A characterization of nef and good divisors by asymptotic multiplier ideals*

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Abstract

A characterization of nef and good divisors is given: a divisor D on a smooth complex projective variety is nef and good if and only if the asymptotic multiplier ideals of sufficiently high multiples of $e(D) \cdot D$ are trivial, where $e(D)$ denotes the exponent of the divisor D. Some results of the same kind are proved in the analytic setting.

Introduction

Let X be a smooth complex quasi-projective variety. One can associate to a $\mathbb{Q}$-divisor D its multiplier ideal sheaf $\mathcal{I}(D) \subseteq \mathcal{O}_X$ whose zero set is the locus at which the pair (X, D) fails to have log-terminal singularities, see [Laz, II.9] and §1 for definitions and notation. The multiplier ideal $\mathcal{I}(D)$ reveals how bad are the singularities of D. To reflect properties of the base locus of the linear systems $|nD|$ for n sufficiently large the notion of asymptotic multiplier ideal has been introduced: the smaller the asymptotic multiplier ideals, the worse the asymptotic base locus of D, see [Laz, II.11] for definitions and also §2. These two concepts and their analytic analogues, which originated the whole theory, play an important role in "correcting" some line bundle in order to have vanishing of cohomology. One can consult [De2] and [Laz, II.9, II.10, II.11] for many applications of the theory of multiplier ideals in analytic and algebraic geometry including results of

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Lelong, Skoda, Siu, Nadel, Demailly, Ein and Lazarsfeld, in chronological order, and also for complete lists of references.

In [Laz, II.11.2.18] it was shown that for a big divisor D on a smooth complex projective variety X nefness is equivalent to the triviality of the asymptotic multiplier ideals of the linear series $|nD|$ for n sufficiently large. The proof in *loc. cit* is obtained via Nadel's Vanishing Theorem for asymptotic multiplier ideals, global generation of asymptotic multiplier ideals, [Laz, II.11.2.13], and boundedness of multiplicities of base loci of nef and big divisors.

Here we prove that if D is a divisor on a smooth projective complex variety X such that $\kappa(X, D) \geq 0$ and if $e(D)$ denotes the exponent of D, then D is nef and good if and only if the asymptotic multiplier ideals of sufficiently high multiples of $|e(D) \cdot D|$ are trivial, i.e. if and only if $\mathcal{I}(n||e(D) \cdot D||) = \mathcal{O}_X$ for n sufficiently large, Theorem 2 (see also [Laz, II.11.2.20]). This generalization shows that the above condition captures the nefness of D and a sort of boundedness of the multiplicities of the fixed components of $|nD|$ as n goes to infinity.

In the last section we recall the analytic definitions of multiplier ideal sheaf and analytic asymptotic multiplier ideal sheaf. After analyzing the relations between the algebraic and analytic settings, we show by an example that the triviality of the analytic asymptotic multiplier ideal implies nefness but not necessarily goodness. Thus there does not exist an analytic characterization of nefness and goodness by analytic asymptotic multiplier ideals because of the existence of "virtual" sections, see [DEL, §1].

1 Notation and definitions

Let X be a normal complex projective variety and let D be a Cartier divisor on X. In [God] Goodman introduced the following definition.

Definition 1. (Almost base point free divisor). A divisor D is said to be *almost base point free* if $\forall \epsilon > 0$ and $\forall x \in X$ (not necessarily closed) there exists $n = n(\epsilon, x)$ and $D_n \in |nD|$ such that $\mathrm{mult}_x(D_n) < n\epsilon$.

Definition 2. A divisor D is said to be *nef* if $(D \cdot C) \geq 0$ for every irreducible curve $C \subset X$.

For a nef divisor D we can define the *numerical dimension of D*:

$$\nu(X, D) := \sup\{\nu \in \mathbb{N} : D^\nu \not\equiv 0\}.$$

It is not difficult to see that $\dim(X) \geq \nu(X, D) \geq \kappa(X, D)$, where $\kappa(X, D)$ is the Kodaira dimension of D.

Definition 3. A nef divisor D is said to be *good* if $\nu(X, D) = \kappa(X, D)$. An arbitrary divisor is said to be *big* if $\kappa(X, D) = \dim(X)$.

By the above inequality, a nef and big divisor is good. Let us describe some examples to clarify the above definitions and to put in evidence some of the significant properties of nef and good divisors.

Example 1. (A nef but not good divisor). Let D be an irreducible curve on a smooth projective surface S such that $D^2 = 0$ and $(D \cdot C) > 0$ for every irreducible curve $C \subset S$ with $C \neq D$. Then D is a nef divisor with $\nu(S, D) = 1$ and such that $|nD| = nD$ for every $n \geq 1$, i.e. $\kappa(S, D) = 0$.

To construct explicit examples one can take as S the blow-up of $\mathbb{P}^2$ in d^2 points, $d \geq 3$, which are general on a smooth curve $H \subset \mathbb{P}^2$ of degree d and take as D the strict transform of H.

Another well known example is constructed by taking as D the "zero section" of $S = \mathbb{P}_F(\mathcal{E})$, where $\mathcal{E}$ is the a rank 2 locally free sheaf on an elliptic curve F, corresponding to (the unique) non-splitting extension $0 \to \mathcal{O}_F \to \mathcal{E} \to \mathcal{O}_F \to 0$.

In [God] it is shown that an almost base point free divisor is nef, see also [Laz, II.11.2.19]. The connection between the above definitions is given by the following result which is a consequence of a theorem proved by Kawamata in [Kaw, §2] (see also [Mor] and [MR]).

Theorem 1. *Let D be a Cartier divisor on a complete normal complex variety X. Then D is almost base point free if and only if D is nef and good.*

Let us recall the definitions of multiplier ideal sheaf associated to an effective $\mathbb{Q}$-divisor D on a smooth complex projective variety X, see also [Laz, II.9]. Let $\mu : X' \to X$ be a log-resolution of D and let $\mathrm{Exc}(\mu)$ be the sum of the exceptional divisors of $\mu : X' \to X$. For a $\mathbb{Q}$-divisor $D = \sum \alpha_i D_i$ with $\alpha_i \in \mathbb{Q}$, we denote by $[D] = \sum [\alpha_i] D_i$ *the integral part of D*, where $[\alpha_i]$ is the integral part of $\alpha_i \in \mathbb{Q}$.

Definition 4. The *multiplier ideal sheaf*

$$\mathcal{I}(D) \subseteq \mathcal{O}_X$$

associated to D is defined to be

$$\mathcal{I}(D) = \mu_*(\mathcal{O}_{X'}(K_{X'/X} - [\mu^*(D)])),$$

where $K_{X'/X} = K_{X'} - \mu^* K_X$ is the relative canonical divisor.

The multiplier ideal sheaf of D does not depend on the log-resolution of D, see for example [Laz, II.9.2.18]. Let now $|V| \subseteq |D|$ be a linear system on X and let $\mu : X' \to X$ be a log-resolution of $|V|$, i.e. $\mu^*(|V|) = |W| + F$ where $F + \mathrm{Exc}(\mu)$ is a divisor with simple normal crossing support and $|W|$ is a base point free linear system, [Laz, II.9.1.11].

Definition 5. Fix a positive rational number $c > 0$. The *multiplier ideal corresponding to c and $|V|$* is

$$\mathcal{I}(c \cdot |V|) = \mu_*(\mathcal{O}_{X'}(K_{X'/X} - [c \cdot F])).$$

Let D an integral Cartier divisor on X with $\kappa(X, D) \geq 0$ and let $e(D)$ be *the exponent of D*, which is by definition the g.c.d. of the semigroup of integers $N(D) = \{m \geq 0 : |mD| \neq \emptyset\}$. Thus there exists a least integer $n_0(D)$, *the Iitaka threshold of D*, such that for every $n \geq n_0(D)$ with $e(D)|n$, $|nD| \neq \emptyset$, see also [Laz, II.11.1.A].

Definition 6. The *asymptotic multiplier ideal sheaf associated to c and $|D|$,*

$$\mathcal{I}(c \cdot ||D||) \subseteq \mathcal{O}_X,$$

is defined to be the unique maximal member among the family of ideals

$$\{\mathcal{I}(\frac{c}{p} \cdot |p \cdot e(D) \cdot D|)\}_{p \cdot e(D) \geq n_0(D)}.$$

In [Laz, II.11.1.A] it is shown that there exists a maximal member in the above family, that it is unique and also that $\mathcal{I}(n||D||) = \mathcal{I}(||nD||)$.

In the next section we need the following properties of multiplier ideals. Let us remember that if $D = \sum \alpha_i \cdot D_i$ is a $\mathbb{Q}$-Cartier divisor and that if $x \in X$, then $\mathrm{mult}_x(D) := \sum \alpha_i \cdot \mathrm{mult}_x(D_i)$.

Proposition 1. *Let D be an effective $\mathbb{Q}$-divisor on X. Suppose there exists a point $x \in X$ such that* $\mathrm{mult}_x(D) < 1$. *Then the multiplier ideal $\mathcal{I}(D)$ is trivial at x, i.e. $\mathcal{I}(D)_x = \mathcal{O}_{X,x}$.*

If there exists a point $x \in X$ such that $\mathrm{mult}_x(D) \geq \dim(X) + n - 1$ *for some integer $n \geq 1$, then $\mathcal{I}(D)_x \subseteq m^n_{X,x} \subset \mathcal{O}_{X,x}$*

For a proof of the first part see [EV] or [Laz, II.9.5.13]. The last part is proved in [Laz, II.9.3.2]. These are algebraic versions of analytic results of Skoda, see [Sko] or [De2, Lemma 5.6].

2 Characterization of nefness and goodness by Asymptotic Multiplier Ideals

Theorem 2. *Let D be a divisor on a smooth proper complex variety X such that $\kappa(X, D) \geq 0$ and let $e(D)$ be the exponent of D. Then D is nef and good if and only if $\mathcal{I}(n||e(D)D||) = \mathcal{O}_X$ for n sufficiently large.*

Proof. By replacing D with $e(D) \cdot D$, we can assume $e(D) = 1$. Let us assume that D is not nef and good. By Theorem 1 there exist $\epsilon > 0$ and $x \in X$, which we can assume to be a closed point, such that for every $m \geq 1$ and for every $D_m \in |mD|$ we have $\mathrm{mult}_x(D_m) \geq m\epsilon$. Choose n such that $[n\epsilon] \geq \dim(X)$ and let k be a sufficiently large integer such that $\mathcal{I}(||nD||) = \mathcal{I}(\frac{1}{k}|knD|)$. Let $\mu : X' \to X$ be a log-resolution of $|knD|$ constructed by first blowing-up X at x. The exceptional divisor of this blow-up determines a prime divisor $E \subset X'$ such that $\mathrm{mult}_x(D_{kn}) = \mathrm{ord}_E(\mu^*(D_{kn}))$ for every $D_{kn} \in |knD|$ and $\mathrm{ord}_E(K_{X'/X}) = \dim(X) - 1$. By definition we have $\mu^*(|knD|) = |W| + F_{kn}$ with $|W|$ base point free. Therefore $kn\epsilon \leq \mathrm{ord}_E(F_{kn})$ and

$$\mathrm{ord}_E(K_{X'/X} - [\frac{1}{k}F_{kn}]) \leq \dim(X) - 1 - [n\epsilon] \leq -1,$$

yielding $\mathcal{I}(||nD||)_x = \mu_*(\mathcal{O}_{X'}(K_{X'/X} - [\frac{1}{k}F_{kn}]))_x \subseteq \mu_*(\mathcal{O}_{X'}(-E))_x = m_{X,x}$. This proves that $\mathcal{I}(n||e(D)D||) = \mathcal{O}_X$ for n sufficiently large implies that D is nef and good.

To prove the other implication, let us assume that there exists a point $x \in X$ such that $\mathcal{I}(||nD||) \subseteq m_{X,x}$ for some $n \geq 1$. For k sufficiently large we have that $\mathcal{I}(||nD||) = \mathcal{I}(\frac{1}{k}|knD|)$ and let $D_{kn} \in |knD|$ be a general divisor. It follows from Proposition 1 that $\text{mult}_x(D_{kn}) \geq k$. Thus D is not almost base point free and hence not nef and good by Theorem 1. ∎

We remark that if D is semiample, i.e. some multiple of D is base point free, then D is nef and good so that $\mathcal{I}(c \cdot ||nD||)$ is trivial for every $n \geq 1$ and for every rational number $c > 0$. Moreover if D is semiample, the associated graded algebra $R(X, D) = \oplus H^0(X, \mathcal{O}_X(nD))$ is finitely generated over the base field by a result of Zariski, see [Zar] and also [Laz, I.2.1.30].

A nef and good divisor on a complete normal variety is semiample if and only if the associated graded algebra $R(X, D)$ is finitely generated over the base field, see for example [MR] (and also [Zar], [Wil], [Rus], [Laz, I.2.3.15] for the case of nef and big divisors). Hence the triviality of the asymptotic multiplier ideals of sufficiently high multiples of a divisor controls the nefness of the divisor and a sort of boundedness of the fixed components but not semiampleness. In [MR] and [Rus] one can find well known examples of nef and good divisors D for which every multiple $|nD|$ has base locus.

3 Analytic analogue

Let now X be a compact complex manifold and let $\mathcal{L}$ be a line bundle on X. Let us recall some definitions. The notation is the same as in [De2].

Definition 7. Let ϕ be a plurisubharmonic function, briefly a psh function, on an open subset $\Omega \subseteq X$. The *Lelong mumber of ϕ in $x \in \Omega$* (or of the hermitian metric h having local expression $e^{-2\phi}$) is

$$\mu(\phi, x) = \liminf_{z \to x} \frac{\phi(z)}{\log(|z - x|)}.$$

For a singular metric $h = e^{-2\phi}$ on $\mathcal{L}$ associated to an effective divisor $D \in |\mathcal{L}|$ we have $\mu(\phi, x) = \text{mult}_x(D)$, that is the Lelong number is the analytic analogue of multiplicity.

The algebraic case suggests the following definition.

Definition 8. A line bundle $\mathcal{L}$ on X is said *analytic almost base point free* if $\forall \epsilon > 0$ and $\forall x \in X$ there exists a possibly singular hermitian metric $h = e^{-2\phi}$ on $\mathcal{L}$, positive in the sense of currents (that is $\frac{i}{2\pi}\Theta_h = dd^c\phi \geq 0$ as a current) and for which $\mu(\phi, x) < \epsilon$.

In the analytic case we have the following definition of nefness.

Definition 9. ([DPS]) Let $\mathcal{L}$ be a line bundle on a complex compact manifold (X, ω), where ω is a hermitian metric on X. Then $\mathcal{L}$ is said to be *nef* if $\forall \epsilon > 0$ there exists a smooth hermitian metric h_ϵ on $\mathcal{L}$ such that $i\Theta_{h_\epsilon} \geq -\epsilon\omega$.

It is easy to see that if X is projective, then the above definition of nefness is equivalent to the previous one, see for example [De2, Proposition 6.2]. The above condition does not imply the existence of a smooth metric with non negative curvature on $\mathcal{L}$, see [DPS, Example 1.7].

If (X, ω) is a compact complex Kähler manifold, Demailly proved in [De1] the following result which is a generalization of Proposition 8 in [God] recalled above in the algebraic setting.

Proposition 2. ([De1]) *Let $\mathcal{L}$ be an analytic almost base point free line bundle on the compact complex Kähler manifold (X, ω). Then $\mathcal{L}$ is nef.*

On a compact Kähler manifold (X, ω) one defines, exactly as in the algebraic case, the notions of Kodaira dimension, $\kappa(X, \mathcal{L})$, of a line bundle $\mathcal{L}$ and, for the nef ones, of numerical dimension, $\nu(X, \mathcal{L})$, see [De2, §6]. Then as in the algebraic case, we have $\dim(X) \geq \nu(X, \mathcal{L}) \geq \kappa(X, \mathcal{L})$ (see [De2]), and one says that a nef line bundle is *good* if $\nu(X, \mathcal{L}) = \kappa(X, \mathcal{L})$.

We have the following relation between the notions of almost base point freeness and analytic almost base point freeness, which is a consequence of the definitions.

Proposition 3. *Let X be a compact complex projective manifold and let $\mathcal{L}$ be an almost base point free line bundle on X. Then $\mathcal{L}$ is analytic almost base point free.*

We now show that the converse does not hold by recalling an example of [DEL].

Example 2. (An analytic almost base point free line bundle is not almost base point free). Let $\mathcal{E}$ be a unitary flat vector bundle on a smooth projective variety Y such that no non-trivial symmetric power of $\mathcal{E}$ or $\mathcal{E}^*$ has sections (such vector bundles exist if for example Y is a curve of genus ≥ 1) and set $\mathcal{F} = \mathcal{O}_C \oplus \mathcal{E}$. Take now $X = \mathbb{P}(\mathcal{F})$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}_Y(\mathcal{F})}(1)$. Then for every $m \geq 1$, $\mathcal{L}^m$ has a unique non trivial section which vanishes to order m along the *divisor at infinity* $H \subset \mathbb{P}(\mathcal{F})$. Then $\mathcal{L}$ is a nef line bundle which has a smooth semipositive metric induced by the flat metric on $\mathcal{E}$, so that it is analytic almost base point free but clearly not almost base point free.

Let us remark that in Example 2 we have $\mathcal{I}(||\mathcal{L}^m||) = \mathcal{O}_X(-mH)$ for every $m \geq 1$ so that analytic base point freeness cannot be characterized by the vanishing of the (algebraic) asymptotic multiplier ideal of sufficiently high multiples of $\mathcal{L}$.

This example suggests that there should exist an analytic analogue of the notion of asymptotic multiplier ideal reflecting the "boundedness of the singularities of the hermitian metrics" on multiples of $\mathcal{L}$. To prove this result in Proposition 4 we introduce the definitions of analytic multiplier ideal and of *metric with minimal singularities* of a line bundle on a compact complex manifold, following Demailly, [De2].

Definition 10. Let ϕ be a psh function on an open subset $\Omega \subset X$ of a complex manifold X. We associate to ϕ the ideal subsheaf $\mathcal{I}(\phi) \subset \mathcal{O}_\Omega$ of germs of holomorphic functions $f \in \mathcal{O}_{\Omega,x}$ such that $|f|^2 e^{-2\phi}$ is integrable with respect to the Lebesgue measure in some local coordinates around x. Then $\mathcal{I}(\phi)$ is said to be the *analytic multiplier ideal sheaf* of ϕ.

In the analytic setting for a line bundle $\mathcal{L}$ on X whose first Chern class lies in the closure of the cone of effective divisors, i.e. for a *pseudoeffective* line bundle, the notion of *singular hermitian metric with minimal singularities* $h_{\min}$ can be defined in the following way (for more details see [De2, §13] and especially the proof of [De2, Theorem 13.1.2]).

Definition 11. Let $\mathcal{L}$ be a pseudoeffective line bundle on X, let h_∞ be any smooth hermitian metric on $\mathcal{L}$ and let $u = i\Theta_{h_\infty}(\mathcal{L})$. Then

$$h_{\min} = h_\infty e^{-\psi_{\max}},$$

where

$$\psi_{\max}(x) = \sup\{\psi(x) \ : \psi \ usc, \ \psi \leq 0, \ i\partial\overline{\partial}\log(\psi) + u \geq 0\}.$$

Then one defines the *analytic asymptotic multiplier ideal sheaf* of $\mathcal{L}$ as the analytic multiplier ideal sheaf of $h_{\min}$, which will be indicated by $\mathcal{I}(h_{\min})$.

The following result follows from the fact that we have $\mathcal{I}(h_{\min}) = \mathcal{O}_X$ if and only if $\mathcal{L}$ is analytic almost base point free by the same argument used in the proof of Theorem 2. We simply replace Proposition 1 by the analytic analogue for Lelong numbers proved by Skoda, see [Sko] or [De2, Lemma 5.6].

Proposition 4. *Let $\mathcal{L}$ be a line bundle on a compact complex manifold X. Then $\mathcal{L}$ is analytic almost base point free if and only if $\mathcal{I}(h_{\min}) = \mathcal{O}_X$ if and only if for every point of X the Lelong numbers of $h_{\min}$ are zero.*

The following is a generalization of [De2, Proposition 13.1.4].

Corollary 1. *Let $\mathcal{L}$ be a nef and good line bundle on a compact complex projective manifold X. Then for m sufficiently large $\mathcal{I}(||\mathcal{L}^{e(\mathcal{L})m}||) = \mathcal{I}(h_{\min}) = \mathcal{O}_X$.*

Let us remark that on a compact complex Kähler manifold X the condition $\mathcal{I}(h_{\min}) = \mathcal{O}_X$ implies nefness but not necessarily goodness, i.e. it does not exist an analytic characterization of nefness and goodness by analytic asymptotic multiplier ideals.

By the above results we know that for a nef and good line bundle on a compact complex projective manifold the algebraic asymptotic multiplier ideal of its high multiples and its analytic asymptotic multiplier ideal are both trivial so that they coincide.

For arbitrary sections $s_1, \ldots, s_N \in H^0(X, \mathcal{L}^m)$ we can take as an admissible ψ function, $\psi(x) = \frac{1}{m}\log\sum_j ||\sigma_j(x)||^2_{h_\infty} + C$, with C a costant. From this it follows, see [DEL] and [De2], that if X is a complex compact projective manifold and if $\kappa(X, \mathcal{L}) \geq 0$, then $\mathcal{I}(||\mathcal{L}^m||) \subseteq \mathcal{I}(h_{\min})$. Example 2 above shows that the inclusion $\mathcal{I}(||\mathcal{L}^m||) \subseteq \mathcal{I}(h_{\min})$ can be strict.

One conjectures that for arbitrary big line bundles the asymptotic algebraic multiplier of its multiples and its analytic multiplier ideal coincide, see [DEL] and also [Laz, II.11.1.11].

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