b)^*$ is the projection to $\mathbb{P}^{a+b+1^*}$ of* $\mathrm{Seg}(1, a+b-1)^*$*, from $\Pi^\perp$, where Π is a subspace as in* (i)*. In particular, $S(a,b)^*$ is a hypersurface in $\mathbb{P}^{a+b+1^*}$ of degree*

$$\deg\bigl(S(a,b)^*\bigr) = \deg\bigl(\mathrm{Seg}(1, a+b-1)^*\bigr) = \deg\bigl(S(a,b)\bigr) = a + b.$$

(iii) *As an abstract surface $S(a,b)$ is isomorphic to the so-called* Hirzebruch surface $\mathbb{F}_{b-a} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a-b))$*, with $\pi : \mathbb{F}_{b-a} \to \mathbb{P}^1$ the structural morphism. In particular, $S(a,b)$ admits a section E of π with $E^2 = a - b \leqslant 0$, which is unique if $a < b$. Moreover, if H is a hyperplane section class of $S(a,b) \subset \mathbb{P}^{a+b+1}$, then $H \equiv E + bF$, where F is the class of the fibers of π.*

(iv) *Let Π be a hyperplane in $\mathbb{P}^r$ tangent to $S(a,b)$ at finitely many points $p_1,\dots,p_m \in S(a,b)$, $m \geqslant 1$. Let $H = H_\Pi$ be the corresponding hyperplane section divisor. Then $H = F_{p_1} + \cdots + F_{p_m} + C$, where F_{p_i} is the ruling of $S(a,b)$ through the point p_i and $C \equiv E + (b-m)F$ is the divisor of a curve in $S(a,b)$ of degree $a+b-m$ passing through $p_1,\dots,p_m$.*

(v) *If $m \leqslant a$ and $3m \leqslant a+b+1$, then the general hyperplane section of $S(a,b)$ tangent at m general points has exactly m ordinary quadratic singularities there and it is smooth elsewhere.*

(vi) *If either $m \leqslant a$ or $m = b$ then the general curve $C \in |E + (b-m)F|$ is smooth and irreducible and, together with m distinct fibres $F_1,\dots,F_m$ of π, gives rise to a hyperplane section tangent at the intersection points p_i of F_i with C, $i = 1,\dots,m$, and nowhere else.*

Proof. (i) This is clear from the above algebraic description of $S(a,b)$ and the corresponding defining equations of the Segre embedding as given by the 2-minors of a generic $2 \times (a+b)$ matrix over k.

(ii) As we pointed out already, these Segre varieties are self-dual, i.e. $\mathrm{Seg}(1, a+b-1)^*$ is projectively equivalent to $\mathrm{Seg}(1, a+b-1)$. Since $\Pi \cap \mathrm{Seg}(1, a+b-1) = S(a,b)$ is reduced and irreducible, then, by Proposition 1.1, $S(a,b)^*$ coincides with the projection of $\mathrm{Seg}(1, a+b-1)^*$ from $\pi^\perp$. Note that $\Pi^\perp \cap \mathrm{Seg}(1, a+b-1)^* = \emptyset$ since $\Pi \cap \mathrm{Seg}(1, a+b-1)$ is smooth. Then the degree of $S(a,b)^*$ is the same as the degree of $\mathrm{Seg}(1, a+b-1)^*$, which is the same as the degree of $S(a,b)$, namely $a+b$.

(iii) The first part is well known (see, e.g., [13]) and the rest follows from this.

(iv) Describing $S(a,b)^* \subset \mathbb{P}^{a+b+1}$ is the same as describing the singular hyperplane sections of $S(a,b)$, i.e. those given by hyperplanes Π containing tangent planes of $S(a,b)$. If $\Pi \supseteq T_{S(a,b),x}$, then $\Pi \supseteq F_x$, the line of the ruling through x. Thus, if Π is a hyperplane tangent to $S(a,b)$ at finitely many points $p_1,\dots,p_m$, $m \geqslant 1$, and $H = H_\Pi$ denotes the corresponding hyperplane section divisor, it is clear that $H = F_{p_1} + \cdots + F_{p_m} + C$, where $C \equiv E + (b-m)F$ is the divisor of a curve of degree $a+b-m$ in $S(a,b)$. Moreover H_Π has to be singular at $p_1,\dots,p_m$, hence C contains $p_1,\dots,p_m$.

(v) If $a \geqslant m$, then $(E + (b-m)F)^2 = b + a - 2m \geqslant 0$. Let C be the general curve in $|E + (b-m)F|$. The exact sequence

$$0 \to \mathcal{O}_{S(a,b)} \to \mathcal{O}_{S(a,b)}(C) \to \mathcal{O}_C(C) \to 0$$

implies that the linear system $|E + (b-m)F|$ is base point free of dimension $b + a - 2m + 1$ and its general curve C is smooth and rational. If $b + a \geqslant 3m - 1$, the general curve in $|E + (b-m)F|$ contains m general points of $S(a,b)$. This proves the assertion.

(vi) If either $m \leqslant a$, or if $m = b$, the general such curve C is smooth and irreducible (see the above argument). The assertion follows. □

Next we highlight the nature of the singularities of the dual $S(a,b)^*$. Let $E \subset S(a,b)$ be as in Proposition 1.5(iii).

Proposition 1.6. *Let $S(a,b) \subset \mathbb{P}^{a+b+1}$, $0 < a \leqslant b$ be as above.*

(i) *The points of $S(a,b)^*$ corresponding to hyperplanes tangent to $S(a,b)$ at m distinct points are points of multiplicity at least m of $S(a,b)^*$.*

(ii) *The singularities* $\mathrm{Sing}(S(a,b)^*)$ *have a natural stratification into locally closed sets* $S^*_\alpha(a,b)$*, with* $2 \leqslant \alpha \leqslant a$ *and* $\alpha = b$*, consisting of points of multiplicity at most* α*; as for* $\alpha = b$*, one has* $S^*_b(a,b) = \langle E\rangle^\perp \subset S(a,b)^*$*, a linear space of dimension* b *contained in* $\mathrm{Sing}(S(a,b)^*)$.

(iii) $(a = 1)$ *The stratum* $\mathrm{Sing}(S(1,d-1)^*) = S^*_{d-1}(1,d-1)$ *is the subspace* $\langle E\rangle^\perp$ *of dimension* $d-1$*, whose general points are points of multiplicity* $d-1$ *of the hypersurface* $S(1,d-1)^* \subset \mathbb{P}^{d+1*}$.

Proof. (i) Quite generally, the points of $S(a,b)^*$ corresponding to hyperplanes tangent to $S(a,b)$ at m distinct points, with $m \leqslant a$ or $m = b$, are points of multiplicity at least m. One sees that $S^*_b(a,b) = \langle E\rangle^\perp \subset S(a,b)^*$ is a linear space of dimension b contained in $\mathrm{Sing}(S(a,b)^*)$. We now suppose that $2 \leqslant m \leqslant a$.

We know that $S(a,b)^*$ is a hypersurface. If a point of $S(a,b)^*$ corresponds to a hyperplane $H = F_{p_1} + \cdots + F_{p_m} + C$ tangent to $S(a,b)$ at the m points $p_1, \ldots, p_m$, one sees that there are at least m distinct branches of $S(a,b)^*$ passing through H, namely the ones corresponding to hyperplane sections of the form $F_{p_i} + C_i$, C_i irreducible and smooth, proving the assertion.

Assertions (ii) and (iii) follow from (i). □

Notice that the scheme structure on $\mathrm{Sing}(S(a,b)^*)$ defined by the partial derivatives of the defining equation of $S(a,b)^*$ has embedded points (see [1] for some interesting considerations on this scheme structure on $\mathrm{Sing}(S(a,b)^*)$).

It is classically known that a non-developable scroll surface is self-dual. We prove this result anew in the case where the scroll is rational, which is our main focus. The proof contains elements for later use.

Proposition 1.7. *Let* $X \subset \mathbb{P}^r$ *be a rational scroll which is not developable. Then* X *is self-dual, i.e. there is a projective transformation sending* X *to* X^*.

Proof. By definition $X \subset \mathbb{P}^r$ is the birational projection to $\mathbb{P}^r$ of a smooth rational normal scroll surface $S(a,b) \subset \mathbb{P}^{a+b+1}$, with $0 < a \leqslant b$, from a subspace Ψ of dimension $a+b-r$ such that $S(a,b) \cap \Psi = \emptyset$.

By part (i) of Proposition 1.1, we have

$$X^* \subseteq \Psi^\perp \cap S(a,b)^*. \tag{1.1}$$

The right-hand side is a hypersurface of degree $a+b$ in $\mathbb{P}^r$. Moreover X^* is also a hypersurface, since X is not developable, and its degree is $a+b$ (see part (i) of Proposition 1.4). Then equality holds in (1.1), i.e.

$$X^* = \Psi^\perp \cap S(a,b)^*. \tag{1.2}$$

By (i) of Proposition 1.5, $S(a,b) = \Pi \cap \mathrm{Seg}(1, a+b-1)$ with Π a subspace of dimension $a+b+1$ of $\mathbb{P}^{2(a+b)-1}$. Thus

$$\begin{aligned} X^* &= \Psi^\perp \cap \big(\mathrm{Seg}(1,a+b-1) \cap \Pi\big)^* \\ &= \Psi^\perp \cap \sigma_{\Pi^\perp}\big(\mathrm{Seg}(1,a+b-1)^*\big). \end{aligned}$$

Therefore, up to a projective transformation

$$\begin{aligned} X^* &= \Psi^\perp \cap \sigma_{\Pi^\perp}\big(\mathrm{Seg}(1, a+b-1)\big) \\ &= \sigma_{\Pi^\perp}\big(\langle \Pi^\perp, \Psi^\perp\rangle \cap \big(\mathrm{Seg}(1, a+b-1)\big)\big) = X. \qquad \square \end{aligned}$$

1.2.2. Multiple line directrix on scrolls

We now consider another interesting class of scroll surfaces. Any non-developable rational scroll surface $X \subset \mathbb{P}^r$ of degree d is a birational external projection of a scroll $S(a,b)$ with $d = a+b$ and X^* is a section of $S(a,b)^*$ by Proposition 1.7. If there is a line $L \subset X$ such that X is smooth at the general point of L and L meets the general ruling of X at one single point, then X is the projection of $S(1, d-1)$. In such a case X^* is a hypersurface of degree d for which the $(r-2)$-dimensional subspace $L^\perp$ has multiplicity $d-1$.

Such a line L is called a *simple line directrix*. More generally, a line $L \subset X$ is a *line directrix of multiplicity* $e := e(X)$ if the general point $x \in L$ has multiplicity e for X and there is some line in $\mathcal{F}(X)$, different from L, passing through x. Note that L may, or may not, belong to $\mathcal{F}(X)$. It is clear that a scroll with a line directrix is not developable, unless it is a plane. Therefore in what follows we will implicitly assume that a scroll with a line directrix is not developable.

Proposition 1.8. *Let L be a line directrix of multiplicity e on a rational scroll surface $X \subset \mathbb{P}^r$ of degree d. Let $\mu := \mu(X)$ denote the number of rulings in $\mathcal{F}(X)$ not coinciding with L and passing through a general point x of L and let $F_{x,i}$, $i = 1, \dots, \mu$, be these rulings. Let $\nu := \nu(X)$ be the dimension of the span $\langle L, F_{x,1}, \dots, F_{x,\mu}\rangle$.*

One has:

- (i) *$\mu \leqslant e$;*
- (ii) *$\mu < e$ if and only if L is a ruling in $\mathcal{F}(X)$;*
- (iii) *The dual $X^* \subset \mathbb{P}^{r*}$ is a hypersurface of degree d and contains the $(r-2)$-dimensional subspace $\Pi = L^\perp$. Moreover X^* has multiplicity $d - \mu$ at the general point of Π and the general hyperplane through Π cuts out on X^* off Π, the union of μ codimension 2 subspaces whose intersection with Π is a subspace of dimension $r - \nu - 1$.*

Proof. (i) Let $f : \bar{X} \to X$ be the normalization morphism. The surface $\bar{X}$ is ruled and its rulings are mapped to the lines in $\mathcal{F}(X)$. Let $p_1, \dots, p_h$ be the points on $\bar{X}$ mapping to x. It is clear that $e \geqslant h$. Note that $\bar{X}$, which is normal and therefore smooth in codimension one, is smooth at $p_1, \dots, p_h$. Hence there is a unique ruling of $\bar{X}$ through each of the points $p_1, \dots, p_h$. Moreover the μ rulings in $\mathcal{F}(X)$, different from L, and passing through x are the image, via f, of rulings on $\bar{X}$ passing through one of the points $p_1, \dots, p_h$. Thus $\mu \leqslant h \leqslant e$.

(ii) Let us prove that, if $\mu < e$, then L is a ruling of $\mathcal{F}(X)$. The converse is similar and can be left to the reader.

Suppose first that $\mu \leqslant h < e$. This is equivalent to say that f is ramified at some of the points $p_1, \dots, p_h$, which we denote by y. Let F be the ruling of $\bar{X}$ passing through y. Since F maps to a line via f, the only possibility is that F maps to L, hence L is a ruling of $\mathcal{F}(X)$ in this case.

Suppose that $h = e$, i.e. f is unramified at a general point $x \in L$. One has therefore e distinct points $p_1, \dots, p_e$ on $\bar{X}$ mapping to x. Let F_i be the ruling through p_i, $i = 1, \dots, e$. If L is not a ruling in $\mathcal{F}(X)$, then the images on X of $F_1, \dots, F_e$ are all distinct from L. Moreover they are also e distinct lines, since f is a finite birational map. Hence $\mu = e$, proving the assertion.

(iii) The general hyperplane $\Xi = x^\perp$ containing $\Pi = L^\perp$ corresponding to the general point $x \in L$ cuts out X^*, off Π, along the union of the μ codimension 2 subspaces $F_{x,i}^\perp$, $i = 1, \dots, \mu$. The intersection

$$\Pi^\perp \cap F_{x,1}^\perp \cap \cdots \cap F_{x,\mu}^\perp = \langle L, F_{x,1}, \dots, F_{x,\mu}\rangle^\perp,$$

has dimension $r - \nu - 1$. □

We will see later how to construct scrolls with $\mu < e$ (see Lemma 3.11 and ff.). As for the case $\mu = e$, the following construct works: consider $S(a,b) \subset \mathbb{P}^{d+1}$, $d = a + b$, $a \geqslant 2$, and project it down to $\mathbb{P}^{b+2}$ from a general linear space of dimension $a - 2$ which sits in $\langle E\rangle$. In this way the image $X(a,b)$ of the projection has still degree d and the image of E is the line Λ to which $\langle E\rangle$ maps. Notice that by projecting $X(a,b)$ from Λ to $\mathbb{P}^b$ one gets a rational normal curve C of degree b. Thus $X(a,b)$ sits on the 3-dimensional cone of degree b projecting C from Λ.

Since E has degree a, one has that Λ is a line directrix of multiplicity a and clearly $\mu = a$. In this case $\nu = a + 1$ (see the argument in the proof of parts (i) and (ii) of Proposition 1.8 above). Notice that $X(a,b)$ is contained in a cone $Z(a,b)$ of dimension $a + 2$ which is swept out by the subspace $\langle \Lambda, F_{x,1}, \dots, F_{x,a}\rangle$ of dimension a as x varies on Λ. The cone $Z(a,b)$ is a rational normal scroll of degree $b - a + 1$ (see [13]).

One can also obtain the previous example in terms of the dual variety of certain projections of more general scrolls, as follows.

Let $1 \leqslant a = a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots \leqslant a_{r-1}$ be integers and set $d = \sum_{i=1}^{r-1} a_i$. Consider the rational normal scroll $X_1 = S(a, a_2, \dots, a_{r-1}) \subset \mathbb{P}^{d+r-2}$ of degree d and dimension $r - 1$, with

$$S(a, a_2, \dots, a_{r-1}) \simeq \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{r-1})\big)$$

embedded via the $\mathcal{O}(1)$ bundle. Algebraically, the homogeneous defining ideal of this embedding is generated by the 2-minors of a multi-piecewise catalecticant matrix as in (1.2.1). Consider also the rational normal scroll $X_2 = S(a_2, \dots, a_{r-1}) \subset \mathbb{P}^{d+r-a-3}$ of degree $d - a$ and dimension $r - 2$. By a suitable identification, one can consider X_2 as a subvariety of X_1. Let Ω be a sufficiently general linear space of dimension $d - 3$ which cuts the linear space $\langle X_2\rangle$ along a subspace of dimension $d - a - 2$, and set $Y = \sigma_\Omega(X_1) \subset \mathbb{P}^r$, where σ_Ω as before stands for the projection from Ω.

Proposition 1.9. *Let the notation be as above, with $Y = \sigma_\Omega(X_1) \subset \mathbb{P}^r$. The dual $Y^* \subset \mathbb{P}^{r*}$ is a scroll surface of degree d, with $e(Y^*) = \mu(Y^*) = a$ and with line directrix $\sigma_\Omega(\langle X_2\rangle)$ of multiplicity a.*

Proof. Clearly $\deg(Y) = \deg(X_1) = d$ and $\mathrm{Sing}(Y)$ contains the linear space $\Pi = \sigma_\Omega(\langle X_2\rangle)$ of dimension $r - 2$. The general point of Π has multiplicity $d - a$ for V because $\deg(X_2) = d - a$.

Since Y is swept out by a 1-dimensional family of projective spaces of dimension $r - 2$, then $X = Y^*$ is a scroll surface of degree d with line directrix $L = \Pi^\perp$. Since Y has multiplicity $d - a$ along Π, we see $\mu(X) = a$. Actually the multiplicity of the line directrix L on Y^* is also a because clearly L is not a ruling in $\mathcal{F}(X)$. □

2. The polar map of a projective hypersurface

Let $f = f(\mathbf{x}) = f(x_0, \dots, x_r) \in k[x_0, \dots, x_r]$ be a nonzero homogeneous polynomial of degree d in the $r + 1$ variables $x_0, \dots, x_r$ over an algebraically closed field k of characteristic zero.

Then $V(f) \subset \mathbb{P}^r$ will denote the hypersurface scheme theoretically defined by the equation $f(x_0, \dots, x_r) = 0$, so $V(f)$ might not be reduced. Its *support* $\mathrm{Supp}(V(f))$ is the set of points of $\mathbb{P}^r$ where f vanishes.

We will often denote by f_i the partial derivative $\frac{\partial f}{\partial x_i}$, $i = 0, \dots, r$.

Let $\mathbf{p} = (p_0, \dots, p_r) \in k^{r+1} \setminus \{0\}$, and let $p = (p_0, \dots, p_r)$ denote the corresponding point in $\mathbb{P}^r$. For every positive integer $s < d$ consider the polynomial

$$\Delta^s_{\mathbf{p}} f(\mathbf{x}) = \left(p_0 \frac{\partial}{\partial x_0} + \cdots + p_r \frac{\partial}{\partial x_r} \right)^{(s)} f(\mathbf{x})$$

where the exponent s in brackets means, as usual, a symbolic power involving products and derivatives. The polynomial $\Delta^s_{\mathbf{p}} f$ has degree $d - s$ and, for any $t \in k^*$, one has:

$$\Delta^s_{t\mathbf{p}} f(\mathbf{x}) = t^s \Delta^s_{\mathbf{p}} f(\mathbf{x}).$$

If $\Delta^s_{\mathbf{p}} f$ is not identically zero, then it makes sense to consider the hypersurface $V(\Delta^s_{\mathbf{p}} f)$ which depends only on p and on $V(f)$ and is called the sth *polar* of $V(f)$ with respect to p. We will denote it by $V^s_p(f)$. If $\Delta^s_{\mathbf{p}} f$ is identically zero, one says that the sth *polar* $V^s_p(f)$ of $V(f)$ with respect to p *vanishes identically*. In this case we consider $V^s_p(f)$ to be the whole $\mathbb{P}^r$.

For general properties of polarity, which we will freely use later on, we refer to [43]. Among these we mention here the so called *reciprocity theorem*:

Proposition 2.1. *Given the hypersurface $V(f)$ in $\mathbb{P}^r$ and two points $p = (p_0, \dots, p_r)$, $q = (q_0, \dots, q_r)$, one has:*

$$\frac{1}{s!} \Delta^s_{\mathbf{p}} f(\mathbf{q}) = \frac{1}{(d-s)!} \Delta^{d-s}_{\mathbf{q}} f(\mathbf{p}).$$

Thus $q \in V^s_p(f)$ if and only if $p \in V^{d-s}_q(f)$.

As p varies in $\mathbb{P}^r$, the polars $V^s_p(f)$ do not vary in a linear system, unless $s = 1$. The base locus scheme of the linear system $\mathcal{P}(f)$ of the first polars of $V(f)$ is the *singular locus* $\mathrm{Sing}(V(f))$ of $V(f)$, defined by the *Jacobian* (or *gradient*) ideal generated by the partial derivatives $f_0(\mathbf{x}), \dots, f_r(\mathbf{x})$.

A consequence of the reciprocity theorem is that the polar hyperplane $\pi_p(f) := V^{d-1}_p(f)$ has equation:

$$f_0(\mathbf{p})x_0 + \cdots + f_r(\mathbf{p})x_r = 0$$

which vanishes identically if and only if $p \in \mathrm{Sing}(V(f))$. If $p \in V(f)$ and it is not singular, then $\pi_p(f)$ is the *tangent hyperplane* $T_{V(f),p}$ to $V(f)$ at p.

The (*first*) *polar map* of f or of $V(f)$ is the rational map

$$\phi_f : x = (\mathbf{x}) \in \mathbb{P}^r \dashrightarrow \big(f_0(\mathbf{x}), \dots, f_r(\mathbf{x})\big) \in \mathbb{P}^r.$$

It can be interpreted as mapping the point p to its polar hyperplane $\pi_p(f)$ and, as such, its target is $\mathbb{P}^{r*}$.

In terms of linear systems ϕ_f is the map defined by the system $\mathcal{P}(f)$ of the first polars. Thus, if $V(f)$ is reduced, as we will now assume, the indeterminacy locus of ϕ_f is $\operatorname{Sing}(V(f))$. The restriction of ϕ_f to $V(f) \setminus \operatorname{Sing}(V(f))$ is the Gauss map of $V(f)$, hence the corresponding image is the dual variety $V(f)^*$ of $V(f)$. We will set $v(f) = \dim(V(f)^*)$.

Denote by $Z(f)$ the closure of the image of $\mathbb{P}^r$ via ϕ_f—called the *polar image* of f—and set $z(f) = \dim(Z(f))$. Clearly $v(f) \leqslant z(f)$, but we shall see in a moment that strict inequality holds (see Remark 2.4).

We denote by $\delta(f)$ the degree of the map ϕ_f, which is meant to be 0 if and only if $z(f) < r$, otherwise it is a positive integer. We will call $\delta(f)$ the *polar degree* of $V(f)$. Let f_{red} be the *radical of* f, i.e. $(f_{\mathrm{red}}) = \sqrt{(f)}$.

We record the following result from [9, Corollary 2] which proves a conjecture stated in [10]:

Theorem 2.2. *Let notation be as above. Then $\delta(f) = \delta(f_{\mathrm{red}})$, i.e. the polar degree of $V(f)$ depends only on* $\operatorname{Supp}(V(f))$.

This result enables us to restrict our attention to reduced hypersurfaces if we are interested in studying the polar degree. The argument in [9] depends on topological considerations. For different proof, see [15], whereas an algebraic proof of the case where the irreducible factors of f are of degree one has been established in [3].

2.1. The Hessian of a projective hypersurface

Consider now the $(r+1) \times (r+1)$ *Hessian matrix* of $f(\mathbf{x})$

$$h(f)(\mathbf{x}) := \det\left(\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}\right)_{i,j=0,\dots,r}.$$

Its determinant $h(f) \in k[x_0, \dots, x_r]$ is the *Hessian polynomial* of $f(\mathbf{x})$.

Sometimes we will abuse notation and denote by $h(f)$ also the Hessian matrix rather than its determinant, hoping no ambiguity will be caused.

We note that the Hessian is covariant by a linear change of variables. If $h(f)$ is a nonzero polynomial, the *Hessian* of the hypersurface $V(f) \subset \mathbb{P}^r$ is the hypersurface $H(f) := V(h(f))$. Otherwise we say that $V(f)$ has *vanishing*, or *indeterminate* Hessian, in which case we consider $H(f)$ to be the whole of $\mathbb{P}^r$.

A couple of basic remarks is in order.

Remark 2.3. A point $p \in \mathbb{P}^r$ belongs to $H(f)$ if and only if the polar quadric $Q_p(f) := V_p^{d-2}(f)$ is either singular or vanishes identically.

Thus, in particular $p \in V(f) \cap H(f)$ if and only if either $p \in \operatorname{Sing}(V(f))$ or p is a *parabolic point* of $V(f)$ in the sense that the tangent cone $A_p(f)$ at p of the intersection of $V(f)$ with the tangent hyperplane $\pi_p(f)$ (necessarily singular at p) has a vertex of positive dimension (see [43,

p. 71]). This cone is called the *asymptotic cone* of $V(f)$ at p. More precisely, a point $p \in V(f)$ is said to be *h-parabolic*, $h \geqslant 0$, if the vertex of the asymptotic cone $A_p(f)$ has dimension h. In that case p is a point of multiplicity h for $H(f)$ (see [39]). Note that 0-parabolic means *non-parabolic*. If f is irreducible and the general point of $V(f)$ is h-parabolic, then f^h divides $h(f)$; in particular, if $h > 0$, then $V(f)$ is contained in $H(f)$.

Conversely, if f is irreducible and $V(f)$ is contained in $H(f)$ then the Gauss map of $V(f)$ is degenerate, i.e. $v(f) < r - 1$ and the general point $p \in V(f)$ is h-parabolic with $h = r - v(f) - 1 > 0$ (see [45, 4–5], [39,6]). In this case, since the general fibre of the Gauss map is a linear space, then $V(f)$ is described by an $(r - h - 1)$-dimensional family of h-dimensional linear subspaces of $\mathbb{P}^r$, parameterized by $V(f)^*$. Moreover $H(f)$ contains $V(f)$ with multiplicity at least $h = r - v(f) - 1$.

The question as to when $H(f)$ contains $V(f)$ with higher multiplicity than the expected value $r - v(f) - 1$ has been considered in [39,42,17,6].

Remark 2.4. A point $p \in \mathbb{P}^r$ belongs to $H(f)$ if and only if the rank of the map ϕ_f at p is not maximal, i.e. if and only if $\mathrm{rk}_p(\phi_f) < r$. Hence $z(f) < r$ if and only if $V(f)$ has vanishing Hessian. Set $\rho(f) := \mathrm{rk}(h(f))$, where the rank of $h(f)$ is computed as a matrix over the field $k(x_0, \dots, x_r)$, or, what is the same, at a general point of $\mathbb{P}^r$. Then one has:

$$z(f) = \rho(f) - 1.$$

Indeed, if $p = (p_0, \dots, p_r)$ is a point in $\mathbb{P}^r$ not on $\mathrm{Sing}(V(f))$, and if $\xi = \phi_f(p)$, then $T_{Z(f),\xi}$ is spanned by ξ and by the points $(f_{i0}(\mathbf{p}), \dots, f_{ir}(\mathbf{p}))$, $i = 0, \dots, r$. Notice that $T^{\perp}_{Z(f),\xi}$ is the vertex of $Q_p(f)$. A vastly more general principle holds in this connection (see [46, Proposition 1.1] for a detailed argument).

Notice that, if $V(f)$ is irreducible and its general point p is h-parabolic, then $v(f) + 2 = r - h + 1 = \mathrm{rk}(Q_p(f)) \leqslant \rho(f) = z(f) + 1$, i.e. $v(f) < z(f)$, namely the dual $V(f)^*$ of $V(f)$ is properly contained in the polar image $Z(f)$.

Note that, by Theorem 2.2, the property of having vanishing Hessian only depends on the support of a hypersurface. Thus, if one is interested in hypersurfaces with vanishing Hessian, one can restrict the attention to the reduced ones.

2.2. *Hypersurfaces with vanishing Hessian*

The hypersurface $V(f)$ has vanishing Hessian if and only the derivatives $f_0, \dots, f_r$ are *algebraically dependent*, i.e. if and only if there is some nonzero polynomial $g(x_0, \dots, x_r) \in k[x_0, \dots, x_r]$ such that $g(f_0, \dots, f_r) = 0$.

Note that $V(f)$ is smooth if and only if $f_0, \dots, f_r$ form a *regular sequence*; in particular, if $V(f)$ is smooth then $h(f) \neq 0$. Thus, having vanishing Hessian implies at least that $\mathrm{Sing}(V(f)) \neq \emptyset$ and one then asks how big is this locus.

The following result due to Zak (see [52, Proposition 4.9]) partially answers this question. Part of it can be traced back to Gordan–Noether (see [20]).

Proposition 2.5. *Let $X = V(f) \subset \mathbb{P}^r$ be a reduced hypersurface with vanishing Hessian and let $Z(f) \subset \mathbb{P}^{r*}$ denote the polar image of f. Then*

(i) *the closure of the fiber of the map* ϕ_f *over a general point* $\xi \in Z(f)$ *is the union of finitely many linear subspaces of dimension* $r - z(f) = r - \rho(f) + 1$*, passing through the subspace* $(T_{Z(f),\xi})^{\perp}$*;*
(ii) $Z(f)^*$ *is contained in* $\mathrm{Sing}(V(f))$.

The careful reader will notice that the argument in [52, Proposition 4.9] actually proves the above statement (i) rather than the corresponding part (ii) of the statement there.

A clear-cut case of vanishing Hessian is when, $f_0, \ldots, f_r$ are linearly dependent, i.e. up to a linear change of variables, f does not depend on all the variables, i.e., when $V(f)$ is a cone (Proposition 1.2). One could naively ask for the converse:

Question 2.6 *(Hesse problem).* Does $h(f) = 0$ imply that the derivatives $f_0, \ldots, f_r$ are linearly dependent?

Hesse claimed this twice (see [22,23]), however the proofs had a gap. The question was taken up by Gordan and Noether in [20], who showed that the question has an affirmative answer for $r \leqslant 3$, but is false in general for $r \geqslant 4$. Their methods have been revisited in more recent times by Permutti in [32,34] and [27].

Using Proposition 2.5 we can give an easy proof of this fact for $r \leqslant 2$. The case $r = 3$ is slightly more complicated and will not be dealt with it here—we refer to [20,17] or [27]. A simple proof is also contained in [19].

Proposition 2.7. *Let* $V(f) \subset \mathbb{P}^r$*,* $1 \leqslant r \leqslant 2$*, be a reduced hypersurface of degree* d*. Then* $V(f)$ *has vanishing Hessian if and only if* $V(f)$ *is a cone. More precisely,* $V(f)$ *has vanishing Hessian if and only if either* $r = 1$ *and* $d = 1$*, or else* $r = 2$ *and* $V(f)$ *consists of* d *distinct lines through a point.*

Proof. If $r = 1$, then $Z(f) \subset \mathbb{P}^1$ must be a point, so the partial derivatives of f are constant and $d = 1$.

Suppose $r = 2$. Then $z(f) \leqslant 1$. As above, $Z(f)$ is a point if and only if $d = 1$. Let $z(f) = 1$. From part (ii) of Proposition 2.5, we have that $Z(f)^* \subset \mathrm{Sing}(V(f))$. Since we are assuming $V(f)$ to be reduced, we have that $Z(f)^*$ is a point, so that $Z(f)$ is a line, hence degenerate. This is equivalent to saying that $V(f)$ is a cone. □

Remark 2.8. It is interesting to note that the only hyperplane arrangements with vanishing Hessian are cones (see [9, Cor. 2 and Cor. 4]).

2.3. Gordan–Noether counterexamples to Hesse's problem

We will now briefly recall the results of Gordan–Noether and Permutti in connection with the Hesse problem, which showed that Hesse's argument was faulty for dimension $r = 4$ and higher.

Thus, assume that $r \geqslant 4$ and fix integers $t \geqslant m + 1$ such that $2 \leqslant t \leqslant r - 2$ and $1 \leqslant m \leqslant r - t - 1$. Consider forms $h_i(y_0, \ldots, y_m) \in k[y_0, \ldots, y_m]$, $i = 0, \ldots, t$, of the same degree, and also forms $\psi_j(x_{t+1},\dots,x_r) \in k[x_{t+1},\dots,x_r]$, $j=0,\dots,m$, of the same degree. Introduce the following homogeneous polynomials all of the same degree:

$$Q_\ell(x_0,\dots,x_r) = \det\begin{pmatrix} x_0 & \dots & x_t \\ \frac{\partial h_0}{\partial \psi_0} & \dots & \frac{\partial h_t}{\partial \psi_0} \\ \dots & \dots & \dots \\ \frac{\partial h_0}{\partial \psi_m} & \dots & \frac{\partial h_t}{\partial \psi_m} \\ a^{(\ell)}_{1,0} & \dots & a^{(\ell)}_{1,t} \\ \dots & \dots & \dots \\ a^{(\ell)}_{t-m-1,0} & \dots & a^{(\ell)}_{t-m-1,t} \end{pmatrix}$$

where $\ell = 1,\dots,t-m$. Here $a^{(\ell)}_{u,v}$ $(u=1,\dots,t-m-1,\ v=0,\dots,t)$ are elements of the base field k, while $\partial h_i/\partial\psi_j$ stands for the derivative $\partial h_i/\partial y_j$ computed at $y_j = \psi_j(x_{t+1},\dots,x_r)$, for $i=0,\dots,t$ and $j=0,\dots,m$. Let n denote the common degree of the polynomials Q_ℓ. Taking Laplace expansion along the first row, one has an expression of the form:

$$Q_\ell = M_{\ell,0}x_0 + \dots + M_{\ell,t}x_t$$

where $M_{\ell,i}$, $\ell = 1,\dots,t-m$, $i=0,\dots,t$, are homogeneous polynomials of degree $n-1$ in $x_{t+1},\dots,x_r$.

Fix an integer $d > n$ and set $\mu = [d/n]$. Fix biforms $P_k(z_1,\dots,z_{t-m}; x_{t+1},\dots,x_r)$ of bidegree $(k, d-kn)$, $k=0,\dots,\mu$. Finally, set

$$f(x_0,\dots,x_r) := \sum_{k=0}^{\mu} P_k(Q_1,\dots,Q_{t-m},x_{t+1},\dots,x_r), \tag{2.1}$$

a form of degree d in $x_0,\dots,x_r$. It will be called a *Gordan–Noether polynomial* (or a GN-*polynomial*) of type (r,t,m,n), and so will also any polynomial which can be obtained from it by a projective change of coordinates. Accordingly, a *Gordan–Noether hypersurface* (or GN-*hypersurface*) of type (r,t,m,n) is the hypersurface $V(f)$ where f is a nonzero GN-polynomial of type (r,t,m,n).

The main point of the Gordan–Noether construction is the following result:

Proposition 2.9. *Every* GN-*polynomial has vanishing Hessian.*

Proof. Let $f(\mathbf{x})$ be a GN-polynomial. Its first $t+1$ partial derivatives

$$f_i = \sum_{\ell=1}^{t-m} \frac{\partial f}{\partial Q_\ell} M_{\ell,i}, \quad i=0,\dots,t,$$

can be expressed in the form of a column vector

$$\begin{pmatrix} f_0 \\ \dots \\ f_t \end{pmatrix} = \begin{pmatrix} M_{1,0} & \dots & M_{t-m,0} \\ \dots & \dots & \dots \\ M_{1,t} & \dots & M_{t-m,t} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f}{\partial Q_1} \\ \dots \\ \frac{\partial f}{\partial Q_{t-m}} \end{pmatrix}. \tag{2.2}$$

Consider the rational map

$$\phi_\ell : (x_{t+1} : \ldots : x_r) \in \mathbb{P}^{r-t-1} \dashrightarrow \big(M_{\ell,0}(x_{t+1}, \ldots, x_r), \ldots, M_{\ell,t}(x_{t+1}, \ldots, x_r)\big) \in \mathbb{P}^t$$

with $\ell = 1, \ldots, t-m$. Its image has dimension at most m since the polynomials h_i, $i = 0, \ldots, t$, appearing in the determinants which define the polynomials $M_{\ell,i}$, $\ell = 1, \ldots, t-m$, $i = 0, \ldots, t$, depend on $m+1$ variables.

The rational map

$$\phi : (x_0 : \ldots : x_r) \in \mathbb{P}^r \dashrightarrow \big(f_0(x_0, \ldots, x_r) : \ldots : f_t(x_0, \ldots, x_r)\big) \in \mathbb{P}^t,$$

is the composite of the polar map ϕ_f with the projection $(x_0 : \ldots : x_r) \dashrightarrow (x_0 : \ldots : x_t)$. Therefore, if we let

$$\sigma : (x_0 : \ldots : x_r) \in \mathbb{P}^r \dashrightarrow (x_{t+1} : \ldots : x_r) \in \mathbb{P}^{r-t-1}$$

denote the complementary projection then, for a general point $p \in \mathbb{P}^r$, Eq. (2.2) shows that $\phi(p)$ sits in the span of $\phi_1(\sigma(p)), \ldots, \phi_{t-m}(\sigma(p))$. Thus we see that the image of ϕ has dimension at most $m + t - m - 1 = t - 1$. This proves that $f_0, \ldots, f_t$ are algebraically dependent, hence so are $f_0, \ldots, f_r$. $\square$

For a proof of the previous proposition which is closer to Gordan–Noether's original approach, see [27].

Following [34] we give a geometric description of a GN-hypersurface of type (r, t, m, n), as follows. For this we introduce the following notion.

Definition 2.10. Let f be GN-hypersurface of type (r, t, m, n). The *core* of $V(f)$ is the t-dimensional subspace $\Pi \subset V(f)$ defined by the equations $x_{t+1} = \cdots = x_r = 0$.

We agree to call a GN-hypersurface of type (r, t, m, n) *general* if the defining data have been chosen generically, namely, the polynomials $h_i(y_0, \ldots, y_m)$, $i = 0, \ldots, t$, the polynomials $\psi_j(x_{t+1}, \ldots, x_r)$, $j = 0, \ldots, m$, the constants $a^{(\ell)}_{u,v}$, $\ell = 1, \ldots, t-m$, $u = 1, \ldots, t-m-1$, $v = 0, \ldots, t$, and the biforms P_k, $k = 0, \ldots, \mu$, are sufficiently general.

Proposition 2.11. *Let $V(f) \subset \mathbb{P}^r$ be a* GN-*hypersurface of type (r, t, m, n) and degree d. Set $\mu = [\frac{d}{n}]$. Then*

(i) *$V(f)$ has multiplicity at least $d - \mu$ at the general point of its core Π.*
(ii) *The general $(t+1)$-dimensional subspace Π' through Π cuts out on $V(f)$, off Π, a cone of degree at most μ whose vertex is an m-dimensional subspace Γ subset Π.*
(iii) *If $V(f)$ is general, then it has multiplicity exactly $d - \mu$ at the general point of Π, the general $(t+1)$-dimensional subspace Π' through Π cuts out on $V(f)$, off Π, a cone of degree exactly μ, and, as Π' varies the corresponding subspace Γ describes the family of tangent spaces to an m-dimensional unirational subvariety $S(f)$ of Π.*
(iv) *If $V(f)$ is general and $\mu > r - t - 2$ then $V(f)$ is not a cone.*
(v) *The general* GN-*hypersurface is irreducible.*

Proof. Let $\bar{\Pi} \subset \mathbb{P}^r$ denote the subspace defined by the equations $x_0 = \cdots = x_t = 0$, the coordinate complementary subspace to Π. For any nonzero $\xi = (0 : \ldots : 0, \xi_{t+1} : \ldots : \xi_r) \in \bar{\Pi}$, set $\Pi_\xi = \langle \Pi, \xi \rangle \subset \mathbb{P}^r$. Then as ξ varies, Π_ξ describes the set of all $(t+1)$-dimensional subspaces containing Π. For a fixed such ξ the points of Π_ξ are parameterizable as $(x_0 : \ldots : x_t : z\xi_{t+1} : \ldots : z\xi_r)$, where z is a parameter. Hence we can take $(x_0 : \ldots : x_t : z)$ as homogeneous coordinates in Π_ξ and $(x_0 : \ldots : x_t)$ as coordinates in Π.

Fix such a ξ. The intersection $V(f) \cap \Pi_\xi$ is a hypersurface V_ξ of Π_ξ with equation:

$$\sum_{k=0}^{\mu} z^{d-k} P_k\big(M_{1,0}(\xi)x_0 + \cdots + M_{1,t}(\xi)x_t, \ldots, M_{t-m,0}(\xi)\,x_0 + \cdots + M_{t-m,t}(\xi)x_t, \xi\big) = 0.$$

The presence of the factor $z^{d-\mu}$ shows that the general point of Π has multiplicity at least $d-\mu$ for $V(f)$. This proves (i). The residual hypersurface W_ξ contains the subspace Γ_ξ of Π with equations:

$$\begin{aligned} &M_{1,0}(\xi)x_0 + \cdots + M_{1,t}(\xi)x_t = 0, \quad \ldots, \\ &M_{t-m,0}(\xi)x_0 + \cdots + M_{t-m,t}(\xi)x_t = 0, \quad z = 0. \end{aligned} \tag{2.3}$$

Furthermore W_ξ is a cone with vertex Γ_ξ. Indeed, if $p = (p_0 : \ldots : p_t : p) \in W_\xi$ and $q = (q_0 : \ldots : q_t : 0) \in \Gamma_\xi$, the line joining p and q is parameterizable by $x_i = \lambda p_i + \nu q_i$, $z = \lambda p$, $i = 0, \ldots, t$, where $(\lambda : \nu) \in \mathbb{P}^1$ is a parameter. By restricting the equation of W_ξ to this line, we find that the resulting equation is identically verified in λ and ν, because $(q_0 : \ldots : q_t)$ is a solution of the system (2.3). This proves (ii).

Assume now $V(f)$ is general. We note that, by setting $x_0 = 1$, $x_2 = \cdots = x_t = 0$, the coefficient of z^{d-k}, $k = 0, \ldots, \mu$, in the resulting polynomial can be seen as a general polynomial of degree $d-k$ in the variables $\xi_{t+1}, \ldots, \xi_r$. By taking into account the proofs of parts (i) and (ii), the first part of (iii) immediately follows.

Next note that $\dim(\Gamma_\xi) \geqslant m$ and the equality holds if $V(f)$ is a general GN-hypersurface. Now Γ_ξ contains the $m+1$ points $p_j(\xi) = (p_{j,0}(\xi) : \ldots : p_{j,t}(\xi))$, where

$$p_{j,i}(\xi) = \frac{\partial h_i}{\partial y_j}\big(\psi_0(\xi), \ldots, \psi_m(\xi)\big), \quad i = 0, \ldots, t, \ j = 0, \ldots, m.$$

Since $V(f)$ is general, the points $p_j(\xi)$, $j = 0, \ldots, m$, are linearly independent. Hence $\Gamma_\xi = \langle p_0(\xi), \ldots, p_m(\xi) \rangle$. Consider the unirational subvariety $S(f)$ of Π which is the image of the map $h : \bar{\Pi} \dashrightarrow \Pi$ sending the general point $\xi \in \bar{\Pi}$ to the point $(\eta_0 : \ldots : \eta_t)$ where

$$\eta_i = h_i\big(\psi_0(\xi), \ldots, \psi_m(\xi)\big), \quad i = 0, \ldots, t.$$

It is clear now that $S(f)$ has dimension m and that Γ_ξ is the tangent space to $S(f)$ at $h(\xi)$. This concludes the proof of (iii).

As for (iv), we notice that, if $\mu + 1 > r - t - 1$, then for no ξ does the hypersurface V_ξ vanish identically. Thus, if $V(f)$ is a cone, the vertex of $V(f)$ should lie on Π. In this case all the tangent spaces to $S(f)$ should contain the vertex of the cone, hence $S(f)$ itself ought to be a cone (cf., e.g., [36, Proposition 1.2.6]). This is clearly not the case for a general GN-hypersurface.

To prove (v) let f be a general GN-polynomial of type (r,t,m,n) and degree d as in (2.1) and let $V(f)\subset\mathbb{P}^r$ be the corresponding hypersurface. Cutting $V(f)$ with $\bar{\Pi}$ gives the hypersurface with equation $P_0(x_{t+1},\dots,x_r)=0$, which does not involve the variables $x_1,\dots,x_{t-m}$. Indeed, for every $k=1,\dots,\mu$, the homogeneous polynomial P_k involves these variables and, being general, must vanish for $x_0=\dots=x_t=0$ as do the Q_ℓ's. Now, since $P_0(x_{t+1},\dots,x_r)$ is also general, the original hypersurface is reduced. In addition, if $t<r-2$, the hypersurface $V(P_0)$ is also irreducible, implying the irreducibility of the original hypersurface. For $t=r-2$, the zero set of the polynomial $P_0(x_{r-1},x_r)$ is a finite set of points, which are the intersection points of the line $\bar{\Pi}$ with $V(f)$. However in this case we can appeal to the fact that the polynomial $P_0(x_{r-1},x_r)=0$ is a general equation of degree d and therefore its Galois group is the full symmetric group. Thus we see that the intersection of $V(f)$ with a general line consists of d distinct points, which are exchanged by monodromy when the line moves, proving the irreducibility also in this case. $\square$

The proposition admits a converse statement to the effect that if $V(f)\subset\mathbb{P}^r$ is a hypersurface of degree d satisfying a suitable reformulation of the above enumerated properties, then it is a GN-hypersurface of type (r,t,m) (see [34, pp. 104–105]).

2.4. Permutti's generalization of Gordan–Noether machine

Permutti (see [34]) has extended Gordan–Noether construction in the case $t=m+1$. Let us briefly recall this too.

Fix integers r,t such that $r\geqslant 2$, $1\leqslant t\leqslant r-2$. Fix $t+1$ homogeneous polynomials $M_0(x_{t+1},\dots,x_r),\dots,M_t(x_{t+1},\dots,x_r)$ of the same degree $n-1$ in the variables $x_{t+1},\dots,x_r$ and assume that they are algebraically dependent over k—which will be automatic if $r\leqslant 2t$ because then the number $r-t$ of variables is smaller than the number $t+1$ of polynomials.

Set $Q=M_0x_0+\dots+M_tx_t$, a form of degree n. Fix an integer $d>n$ and set $\mu=[\frac{d}{n}]$. Further fix forms $P_k(x_{t+1},\dots,x_r)$ of degree $d-kn$ in $x_{t+1},\dots,x_r$, $k=0,\dots,\mu$. The form of degree d

$$f(x_0,\dots,x_r)=\sum_{k=0}^{\mu}Q^kP_k(x_{t+1},\dots,x_r),$$

or any form obtained thereof by a linear change of variables, will be called a *Permutti polynomial*, or a P-*polynomial* of type (r,t,n). Accordingly, the corresponding hypersurface $V(f)\subset\mathbb{P}^r$ will be called a *Permutti hypersurface* or P-*hypersurface* of type (r,t,n), with *core* the t-dimensional subspace Π with equations $x_{t+1}=\dots=x_r=0$. It is immediate to see that a GN-polynomial of type $(r,t,t-1,n)$ is a P-polynomial of type (r,t,n).

Proposition 2.12. *Every* P-*hypersurface has vanishing Hessian.*

Proof. Let f be a P-polynomial. Then it is immediate to see that

$$\frac{\partial f}{\partial x_i}=\frac{\partial f}{\partial Q}M_i,\quad i=0,\dots,t,$$

where $\partial f/\partial Q$ denotes the formal derivative of f with respect to Q. Since by assumption $M_0,\dots,M_t$ are algebraically dependent, it is clear that $\partial f/\partial x_0,\dots,\partial f/\partial x_r$ are algebraically dependent too. $\square$

One can easily prove the analogue of Proposition 2.11 in Permutti's setup. We use the same terminology and notation employed in the previous section.

Proposition 2.13. *Let $V(f) \subset \mathbb{P}^r$ be a general* P*-hypersurface of type (r,t,n) and degree d. Set $\mu = [\frac{d}{n}]$. Then*

(i) *$V(f)$ has multiplicity $d-\mu$ at the general point of its core Π.*
(ii) *The general $(t+1)$-dimensional subspace Π' through Π cuts out on $V(f)$, off Π, a cone of degree at most μ, consisting of μ subspaces of dimension t which all pass through a subspace Γ of Π' of dimension $t-1$.*
(iii) *As Π' varies the corresponding Γ describes a unirational family of dimension $\chi \leqslant \min\{t-1, r-t-1\}$.*
(iv) *If $\mu > r-t-2$, then $V(f)$ is a cone if and only if the forms $M_0, \ldots, M_t$ are linearly dependent over k. This in turn happens as soon as either $t=1$, or $n=1,2$.*
(v) *$V(f)$ is irreducible.*

Proof. One verifies that the general subspace Π_ξ cuts out on $V(f)$ a hypersurface V_ξ which contains Π with multiplicity $d-\mu$. The residual hypersurface W_ξ is the union of μ subspaces of dimension t which all pass through the subspace Γ_ξ of Π_ξ of dimension $t-1$ with equation:

$$M_0(\xi)x_0 + \cdots + M_t(\xi)x_t = 0.$$

Note that, since $M_0, \ldots, M_t$ are algebraically dependent, then $\chi \leqslant t-1$. The inequality $\chi \leqslant r-t-1$ is obvious. Parts (i)–(iii) follow by these considerations.

As for part (iv), like in the proof of Proposition 2.11, we see that the hypersurface $V(f)$ is a cone if and only if, as ξ varies, the subspace Γ_ξ contains a fixed point. This happens if and only if the polynomials $M_0, \ldots, M_t$ are linearly dependent. The rest of the assertion is trivial.

The proof of (v) is completely analogous to the proof of the corresponding statement in Proposition 2.11 and shall be omitted. □

It has been proved in [34, pp. 100–101] a converse to the effect that if $V(f) \subset \mathbb{P}^r$ is a hypersurface of degree d enjoying the above properties—with the core replaced by a subspace with the same property—then it is a P-hypersurface of type (r,t,n).

For P-hypersurfaces $V(f) \subset \mathbb{P}^r$ one can describe the dual variety $V(f)^* \subset \mathbb{P}^{r*}$. Note that, as ξ varies in the subspace $\bar{\Pi}$ with equations $x_0 = \cdots = x_t = 0$, then the subspace $\Gamma_\xi^\perp$ of dimension $r-t$ varies describing a cone $W(f) \subset \mathbb{P}^{r*}$, of dimension $r-t-1$ with vertex $\Pi^\perp$ which contains the subspace $\Pi_\xi^\perp$ of dimension $r-t-2$. More precisely, we have the:

Proposition 2.14. *Let $V(f) \subset \mathbb{P}^r$ be a general* P*-hypersurface of type (r,t,n) and degree d. Let $\mu = [\frac{d}{n}]$. Then:*

(i) *$V(f)^* \subset W(f)$, where $W(f) \subset \mathbb{P}^{r*}$ is a cone over a unirational variety of dimension $\chi \leqslant \min\{t-1, r-t-1\}$ whose vertex is the orthogonal of the core Π of $V(f)$.*
(ii) *The general ruling of the cone $W(f) \subset \mathbb{P}^{r*}$ is an $(r-t)$-dimensional subspace through $\Pi^\perp$ which cuts $V(f)^*$, off $\Pi^\perp$, in μ subspaces of dimension $r-t-1$ all passing through the same subspace of $\Pi^\perp$ of dimension $r-t-2$. Hence $v(f) = \min\{r-2, 2(r-t-1)\}$.*

Conversely, if $V(f) \subset \mathbb{P}^r$ is the dual of such a variety, then $V(f) \subset \mathbb{P}^r$ is a P*-hypersurface.*

Proof. It follows by dualizing the contents of Proposition 2.13. □

From this we also see that a general P-hypersurface is not a cone. In addition, one has:

Proposition 2.15. *Let $V(f) \subset \mathbb{P}^r$ be a general* P*-hypersurface of type* (r,t,n)*. Then* $Z(f) = W(f) \subset \mathbb{P}^{r*}$*, and therefore* $z(f) = \min\{r-1, 2(r-t)-1\}$.

Proof. For $\xi \in \bar{\Pi}$ general, Π_ξ cuts out on $V(f)$ a hypersurface V_ξ which is a union of hyperplanes of Π_ξ and is a cone with vertex Γ_ξ. If $p \in \Pi_\xi$ is a general point, then the polar hyperplane $\pi_{\xi,p}$ of p with respect to V_ξ contains Γ_ξ. By Remark 2.8, when p varies in Π_ξ, then $\pi_{\xi,p}$ varies describing an open dense subset of the set of all hyperplanes of Π_ξ containing Γ_ξ. If $\pi_p(f)$ is the polar hyperplane of p with respect to $V(f)$, then π_p cuts out $\pi_{\xi,p}$ on Π_ξ. Hence the subspace $\langle \phi_f(p), \Pi_\xi^\perp \rangle$ sits in the ruling $\Gamma_\xi^\perp$ of $W(f)$ and, as p varies, it describes a dense open subset of $\Gamma_\xi^\perp$. This proves that $W(f) = Z(f)$. □

Remark 2.16. The case $t = r-2$ is particularly interesting. Then $V(f)^*$ is a scroll surface with a line directrix $L = \Pi^\perp$ of multiplicity $e \geqslant \mu$, where μ is the invariant introduced in Section 1.2.2. It is a subvariety of the 3-dimensional rational cone $W(f)$ over a curve with vertex L, and the general plane ruling of the cone cuts $V(f)^*$ along μ lines of $V(f)^*$, all passing through the same point of L. In particular, for $\mu = 1$, the dual $V(f)^*$ is a rational scroll (see Sections 1.2.1 and 1.2.2). According to Proposition 2.15, we have $Z(f) = W(f)$, hence $z(f) = 3$.

If $t = 2$ the two constructs of GN-hypersurfaces and P-hypersurfaces coincide. For $r = 4$ this is the only value of t which leads to hypersurfaces which are not cones. The case $r = 4$ is well understood due to a result of Franchetta (see [18]; see also Proposition 2.14; according to [27] this result is contained in [20]; for another proof see [19]):

Theorem 2.17. *Let $V(f) \subset \mathbb{P}^4$ be a reduced hypersurfaces of degree d. The following conditions are equivalent:*

(i) *$V(f)$ has vanishing Hessian;*
(ii) *$V(f)$ is a* GN*-hypersurface of type* $(4,2,1,n)$*, with* $\mu = [\frac{d}{n}]$*, which has a plane of multiplicity* $d-\mu$*;*
(iii) *$V(f)^*$ is a scroll surface of degree d, having a line directrix L of multiplicity e, sitting in a rational cone $W(f)$ of dimension* 3 *with vertex L, and the general plane ruling of the cone cuts $V(f)^*$ off L along $\mu \leqslant e$ lines of the scroll, all passing through the same point of L.*

In particular, $V(f)^*$ is smooth if and only if $d = 3$, $V(f)^*$ is a rational normal scroll and $V(f)$ contains a plane, the orthogonal to the line directrix of $V(f)^*$, with multiplicity 2.

2.5. Variations on some results of Perazzo

Let $V(f) \subset \mathbb{P}^r$ be a hypersurface of degree d with $r \geqslant 4$. If $d = 2$, it is clear that $V(f)$ has vanishing Hessian if and only if it is a cone. So the first meaningful case is the one $d = 3$, in which, as we saw, there are examples which are not cones (see Theorem 2.17). The case of cubic hypersurfaces has been studied in some detail by U. Perazzo (see [31]). We will partly generalize Perazzo's results. Inspired by the construction of P-hypersurfaces and by Perazzo's results, we will give new examples of hypersurfaces with vanishing Hessian, which are extensions of some P-hypersurfaces.

Consider a hypersurface $V(f) \subset \mathbb{P}^r$ which contains a subspace Π of dimension t such that the general subspace Π_ξ of dimension $t+1$ through Π cuts out on $V(f)$ a cone with a vertex Γ_ξ of dimension s. Assume that $s \geqslant r-t-1$. By extended analogy, we will call Π the *core* of $V(f)$ and call $V(f)$ an H-*hypersurface* of type (r,t,s). Notice that a P-hypersurface of type (r,t,n) with $r \leqslant 2t$ is also an H-hypersurface of type $(r,t,t-1)$.

As for P-hypersurfaces, we can introduce the cone $W(f) \subset \mathbb{P}^{r*}$ with vertex $\Pi^\perp$, which is swept out by the $(r-s-1)$-dimensional subspaces $\Gamma_\xi^\perp$ as Π_ξ varies among all subspaces of dimension $t+1$ containing Π.

A special case of an H-hypersurface is that of a hypersurface $V(f) \subset \mathbb{P}^r$ of degree d containing a subspace Π of dimension t whose general point has multiplicity $d-\mu > 0$ for $V(f)$, such that the general subspace Π_ξ of dimension $t+1$ through Π cuts out on $V(f)$, off Π, a union of μ subspaces of dimension t, with $\mu \leqslant 2t-r+1$. In this situation, we will call $V(f) \subset \mathbb{P}^r$ an R-*hypersurface* of type (r,t,μ).

Proposition 2.18. *An* H-*hypersurface* $V(f) \subset \mathbb{P}^r$ *of type* (r,t,s) *has vanishing Hessian. Moreover* $Z(f) = W(f) \subset \mathbb{P}^{r*}$.

Proof. Let p be a general point in $\mathbb{P}^r$ and let Π' be the span of Π and p. Since the intersection of $V(f)$ with Π' is a cone with vertex a subspace Γ of dimension s, the polar quadric $Q_p(f)$ cuts out on Π' a quadric singular along Γ. If $Q_p(f)$ is smooth we have $s = \dim(\Gamma) \leqslant r-t-2$, a contradiction. This proves that $Q_p(f)$ is singular hence $V(f)$ has vanishing Hessian.

The argument for the second assertion is similar to the one in the proof of Proposition 2.15 and therefore can be omitted. □

Remark 2.19. It is interesting to look at duals of R-hypersurfaces of degree d and type $(r, r-2, \mu)$. If $V(f) \subset \mathbb{P}^r$ is such a hypersurface, its dual $V(f)^* \subset \mathbb{P}^{r*}$ is a scroll surface with a line directrix L of multiplicity $e \geqslant \mu$, where $\mu \leqslant r-3$ is as in Section 1.2.2. We assume $V(f) \subset \mathbb{P}^r$ not a cone and therefore $V(f)^* \subset \mathbb{P}^{r*}$ is non-degenerate.

In this case the invariant s is related to the number ν introduced in Section 1.2.2: $\nu = r - s - 1$ and, moreover, one has $Z = W(f) \subset \mathbb{P}^{r*}$ where Z is the cone in the same section, and $\dim W(f) = r-s$.

By Proposition 2.18, one has $Z(f) = W(f) = Z$. This means that $\rho(f) = r-s+1$, hence the vertex of the general polar quadric has dimension $s-1$.

Let $p \in \mathbb{P}^r$ be a general point. The quadric $Q_p(f)$ cuts the hyperplane $\Pi' = \langle \Pi, p\rangle$ in a quadric singular along the subspace Γ of dimension s. Set $\xi = \phi_f(p)$. The vertex of $Q_p(f)$, which coincides with $T^\perp_{Z(f),\xi}$ (see Remark 2.4), has dimension $s-1$, hence it is contained in Γ.

An R-hypersurface with $\mu = 1$ is a hypersurface of degree d with a core Π of dimension t whose general point has multiplicity $d-1$ for the hypersurface, and moreover $2t \geqslant r$. This is the case considered by Perazzo in [31, p. 343], where he proves that these hypersurfaces have vanishing Hessian.

3. Homaloidal polynomials

A hypersurface $V(f) \subset \mathbb{P}^r$, or the form f, of degree d is said to be *homaloidal* if $\delta(f) = 1$, i.e. if the polar map ϕ_f is birational. According to Theorem 2.2, this property depends only on $\mathrm{Supp}(V(f))$, therefore we will mainly refer to the case $V(f)$ reduced.

The simplest example is when $V(f)$ is a smooth quadric: in this case the polar map ϕ_f is the usual polarity, which is an invertible linear map. This is also the only case of a reduced homaloidal polynomial if $r = 1$.

Reduced homaloidal curves in $\mathbb{P}^2$ have been classified by Dolgachev in [10]:

Theorem 3.1. *A reduced plane curve $V(f) \subset \mathbb{P}^2$ of degree d is homaloidal if and only if either*

(i) *$V(f)$ is a smooth conic, or*
(ii) *$d = 3$ and $V(f)$ consists of three non-concurrent lines, or*
(iii) *$d = 3$ and $V(f)$ consists of the union of a smooth conic with one of its tangent lines.*

Note that in case (ii) the polar map ϕ_f is a *standard quadratic transformation* based at three distinct points, whereas in case (iii) the map ϕ_f is a *special quadratic transformation* based at a curvilinear scheme of length three supported at one single point. More algebraically, in cases (ii) and (iii) the base locus ideal of ϕ_f is a codimension 2 perfect ideal (Hilbert–Burch)—see 4.1 and also [37,47] for the ubiquitous role of Hilbert–Burch ideals in the theory of Cremona transformations.

Remark 3.2. We note that the three cases in Theorem 3.1 can be naturally extended to any dimension $r \geqslant 2$, thus yielding an infinite series of homaloidal hypersurfaces in $\mathbb{P}^r$, with $r \geqslant 2$ (see [10]). Namely, the following reduced hypersurfaces $V(f) \subset \mathbb{P}^r$ of degree d are homaloidal in $\mathbb{P}^r$ for any $r \geqslant 2$:

(i) a smooth quadric;
(ii) the union of $r + 1$ independent hyperplanes;
(iii) the union of a smooth quadric with one of its tangent hyperplanes.

Note that (ii) gives the only example of arrangements of hyperplanes which are homaloidal (see [9,10]).

There is a general principle for rational maps $\phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$. In what follows, we adopt the terminology *the image of* ϕ to mean the closure in the target of the image of the points of the source $\mathbb{P}^r$ at which ϕ is well-defined. Accordingly, we use the notation $\phi(\mathbb{P}^r)$. This convention sticks to subvarieties as well.

Proposition 3.3. *Let $\phi = (F_0 : \ldots : F_r) : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ denote a rational map, where $F_i \in k[x_0, \ldots, x_r]$ are forms of the same degree without proper common factor. Let $J \in k[x_0, \ldots, x_r]$ denote the Jacobian determinant of these forms. Consider the following conditions:*

(i) *$J \neq 0$;*
(ii) $\dim(\phi(\mathbb{P}^r)) = r$.

Then (i) $\Leftrightarrow$ (ii).

If ϕ is birational, then $\dim \phi(V(J)) \leqslant r - 2$.

Proof. First note that ϕ is well defined at a general point of $V(J)$, otherwise $F_0, \ldots, F_r$ would be multiples of a single form which contradicts the assumption on these forms.

(i) $\Leftrightarrow$ (ii) Note that, up to a degree renormalization, the homogeneous coordinate ring of $\phi(\mathbb{P}^r) \subset \mathbb{P}^r$ is $k[F_0, \ldots, F_r]$. One then draws upon the known fact saying that, in characteristic zero, the dimension of $k[F_0, \ldots, F_r]$ is the rank of the Jacobian matrix of $F_0, \ldots, F_r$ (see, e.g., [46]).

If ϕ is birational, it is dominant so that $J \neq 0$. Moreover, it has to contract the hypersurface $V(J)$ since this is the locus where ϕ drops rank. $\square$

If $\dim(V(J)) \leqslant r-2$, we shall say that $V(J)$ is *contracted by* ϕ.

Corollary 3.4. *If a hypersurface $V(f) \subset \mathbb{P}^r$ is homaloidal, then $h(f)$ does not vanish identically and $H(f)$ is contracted by the polar map ϕ_f.*

Remark 3.5. An interesting case of Corollary 3.4 is when $f \in k[x_0, \ldots, x_r]$ is a nonzero reduced, homaloidal form of degree d such that $h(f) = cf^{\frac{(d-2)(r+1)}{d}}$ with $c \in k^*$. In this case we will say that such an f is *totally Hessian* and use the same terminology for the corresponding hypersurface. Note that it entails the equality $\mathrm{Supp}(V(f)) = \mathrm{Supp}(H(f))$—hence $V(f)$ is also contracted by ϕ_f—and any smooth point of $V(f)$ is parabolic (see Remark 2.3). It would be interesting to find whether a totally Hessian form is homaloidal.

A good deal of examples of totally Hessian forms arises from the theory of pre-homogeneous vector spaces (see [39,42], also [28]), a notion introduced by Kimura and Sato (see [24], see also [11,10,14]), which we now briefly recall for the reader's convenience.

A *pre-homogeneous vector space* is a triple (V, G, χ) where V is a complex vector space of finite dimension, G is a complex algebraic group, V is an algebraic linear representation of G, $\chi : G \to \mathbb{C}^*$ is a non-trivial character, and there is a nonzero homogeneous polynomial $f : V \to \mathbb{C}$, with no multiple factors, such that $f(g \cdot v) = \chi(g) f(v)$ for all $g \in G$ and $v \in V$, and such that the complement of the hypersurface $\{f = 0\}$ is a G-orbit.

The polynomial f, called the *relative invariant* of the pre-homogeneous space, is unique up to a nonzero factor from $\mathbb{C}$. The pre-homogeneous vector space (V, G, χ) is said to be *regular* if $h(f) \neq 0$. In this case the relative invariant f is totally Hessian (see [24,11]) and f is a homaloidal polynomial such that ϕ_f coincides with its inverse, modulo a projective transformation (see [11, Theorem 2.8]). In [14] (see also in [28], [51, Ch. III] and [11]), there is a description of several regular homogeneous vector spaces related to smooth projective varieties with extremal geometric properties (Severi and Scorza varieties, some varieties with one apparent double point, varieties whose dual is small, see [14]). The first instances among these examples were described in the classic literature (see [5,39,17]).

Being homaloidal or having vanishing Hessian implies strong constraints on the singularities of the hypersurface $V(f)$. Thus, if $\dim V(f) \geqslant 2$ and if $V(f) \subset \mathbb{P}^r$ has vanishing Hessian, then $V(f)$ cannot have isolated singularities. Also there is a conjecture in [9] to the effect that a hypersurface of dimension at least 2 with isolated singularities cannot be homaloidal. We now prove a result which points somewhat in this direction.

First we need to introduce some notation. Suppose $V(f) \subset \mathbb{P}^r$ is a reduced hypersurface of degree d. Let us resolve the indeterminacies of the polar map ϕ_f by iteratively blowing up $\mathbb{P}^r$

$$X := X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = \mathbb{P}^r,$$

thus getting $p: X \to \mathbb{P}^r$ so that $\phi_f \circ p: X \to \mathbb{P}^{r*}$ is a morphism. Here the map $X_i \to X_{i-1}$, $i = 1, \dots, n$, is a blowup with center a smooth variety of codimension $a_i + 1$, with $1 \leqslant a_i \leqslant r - 1$. We denote by E_i the total transform on X of the exceptional divisors of the blowup $X_i \to X_{i-1}$, $i = 1, \dots, n$. Further let H stand for the proper transform on X of a general hyperplane of $\mathbb{P}^r$ and Φ for the proper transform on X of the first polar hypersurface of $V(f)$ with respect to a general point of $\mathbb{P}^r$. Then

$$\Phi \equiv (d-1)H - \sum_{i=1}^{n} \mu_i E_i,$$

where the μ_i's are the *multiplicities* of Φ along the various centers of the iterated blowups. By an obvious minimality assumption, we may assume $\mu_i > 0$, $i = 1, \dots, n$.

The following result can be seen as a consequence of the so-called *Noether–Fano inequality* for Mori fibre spaces (see [8]). We give here a short direct proof. Let us recall that $\delta(f) = \deg(\phi_f)$ with the usual convention that $\deg(\phi_f) = 0$ if and only if ϕ_f is not dominant.

Proposition 3.6. *In the above setting, if $\delta(f) \leqslant 1$ then either $d \leqslant r+1$ or $\mu_i > a_i$ for some $i = 1, \dots, n$, i.e. either $d \leqslant r+1$ or the singularities of the general first polar of $V(f)$ are not* log-canonical *(see* [26, p. 56]*).*

Proof. As above, let Φ denote the proper transform on X of the general first polar hypersurface of $V(f)$. Note that Φ is smooth, because the linear system $|\Phi|$ is base point free. If $\delta(f) \leqslant 1$ then Φ is either rational or ruled (see Proposition 2.5). Since

$$K_X \equiv -(r+1)H + \sum_{i=1}^{n} a_i E_i,$$

one has

$$K_\Phi \equiv (d-r-2)H_{|\Phi} + \sum_{i=1}^{n} (a_i - \mu_i) E_{i|\Phi}.$$

If $d > r+1$ and $a_i \geqslant \mu_i$ for every $i = 1, \dots, n$, this divisor is effective, which would contradict the ruledness of Φ. □

Remark 3.7. Although the proper transform $\tilde{V}$ of $V(f)$ on X admits a similar expression

$$\tilde{V} \equiv dH - \sum_{i=1}^{n} m_i E_i,$$

here, in spite of the previous minimality assumption, some of the m_i's may vanish (see Section 3.2 below).

The problem of understanding the relationship between the m_i's and the μ_i's is longstanding, dating back to M. Noether, and is far from being solved in general. For further contributions in the plane case see [40] and [50] (see also Remark 3.20 below). Roughly speaking, one would expect $\mu_i = m_i - 1$ but this is not always the case.

Corollary 3.8. *In the above setting suppose that $\mu_i = m_i - 1$ for all $i = 1, \ldots, n$. If $d \geqslant r + 2$ and $\delta(f) \leqslant 1$, then $m_i > a_i + 1$ for some $i = 1, \ldots, n$. In particular, a surface $V(f) \subset \mathbb{P}^3$ with $d \geqslant 5$ and $\delta(f) \leqslant 1$ cannot have ordinary singularities.*

3.1. Irreducible homaloidal polynomials of arbitrarily large degrees

In this section we produce, for every $r \geqslant 3$, an infinite series of irreducible homaloidal hypersurfaces in $\mathbb{P}^r$ of arbitrarily large degree, thus settling a question that has been going around for some time. These polynomials are the dual hypersurfaces to certain scroll surfaces. It is relevant to observe, as we indicate below, that these examples are not related to the ones based on pre-homogeneous vector spaces as in [14] and in [11].

The examples show that, perhaps opposite to the ongoing folklore, there are plenty of homaloidal polynomials around. They even seem to be in majority as compared to polynomials with vanishing Hessian, though a complete classification does not seem to be presently at hand.

In this respect Dolgachev's classification Theorem 3.1 might be considered in counterpoint to Hesse's result to the effect that the only hypersurfaces with vanishing Hessian in $\mathbb{P}^r$, $r \leqslant 3$, are cones (see Section 2.2).

We wonder whether a counterpart of Franchetta's Theorem 2.17 could be a result to the effect that in $\mathbb{P}^3$ there are only finitely many projectively distinct types of (irreducible) homaloidal polynomials, apart from the ones constructed in this section.

We start with lemmas of general content.

Lemma 3.9. *Let $V(f)$ be a hypersurface in $\mathbb{P}^r$. Suppose there is a point $p \in V(f)$ and s linearly independent hyperplanes H_i, $i = 1, \ldots, s$, passing through p and each cutting $V(f)$ in a hypersurface having a point of multiplicity at least s in p. Then $V(f)$ has multiplicity at least s in p.*

Proof. Assume p is the origin in affine coordinates and that H_i has equation $x_i = 0$, $i = 1, \ldots, s$. Write $f = f_0 + f_1 + \cdots + f_d$, where f_j is the homogeneous component of degree j, with $j = 0, \ldots, d$. By the assumption $f_0, \ldots, f_{s-1}$ have to be divisible by x_i, $i = 1, \ldots, s$. Hence $f_0, \ldots, f_{s-1}$ are identically zero, proving the assertion. □

Recall now that the polar map of a form $f \in k[x_0, \ldots, x_r]$ is denoted ϕ_f or $\phi_{V(f)}$ to stress the corresponding hypersurface $V(f) \subset \mathbb{P}^r$.

Lemma 3.10. *Let $V(f) \subset \mathbb{P}^r$ be a hypersurface. Let $H \subset \mathbb{P}^r$ be a hyperplane not contained in $V(f)$, let $\xi = H^\perp$ be the corresponding point in $\mathbb{P}^{r*}$ and let σ_ξ denote the projection from ξ. Then*

$$\phi_{V(f) \cap H} = \sigma_\xi \circ (\phi_{V(f)})_{|H}.$$

Proof. The proof is straightforward by assuming, as one can, that H is a coordinate hyperplane. □

Lemma 3.11. *Let C be a rational normal curve in $\mathbb{P}^n$. Let $\mathcal{L}$ be a g^1_m, with $m < n$ and consider the rational normal scroll $W(\mathcal{L}) = \bigcup_{D\in\mathcal{L}}\langle D\rangle$, of dimension m and degree $n - m + 1$ (see, e.g., [13]). Let $p_1, \dots, p_{n-m}$ be points of C. Then:*

(i) *$W(\mathcal{L})$ intersects the $(n - m - 1)$-dimensional subspace $\Pi = \langle p_1, \dots, p_{n-m}\rangle$ transversally only at $p_1, \dots, p_{n-m}$;*
(ii) *the general tangent space to $W(\mathcal{L})$ does not intersect the $(n - m - 1)$-dimensional subspace $\Pi = \langle p_1, \dots, p_{n-m}\rangle$.*

Proof. Project from Π to $\mathbb{P}^m$. The image of C is a rational normal curve C' and the image of $W(\mathcal{L})$ is the analogous scroll $W(\mathcal{L}') = \bigcup_{D\in\mathcal{L}'}\langle D\rangle$, where $\mathcal{L}'$ is the g^1_m on C' which is the image of $\mathcal{L}$. Since this fills up $\mathbb{P}^m$, both assertions follow. $\square$

We introduce now the promised examples. Recall from Section 1.2.2 that we have rational scrolls $X := X(a, b)$ of degree $d = a + b$ with a multiple line directrix Λ of multiplicity a in $\mathbb{P}^{b+2}$, for $1 \leqslant a \leqslant b$. The dual hypersurface $X^* = X(a, b)^*$ has vanishing Hessian as soon as $1 \leqslant a < b$ and the image of the corresponding polar map is the cone $Z := Z(a, b)$ containing X, introduced in Section 1.2.2 (see Remark 2.19). This is a rational normal cone of degree $b - a + 1$ and dimension $a + 2$ with vertex Λ. More specifically, let C be the rational normal curve in $\mathbb{P}^b$ which is the projection of X from Λ. One has the general linear series $\mathcal{L} = g^1_a$ on C whose general divisor is the projection on C of the a lines of X passing through the general point of Λ. The scroll Z is the cone with vertex Λ over $W(\mathcal{L})$, which, by the generality assumption about $X(a, b)$, is a general rational normal scroll of degree $b - a + 1$ and dimension a in $\mathbb{P}^b$.

An essential piece of information for the construction of our examples is the following:

Theorem 3.12. *If $1 \leqslant a < b$, the closure of the general fibre of the polar map $\phi := \phi_{X(a,b)^*}$ is a projective subspace of dimension $b - a$ of $\mathbb{P}^{b+2}$.*

Proof. Let p be a general point of $\mathbb{P}^{b+2}$. Then $\xi = \phi(p)$ is a general point of Z. Recall that the closure F_p of the fibre of ϕ over ξ is the union of finitely many $(b - a)$-dimensional subspaces containing $T^{\perp}_{Z,\xi}$, which in turn is the $(b - a - 1)$-dimensional vertex V_p of the polar quadric Q_p of p with respect to $X^* = X(a, b)^*$ (see Proposition 2.5 and Remark 2.4). What we have to prove is that F_p consists of the single subspace $\langle p, V_p\rangle$.

Recall Proposition 1.8 and Remark 2.19 and keep the notation introduced therein. In particular $\Pi = \Lambda^{\perp}$ is a subspace of dimension b in $\mathbb{P}^{b+2^*}$, which has multiplicity b for X^*. The hyperplane $\Pi' := \Pi'_p = \langle \Pi, p\rangle$ is dual to the general point $x \in \Lambda$. Let $F_{x,1}, \dots, F_{x,a}$ be the rulings of X passing through x, hence Π' cuts X^* along Π, with multiplicity b, and along the a subspaces $\Sigma_i := F^{\perp}_{x,i}$, $i = 1, \dots, a$, of dimension b. The intersection

$$\Gamma_p = \Pi \cap \Sigma_1 \cap \cdots \cap \Sigma_a = \langle \Lambda, F_{x,1}, \dots, F_{x,a}\rangle^{\perp}$$

has dimension $b - a$. Note that $\Gamma_p = W^{\perp}_{\xi}$, where $W_{\xi} = \langle \Lambda, F_{x,1}, \dots, F_{x,a}\rangle$ is the ruling of Z containing ξ.

Let p' be another point in $\mathbb{P}^{b+2}$ where ϕ is defined, and set $\xi' = \phi(p')$. The above description implies that $\Gamma_p = \Gamma_{p'}$ if and only if $W_{\xi} = W_{\xi'}$. By recalling the structure of the scroll Z we see that this happens if and only if $\Pi'_p = \Pi'_{p'}$.

As we saw in Remark 2.19, the vertex V_p of the quadric Q_p is contained in Γ_p because $W_\xi \subseteq T_{Z,\xi}$. We claim now that there is no point p' such that $\Gamma_{p'} \neq \Gamma_p$ and $\Gamma_p \cap \Gamma_{p'} = V_p$. In fact if this happens, then $T_{Z,\xi} = V_p^\perp$ contains $W_{\xi'} = \Gamma_{p'}^\perp$. This means that, if W is a general ruling of Z, then the tangent space to Z at the general point of W contains some other ruling W' of Z. By projecting Z from Λ onto the a-dimensional rational normal scroll $W(\mathcal{L}) \subset \mathbb{P}^b$, we would have that, for a general point $q \in W(\mathcal{L})$, the tangent space $T_{W(\mathcal{L}),q}$ would contain some ruling of $W(\mathcal{L})$ different from the one of q. This is impossible. Indeed, by cutting with $a-1$ general hyperplanes, we would have the general curve section C of $W(\mathcal{L})$, a rational normal curve, with the property that its general tangent line $T_{C,q}$ intersects C at a point $q' \neq q$, which is clearly not the case.

Let now p, p' be points such that $\phi(p) = \phi(p')$. Then $V_p = V_{p'}$, therefore $\Gamma_p = \Gamma_{p'}$ and $\Pi'_p = \Pi'_{p'}$. Thus $F_p = F_{p'}$ is contained in Π'_p. To simplify notation, we set $V = V_p$, $\Gamma = \Gamma_p$, $\Pi' = \Pi'_p$, $F = F_p$.

We claim next that if $\phi(p) = \phi(p')$, then $\langle p, \Gamma\rangle = \langle p', \Gamma\rangle$ and this $(b-a+1)$-dimensional subspace Γ' contains F. In fact, F is the closure of the intersection, off the singular locus of X^*, of all first polars of X^* containing p. In particular F is contained in the intersection of Π' with of all first polars of points of Π' containing p (or p'). Remarks 2.8 and 3.2, imply that this intersection is exactly Γ'. Our claim thus follows.

Note now that the linear system cut out on Γ' by of all first polars of the points of Π' is 0-dimensional, consisting of Γ, counted with multiplicity $a+b-1$. Thus the linear system $\mathcal{N}$ of hypersurfaces of degree $a+b-1$ cut out on Γ' by all first polars of X^* is a pencil, i.e. $\dim(\mathcal{N}) = 1$. Note that the fixed locus of $\mathcal{N}$ certainly contains Γ with multiplicity $b-1$, since the general first polar contains Π with this multiplicity. To finish our proof, we have to show that the movable part of $\mathcal{N}$, whose degree is bounded by a, is actually a pencil of hyperplanes.

To see this, look at the linear system $\mathcal{M}$ cut out by all first polars on Π' off Π, which, as we said, appears with multiplicity $b-1$ in the base locus. The general member M of $\mathcal{M}$ is a hypersurface of degree a. Let us consider its intersection with the hyperplanes Σ_i, $i = 1, \ldots, a$. Note that the intersections $\Sigma_i \cap \Sigma_j$, $1 \leqslant i < j \leqslant a$, all of dimension $b-1$ and containing Γ, sit in the singular locus of X^*, since they are intersection of rulings of the scroll X^*. Hence the intersection of M with Σ_i has multiplicity $a-1$ along Γ for all $i = 1, \ldots, a$. By Lemma 3.9, M has multiplicity $a-1$ along Γ. This implies that the movable part of $\mathcal{N}$ has degree one, thus ending the proof of the theorem. □

Let now $F_1, \ldots, F_{b-a}$ be general rulings of $X(a,b)$. Together with Λ they span a projective space Φ of dimension $b-a+1$. Choose a general subspace Ψ of dimension $b-a-1$ in Φ and project down $X(a,b)$ from Ψ to $\mathbb{P}^{a+2}$. The projection is a scroll surface $Y(a,b) \subset \mathbb{P}^{a+2}$ of degree $d = a+b \geqslant 2a+1$ which has a directrix L, the image of Λ, of multiplicity $e = b$. However we have here $\mu = a$ because, if $x \in L$ is the general point, only a among the b lines of the ruling through x vary, the other $b-a$ stay fixed and coincide with L.

Theorem 3.13. *For every $r \geqslant 3$ and for every $d \geqslant 2r-3$ the hypersurface $Y(r-2, d-r+2)^* \subset \mathbb{P}^{r*}$ of degree d is homaloidal.*

Proof. We keep the above notation. A repeated use of (1.2) gives

$$Y(r-2, d-r+2)^* = X(r-2, d-r+2)^* \cap \Psi^\perp.$$

To simplify the notation, set $X = X(r-2, d-r+2)$ and $Y = Y(r-2, d-r+2)$.

According to Remark 2.19, X^* has vanishing Hessian and the image of its polar map is a rational normal scroll $Z = Z(r-2, d-r+2)$ of dimension r and degree $d-2r+5$. By Theorem 3.12 the general fibre of the polar map is a projective subspace of dimension $d-2r+4$.

Let us repeat all pertinent dimensions translating from above a, b to present d, r:

$$\dim(X) = 2, \qquad \dim(X^*) = d-r+3 \geqslant r \quad \text{(from the assumed inequality)}$$

$$\dim(Y) = 2, \qquad \dim(Y^*) = r-1, \qquad \dim(\Phi) = d-2r+5, \qquad \dim(\Psi) = d-2r+3,$$

$$\begin{aligned}\dim(Z) = \dim\bigl(\Psi^\perp\bigr) &= d-r+4-\dim(\Psi)-1 \\ &= d-r+4-\bigl(d-r+2-(r-2)-1\bigr)-1 = r.\end{aligned}$$

By a repeated use of Lemma 3.10 in a dual form, one has:

$$\phi_{Y^*} = \sigma_\Psi \circ (\phi_{X^*})_{|\Psi^\perp}. \tag{3.1}$$

We claim that the map $(\phi_{X^*})_{|\Psi^\perp} : \Psi^\perp \dashrightarrow Z$ is birational. By part (ii) of Lemma 3.11 and duality, if $z \in Z$ is a general point, then $\Phi^\perp \cap T^\perp_{Z,z} = \emptyset$. Let $\xi \in \mathbb{P}^{r*}$ be an inverse image of z by ϕ_{X^*}. Then $\langle \xi, \Phi^\perp \rangle \cap \langle \xi, T^\perp_{Z,z} \rangle = \{\xi\}$. Assuming, as we may, that $\Psi^\perp$ is a general subspace of dimension r through $\langle \xi, \Phi \rangle$, then $\Psi^\perp \cap \langle \xi, T^\perp_{Z,z} \rangle = \{\xi\}$ and moreover $\Psi^\perp$ intersects the fiber of ϕ_{X^*} over z only at ξ (see Propositions 2.5 and Theorem 3.12).

By (3.1) and Theorem 3.12, the degree of the polar map ϕ_{Y^*} is the same as the degree of the restriction of the projection σ_Ψ to Z. To compute this latter degree, note that Ψ intersects Z exactly in $d-2r+4$ distinct points, namely the intersections of Ψ with each one of the $d-2r+4$ planes spanned by Λ and by one of the $d-2r+4$ rulings spanning Ψ together with Λ. We claim that the intersection of Ψ with Z at these points is transversal. Indeed, by projecting from Λ to $\mathbb{P}^{d-r+2}$, we see that X maps to a rational normal curve C, the lines $F_1, \ldots, F_{d-2r+4}$ map to points $p_1, \ldots, p_{d-2r+4}$ on C and Ψ maps to $\Pi = \langle p_1, \ldots, p_{d-2r+4} \rangle$. By part (i) of Lemma 3.11, Π intersects the projection of Z transversally at $p_1, \ldots, , p_{d-2r+4}$. The claim follows.

Thus the restriction of the projection σ_Ψ to Z coincides with the projection of Z from $d-2r+4$ independent points on it. Since, as seen, $\deg(Z) = d-2r+5$, the restriction of the projection σ_Ψ to Z is a birational map of Z to $\mathbb{P}^r$, thus completing proof. $\square$

Remark 3.14. As in Section 1.2.2 and in the description before Proposition 1.9, one can take the dual viewpoint to describe the homaloidal hypersurfaces we constructed above.

More precisely, to obtain the dual of $Y(a, b)$, one can proceeds as follows. Consider the scroll $X_1 = S(1^a, b) \subset \mathbb{P}^{b+2a}$ of degree $d = a+b$ and dimension $a+1$, with $S(1^a, b) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus a} \oplus \mathcal{O}_{\mathbb{P}^1}(b))$ embedded via the $\mathcal{O}(1)$ bundle. Consider also the rational normal scroll $X_2 = S(1^a) \subset \mathbb{P}^{2a-1}$ of degree a and dimension a. Clearly $X_2 \subset X_1$. Take $b-a$ general rulings $F_1, \ldots, F_{b-a}$ of X_1. The span $\Sigma = \langle X_2, F_1, \ldots, F_{b-a} \rangle$ has dimension $2a-1+b-a = a+b-1$. Take a sufficiently general subspace Σ' of dimension $b+a-3$ intersecting Σ in a general subspace of dimension $b-2$, and project form Σ' down to $\mathbb{P}^{a+2}$. The image of X_1 is a hypersurface V with a subspace Π of dimension a, the image of Σ, of multiplicity b, since it is the image of X_2 and of $F_1, \ldots, F_{b-a}$. The hypersurface V is the dual of $Y(a, b)$.

In this way we see that $Y(a,b)$ is a section of $S(1^a,b)^*$ made with a suitable linear space of dimension $a+2$. Since $S(1^a,b)$ is a suitable linear section of $S(1^{a+b}) = \mathrm{Seg}(1, a+b-1)$, which is self-dual, we see that $S(1^a,b)^*$ is a suitable projection in $\mathbb{P}^{b+2a}$ of $\mathrm{Seg}(1, a+b-1)$.

Remark 3.15. In the construction of $Y(a,b)$, it is not necessary that the rulings $F_1, \dots, F_{b-a}$ be distinct. Indeed, one can consider an effective divisor $D = m_1F_1 + \dots + m_hF_h$ of degree $b-a$ formed by lines of the ruling of $X(a,b)$. Then in the above construction one replaces the subspace Φ with the span of Λ and of the osculating spaces of order m_i at the points $p_i \in C$ projections of the lines F_i, $i = 1, \dots, h$. We will denote the resulting surface by $Y(a, b; m_1, \dots, m_h)$.

The corresponding hypersurfaces in Theorem 3.13 are still homaloidal, since the proof of the theorem works even in this special situation: indeed the intersection of Ψ with $Z(r-2, d-r+2)$ is no longer formed by $d-2r+4$ distinct points, but by a 0-dimensional scheme of length $d-2r+4$, formed by h points $x_1, \dots, x_h$, with length $m_1, \dots, m_h$ respectively, hence the projection of $Z(r-2, d-r+2)$ to $\mathbb{P}^r$ from Ψ is still birational.

As we will see in the next section however, this specialization influences the degree of the inverse of the resulting polar map.

Remark 3.16. The scroll $S(a,b)$, with $0 < a < b$, has a group of dimension $b-a+5$ of projective transformations which fixes it and all scrolls $S(a,b)$ are projectively equivalent, i.e. $S(a,b)$ has no projective moduli.

The scroll $X(a,b)$ has a group of dimension $\max\{0, b-3a+7\}$ of projective transformations which fixes it, and the scrolls $X(a,b)$ have no projective moduli if and only if $b-3a+7 \geqslant 0$.

Assume $b-a \leqslant 3$. Then the subgroup fixing $F_1, \dots, F_{b-a}$ has dimension $\max\{0, 7-2a\}$. In conclusion $Y(a,b)$ has a group of dimension $\max\{0, 7-2b\}$ of projective transformations which fixes it, and there are no projective moduli if and only if $2b \leqslant 7$, i.e. $b \leqslant 3$.

If $b-a \geqslant 4$ the subgroup fixing $F_1, \dots, F_{b-a}$ has dimension $\max\{0, b-3a+4\}$, and $Y(1,b)$ has no continuous group of projective transformations because $b-3a+4-2(b-a) = 4-a-b < 0$. In this case $Y(a,b)$ has projective moduli.

This implies that, except for $Y(1,2)^*$, the homaloidal hypersurfaces we constructed here cannot be related to pre-homogeneous vector spaces. The same holds for $Y(1,2)^*$, as we will see later (see Theorem 4.4(iii) and (iv), and Example 4.7).

We finish this section by producing families of homaloidal hypersurfaces in $\mathbb{P}^r$, which are different from the above ones as soon as $r \geqslant 4$. They do not seem to be related in general to hypersurfaces with vanishing Hessian. For $r = 3$ instead one essentially recovers the above examples.

Let $X \subset \mathbb{P}^r$ be a non-degenerate scroll surface of degree d with a line directrix L of multiplicity $e = r-2$, with $\mu = e = r-2$, i.e. such that there are $r-2$ variable rulings $F_{x,1}, \dots, F_{x,r-2}$ of X passing through the general point $x \in L$. According to Proposition 2.18, if $\nu \leqslant r-2$, i.e. if $F_{x,1}, \dots, F_{x,r-2}$ and L do not span a hyperplane, then X^* has vanishing Hessian. We will assume instead that $\nu = r-1$ and that the hyperplane $\langle L, F_{x,1}, \dots, F_{x,r-2}\rangle$ homographically varies in a pencil when x moves on L.

Example 3.17 *(Scrolls in $\mathbb{P}^r$ with line directrix, having $e = \mu = r-2$ and $\nu = r-1$).* Take a curve C of degree $n \geqslant 2r-5$ in a $(r-2)$-dimensional subspace Π of $\mathbb{P}^r$, having a $(n-r+2)$-secant $(r-4)$-dimensional subspace Π'. Assume also that the general hyperplane in Π through Π' cuts C, off Π, in $r-2$ independent points. Curves of this sort are not difficult to construct.

The first instance, is for $r=4$ in which case C is a plane curve of degree $n \geqslant 3$ with a point O of multiplicity $n-2$. Note that, for $r \geqslant 4$ these curves need not to be rational.

Take a line L in $\mathbb{P}^r$ skew with Π and set up an isomorphism between L and the pencil of hyperplanes through Π' in Π. Fix a general point $x \in L$, let Π_x be the corresponding hyperplane in Π through Π' and let $x_1, \dots, x_{r-2}$ be the intersection points of Π_x with C off Π. Then let $F_{x,i}$ be the line joining x with x_i, $i = 1, \dots, r-2$. As x varies on L, the lines $F_{x,1}, \dots, F_{x,r-2}$ describe a scroll X of the aforementioned type: the hyperplane $\langle L, F_{x,1}, \dots, F_{x,r-2}\rangle = \langle L, \Pi_x\rangle$ varies in the pencil of hyperplanes through $\langle L, \Pi'\rangle$.

The degree of such a scroll is $d = n+r-2$, as one sees by cutting it with a general hyperplane through L.

Theorem 3.18. *Let $X \subset \mathbb{P}^r$, be a non-degenerate scroll surface of degree d with a line directrix L of multiplicity $e = r-2$ and with $\mu = e = r-2$. Let $F_{x,1}, \dots, F_{x,r-2}$ be the variables rulings of X passing through the general point $x \in L$. Suppose that $\langle L, F_{x,1}, \dots, F_{x,r-2}\rangle$ is a hyperplane in $\mathbb{P}^r$ varying homographically in a pencil when x moves on L. Then $X^* \subset \mathbb{P}^{r*}$ is a homaloidal hypersurface.*

Proof. The space $\Sigma = L^\perp$ has multiplicity $d-r+2$ for $V = X^*$ and the general hyperplane $\xi = x^\perp$, $x \in L$, through Σ, cuts out on V a hypersurface V_ξ formed by Σ with multiplicity $d-r+2$ and $r-2$ more $(r-2)$-dimensional subspaces $\Sigma_i := F_{x,i}^\perp$, $i = 1, \dots, r-2$, such that $\Sigma \cap \Sigma_1 \cap \dots \cap \Sigma_{r-2} = \{p\}$, where $p = \langle L, F_{x,1}, \dots, F_{x,r-2}\rangle^\perp$. Hence, as ξ varies, p homographically describes a line Λ in Σ.

Consider the subspaces $T_{i,j} = \Sigma_i \cap \Sigma_j$, $1 \leqslant i < j \leqslant r-2$, which have multiplicity 2 for V_ξ, whereas p has multiplicity d for V_ξ. Note that all $T_{i,j}$, with $1 \leqslant i < j \leqslant r-2$, belong to the singular locus of V since they are intersections of two rulings of the scroll V. Furthermore, they all contain p.

Let now z be a general point in ξ, hence a general point in $\mathbb{P}^{r*}$. The polar hyperplane π_z of z with respect to X^* contains p. However it cannot contain the line Λ, otherwise all the polar hyperplanes would contain Λ and, by the reciprocity theorem, the first polars of X^* with respect to the points of Λ would vanish identically, i.e. the points of Λ would all have multiplicity d for X^*, which would be a cone, a contradiction, because X is non-degenerate.

This proves that if $\pi_z = \pi_{z'}$ then z' lies in $\xi = \langle z, \Sigma\rangle$. To finish our proof, we have to prove that all first polars through z intersect ξ only at z, off the singular locus of V. To see this, first consider the polars with respect to points $y \in \xi$, and containing z. By Remarks 2.8 and 3.2, the closure of their intersection off the singular locus of V is the line $\ell = \langle z, p\rangle$. There is finally one more independent polar through z which we have to take into account. It passes however with multiplicity $d-r+1$ through Σ, hence it cuts ξ along Σ counted with multiplicity $d-r+1$, plus another hypersurface V' of degree $r-2$, which, as we saw, contains all the subspaces $T_{i,j}$, $1 \leqslant i < j \leqslant r-2$. By applying Lemma 3.9, we see that V' has multiplicity $r-3$ at p. Hence V' intersects ℓ only in z and p, thus proving that z is the only point having the polar hyperplane π_z, i.e. the assertion. □

Remark 3.19. Note that the homaloidal hypersurfaces in Proposition 3.18 are reminiscent, in its structure, to the homaloidal hypersurface $F_4 = V(f^{(4)})$ in Theorem 4.4 below.

3.2. Examples in $\mathbb{P}^3$ revisited

In this section we want to revisit the examples of homaloidal surfaces $Y(1, d-1)^*$ in $\mathbb{P}^3$ of degree $d \geqslant 3$ constructed in Theorem 3.13. We want to analyze the singularities of these surfaces and understand the degree of the inverse of the polar map.

First of all, the scrolls $Y(1, d-1)$ are self-dual, i.e. $Y(1, d-1)^*$ is projectively equivalent to $Y(1, d-1)$ (see Proposition 1.7). The surface $Y(1, d-1)$ has a line L of multiplicity $d-1$ and no other singularity. One obtains the desingularization $S(1, d-1)$ of $Y(1, d-1)$ by simply blowing up L. The pull-back of L on $S(1, d-1)$ consists of the line directrix E plus $F_1, \dots, F_{d-2}$ rulings. This means that L is the intersection of $d-1$ distinct, generically smooth, branches, $X, X_1, \dots, X_{d-2}$ respectively corresponding to $E, F_1, \dots, F_{d-2}$. The branches $F_1, \dots, F_{d-2}$ intersect transversally at a general point of L, whereas the branch X glues with the branch X_i at the point O_i, which is the image of the intersection point O'_i of E with F_i, $i = 1, \dots, d-2$.

We want to resolve the singularities of the polar maps. We will see that, in order to do so, it is not sufficient to blow up L, but one has to perform further blowups.

In order to illustrate this, we analyze in detail the case $d = 3$. The other cases can be treated similarly, and we will briefly discuss them later.

Consider the surface F with equation:

$$f = x_2^3 - 2x_1x_2x_3 + x_0x_3^2 = 0.$$

It will be shown later that $F = Y(1, 2)$ (see Example 4.7). The partial derivatives of f are:

$$f_0 = x_3^2, \qquad f_1 = x_2x_3, \qquad f_2 = 3x_2^2 - 2x_1x_3, \qquad f_3 = -2x_1x_2 + 2x_0x_3.$$

The double line L has equation $x_2 = x_3 = 0$. Now we pass to affine coordinates $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$, $z = \frac{x_3}{x_0}$, so that the equation of the surface becomes:

$$z^2 + y^3 - 2xyz = 0,$$

whereas the first polars $V(f_i)$, $i = 0, \dots, 3$, become:

$$z^2 = 0, \qquad yz = 0, \qquad 3y^2 - 2xz = 0, \qquad z - xy = 0 \tag{3.2}$$

and L is the line $y = z = 0$. Blow up this line. To do this, introduce coordinates (x, y, ξ), the blowup map being:

$$(x, y, \xi) \to (x, y, y\xi).$$

The exceptional divisor M of the blowup has equation $y = 0$. The strict transform F' of the surface $F = Y(1, 2)$ has equation:

$$\xi^2 + y - 2x\xi = 0$$

which is smooth. A similar analysis at the infinity, shows that the singularities of $Y(1, 2)$ can be resolved with one single blowup along the double line L. The proper transform of L has now the equation:

$$y = 0, \qquad \xi(\xi - 2x) = 0,$$

which is the union of two smooth rational curves on M, meeting at the point $O' = (0,0,0)$ which maps to the origin O in $\mathbb{A}^3$, which is where the two branches of $Y(1,2)$ through L glue. Consider now the proper transform of the first polars

$$y\xi^2, \quad y\xi, \quad 3y - 2x\xi, \quad \xi - x. \tag{3.3}$$

We see that all these pass through O'. Thus, in order to resolve the singularities of the polar map, one still has to blow up O'—though, we emphasize, this is no longer a singular point of F'.

This tells us that the scheme $S = \mathrm{Sing}(Y(1,2))$ is not reduced: it consists of the line L with an embedded point at O. There is no need to blow up in order to understand the structure of this embedded point—it suffices to analyze the affine equations (3.2) of the first polars. The scheme in question is a subscheme of the surface of equation $z = xy$, whose coordinate ring is $k[x,y,z]/(z - xy) \simeq k[x,y]$. Hence we interpret the scheme S as the subscheme of $\mathbb{A}^2$ defined by the equations:

$$x^2y^2 = 0, \qquad xy^2 = 0, \qquad 3y^2 - 2x^2y = 0.$$

The line L, which has now equation $y = 0$ splits off, leaving a zero-dimensional scheme S' supported at the origin O, which is responsible for the embedded point of S. The equations of S' are:

$$x^2y = 0, \qquad xy = 0, \qquad 3y - 2x^2 = 0.$$

This is now a subscheme of the smooth curve C of equation $3y - 2x^2 = 0$, which is simply tangent to L at O. The coordinate ring of C is $k[x,y]/(3y - 2x^2) \simeq k[x]$ and the scheme S' has now the equations $x^3 = 0$. Summing up, the embedded point at the origin is due to the fact that all the polars have multiplicity of intersection 4 with the curve C at O. We thus see that we will have to blow up along L and then three more times at subsequent infinitely near points to resolve the singularities of the polar map.

Remark 3.20. Note that, after the first blowup, the polar system is given by the system (3.3). The base point scheme is now zero-dimensional supported at O'. Indeed it is a curvilinear scheme T of length 3 along the proper transform C' of the curve C, defined by the equations $x = \xi$, $3y = 2x^2$. Note however that F' has only intersection multiplicity 2 with C' at O'. This means that F' does not contain T. In other words, the fourth (and last) point, infinitely near to O, to be blown up in order to resolve the singularities of the polar map, does not even belong to the original surface F.

The above analysis gives another reason why $Y(1,2)$ is a homaloidal surface. Indeed the polar system is formed by quadrics through L. The general such quadric is smooth, as we see from the equations of the polars or from (3.2). The residual intersections of two general polars off L are rational normal cubics, i.e., curves of type $(1,2)$ on the general such quadric. The self-intersection of these curves is therefore 4. However, the curves in question have to contain the 0-dimensional scheme of length 3 supported at O, which is responsible for the embedded point of $\mathrm{Sing}(Y(1,2))$ on L. This drops the self-intersection of the system of cubic curves to 1 and explains why the polar map is birational.

We emphasize that the polar map is a quadratic transformation of $\mathbb{P}^3$ which is a degenerate case of the well-known quadratic transformation defined by all quadrics passing through a given line L and three distinct general points p_1, p_2, p_3 (see [7,30]).

Remark 3.21. It is worth comparing the behavior of the polar map of $Y(1,2)$ with the one of the general projection Y of $S(1,2)$ to $\mathbb{P}^3$. We may think of Y as the surface defined by the equation $x_1(x_2^2+x_3^2)-2x_0x_2x_3=0$, whose double line L has the equations $x_2=x_3=0$. The resolution of the singularities of Y is obtained by blowing up along L. In this way one recovers $S(1,2)$, and the proper transform of L is a conic C, which projects $2:1$ to L, with two branch points, located at the points O_1, O_2 with affine coordinates $(0,0,0)$ and $(1,0,0)$. The scheme $\mathrm{Sing}(Y)$ consists of L with two embedded points of length 2 at O_1 and O_2. This yields degree 2 for the polar map. The surface $Y(1,2)$ can be thought of as obtained from Y when O_1 and O_2 collapse together. Indeed the conic C then splits as the union of the line directrix E of $S(1,2)$ and a ruling. This also clarifies why $Y(1,2)$ coincides with the surface F, which is a member of a series of homaloidal hypersurfaces under the general name of sub-Hankel hypersurfaces, to be dealt with in the next section—in the notation of that subsection, one has $F=V(f^{(3)})$ (see Example 4.7).

The analysis of the general case $Y(1,d-1)$ is similar. The general point p of L has multiplicity $d-1$ and it is the intersection of $d-1$ smooth branches of $Y(1,d-1)$ containing L and pairwise intersecting transversally along L around p. There are however $d-2$ points $O_1,\dots,O_{d-2}$ on L around which $Y(1,d-1)$ looks like the union of $d-3$ branches which intersect transversally along L around O_i, plus another branch which is analytically equivalent to $Y(1,2)$ at O and which is generically located with respect to the previous $d-3$ branches. The singularities of $Y(1,d-1)$ can be resolved by blowing up along L: in this way one obtains $S(1,d-1)$ and the blowingup map is nothing but the projection $S(1,d-1)\to Y(1,d-1)$.

The general polar has a point of multiplicity $d-2$ at a general point of L. It is again resolved when we blow up L. However, for the same reason as in the case of $Y(1,2)$, after blowing up, there are $d-2$ curvilinear schemes of length 3 supported at each of the points $O'_1,\dots,O'_{d-2}$, which belong to the base locus of the proper transform of the polar system. Another way of saying this is that there are $d-2$ embedded points $O_1,\dots,O_{d-2}$ in the scheme structure of $\mathrm{Sing}(Y(1,d-1))$ along L supported at $O_1,\dots,O_{d-2}$.

Again this explains the reason why the polar map is birational. Let Φ be the proper transform of the general first polar after having blown up L. This is a rational scroll. Let us denote by R the general ruling and by D the proper transform of L, which is a section. If H is the pull-back of a general plane section, we have $H\equiv D+R$. Since $H^2=d-1$, we find $D^2=d-3$. If Γ is the trace on Φ of the proper transform $\mathcal{L}$ of the polar system, we have $\Gamma\equiv(d-1)H-(d-2)D\equiv(d-1)R+D$, thus $\Gamma^2=3d-5$. Notice however that the trace of $\mathcal{L}$ on Φ has $d-2$ base point schemes each of length 3. After having further blown up these base point schemes, this reduces the self-intersection of $\mathcal{L}$ to 1.

The analysis is more complicated for the surfaces $Y(a,b;m_1,\dots,m_h)$, $m_1+\cdots+m_h=d-2$, described in Remark 3.15. We will merely outline the results that can be checked by a careful treatment. The singularity can still be resolved with a simple blowup along L thus getting $S(1,d-1)$. The proper transform of L is now $E+m_1F_1+\cdots+m_hF_h$. This means that L is the intersection of $h+1$ branches, a smooth one X, corresponding to the line directrix E of $S(1,d-1)$, the other branches $X_1,\dots,X_h$ are instead cuspidal of orders $m_1,\dots,m_h$ corresponding to the rulings $F_1,\dots,F_h$ respectively.

As for the degree of the inverse map, one has the following. First, the degree of the inverse map of the polar map ϕ of $Y(1, d-1)$ coincides with the degree of the image of a general plane π via ϕ. The linear system cut out on π by the system of the first polars, is a 3-dimensional linear system of curves of degree $d-1$ with only one ordinary base point x of multiplicity $d-2$, i.e. x is the intersection of L with π. Thus the image of π has degree $(d-1)^2-(d-2)^2=2d-3$. Note that, for $d=3$, one retrieves the expected degree of the inverse to the polar map of the specialized Hankel determinant (see Remark 4.6(c))

Consider now the surface $Y(a,b;m_1,\dots,m_h)$ with $m_1+\cdots+m_h=d-2$. The general first polar has again a point of multiplicity $d-2$ at a general point $x\in L$. Moreover, a local computation shows that it has tangency of order m_i-1 along the plane π_i tangent to the branch X_i, $i=1,\dots,h$. Arguing as above, we see that this decreases the degree of the inverse of ϕ by $\sum_{i=1}^{h}(m_i-1)=d-2-h$. In particular, if $h=1, m_1=d-2$, then we have the maximal drop of the degree of the inverse, namely $d-3$, i.e. the degree of the inverse of ϕ is d.

It would be interesting to have a similar analysis in $\mathbb{P}^r$, for $r>3$.

4. Some determinantal homaloidal polynomials

In this section we bring up a series of examples of homaloidal polynomials which can be treated in an algebraic fashion. Some of the proofs, though elementary in spirit, are nevertheless quite involved.

4.1. Degenerations of Hankel matrices

First we need a few algebraic concepts (see [37] for more contextual details).

Definition 4.1. Let R be a Noetherian ring and let $I\subset R$ be an ideal.

(1) Let $\mathcal{S}_R(I)\twoheadrightarrow\mathcal{R}_R(I)$ denote the structural graded R-algebra homomorphism from the symmetric algebra of I to its *Rees algebra*, i.e. the R-algebra that defines the blowup along the subscheme corresponding to the ideal I (see [12, Section 5.2]). We say that I is of *linear type* if this map is injective.
(2) If R is a Noetherian local ring (or a standard graded ring over a field) the ideal I is said to be *perfect* if it has finite homological (i.e., projective) dimension over R and this attains its minimal possible value, namely, the codimension of I (see [12, p. 485]). It is known that if R is moreover a Cohen–Macaulay ring (e.g., regular) then an ideal I is perfect if and only if R/I is Cohen–Macaulay.
(3) An ideal $I\subset R$ of linear type satisfies the *Artin–Nagata condition* G_∞ (see [2]) which states that the minimal number of generators of I locally at any prime $p\in\operatorname{Spec}(R)$ is at most the codimension of p. This condition is equivalent to a condition in terms of a free presentation

$$R^m\xrightarrow{\varphi}R^n\longrightarrow I\longrightarrow 0$$

of I, namely:

$$\operatorname{cod}\bigl(I_t(\varphi)\bigr)\geqslant\operatorname{rank}(\varphi)-t+2,\quad\text{for } 1\leqslant t\leqslant\operatorname{rank}(\varphi),\tag{4.1}$$

where $I_t(\varphi)$ denotes the determinantal ideal of the $t\times t$ minors of a representative matrix of φ (see, e.g., [49, Section 1.3]).

(4) Suppose that R is standard graded over a field k and I is generated by forms of a given degree s. In this case, I is more precisely given by means of a free graded presentation

$$R\bigl(-(s+1)\bigr)^{\ell} \oplus \sum_{j\geqslant 2} R\bigl(-(s+j)\bigr) \xrightarrow{\varphi} R(-s)^n \to I \to 0$$

for suitable ℓ. We call the image of $R(-(s+1))^{\ell}$ by φ the *linear part* of φ and say that the corresponding submatrix φ_1 has *maximal rank* if its rank is $n-1$ $(=\operatorname{rank}(\varphi))$. Clearly, the latter condition is trivially satisfied if $\varphi_1 = \varphi$, in which case I is said to have *linear presentation* (or is *linearly presented*).

We remark that such an ideal, if it is of linear type, then it is generated by algebraically independent elements over k. In particular, if $R = k[\mathbf{x}] = k[x_0, \ldots, x_r]$ and I happens to be of linear type and generated by $r+1$ forms of the same degree then these forms define a dominant rational map $\mathbb{P}^r \dashrightarrow \mathbb{P}^r$.

4.1.1. Arithmetic of sub-Hankel matrices

So much for generalities. We now introduce the main object of this part, which is a degeneration of a generic Hankel matrix over a polynomial ring by specializing convenient entries to zero (see [38] for further classes of specializations of square generic matrices whose determinants are often homaloidal, treated within the general framework of the theory of ideals).

Let $y_1, \ldots, y_{r+1}$ be variables over a field k and set

$$M^{(r)} = M^{(r)}(y_1, \ldots, y_{r+1}) = \begin{pmatrix} y_1 & y_2 & y_3 & \ldots & y_{r-1} & y_r \\ y_2 & y_3 & y_4 & \ldots & y_r & y_{r+1} \\ y_3 & y_4 & y_5 & \ldots & y_{r+1} & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ y_{r-1} & y_r & y_{r+1} & \ldots & 0 & 0 \\ y_r & y_{r+1} & 0 & \ldots & 0 & 0 \end{pmatrix}$$

Note that the matrix has two tags: the upper index (r) indicates the size of the matrix, while the variables enclosed in parentheses are the total set of variables used in the matrix. We call attention to the notation as several of these matrices will be considered with variable tags throughout, though we will often omit the list of variables if they are sufficiently clear from the context.

This matrix will be called a *generic sub-Hankel matrix*; more precisely, $M^{(r)}$ is the *generic sub-Hankel matrix of order r on the variables* $y_1, \ldots, y_{r+1}$. Its determinant, a form of degree r, will be the central object of this section. Throughout we fix a polynomial ring $k[x_0, \ldots, x_r]$ which will be the source of all lists of variables appearing in the various such matrices considered heretofore. We will denote by $f^{(r)}(x_0, \ldots, x_r)$ the determinant of $M^{(r)}(x_0, \ldots, x_r)$ for any $r \geqslant 1$, and we set $f^{(0)} = 1$. We also set $\phi^{(j)} = \phi^{(j)}(x_{r-j}, \ldots, x_r) := f^{(j)}(x_{r-j}, \ldots, x_r)$.

We now head on to the main result concerning generic sub-Hankel matrices. First we need the following algebraic structural lemmas about the partial derivatives of $f^{(r)}$.

Lemma 4.2. *Let $r \geqslant 2$. Then:*

(i) *One has*

$$\frac{\partial f^{(r)}}{\partial x_i} = (-1)^r x_r \frac{\partial \phi^{(r-1)}}{\partial x_{i+1}}, \quad 0 \leqslant i \leqslant r-2. \tag{4.2}$$

(ii) *For* $0 \leqslant i \leqslant r-1$, *one has*

$$\frac{\partial f^{(r)}}{\partial x_0}, \dots, \frac{\partial f^{(r)}}{\partial x_i} \in k[x_{r-i}, \dots, x_r] \tag{4.3}$$

and the g.c.d. of these partial derivatives is x_r^{r-i-1}.

(iii) *For any* i *in the range* $1 \leqslant i \leqslant r-1$, *the following holds:*

$$x_r \frac{\partial f^{(r)}}{\partial x_i} = -\sum_{k=0}^{i-1} \frac{2i-k}{i} x_{r-i+k} \frac{\partial f^{(r)}}{\partial x_k}. \tag{4.4}$$

Moreover,

$$x_r \frac{\partial f^{(r)}}{\partial x_r} = (r-1)x_0 \frac{\partial f^{(r)}}{\partial x_0} + (r-2)x_1 \frac{\partial f^{(r)}}{\partial x_1} + \cdots + x_{r-2} \frac{\partial f^{(r)}}{\partial x_{r-2}}. \tag{4.5}$$

Proof. (i) We induct on r. For $r = 2$, the relation is readily seen to hold. To proceed, introduce the following sign function on integers: $\xi(r) = 1$ if $r \equiv 1, 2 \pmod 4$ and $\xi(r) = -1$ if $r \equiv 0, 3 \pmod 4$. The following identity is easily established:

$$\xi(j)\xi(j-1) = (-1)^j. \tag{4.6}$$

Equivalently one has

$$(-1)^{j+1}\xi(j) = -\xi(j-1). \tag{4.7}$$

Assume that $r \geqslant 3$. Expanding $f^{(r)}$ by Laplace along the first row one finds

$$f^{(r)} = -\xi(r) \sum_{j=0}^{r-1} \xi(j) x_j x_r^{r-j-1} \phi^{(j)}. \tag{4.8}$$

By the same token, expanding $\phi^{(r-1)}$ by Laplace along the first row one finds

$$\phi^{(r-1)} = -\xi(r-1) \sum_{j=1}^{r-1} \xi(j-1) x_j x_r^{r-j-1} \phi^{(j-1)}. \tag{4.9}$$

Suppose now $0 \leqslant i \leqslant r-1$. Taking x_i-derivatives of both sides of (4.8), for i in this range, yields

$$\frac{\partial f^{(r)}}{\partial x_i} = -\xi(r) \left(\xi(i) x_r^{r-i-1} \phi^{(i)} + \sum_{j=1}^{r-1} \xi(j) x_j x_r^{r-j-1} \frac{\partial \phi^{(j)}}{\partial x_i} \right). \tag{4.10}$$

Similarly, taking x_{i+1}-derivatives of both sides of (4.9) in the range $0 \leqslant i \leqslant r-2$, yields

$$\frac{\partial \phi^{(r-1)}}{\partial x_{i+1}} = -\xi(r-1)\left(\xi(i)x_r^{r-i-2}\phi^{(i)} + \sum_{j=1}^{r-1}\xi(j-1)x_j x_r^{r-j-1}\frac{\partial \phi^{(j-1)}}{\partial x_{i+1}}\right). \tag{4.11}$$

Thus, by the inductive hypothesis applied to $f^{(i)}$, with $i<r$, hence to $\phi^{(i)}$, with $i<r$, and by the identity (4.6), we find

$$\begin{aligned} x_r\frac{\partial \phi^{(r-1)}}{\partial x_{i+1}} &= -\xi(r-1)\left(\xi(i)x_r^{r-i-1}\phi^{(i)} + \sum_{j=1}^{r-2}\xi(j)x_j x_r^{r-j-1}\left[\xi(j)\xi(j-1)x_r\frac{\partial \phi^{(j-1)}}{\partial x_{i+1}}\right]\right) \\ &= -\xi(r-1)\left(\xi(i)x_r^{r-i-1}\phi^{(i)} + \sum_{j=1}^{r-1}\xi(j)x_j x_r^{r-j-1}\frac{\partial \phi^{(j)}}{\partial x_i}\right). \end{aligned}$$

Multiplying the last line above by $\xi(r-1)$, using (4.7), and drawing upon (4.10) as multiplied by $\xi(r)$, one obtains

$$\xi(r)\frac{\partial f^{(r)}}{\partial x_i} = \xi(r-1)x_r\frac{\partial \phi^{(r-1)}}{\partial x_{i+1}}. \tag{4.12}$$

Since $\xi(r)\xi(r-1)=(-1)^r$ for every $r \geqslant 1$ by (4.6), Eq. (4.12) yields (4.2).

(ii) We induct on r. Both assertions are readily verified for $r=2$ since $f^{(2)} = x_0x_2 - x_1^2$. Thus, assume that $r \geqslant 3$. Note that if $i<r-j$, the form $\phi^{(j)}$ does not involve the variable x_i, hence all its derivatives with respect to x_i vanish, for $0 \leqslant i \leqslant r-2$ and $j<r-i$. Thus, using (4.10) we immediately see that (4.3) holds. As for the assertion about the g.c.d., this is easy in the range $0 \leqslant j \leqslant r-2$, since it follows form the expressions (4.10) and the inductive hypothesis applied to $f^{(i)}$, hence to $\phi^{(i)}$, with $i<r$.

As for $i=r-1$, we still have the expression

$$\frac{\partial f^{(r)}}{\partial x_{r-1}} = -\xi(r)\left(\xi(r-1)\phi^{(r-1)} + \sum_{j=1}^{r-1}\xi(j)x_j x_r^{r-j-1}\frac{\partial \phi^{(j)}}{\partial x_{r-1}}\right), \tag{4.13}$$

coming from (4.10). To prove the assertion about the *g.c.d.*, replace $\phi^{(r-1)}$ by its Euler expansion in (4.13) and collect the two terms in $x_{r-1}\,(\partial\phi^{(r-1)}/\partial x_{r-1})$. We get

$$\begin{aligned} \frac{\partial f^{(r)}}{\partial x_{r-1}} &= \xi(r)\left(\sum_{j=1}^{r-2}\xi(j)x_j x_r^{r-j-1}\frac{\partial \phi^{(j)}}{\partial x_{r-1}}\right) \\ &\quad + (-1)^{r+1}\frac{1}{r-1}\left(\sum_{j=1}^{r-2}x_j\frac{\partial \phi^{(r-1)}}{\partial x_j} + (r-1)x_{r-1}\frac{\partial \phi^{(r-1)}}{\partial x_{r-1}} + x_r\frac{\partial \phi^{(r-1)}}{\partial x_r}\right). \end{aligned} \tag{4.14}$$

Now the g.c.d. of the derivatives up to order $i=r-2$ was found to be $x_r^{r-i-1}=x_r$. If the derivatives up to order $i=r-1$ would have a nonunit g.c.d. then it had to be x_r. Thus, assume as if it were that x_r divides the left-hand side in (4.14). Since x_r divides the first summand in the right-hand side of (4.14) and, by the inductive hypothesis applied to $f^{(r-1)}$, x_r divides the summands $x_j(\partial\phi^{(r-1)}/\partial x_j)$, for $1 \leqslant j \leqslant r-2$, then x_r would divide the derivative $\partial\phi^{(r-1)}/\partial x_{r-1}$, which would contradict the inductive hypothesis as applied to $f^{(r-1)}$.

(iii) We begin with (4.5). The formula is readily verified for $r=2$ so we induct on $r \geqslant 3$. Taking x_r-derivatives in (4.8), multiplying by x_r we get

$$\begin{aligned}
&x_r\frac{\partial f^{(r)}}{\partial x_r} - \xi(r)\left(\sum_{j=0}^{r-2}(r-j-1)\xi(j)x_jx_r^{r-j-1}\phi^{(j)} + \sum_{j=0}^{r-1}\xi(j)x_jx_r^{r-j-1}\left(x_r\frac{\partial\phi^{(j)}}{\partial x_r}\right)\right)\\
&\quad = -\xi(r)\left(\sum_{i=0}^{r-2}(r-i-1)\xi(i)x_ix_r^{r-i-1}\phi^{(i)} + \sum_{j=0}^{r-1}\xi(j)x_jx_r^{r-j-1}\left(\sum_{i=0}^{r-2}(r-i-1)x_i\frac{\partial\phi^{(j)}}{\partial x_i}\right)\right)\\
&\quad = -\xi(r)\sum_{i=0}^{r-2}(r-i-1)x_i\left(\xi(i)x_r^{r-i-1}\phi^{(i)} + \sum_{j=0}^{r-1}\xi(j)x_jx_r^{r-j-1}\frac{\partial\phi^{(j)}}{\partial x_i}\right)\\
&\quad = \sum_{i=0}^{r-2}(r-i-1)x_i\frac{\partial f^{(r)}}{\partial x_i}
\end{aligned}$$

where in the second line we applied the inductive hypothesis to $f^{(\ell)}$, for every $l = 1,\dots,r-1$, to wit

$$x_r\frac{\partial\phi^{(l)}}{\partial x_r} = \sum_{j=r-l}^{r-2}(r-j-1)x_j\frac{\partial\phi^{(l)}}{\partial x_j} = \sum_{j=0}^{r-2}(r-j-1)x_j\frac{\partial\phi^{(l)}}{\partial x_j} \tag{4.15}$$

and in the fourth line we used the expression obtained from multiplying (4.10) both sides by x_i, for $i = 0,\dots,r-1$.

We now prove formula (4.4). In the range $0 \leqslant i \leqslant r-2$ the formula follows from (4.2). Indeed, the formula is easily obtained for $r=2$. Inducting on r in this range, we assume that

$$x_r\frac{\partial\phi^{(r-1)}}{\partial x_{i+1}} = -\sum_{j=1}^{i}\frac{2i+1-j}{i}x_{r-i-1+j}\frac{\partial\phi^{(r-1)}}{\partial x_j} \tag{4.16}$$

holds. Therefore

$$\begin{aligned}
x_r\frac{\partial f^{(r)}}{\partial x_i} &= (-1)^r x_r\left(x_r\frac{\partial\phi^{(r-1)}}{\partial x_{i+1}}\right)\\
&= -\left(\sum_{j=1}^{i}\frac{2i+1-j}{i}x_{r-i-1+j}\left[(-1)^r x_r\frac{\partial\phi^{(r-1)}}{\partial x_j}\right]\right)\\
&= -\left(\sum_{j=1}^{i}\frac{2i+1-j}{i}x_{r-i-1+j}\frac{\partial f^{(r)}}{\partial x_{j-1}}\right)
\end{aligned}$$

$$= -\left(\sum_{j=0}^{i-1} \frac{2i-j}{i} x_{r-i+j} \frac{\partial f^{(r)}}{\partial x_j}\right),$$

as was to be shown.

It remains to get the case where $i = r-1$. For this first note that a repeated use of (4.2) yields, for every $j = 1, \dots, r-1$, the relation

$$\xi(r)\frac{\partial f^{(r)}}{\partial x_{j-1}} = \xi(j) x_r^{r-j} \frac{\partial \phi^{(j)}}{\partial x_{r-1}}. \tag{4.17}$$

Applying (4.5) to $\phi^{(r-1)}$ and using Euler's formula yields

$$\begin{aligned}
(-1)^r x_r\big[(r-1)\phi^{(r-1)}\big] &= \sum_{k=1}^{r-1} x_k \left[(-1)^r x_r \frac{\partial \phi^{(r-1)}}{\partial x_k}\right] + \big((-1)^r x_r\big)\left[x_r \frac{\partial \phi^{(r-1)}}{\partial x_r}\right] \\
&= \sum_{k=1}^{r-1} x_k \frac{\partial f^{(r)}}{\partial x_{k-1}} + \sum_{k=1}^{r-1} (r-1-k) x_k \left[(-1)^r x_r \frac{\partial \phi^{(r-1)}}{\partial x_k}\right] \\
&= \sum_{k=1}^{r-1} (r-k) x_k \frac{\partial f^{(r)}}{\partial x_{k-1}}. \qquad (4.18)
\end{aligned}$$

Combining (4.13) with (4.17) and (4.18), we get

$$\begin{aligned}
(r-1)x_r \frac{\partial f^{(r)}}{\partial x_{r-1}} &= -\xi(r)\left(\xi(r-1)x_r\big[(r-1)\phi^{(r-1)}\big] + (r-1)\sum_{j=1}^{r-1} x_j \left[\xi(j) x_r^{r-j} \frac{\partial \phi^{(j)}}{\partial x_{r-1}}\right]\right) \\
&= -\left((-1)^r x_r\big[(r-1)\phi^{(r-1)}\big] + (r-1)\sum_{k=1}^{r-1} x_k \frac{\partial f^{(r)}}{\partial x_{k-1}}\right) \\
&= -\sum_{k=1}^{r-1} (2r-1-k) x_k \frac{\partial f^{(r)}}{\partial x_{k-1}},
\end{aligned}$$

proving (4.4) also in this case.

This completes the proof of the lemma. □

Proposition 4.3. *Let $r \geqslant 2$. Set $f = f^{(r)}$. Then upon factoring out the g.c.d. of $\partial f/\partial x_0, \dots, \partial f/\partial x_i$, the resulting polynomials generate a codimension two perfect ideal $J_i \subset k[x_{r-i}, \dots, x_r]$ of linear type with linear presentation.*

Proof. The assertion is readily checked for $r = 2$ since $f^{(2)} = x_0x_2 - x_1^2$. Thus, we assume henceforth that $r \geqslant 3$.

Fix an i in the range $1 \leqslant i \leqslant r-1$. Let J_i denote the ideal of the ring $R^{[i]} = k[x_{r-i}, \dots, x_r]$ generated by the partial derivatives of $f = f^{(r)}$ with respect to $x_0, \dots, x_i$ divided by x_r^{r-i-1}.

We claim that the presentation matrix of J_i is an $(i+1)\times i$ recurrent matrix having the form

$$\Phi^{[i]} = \left(\begin{array}{c|c} 2\,x_{r-i} & \\ \frac{2i-1}{i}\,x_{r-i+1} & \\ \vdots & \Phi^{[i-1]} \\ \frac{i+1}{i}\,x_{r-1} & \\ x_r & \mathbf{0} \end{array}\right), \qquad \mathbf{0} = \underbrace{(0,\ldots,0)}_{i-1}, \tag{4.19}$$

where the first column comes from (4.4). By induction on i, one has that the last $i-1$ columns of $\Phi^{[i]}$ are relations of J_i, hence the full matrix $\Phi^{[i]}$ is a matrix of relations of J_i and, moreover, its linear part has maximal rank $(=i)$.

On the other hand, by a well-known acyclicity criterion (see, e.g., [4]), in this case it suffices to check that the columns of $\Phi^{[i]}$ are relations of the generators of J_i and the determinantal ideal $I_i(\Phi^{[i]})$ has codimension at least 2. Thus, we are left with finding two i-minors of $\Phi^{[i]}$ without non-trivial common factor. Let δ_1 (respectively, δ_2) denote the minor obtained by deleting the first (respectively, the last) row of $\Phi^{[i]}$. By induction on i, $\delta_1 = \pm x_r^i$ and δ_2 admits a summand of the form $\pm(i+1)x_{r-1}^i$ that results from multiplying the entries along the anti-diagonal of the first i rows—indeed, by (4.4), taking $k = i-1$, the coefficient of the $(i, i-1)$ entry is $(i+1)/i$, hence the product is $(i+1/i)(i/i-1)\cdots(3/2)(2/1) = i+1$. It follows that δ_1 is not divisible by x_r. Therefore, $I_i(\Phi^{[i]})$ has codimension at least 2, as required.

What we have proved so far is that J_i has a Hilbert–Burch resolution, and since it has codimension at least 2 then it is a codimension two perfect ideal. So, it remains to prove the last contention of this item, namely, that J_i is an ideal of linear type. By [49, Corollary 1.4.2 and Theorem 3.1.6] this will be the case if the inequalities in (4.1) are fulfilled.

Since $\operatorname{cod}(I_i(\Phi^{[i]})) \geqslant 2 = i - i + 2 = \operatorname{rank}(\Phi^{[i]}) - i + 2$, we only have to check that

$$\operatorname{cod}\bigl(I_t\bigl(\Phi^{[i]}\bigr)\bigr) \geqslant i - t + 2, \quad \text{for } 1 \leqslant t \leqslant i-1. \tag{4.20}$$

We proceed by induction on i, so $\operatorname{cod}(I_t(\Phi^{[i-1]})) \geqslant i-1-t+2 = i-t+1$ holds true in the range $1 \leqslant t \leqslant i-2$. Therefore, one needs, for every t in the range $1 \leqslant t \leqslant i-1$, an additional t-minor of $\Phi^{[i]}$ which is a nonzero-divisor on the ideal $I_t(\Phi^{[i-1]})$. Since $\Phi^{[i-1]}$ has entries in the polynomial ring $R^{[i-1]} = k[x_{r-i+1},\ldots,x_r]$, it suffices to show that there exists such a minor effectively involving the extra variable x_{r-i}. Supposing this were not the case, the full matrix of relations $\Phi^{[i]}$ would have entries entirely contained in the ring $R^{[i-1]}$ and, since the generators of J_i are the maximal minors of $\Phi^{[i]}$, they would all belong to $R^{[i-1]}$, which is not the case.

This finishes the proof of the last statement. □

4.1.2. Sub-Hankel polynomials are homaloidal

From the previous lemma ensues the following geometric result.

Theorem 4.4. *Let $r \geqslant 2$. Set $f = f^{(r)}$ and $J = (\partial f/\partial x_0,\ldots,\partial f/\partial x_r)$. Then:*

(i) *For every value of i in the range $1 \leqslant i \leqslant r-1$, the partial derivatives $\partial f/\partial x_0,\ldots,\partial f/\partial x_i$ divided by their common g.c.d. define a Cremona transformation of $\mathbb{P}^i$; in addition, the base ideal of the inverse map is also a codimension two perfect ideal of linear type and both ideals are generated in degree i.*

(ii) *The linear part of the graded presentation matrix of J has maximal rank.*
(iii) *The Hessian of f has the form $h(f) = c\,x_r^{(r+1)(r-2)}$, $c \in k, c \neq 0$.*
(iv) *f is homaloidal.*

Proof. (i) This follows from Proposition 4.3 and [37, Example 2.4] (see also Proposition 4.5(iii)).

(ii) Again from Proposition 4.3 we know that J_{r-1} is linearly presented, generated by $\partial f/\partial x_0, \ldots, \partial f/\partial x_{r-1}$ (since for the value $i = r-1$, the g.c.d. is 1), hence it yields a chunk of rank $r-1$ of the linear part of the graded presentation matrix of J.

In addition, by (4.4) there is a linear relation with last coordinate x_r—hence, nonzero. Clearly then the rank of the full presentation matrix of J has rank at least r. Since this is the maximal possible value of the rank, the linear part of the matrix has maximal rank.

(iii) By (4.3) one had $\partial f/\partial x_i \in k[x_{r-i}, \ldots, x_r]$, hence $\partial^2 f/\partial x_i \partial x_j = 0$ for every $j < r-i$, or equivalently, $\partial^2 f/\partial x_i \partial x_j = 0$ for all pairs i, j such that $i + j \leqslant r-1$. This means that the Hessian matrix is anti-lower triangular (i.e., all zeros below the anti-diagonal). Therefore the determinant is the product of the entries along its anti-diagonal, namely, $\partial^2 f/\partial x_i \partial x_{r-i}$, for $i = 0, \ldots, r$.

We now see that

$$\frac{\partial^2 f}{\partial x_i \partial x_{r-i}} = c_i x_r^{r-2},$$

for suitable nonzero constants $c_i \in k$. To calculate these derivatives, we induct on r. One may assume at the outset that $0 \leqslant i \leqslant r-1$ as otherwise one changes the roles of i and $r-i$ not affecting the result except for the value of the nonzero coefficient. Applying (4.2) in this range we obtain

$$\frac{\partial^2 f}{\partial x_i \partial x_{r-i}} = \pm x_r \frac{\partial^2 \phi^{(r-1)}(x_1, \ldots, x_r)}{\partial x_{i+1} \partial x_{r-i}}. \tag{4.21}$$

By the inductive hypothesis applied to $f^{(r-1)}$, we deduce that

$$\frac{\partial^2 \phi^{(r-1)}(x_1, \ldots, x_r)}{\partial x_{i+1} \partial x_{r-i}} = c_i x_r^{r-3}.$$

Substituting in (4.21), we get the stated values.

It now follows that $h(f) = c x_r^{(r+1)(r-2)}$, $c = \Pi c_i \neq 0$.

(iv) We will show that the polar map ϕ_f is a Cremona map by drawing upon results from [47] and [38]. Since the latter is not yet published, we choose to state the method *ab initio*, in the form of a self-contained proposition adapted to our present purpose.

Proposition 4.5. *Let $\phi = (F_0 : \cdots : F_r) : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ be a rational map where $F_0, \ldots, F_r$ are forms of the same degree generating an ideal $I \subset k[\mathbf{x}]$ of codimension $\geqslant 2$. Set $k[\mathbf{x}, \mathbf{y}]$ for the bihomogeneous coordinate ring of $\mathbb{P}^r \times \mathbb{P}^r$ and consider the bigraded incidence k-algebra*

$$\mathcal{A} = k[\mathbf{x}, \mathbf{y}]/I_1(\mathbf{y} \cdot \varphi_1),$$

defined by the ideal of entries of the product matrix $(\mathbf{y}) \cdot \varphi_1$*, where* φ *denotes a graded presentation matrix of* I *over* $k[\mathbf{x}]$*. Finally, let* $\mathcal{R} = \mathcal{R}_{k[\mathbf{x}]}(I)$ *stand for the Rees algebra of the ideal* $I \subset k[\mathbf{x}]$*. Then:*

(i) *There is a surjective map of bigraded* k*-algebras* $\rho : \mathcal{A} \twoheadrightarrow \mathcal{R}$*.*
(ii) *If the Jacobian determinant of* $F_0, \dots, F_r$ *is nonzero and if* $\ker(\rho)$ *is a minimal prime of* $\mathcal{A}$ *then* ϕ *is a Cremona map.*
(iii) *If the Jacobian determinant of* $F_0, \dots, F_r$ *is nonzero and if* φ_1 *has maximal rank* r *then* ϕ *is a Cremona map.*

Proof. (i) This is a general algebraic fact: there is a structural surjection $\mathcal{S} \twoheadrightarrow \mathcal{R}$ where $\mathcal{S}$ stands for the symmetric algebra of I. Since $\mathcal{S} \simeq k[\mathbf{x}, \mathbf{y}]/I_1(\mathbf{y} \cdot \varphi)$, where φ is the full presentation matrix of I, there is a natural surjection $\mathcal{A} \twoheadrightarrow \mathcal{S}$.

(ii) Let $V \subset \mathbb{P}^r \times \mathbb{P}^r$ stand for the subscheme whose bihomogeneous coordinate ring is $\mathcal{A}$ and let $\Gamma \subset \mathbb{P}^r \times \mathbb{P}^r$ stand for the irreducible subvariety whose bihomogeneous coordinate ring is $\mathcal{R}$, i.e. Γ is the closure of the graph of ϕ. Let $V_1, \dots, V_r$ denote the distinct irreducible components of V_{red} where, say, $V_1 = \Gamma$. Let $\pi_2 : V \to \mathbb{P}^r$ denote the second projection restricted to V and let $p_2 : \Gamma \to \mathbb{P}^r$ stand for its restriction to Γ. Since $p_2(\Gamma) = \mathbb{P}^r$, we have $\pi_2^{-1}(p) \neq \emptyset$ for every $p \in \mathbb{P}^r$. By the nature of V, given any point $p \in \mathbb{P}^r$, there is a non-negative integer $s(p)$ such that the scheme theoretic fiber $\pi_2^{-1}(p)$ is of the form $\mathbb{P}^{s(p)} \times \{p\}$, linearly embedded in $\mathbb{P}^r \times \{p\}$. Since $\mathbb{P}^{s(p)} \times \{p\}$ is irreducible and reduced, for every $p \in \mathbb{P}^r$ one has $\pi_2^{-1}(p) = \mathbb{P}^{s(p)} \times \{p\} \subseteq V_i$ for some $i = i(p)$. Moreover,

$$p_2^{-1}(p) = \Gamma \cap \pi_2^{-1}(p) \subseteq V_1 \cap V_{i(p)},$$

as schemes. On the other hand, for every $i \geqslant 2$ we have $\dim(V_1 \cap V_i) < \dim(V_1) = r$ so that $\dim(p_2(V_1 \cap V_i)) < r$ for every $i \geqslant 2$. Let

$$W = \bigcup_{i \geqslant 2} p_2(V_1 \cap V_i) \subsetneq \mathbb{P}^r.$$

Then for every $p \in \mathbb{P}^r \setminus W$ we have

$$p_2^{-1}(p) = \pi_2^{-1}(p) = \mathbb{P}^{s(p)} \times \{p\},$$

as schemes. By the theorem on the dimension of the fibers of a morphism, there exists an open subset $U \subseteq \mathbb{P}^r$ such that $\dim(p_2^{-1}(p)) = 0$ for every $p \in U$. Thus for every $p \in U \cap (\mathbb{P}^r \setminus W)$ we get $s(p) = 0$ and scheme theoretically $p_2^{-1}(p)$ reduces to a point, yielding the birationality of p_2 and hence of ϕ.

(iii) One shows that the maximal rank condition implies the condition on $\ker(\rho)$ in (ii). Namely, note that the incidence algebra $\mathcal{A}$ is isomorphic, as a bigraded algebra, to the symmetric algebra $\mathcal{S}(E)$ of the $k[\mathbf{x}]$-module $E = \operatorname{coker}(\varphi_1)$. The assumption on the rank of φ_1 then says that $I \simeq E/(k[\mathbf{x}]\text{-torsion})$. By definition, $\mathcal{R} \simeq \mathcal{R}_{k[\mathbf{x}]}(E)$, where the latter is understood as $\mathcal{S}(E)/(k[\mathbf{x}]\text{-torsion})$ (cf., e.g., [48]). Therefore $\ker(\rho)$ is actually the $k[\mathbf{x}]$-torsion $\tau(\mathcal{S}(E))$ of $\mathcal{S}(E)$. If we show that the torsion is a minimal prime of $\mathcal{S}(E)$, we will be done. Now, one has by definition

$$\tau\big(\mathcal{S}(E)\big) = \ker\big(\mathcal{S}(E) \to \mathcal{S}(E) \otimes_{k[\mathbf{x}]} k(\mathbf{x})\big),$$

hence $\tau(\mathcal{S}(E))$ is a prime ideal and moreover it is annihilated by some nonzero $g \in k[\mathbf{x}]$. It follows that any graded prime ideal of $\mathcal{S}(E)$ whose degree zero part vanishes must contain $\tau(\mathcal{S}(E))$, because it contains $(0) = (g) \cdot \tau(\mathcal{S}(E))$ and does not contain (g). Since $\tau(\mathcal{S}(E))$ itself is one such prime—because $k[\mathbf{x}]$ is a domain—it cannot properly contain a minimal prime of $\mathcal{S}(E)$ (necessarily graded) whose degree zero part is nonzero. Therefore, $\tau(\mathcal{S}(E))$ has to be a minimal prime itself. □

To conclude the proof of part (3) of the theorem we apply Proposition 4.5(iii), and parts (2) and (3) of the theorem. □

Remark 4.6. (a) The proof of part (ii) of the proposition is a more geometric formulation of the argument in [47, Theorem 4.1] which imprecisely claims that *every* fiber of p_2 is linear. This is true if V is irreducible, but not otherwise in general: some special fibers of π_2 may cut Γ along non-linear varieties.

(b) We note that the recurrence ideals J_i $(1 \leqslant i \leqslant r-1)$ are (x_{r-1}, x_r)-primary ideals, however the full Jacobian ideal J is not. Geometrically, it obtains that the singular locus of the sub-Hankel hypersurface is a multiple structure over the codimension 2 linear subspace $x_{r-1} = x_r = 0$ off the codimension 3 subspace $x_{r-2} = x_{r-1} = x_r = 0$. One can see that the generators of J_i define a generalized de Jonquières transformation as introduced by Pan under the designation of *stellar Cremona maps* (cf. [29]). This would give a different proof that J_i is the base ideal of a Cremona map, yet the structure of J_i might not follow immediately from [29], let alone that of J.

(c) A sub-Hankel determinant $f^{(r)}$ is irreducible. Indeed, we can readily see that

$$f^{(r)} = -\xi(r)x_r^{r-1}x_0 + g(x_1, \ldots, x_r),$$

where $g(x_1, \ldots, x_r) \subset k[x_1, \ldots, x_r]$ is monic in x_{r-1}. Therefore, as a polynomial in the ring $(k[x_1, \ldots, x_r])[x_0]$ it is primitive and of degree one in x_0, hence is irreducible.

(d) There is enough evidence to conjecture that the ideal J is also of linear type. For one thing, the computation for various values of r corroborates the conjecture. Knowing that J is of linear type would shorten by quite a bit the proof of part (3) of the theorem and circumvent the need for the full apparatus of Proposition 4.5 and, moreover, it would give immediately the dominance of the polar map ϕ_f. Finally, it would also imply that the inverse map to the polar map is defined by forms of degree r generating a codimension two perfect ideal—this has also been computationally checked for various values of r. For a fuller coverage of the syzygy theoretic properties of J see [38].

Example 4.7. It is interesting to consider, in particular, the first two cases $r = 3, 4$. The sub-Hankel surface $V(f^{(3)})$ has degree 3 and it has the double line L defined by $x_3 = x_2 = 0$. Hence it is a rational scroll which is a projection of $S(1,2) \subset \mathbb{P}^4$.

Consider the general plane π_λ, with $\lambda = (\lambda_2, \lambda_3) \neq \mathbf{0}$, through L, defined by the equation $\lambda_2 x_3 = \lambda_3 x_2$. By introducing a parameter t and by taking x_0, x_1, t as homogeneous coordinates in π_λ, the equation of the intersection R_λ of π_λ with $V(f^{(3)})$ off L is:

$$\det\begin{pmatrix} x_0 & x_1 & \lambda_2 \\ x_1 & \lambda_2 t & \lambda_3 \\ \lambda_2 & \lambda_3 & 0 \end{pmatrix} = 0.$$

Hence R_λ is a line which varies linearly with λ. In particular, when $\lambda_3 = 0$, R_λ coincides with L. Thus we see that L is a line directrix of multiplicity $e = 2$, but $\mu = 1$ (see Section 1.2.2). This shows that $V(f^{(3)})$ is the projection of $S(1,2)$ from a point which lies in a plane spanned by the (-1)-section E and by a ruling F, precisely the one corresponding to $R_{(1,0)}$—see Remark 3.21.

The threefold $V(f^{(4)})$ has degree 4 and it has the double plane Π defined by $x_3 = x_4 = 0$.

As above, consider the general hyperplane π_λ through Π, defined by $\lambda_4 x_3 = \lambda_4 x_3$. By introducing a parameter t and by taking x_0, x_1, x_3, t as homogeneous coordinates in π_λ, the equation of the intersection Q_λ of π_λ with $V(f^{(4)})$ off Π is:

$$\det\begin{pmatrix} x_0 & x_1 & x_2 & \lambda_3 \\ x_1 & x_2 & \lambda_3 t & \lambda_4 \\ x_2 & \lambda_3 t & \lambda_4 & 0 \\ \lambda_3 & \lambda_4 & 0 & 0 \end{pmatrix} = 0.$$

One sees that Q_λ is a quadric cone with vertex $P_\lambda = [2\lambda_3, -\lambda_4, 0, 0]$, thus P_λ sits on Π and linearly moves on a line as λ varies.

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