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REGULARITY OF SOLUTIONS OF DIVERGENCE FORM ELLIPTIC EQUATIONS

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Dedicated to the memory of two friends Filippo Chiarenza and Gene Fabes

Abstract. The aim of this paper is to study local regularity in the Morrey spaces $L^{p,\lambda}$ of the first derivatives of the solutions of an elliptic second order equation in divergence form

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j} = divf(x) \quad \text{for a.a. } x \in \Omega,$$

where f is assumed to be in some $L^{p,\lambda}$ spaces and the coefficients a_{ij} belong to the space VMO.

1. Introduction

In this note we consider the divergence form elliptic equation

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j} = divf(x) \quad \text{for almost all } x \in \Omega, \tag{1.1}$$

in a bounded open set $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

We assume $\mathcal{L}$ to be a linear elliptic operator with coefficients a_{ij} taken in the space VMO and $f = (f_1, f_2, ..., f_n)$ is assumed such that f_i belongs to some Morrey spaces $L^{p,\lambda}$, $1 < p < \infty, 0 < \lambda < n, \quad \forall i = 1, ..., n$ (see section 2 below for definitions). The space VMO was introduced by Sarason in [10] and it's the subspace of the John-Nirenberg space BMO (see [7]) whose BMO norm over a ball vanishes as the radius of the ball tends to zero.

Our main result in this note is local regularity in the Morrey spaces $L^{p,\lambda}$ of the first derivatives of the solutions of the equation with discontinuous coefficients (1.1) (Theorem 4.3) as in [5] has been done in L^p-spaces for equations in divergence form and in [6] has been done in the nondivergence form.

As a consequence of the main result we prove (Theorem 4.4) a $C^{0,\alpha}_{loc}$-regularity result. This generalizes Stampacchia's theorem 7.2 [11] in the framework of Morrey spaces.

Our argument rests on an integral representation formula for the first derivatives of the solutions of equation (1.1) and the boundedness in $L^{p,\lambda}$ of some integral

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operators and commutators appearing in this formula. This method was used to obtain L^p regularity results in the constant coefficient case. It's associated with the names of A. Calderón and A. Zygmund and had been used by many authors (G. Giraud, C. Miranda and others) to study the regularity properties in the Hölder spaces.

This technique of Calderón and Zygmund was used in the deep result [3] by Chiarenza, Frasca and Longo for the nondivergence form equation and made it possible to pass from the coefficient case $a_{ij} \in C^0(\overline{\Omega})$ to the discontinuous coefficient case.

In the noteworthy paper [1] L^p results are obtained for linear elliptic systems which however cannot be used in the case of VMO coefficients.

In a subsequent paper the author will study the boundary estimates and the solvability of the Dirichlet problem for equation (1.1) as Chiarenza, Frasca and Longo have done in [4] for the nondivergence form equation.

2. Some definitions

Let Ω be an open bounded set in $\mathbb{R}^n$ $(n \geq 3)$ and f be a locally integrable function in Ω. We say that f is in the John-Nirenberg space BMO of the functions of bounded mean oscillation if

$$\sup_B \frac{1}{|B|} \int_B |f(x) - f_B| dx \equiv \|f\|_* < \infty,$$

where B ranges in the class of the balls contained in Ω and

$$f_B \equiv \frac{1}{|B|} \int_B f(x) dx.$$

We denote the number $\|f\|_*$ as the norm of f in BMO.

If $f \in BMO$ and r is a positive number we set

$$\sup_{x \in \mathbb{R}^n,\ \rho \leq r} \frac{1}{|B_\rho|} \int_{B_\rho} |f(x) - f_{B_\rho}| dx \equiv \eta(r)$$

where B_ρ stands for a ball with radius ρ less than or equal to r.

We say that $f \in BMO$ is in the space VMO if $\lim_{r \to 0^+} \eta(r) = 0$.

Furthermore we denote η_{ij} the VMO modulus of the function a_{ij} and $\eta(r) = (\sum_{i,j=1}^n \eta_{ij}^2(r))^{\frac{1}{2}}$.

Also, for $1 < p < \infty$, $0 < \lambda < n$ we set

$$\|f\|^p_{L^{p,\lambda}(\Omega)} \equiv \sup_{x \in \Omega,\ \rho > 0} \frac{1}{\rho^\lambda} \int_{B_\rho(x) \cap \Omega} |f(y)|^p dy < \infty$$

and we call Morrey space $L^{p,\lambda}(\Omega)$ the set of those measurable f for which the norm of f is finite.

Remark 2.1 ([9, Remark 2.2]). Let Ω an open subset of $\mathbb{R}^n$. Then $L^{p,\lambda}(\Omega)$ is continuously imbedded in $L^{p',\lambda'}(\Omega)$ for $p' \leq p$ and $\lambda' \leq \lambda \frac{p'}{p} + n(1 - \frac{p'}{p})$.

Let Ω be a bounded open set in $\mathbb{R}^n$ $(n \geq 3)$ and let us consider

$$\mathcal{L}u \equiv - \sum_{i,j=1}^n (a_{ij}(x) u_{x_i})_{x_j} = div f(x) \quad \text{a.a. } x \in \Omega,$$

where we assume that

$$\exists p \in]1, +\infty[,\ \lambda \in]0, n[\text{ such that}$$

$$f = (f_1, ..., f_n) \in [L^{p,\lambda}(\Omega)]^n; \tag{2.1}$$

$$a_{ij}(x) \in L^\infty(\Omega) \cap VMO, \quad \forall i, j = 1, ..., n; \tag{2.2}$$

$$a_{ij}(x) = a_{ji}(x), \quad \forall i, j = 1, ..., n \quad \text{a.a. } x \in \Omega; \tag{2.3}$$

$$\exists \sigma > 0 :\ \sigma^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \sigma|\xi|^2 \quad \forall \xi \in \mathbb{R}^n,\ \text{a.a. } x \in \Omega. \tag{2.4}$$

We say that u is a solution of (1.1) if u and $\partial_{x_i} u \in L^p(\Omega)$ $\forall i = 1, \dots, n$ and for some $1 < p < \infty$

$$\int_\Omega a_{ij} u_{x_i} \phi_{x_j} dx = -\int_\Omega f_i \phi_{x_i} dx \qquad \forall \phi \in C_0^\infty(\Omega).$$

3. Preliminary tools

Theorem 3.1 ([9, Lemma 2.8])**.** *Let Ω be an open subset of $\mathbb{R}^n$, $0 < \alpha < n$, $1 < q < \infty$, $0 < \mu < n$ such that $\mu + \alpha p < n$ and $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ a homogeneous function of degree $-\alpha$. Then, for every $f \in L^{q,\mu}(\Omega)$ the operator*

$$T f(x) = \int_\Omega K(x-y) f(y) dy$$

is a.e. defined, belongs to $L^{p,\lambda}(\Omega)$ where $\frac{1}{p} = \frac{1}{q} - \frac{\alpha}{n}$, $\lambda = \mu\frac{p}{q}$ and exists $c = c(q, \mu, \alpha) > 0$ such that

$$\|Tf\|_{L^{p,\lambda}(\Omega)} \leq c\|f\|_{L^{q,\mu}(\Omega)}.$$

We recall the definition and some useful results on singular integrals.

Definition 3.2. Let $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$. We say that $k(x)$ is a Calderón-Zygmund kernel (C-Z kernel) if

i) $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$;

ii) $k(x)$ is homogeneous of degree $-n$;

iii) $\int_\Sigma k(x)\, dx = 0$, where $\Sigma = \{\, x \in \mathbb{R}^n : |x| = 1 \,\}$.

Theorem 3.3 ([6, Theorem 2.3])**.** *Let B be an open ball in $\mathbb{R}^n$, $f \in L^{p,\lambda}(B)$, $1 < p < \infty$, $0 < \lambda < n$, $a \in BMO(B)$. Let $k(x,z)$ be a real measurable function in $B \times (\mathbb{R}^n \setminus \{0\})$ such that*

(i) $k(x,.)$ is a Calderón–Zygmund kernel for a. a. $x \in B$;

(ii) $\max_{|j| \leq 2n} \left\| \frac{\partial^j}{\partial z^j} k(x,z) \right\|_{L^\infty(B\times\Sigma)} = M < +\infty$.

Let also, for any $\varepsilon > 0$,

$$K_\varepsilon f(x) = \int_{|x-y|>\varepsilon,\ x\in B} k(x, x-y) f(y) dy,$$

$$C_\varepsilon(a, f)(x) = \int_{|x-y|>\varepsilon,\ x\in B} k(x, x-y)(a(x) - a(y)) f(y) dy.$$

Then, there exist Kf, $C(a,f) \in L^{p,\lambda}(B)$ *such that*

$$\lim_{\varepsilon\to 0} \|K_\varepsilon f - Kf\|_{L^{p,\lambda}(B)} = 0, \quad \lim_{\varepsilon\to 0} \|C_\varepsilon(a,f) - C(a,f)\|_{L^{p,\lambda}(B)} = 0$$

and there exists a constant $c = c(n,p,\lambda,M)$ *such that*

$$\|Kf\|_{L^{p,\lambda}(B)} \le c\|f\|_{L^{p,\lambda}(B)}\,, \quad \|C(a,f)\|_{L^{p,\lambda}(B)} \le c\|a\|_*\|f\|_{L^{p,\lambda}(B)}.$$

The next theorem will be crucial in the following.

Theorem 3.4 ([6, Theorem 2.4])**.** *Let* $a \in VMO \cap L^\infty(\mathbb{R}^n)$ *and* $k(x,z)$ *satisfying the hypothesis of Theorem 3.3. Then for any* $\epsilon > 0$, *there exists* $\rho_0 > 0$ *such that for any ball* B_r *of radius* $r \in]\,0,\rho_0\,[$ *and* $f \in L^{p,\lambda}(B_r)$, *with* $1 < p < \infty$ *and* $0 < \lambda < n$, *we have*

$$\|C(a,f)\|_{L^{p,\lambda}(B_r)} \le c\,\epsilon\,\|f\|_{L^{p,\lambda}(B_r)}. \tag{3.1}$$

4. A representation formula and regularity results

In the following we set

$$\Gamma(x,t) = \frac{1}{n(2-n)\omega_n\sqrt{\det\{a_{ij}(x)\}}}\left(\sum_{i,j=1}^n A_{ij}(x)t_it_j\right)^{(2-n)/2},$$

$$\Gamma_i(x,t) = \frac{\partial}{\partial t_i}\Gamma(x,t), \quad \Gamma_{ij}(x,t) = \frac{\partial}{\partial t_i\partial t_j}\Gamma(x,t),$$

$$M = \max_{i,j=1,\dots,n}\max_{|\alpha|\le 2n}\left\|\frac{\partial^\alpha\Gamma_{ij}(x,t)}{\partial t^\alpha}\right\|_{L^\infty(\Omega\times\Sigma)}$$

for a.a. $x \in B$ and $\forall t \in \mathbb{R}^n\setminus\{0\}$ where $A_{ij}(x)$ stand for the entries of the inverse matrix of the matrix $\{a_{ij}(x)\}_{i,j=1,\dots,n}$, and ω_n is the measure of the unit ball in $\mathbb{R}^n$.

It is well known that $\Gamma_{ij}(x,t)$ are Calderón–Zygmund kernels in the t variable.

Let $r, R \in \mathbb{R}^+$, $r < R$ and $\phi \in C^\infty(\mathbb{R})$ be a standard cut-off function such that for every $B_R \subset \Omega$

$$\phi(x) = 1 \quad \text{in } B_r, \qquad \phi(x) = 0, \quad \forall x \notin B_R.$$

Then if u is a solution of (1.1) and $v = \phi u$ we have

$$L(v) = div\,G + g \tag{4.1}$$

where we set

$$\begin{aligned} G &= \phi f + u\,A\,D\phi \\ g &= \langle A\,Du\,,\,D\phi\rangle - \langle f\,,\,D\phi\rangle. \end{aligned} \tag{4.2}$$

We recall an integral representation formula for the first derivative of a solution u of (1.1).

Lemma 4.1 ([8, Theorem 3.1])**.** *Let,* $\forall i = 1,\dots,n$, $a_{ij} \in VMO \cap L^\infty(\mathbb{R}^n)$ *satisfy* (2.3) *and* (2.4)*, let* u *be a solution of* (1.1) *and let* ϕ, g *and* G *be defined as above.*

Then, for every $i = 1, \dots, n$ we have

(4.3)

$$\partial_{x_i}(\phi u)(x) = \sum_{h,j=1}^{n} P.V. \int_{B_R} \Gamma_{ij}(x, x-y) \left\{(a_{jh}(x) - a_{jh}(y))\partial_{x_h}(\phi u)(y) - G_j(y)\right\} dy$$

$$- \int_{B_R} \Gamma_i(x, x-y)\, g(y) dy + \sum_{h=1}^{n} c_{ih}(x) G_h(x)\,,\ \forall x \in B_R$$

setting $c_{ih} = \int\limits_{|t|=1} \Gamma_i(x,t)\, t_h d\sigma_t$.

Now as an application of the previous lemma we prove an a priori estimate for the first derivatives of a solution of equation (1.1).

Theorem 4.2. *Under the assumptions of Lemma 4.1, u and $\partial_{x_i} u \in L^p(\Omega), \forall i = 1, \dots, n$, $f \in \left[L^{p,\lambda}(\Omega)\right]^n$ $0 < \lambda < n$, $1 < p < \infty$, $\forall K \subset \Omega$, there exists a constant $c = c(n, p, \lambda, \sigma, \eta, dist(K, \partial\Omega))$ such that*

$$i)\ \partial_{x_i} u \in L^{p,\lambda}(K)\,,\ \forall i = 1, \dots, n;$$

$$ii)\ \|\partial_{x_i} u\|_{L^{p,\lambda}(K)} \le c\left(\|u\|_{L^{p,\lambda}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)} + \|\partial_{x_i} u\|_{L^{q,\mu}(\Omega)}\right)$$

where $q = p_*$ is such that $\frac{1}{p_*} = \frac{1}{p} + \frac{1}{n}$ and $\mu = \frac{\lambda p_*}{p}$.

Proof. First we prove i). Let $v = \phi u$ from Lemma 4.1; it follows that

$$v_{x_i} = S_{ijh}(v_h) + h_i \tag{4.4}$$

where

$$S_{ijh}(v_h) = \sum_{h,j=1}^{n} \text{P.V.} \int_{\mathbb{R}^n} \Gamma_{ij}(x, x-y) \left\{(a_{jh}(x) - a_{jh}(y)) v_{x_h}(y)\right\} dy$$

and

$$h_i = -\left(\sum_{i=1}^{n} \text{P.V.} \int_{\mathbb{R}^n} \Gamma_{ij}(x, x-y) G_j(y) dy + \int_{\mathbb{R}^n} \Gamma_i(x, x-y)\, g(y) dy\right)$$

$$+ \sum_{h=1}^{n} c_{ih}(x) G_h(x) \quad \forall i = 1, ..., n.$$

We will need to prove that $h_i \in L^{p,\lambda_1}(\Omega)$, $\lambda_1 \le \lambda$, for every $i = 1, \dots, n$.

By hypotheses for every $i = 1, \dots, n$, $\partial_{x_i} u \in L^p(\Omega)$ and applying Remark 2.1 $\partial_{x_i} u \in L^{p_*, p_*}(\Omega)$, $\forall i = 1, \dots, n$.

For the sake of brevity we set $Tg(x) = \int\limits_{B} \Gamma_i(x, x-y)\, g(y) dy$.

If $\frac{\lambda p_*}{p} \le p_*$, i.e., $\lambda \le p$, we have $g \in L^{p_*, \frac{\lambda p_*}{p}}(\Omega)$, and by Theorem 3.1 $Tg \in L^{p,\lambda}(\Omega)$.

Otherwise if $\lambda > p$, then $g \in L^{p_*, p_*}(\Omega)$ and then $Tg \in L^{p,p}(\Omega)$.

In both cases $Tg \in L^{p,\lambda_1}(\Omega)$, $\quad \lambda_1 = \min(\lambda, p)$.

We see that also $G \in L^{p,\lambda_1}(\Omega)$ with $\lambda_1 = \min(\lambda, p)$ because $\partial_{x_i} u \in L^p(\Omega)$ and then $u \in L^{p,p}(\Omega)$ and $f \in L^{p,\lambda}(\Omega)$. We have therefore demonstrated that $h_i \in L^{p,\lambda_1}(\Omega)$.

Let us define the map

$$\mathcal{T}: \quad \left[L^{p,\lambda_1}(\Omega)\right]^n \to \left[L^{p,\lambda_1}(\Omega)\right]^n$$

as

$$\mathcal{T}w \equiv \{(\mathcal{T}w)_i\}_{i=1,\dots,n} = \left(\sum_{j,h=1}^{n} S_{ijh}(w_i) + h_i\right)_{i=1,\dots,n}.$$

The operator $\mathcal{T}$ is a contraction in $\left[L^{p,\lambda_1}(\Omega)\right]^n$ and then $\mathcal{T}$ has a unique fixed point. Since, by (4.4), $(v_{x_i})_{i=1,\dots,n}$ is also a fixed point in $[L^p(\Omega)]^n$ by the uniqueness of a fixed point $v_{x_i} = w_i$, $\forall i = 1,\dots,n$.

If $\lambda_1 = \lambda$, then we have $v_{x_i} \in L^{p,\lambda}(\Omega)$ $\forall i = 1,\dots,n$ and then $\partial_{x_i} u \in L^{p,\lambda}_{loc}(\Omega)$, $\forall i = 1,\dots,n$.

If $\lambda_1 < \lambda$, the same result is obtained iterating this procedure a finite number of times.

Now we prove the estimate ii).

It's sufficient to prove the theorem when the open set Ω is a ball B_R and the compact set K is $\overline{B}_r$ with $r \in]0, R[$.

Let $B_\rho \equiv B_\rho(x)$ in the class of balls centered in $x \in \Omega$ with radius $\rho > 0$. Integrating over $B_R \cap B_\rho$ the representation formula (4.3) and using Theorem 3.1 and Theorem 3.4 we find

$$\begin{aligned}\left(\frac{1}{\rho^\lambda}\int_{B_r\cap B_\rho} |\partial_{x_i} u|^p\right)^{\frac{1}{p}} &\le \left(\frac{1}{\rho^\lambda}\int_{B_R\cap B_\rho} |\partial_{x_i} v|^p\right)^{\frac{1}{p}} \\ &\le c\left(\|a\|_* \, \|\partial_{x_i} v\|_{L^{p,\lambda}(\Omega)} + \|G\|_{L^{p,\lambda}(\Omega)} + \|g\|_{L^{p_*,\frac{\lambda p_*}{p}}(\Omega)}\right)\end{aligned}$$

where the norm $\|a\|_*$ is taken in the set B_R.

Using Theorem 3.4 and just observing that $L^{p,\lambda}(\Omega) \subset L^{p_*,\frac{\lambda p_*}{p}}(\Omega)$ the estimate requested taking B_R with radius R such that $c\|a\|_* < \frac{1}{2}$ follows immediately. $\square$

Theorem 4.3. *Let $a \in VMO \cap L^\infty(\mathbb{R}^n)$ such that (2.3) and (2.4) hold true. Let also $f \in \left[L^{p,\lambda}(\Omega)\right]^n$ $0 < \lambda < n, 1 < p < \infty$, $u \in L^p(\Omega)$, the solution of $\mathcal{L}u = div\, f$ with $\partial_{x_i} u \in L^p(\Omega)$ $\forall i = 1,\dots,n$.*

Then for any compact set $K \subset \Omega$ there exists $c = c(n,p,\lambda,\sigma,\eta,K,\Omega)$ such that

$$\|\partial_{x_i} u\|_{L^{p,\lambda}(K)} \le c\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\right)$$

for every $i = 1,\dots,n$, $0 < \lambda < n$, $2 < p < +\infty$.

Proof. From ii) of the previous theorem we have

(4.5)

$$\|\partial_{x_i} u\|_{L^{p,\lambda}(K)} \le c\left(\|u\|_{L^{p,\lambda}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)} + \|\partial_{x_i} u\|_{L^{p_*,\frac{\lambda p_*}{p}}(\Omega)}\right).$$

Let $2 < p \le 2^*$ ($p_* \le 2$), where 2^* is such that $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$. Then using [2, Corollary, p. 277]

$$\|u\|_{L^{p,\lambda}(\Omega)} \le c\left(\|\partial_{x_i} u\|_{L^{q,\mu}(\Omega)}\right)$$

with $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$, $\mu = \frac{(n-1)\lambda}{n}$. Moreover from Remark 2.1 we get $L^{2,\lambda} \subset L^{p_*,\frac{(n-1)\lambda}{n}}$; then

$$\|u\|_{L^{p,\lambda}(\Omega)} \leq c\|\partial_{x_i} u\|_{L^{p_*,\frac{(n-1)\lambda}{n}}(\Omega)} \leq \|\partial_{x_i} u\|_{L^{2,\lambda}(\Omega)}.$$

Using again Remark 2.1 we obtain $L^{2,\lambda} \subset L^{p_*,\frac{\lambda p_*}{p}}$; then

$$\|\partial_{x_i} u\|_{L^{p_*,\frac{\lambda p_*}{p}}(\Omega)} \leq \|\partial_{x_i} u\|_{L^{2,\lambda}(\Omega)}.$$

Then we find from (4.5) and the classical $L^{2,\lambda}$ theory

$$\|\partial_{x_i} u\|_{L^{p,\lambda}(K)} \leq c\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\right). \tag{4.6}$$

Let $2^* < p \leq 2^{**}$ ($p_* \leq 2^*$). Using Remark 2.1 we have $L^{2^*,\lambda} \subset L^{p_*,\frac{(n-1)\lambda}{n}}$; then

$$\|u\|_{L^{p,\lambda}(\Omega)} \leq c\|\partial_{x_i} u\|_{L^{p_*,\frac{(n-1)\lambda}{n}}(\Omega)} \leq \|\partial_{x_i} u\|_{L^{2^*,\lambda}(\Omega)}.$$

Because of $L^{2^*,\lambda} \subset L^{p_*,\frac{\lambda p_*}{p}}$ we have

$$\|\partial_{x_i} u\|_{L^{p_*,\frac{\lambda p_*}{p}}(\Omega)} \leq \|\partial_{x_i} u\|_{L^{2^*,\lambda}(\Omega)}.$$

We deduce that

$$\begin{aligned} \|\partial_{x_i} u\|_{L^{p,\lambda}(K)} &\leq c\left(\|f\|_{L^{p,\lambda}(\Omega)} + \|\partial_{x_i} u\|_{L^{2^*,\lambda}(\Omega)}\right) \\ &\quad \text{applying (4.6) with } p = 2^* \\ &\leq c\left(\|f\|_{L^{p,\lambda}(\Omega)} + \|f\|_{L^{2^*,\lambda}(\Omega)} + \|u\|_{L^2(\Omega)}\right) \leq c\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\right). \end{aligned}$$

Iterating this technique we deduce the result for every $p > 2$. □

Theorem 4.4. *Let* $0 < \lambda < n$, $2 < p < \infty$, *such that* $p > n - \lambda$, $a \in VMO \cap L^\infty(\mathbb{R}^n)$ *satisfying* (2.3) *and* (2.4). *Then if* $u \in L^p(\Omega)$ *is solution of* $\mathcal{L}u = div\, f$ *with* $\partial_{x_i} u \in L^p(\Omega)$ *for every* $i = 1, \dots, n$ *and* $f \in \left[L^{p,\lambda}(\Omega)\right]^n$ *we have that for any compact set* $K \subset \Omega$, $u \in C^{(0,\alpha)}(K)$ *and*

$$\|u\|_{C^{(0,\alpha)}(K)} \leq c\left(\|f\|_{L^{p,\lambda}(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

where $\alpha = 1 - \frac{n}{p} + \frac{\lambda}{p}$ *and* c *has the same meaning as Theorem 4.3.*

Proof. To prove the above inequality we argue using Theorem 4.3 (see, e.g., [8, Theorem 1.2]). □

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