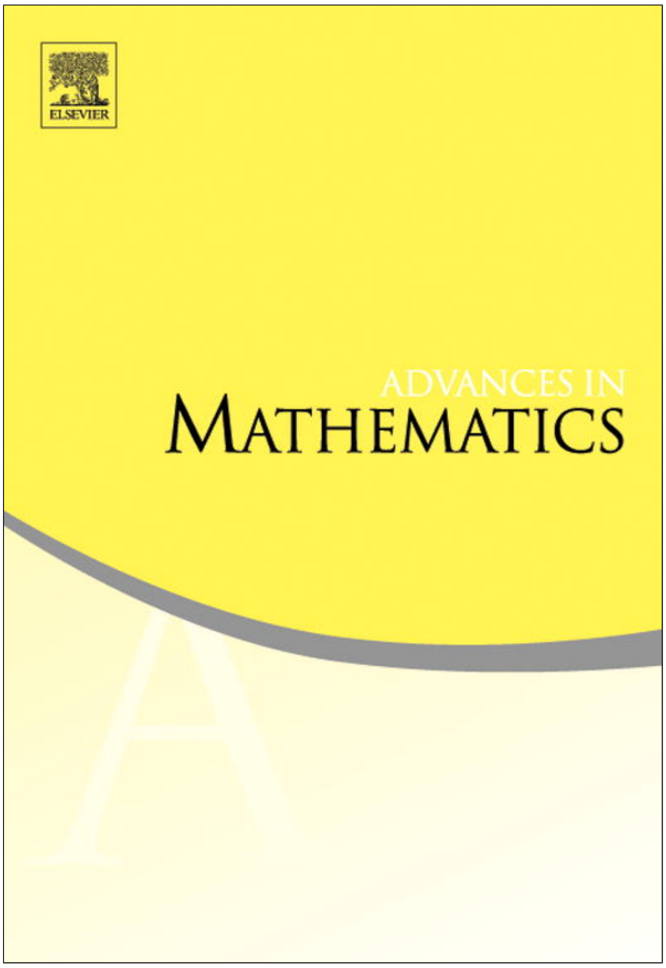
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Asymptotic resurgences for ideals of positive dimensional subschemes of projective space

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Abstract

Recent work of Ein–Lazarsfeld–Smith and Hochster–Huneke raised the containment problem of determining which symbolic powers of an ideal are contained in a given ordinary power of the ideal. Bocci–Harbourne defined a quantity called the resurgence to address this problem for homogeneous ideals in polynomial rings, with a focus on zero-dimensional subschemes of projective space. Here we take the first steps toward extending this work to higher dimensional subschemes. We introduce new asymptotic versions of the resurgence and obtain upper and lower bounds on them for ideals I of smooth subschemes, generalizing what is done in Bocci and Harbourne (2010) [5]. We apply these bounds to ideals of unions of general lines in $\mathbb{P}^N$. We also pose a Nagata type conjecture for symbolic powers of ideals of lines in $\mathbb{P}^3$.

MSC: 13F20; 13A15; 14C20

Keywords: Symbolic powers; Monomial ideals; Points; Lines; Resurgence

1. Introduction

Given a homogeneous ideal I in the homogeneous coordinate ring $k[\mathbb{P}^N]$ of projective space over a field k, the containment problem is to determine the set S_I of pairs (r, m) for which

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the symbolic power $I^{(m)}$ is contained in the ordinary power I^r. The groundbreaking results of [13,23] show that $I^{(m)} \subseteq I^r$ whenever $m \geq Nr$ and hence $\{(r, m) : m \geq rN\} \subseteq S_I$. Refinements of this have recently been given in [5,6,4,11,21] but there are few cases for which S_I is known completely. The resurgence $\rho(I)$, introduced in [5], is a way of characterizing S_I numerically (see Section 1.1), but the resurgence itself has been determined only in very special cases. For example, $S_I = \{(r, m) : m \geq r\}$ if I is generated by a regular sequence [5], in which case $\rho(I) = 1$. Other examples for which S_I is known include certain cases for which I is a monomial ideal [17, Theorem 4.11] or the ideal of a projective cone [5, Proposition 2.5.1]. However, there are very few other cases of ideals I of positive dimensional subschemes for which either S_I or the resurgence have been determined. Somewhat more is known for ideals I defining zero-dimensional subschemes of projective space [5,6,12]. For example, if I defines a zero-dimensional subscheme such that $\alpha(I) = \mathrm{reg}(I)$, where $\mathrm{reg}(I)$ is the Castelnuovo–Mumford regularity of I and $\alpha(I)$ is the degree of a nonzero element of I of least degree, then $\rho(I)$ and S_I can be completely described in terms of numerical invariants of I; see [5, Corollary 2.3.7] and [6, Corollary 1.2].

In order to extend this last result to higher dimensional subschemes, we introduce asymptotic refinements $\rho_a(I)$ and $\rho'_a(I)$ of the resurgence better adapted to studying ideals $I = I(Z)$ defining higher dimensional subschemes Z of projective space $\mathbb{P}^N$. As a tradeoff we require that the subschemes Z be smooth. Our main result, Theorem 1.2, gives upper and lower bounds on $\rho_a(I)$ in terms of four numerical invariants of I (viz., $\alpha(I)$, $\omega(I)$, $\mathrm{reg}(I)$ and $\gamma(I)$, where $\omega(I)$ is the largest degree in a minimal homogeneous set of generators of I and $\gamma(I) = \lim_{m\to\infty} \alpha(I^{(m)})/m$). When these bounds agree, we get $\rho_a(I)$ exactly and, by Proposition 1.1, we get a nearly complete asymptotic determination of those m and r such that $I^{(m)} \subseteq I^r$. As an application, we consider ideals $I = I(Y)$ of unions Y of general lines in $\mathbb{P}^N$ for $N \geq 3$. We determine the asymptotic resurgence in certain cases, sometimes in terms of $\gamma(I)$ (see Corollary 1.3 and Theorem 1.5). As discussed in Section 2, it is in principle possible to compute $\gamma(I)$ arbitrarily accurately, but in most cases the exact value of $\gamma(I)$ is not known. Thus in Section 5 we develop a new method for obtaining bounds on $\gamma(I)$; this leads us to propose an approach for formulating conjectures for the value of $\gamma(I)$ for generic lines in $\mathbb{P}^N$, $N \geq 3$. The resulting conjectures generalize a famous conjecture of M. Nagata [27] arising out of his resolution of Hilbert's 14th problem. This conjecture, which asserts that $\gamma(I) = \sqrt{s}$ for the ideal I of a set of $s \geq 10$ generic points in $\mathbb{P}^2$, has seen a lot of attention (see [3], [8, Theorem 9], [9, Conjecture 5.11], [10,20], [26, Remark 5.1.14] and [31], to mention just a few references). Versions of Nagata's conjecture have also been formulated for ideals of sets of sufficiently many generic points in $\mathbb{P}^N$, $N > 2$, by A. Iarrobino and L. Evain [24,16]. We demonstrate our approach by stating an explicit conjecture for $\gamma(I)$ for ideals I of s generic lines in $\mathbb{P}^3$ for $s \gg 0$.

Throughout this paper, we work over an algebraically closed field k, of arbitrary characteristic. We now recall the definition and significance of the resurgence. Let $(0) \subsetneq I \subsetneq (1)$ be a homogeneous ideal in the homogeneous coordinate ring $k[\mathbb{P}^N] = k[x_0, \dots, x_N]$ of $\mathbb{P}^N$. Then

$$\rho(I) = \sup\left\{m/r : I^{(m)} \not\subseteq I^r\right\}$$

is called the *resurgence* of I [5] (see Section 2 below for the definition of the symbolic power $I^{(m)}$). It follows from [13,23] that $\rho(I) \leq N$, and more generally that $\rho(I) \leq h_I$ where h_I is the big height of I (i.e., the maximum of the heights of associated primes of I).

By definition, if m and r are positive integers such that $m/r > \rho(I)$, then $I^{(m)} \subseteq I^r$. A lower bound, $\alpha(I)/\gamma(I) \leq \rho(I)$, is given in [5]. If $m/r < \alpha(I)/\gamma(I)$, then it follows from [5, Lemma 2.3.2(b)] that $I^{(m')} \not\subseteq I^{r'}$ for all but finitely many m', r' with $m'/r' = m/r$. In case I defines a zero-dimensional subscheme, then there is also an upper bound, $\rho(I) \leq \mathrm{reg}(I)/\gamma(I)$[5, Corollary 2.3.5]. If in addition $\alpha(I) = \mathrm{reg}(I)$, then not only do we obviously have $\rho(I) = \alpha(I)/\gamma(I)$, but by [6, Corollary 1.2] we have a complete solution to the containment problem in terms of numerical invariants: that is, $I^{(m)} \subseteq I^r$ if and only if $\alpha(I^{(m)}) \geq r\alpha(I)$.

The restriction of the upper bound $\rho(I) \leq \mathrm{reg}(I)/\gamma(I)$ to ideals $I = I(Z)$ of zero-dimensional subschemes $Z \subset \mathbb{P}^N$ involves two issues. The simpler issue is that the proof uses the fact that $I^{(m)}$ is the saturation of I^m for the ideal $I = I(Z)$ of any zero-dimensional subscheme Z, but this is also true for smooth Z subschemes of any dimension. So we can overcome this hurdle by restricting attention to smooth Z, which is the case of main interest. The second and more serious issue is that the proof uses the fact that $\mathrm{reg}(I^r) \leq r\,\mathrm{reg}(I)$. This holds for ideals $I = I(Z)$ of zero-dimensional subschemes Z, but it does not hold in general. What is true in general is only that $\mathrm{reg}(I^r) \leq r\,\mathrm{reg}(I) + c_I$ for some constant c_I depending on I [25]. We deal with this issue by focusing our attention on asymptotic versions of $\rho(I)$. This allows us to obtain for smooth subschemes Z of any dimension asymptotic versions of the results discussed above which had been shown only for Z of dimension zero. In particular, we obtain $\mathrm{reg}(I)/\gamma(I)$ as an upper bound for $\rho_a(I)$ for the ideal I of any smooth subscheme Z (see Theorem 1.2), and when $\alpha(I) = \mathrm{reg}(I)$ we obtain a nearly complete asymptotic solution to the containment problem in terms of numerical invariants (see Proposition 1.1).

1.1. Discussion of results

For a homogeneous ideal $(0) \subsetneq I \subsetneq k[\mathbb{P}^N]$, we define an *asymptotic resurgence* as follows:

$$\rho_a(I) = \sup\{m/r : I^{(mt)} \not\subseteq I^{rt} \text{ for all } t \gg 0\}.$$

We will also consider an additional asymptotic version of the resurgence. We define

$$\rho'_a(I) = \limsup_{t\to\infty} \rho(I, t),$$

where $\rho(I, t) = \sup\{m/r : I^{(m)} \not\subseteq I^r, m \geq t, r \geq t\}$.

Our underlying motivation is to understand the set S_I. No algorithm is known for computing S_I in general (in finite time that is), but numerical invariants can give some insight into the structure of S_I. By the results of [13,23], $m/r > N$ guarantees that $I^{(m)} \subseteq I^r$. The resurgence $\rho(I)$ is the least value c such that $m/r > c$ guarantees the containment $I^{(m)} \subseteq I^r$, and thus $\rho(I)$ gives significant (but of course incomplete) information about S_I: $\rho(I)$ is the least slope λ such that all lattice points (r, m) above the line $m = \lambda r$ lie in S_I.

The asymptotic resurgences give similar but asymptotic information about S_I. For example, $\rho_a(I)$ is the least value c such that $m/r > c$ implies $I^{(mt)} \subseteq I^{rt}$ for infinitely many $t > 0$. We also note that $m/r < \rho_a(I)$ implies that $I^{(mt)} \not\subseteq I^{rt}$ for infinitely many $t > 0$. (If $m/r < \rho_a(I)$, then there is a pair (m', r') such that $m/r \leq m'/r' \leq \rho_a(I)$ with $I^{(m't)} \not\subseteq I^{r't}$ for all $t \gg 0$, and so $I^{(mm't)} \not\subseteq I^{r'mt}$ for all $t \gg 0$. But $r'm \leq m'r$ so $I^{rm't} \subseteq I^{r'mt}$, hence $I^{(mm't)} \not\subseteq I^{rm't}$ for all $t \gg 0$, so $I^{(mt)} \not\subseteq I^{rt}$ for infinitely many $t > 0$.) Moreover, $m/r > \rho'_a(I)$ implies $I^{(m')} \subseteq I^{r'}$ for all $m' \gg 0$, $r' \gg 0$ such that $m'/r' \geq m/r$, while $m/r < \rho'_a(I)$ implies there are infinitely many pairs (m', r') with $m/r \leq m'/r' \leq \rho'_a(I)$ such that $I^{(m')} \not\subseteq I^{r'}$.

However, no general algorithm is known for computing $\rho_a(I)$, $\rho'_a(I)$ or $\rho(I)$. Thus, it is also helpful to be able to relate the structure of S_I to numerical invariants that are more easily computable, as the following proposition does. Note that this proposition provides a nearly complete asymptotic solution for which symbolic powers of the ideal I of a smooth subscheme are contained in any given ordinary power, in the case that $\alpha(I) = \mathrm{reg}(I)$ (a complete solution would also deal with the case that $m/r = \alpha(I)/\gamma(I)$). As an added benefit, we derive from this proposition the bounds on $\rho_a(I)$ stated in our main result, Theorem 1.2.

Proposition 1.1. *Consider a homogeneous ideal* $(0) \neq I \subsetneq k[\mathbb{P}^N]$. *Let* m *and* r *be positive integers.*

(1) *If* $m/r < \alpha(I)/\gamma(I)$, *then* $I^{(m')} \not\subseteq I^{r'}$ *for all but finitely many integers* $m', r' > 0$ *with* $m'/r' = m/r$.
(2) *If in addition* I *is the ideal of a (non-empty) smooth subscheme* $Z \subsetneq \mathbb{P}^N$ *and* $m/r > \mathrm{reg}(I)/\gamma(I)$, *then* $I^{(m')} \subseteq I^{r'}$ *for all but finitely many integers* $m', r' > 0$ *with* $m'/r' = m/r$. *(This statement also holds with* $\mathrm{reg}(I)$ *replaced by* $\omega(I)$*.)*

Note that $\omega(I) \leq \mathrm{reg}(I)$ so Proposition 1.1(2) is stronger with $\omega(I)$ in place of $\mathrm{reg}(I)$, but $\mathrm{reg}(I)$ is sometimes easier to compute than $\omega(I)$.

It is not hard to see that Proposition 1.1(1) implies the bound $\alpha(I)/\gamma(I) \leq \rho_a(I)$ given in Theorem 1.2(1) of our main result, and Proposition 1.1(2) implies the bound $\rho_a(I) \leq \mathrm{reg}(I)/\gamma(I)$ given in Theorem 1.2(2). The lower bound is in fact what was actually proved (but not stated) in [5], and applies to any nontrivial homogeneous ideal. Our upper bound on $\rho_a(I)$ is closely related to that of [5] but applies to the ideal of any smooth subscheme of $\mathbb{P}^N$, whereas the upper bound on the resurgence given in [5] applies only to ideals defining zero-dimensional subschemes (although the subschemes do not need to be smooth). As for Theorem 1.2(3), note by [5, Lemma 2.3.3(a)] that $I^{(m)} \not\subseteq I^r$ if $m < r$ for a homogeneous ideal $(0) \neq I \neq (1)$, and hence if $I^{(m)} = I^m$ for all $m \geq 1$, then $I^{(m)} = I^m \subseteq I^r$ if and only if $r \leq m$, so $\rho_a(I) = \rho_a'(I) = \rho(I) = 1$. Theorem 1.2(3) generalizes this fact. Its hypothesis that there is a c such that $I^{(cm)} = (I^{(c)})^m$ for all $m \geq 1$, applies, for example, whenever the symbolic power algebra $\bigoplus_j I^{(j)}$ is Noetherian; see [29, Proposition 2.1] or [28].

Theorem 1.2. *Consider a homogeneous ideal* $(0) \neq I \subsetneq k[\mathbb{P}^N]$. *Let* $h = \min(N, h_I)$ *where* h_I *is the maximum of the heights of the associated primes of* I.

(1) *We have* $1 \leq \alpha(I)/\gamma(I) \leq \rho_a(I) \leq \rho_a'(I) \leq \rho(I) \leq h$.
(2) *If* I *is the ideal of a (non-empty) smooth subscheme of* $\mathbb{P}^N$, *then*

$$\rho_a(I) \leq \frac{\omega(I)}{\gamma(I)} \leq \frac{\mathrm{reg}(I)}{\gamma(I)}.$$

(3) *If for some positive integer* c *we have* $I^{(cm)} = (I^{(c)})^m$ *for all* $m \geq 1$, *and if* $I^{(c)} \subseteq I^b$ *for some positive integer* b, *then*

$$\rho_a'(I) \leq \frac{c}{b}.$$

As an application of Theorem 1.2(1, 2) showing that our results can be applied in interesting cases, we have the following:

Corollary 1.3. *Let* I *be the ideal of* s *general lines in* $\mathbb{P}^N$ *for* $N \geq 3$, *where* $s = \binom{t+N}{N}/(t+1)$ *for any integer* $t \geq 0$ *such that* s *is an integer (there are always infinitely many such* t*; for example, let* $t = p - 1$ *for a prime* $p > N$*). Then* $\rho_a(I) = (t+1)/\gamma(I)$.

Here are examples demonstrating Theorem 1.2(3).

Example 1.4. Let I be the ideal of n general points in $\mathbb{P}^2$. If $n = 6$, we have $I^{(10m)} = (I^{(10)})^m$ [21, end of Section 3] and $I^{(10)} \subseteq I^8$ [6, Proposition 4.1] so $\rho'_a(I) \le 10/8 = 5/4$ by Theorem 1.2(3). In fact, $\rho(I) = 5/4$ by [6, Proposition 4.1], so this also follows from Theorem 1.2(1). Moreover, $I^{(m)} \not\subseteq I^r$ if and only if $m < 5r/4 - 5/12$ by [6, Proposition 4.1]. It follows that $I^{(mt)} \not\subseteq I^{rt}$ for all $t \gg 0$ whenever $m/r < 5/4$ so $5/4 \le \rho_a(I)$ and hence we have $\rho_a(I) = \rho'_a(I) = \rho(I) = 5/4$. Similarly, if $n = 7$ we have $I^{(24m)} = (I^{(24)})^m$ [21] and $I^{(24)} \subseteq I^{21}$ [6, Proposition 4.3] so $\rho'_a(I) \le 24/21 = 8/7$ (here also we have $\rho_a(I) = \rho'_a(I) = \rho(I) = 8/7$ since $I^{(m)} \not\subseteq I^r$ if $m < 8r/7$ for $m > 1$ by [6, Proposition 4.3], so it follows that $I^{(mt)} \not\subseteq I^{rt}$ for all $t \gg 0$ whenever $m/r < 8/7$). Finally, if $n = 8$, then $I^{(102m)} = (I^{(102)})^m$ for all $m \ge 1$ by Harbourne and Huneke [21] and $I^{(102)} \subseteq I^{72}$ by [6, Proposition 4.4], so $\rho'_a(I) \le 102/72 = 17/12$ (and again $\rho_a(I) = \rho'_a(I) = \rho(I) = 17/12$, since $I^{(m)} \not\subseteq I^r$ if $m < 17r/12 - 1/3$ for $m > 1$ by [6, Proposition 4.4]).

It is in general an open problem to compute $\gamma(I)$. It is often challenging just to find good bounds on $\gamma(I)$; see [15,21,30]. Thus Corollary 1.3 shows that the problem of computing $\rho_a(I)$ is related to a significant open problem. Although it can be difficult to determine $\gamma(I)$ exactly, it is possible in principle to estimate it to any desired precision; see Section 2. In some cases, however, such as for ideals of s general lines in $\mathbb{P}^N$ for small values of s, determining $\gamma(I)$ and $\rho(I)$ (and also $\rho_a(I)$ and $\rho'_a(I)$) is much easier:

Theorem 1.5. *Let I be the ideal of s general lines in $\mathbb{P}^N$ for $N \ge 2$ and $s \le (N+1)/2$. Then $\rho(I) = \rho'_a(I) = \rho_a(I) = \max(1, 2\frac{s-1}{s})$. Moreover, if $2s < N+1$, then $\gamma(I) = 1$, while if $2s = N+1$, then $\gamma(I) = \frac{N+1}{N-1}$.*

In Section 2 we recall basic facts that we will need for the proofs. We give the proofs of Proposition 1.1, Theorem 1.2 and Corollary 1.3 in Section 3 and the proof of Theorem 1.5 in Section 4. We discuss the problem of computing $\gamma(I)$ in more detail in Section 5. In Section 6 we include some closing comments and open questions.

2. Preliminaries

Let $I \subsetneq R = k[\mathbb{P}^N] = k[x_0, \dots, x_N]$ be a homogeneous ideal. Then I has a homogeneous primary decomposition, i.e., a primary decomposition $I = \bigcap_i Q_i$ where each $\sqrt{Q_i}$ is a homogeneous prime ideal, and Q_i is homogeneous and $\sqrt{Q_i}$-primary [32, Theorem 9, p. 153]. We define the m-th *symbolic power* of I to be the ideal $I^{(m)} = \bigcap_j P_{i_j}$, where $I^m = \bigcap_i P_i$ is a homogeneous primary decomposition, and the intersection $\bigcap_j P_{i_j}$ is over all primary components P_i such that $\sqrt{P_i}$ is contained in an associated prime of I. Thus $I^{(m)} = \bigcap_{P \in \mathrm{Ass}(I)} (I^m R_P \cap R)$, where R_P is the localization at P and the intersection is taken over the associated primes P of I. In particular, we see that $I^{(1)} = I$ and that $I^m \subseteq I^{(m)}$.

In Corollary 1.3, we are interested in the ideal I of a scheme X which is a union of s disjoint lines $L_i \subset \mathbb{P}^N$ with $N \ge 3$. The m-th symbolic power of I in this case is $I^{(m)} = \bigcap_{i=1}^s I(L_i)^m$.

If I is a complete intersection (i.e., generated by a regular sequence), such as is the case for the ideal of a single line in $\mathbb{P}^N$ for $N > 1$, then $I^{(r)} = I^r$ for all $r \ge 0$ (see [32, Lemma 5, Appendix 6]), hence $\gamma(I) = 1$, and, as noted in the discussion before Theorem 1.2, $\rho_a(I) = \rho'_a(I) = \rho(I) = 1$. More generally, if I is the ideal of a smooth subscheme $X \subset \mathbb{P}^N$, then X is locally a complete intersection, but powers of ideals which are complete intersections are unmixed (see [32, Appendix 6]) so the only possible associated primes for powers of I are the minimal primes and the irrelevant prime $M = (x_0, \dots, x_N)$. But M is never an associated prime for the ideal of a subscheme, so $I^{(m)}$ is obtained from I^m by removing the M-primary component (if any); i.e., $I^{(m)}$ is the *saturation* $\operatorname{sat}(I^m)$ of I^m. In particular, the degree t homogeneous components $(I^m)_t$ of I^m and $(I^{(m)})_t$ of $I^{(m)}$ agree for $t \gg 0$. The least s for which $(I^m)_t = (I^{(m)})_t$ for all $t \geq s$ is called the *saturation degree* of I, denoted $\operatorname{satdeg}(I^m)$. We emphasize that the reason the smoothness hypothesis is required in Theorem 1.2 is to allow us to equate symbolic powers with saturations of ordinary powers. For singular varieties, more delicate methods will be required since $\operatorname{sat}(I^m)$ need not equal $I^{(m)}$.

It is known that $\operatorname{satdeg}(I^r) \leq \operatorname{reg}(I^r)$; see [2, Remark 1.3]. Moreover, by Kodiyalam [25, Corollary 3, Proposition 4], the regularity of I^r is bounded above by a linear function $\lambda_I r + c_I$ of r and moreover $\lambda_I \leq \omega(I) \leq \operatorname{reg}(I)$. (Unfortunately the constant term c_I may be positive so we know only that the regularity is bounded above by $\lambda_I r + c_I$ for some c_I; we do not know that $\lambda_I r$ is an upper bound.)

Given homogeneous ideals $(0) \neq I \subseteq J$ we clearly have $\alpha(I) \geq \alpha(J)$. This, and the easy fact that $\alpha(I^r) = r\alpha(I)$, give a useful criterion for showing $I^{(m)} \not\subseteq I^r$, namely, if $\alpha(I^{(m)}) < r\alpha(I)$, then $I^{(m)} \not\subseteq I^r$.

The bounds given in Theorem 1.2 involve the asymptotic quantity $\gamma(I)$, defined above as a limit. For the fact that this limit exists, see [5, Lemma 2.3.1]. It is also known that $\gamma(I) \leq \alpha(I^{(m)})/m$ for all $m \geq 1$. (This is because $\gamma(I) = \lim_{t\to\infty} \alpha(I^{(tm)})/(tm)$, but $(I^{(m)})^t \subseteq I^{(mt)}$, hence $\alpha(I^{(tm)}) \leq \alpha((I^{(m)})^t) = t\alpha(I^{(m)})$, so $\alpha(I^{(tm)})/(tm) \leq \alpha(I^{(m)})/m$.) There are also lower bounds for $\gamma(I)$; indeed, $\alpha(I^{(m)})/(m + N - 1) \leq \gamma(I)$ (see [21, Section 4.2]). In fact, the proof given there works with N replaced by the big height h_I of I; i.e.,

$$\frac{\alpha(I^{(m)})}{m + h_I - 1} \leq \gamma(I). \tag{2.1}$$

Note that $h_I = N - 1$ for the case of a radical ideal defining lines in $\mathbb{P}^N$. Thus while $\gamma(I)$ is hard to compute, one can estimate $\gamma(I)$ arbitrarily accurately by computing values of $\alpha(I^{(m)})$.

In the statement of Theorem 1.2 we divide by $\gamma(I)$. This is allowed since $\gamma(I) > 0$. In fact, $\gamma(I) \geq 1$ when I is not (0) and not (1); see [1, Lemma 8.2.2].

3. Proofs of Proposition 1.1, Theorem 1.2 and Corollary 1.3

Proof of Proposition 1.1. (1) By [5, Lemma 2.3.2(b)], if $m/r < \alpha(I)/\gamma(I)$, then $I^{(mt)} \not\subseteq I^{rt}$ for all $t \gg 0$. We may as well assume that m and r are relatively prime, in which case $m'/r' = m/r$ if and only if $m' = mt$ and $r' = rt$ for some integer $t > 0$, so the result follows.

(2) As is well known, $\omega(I) \leq \operatorname{reg}(I)$, so it is enough to prove the statement with $\omega(I)$ in place of $\operatorname{reg}(I)$. Note that $\omega(I) \geq \alpha(I)$, so $\omega(I)/\gamma(I) \geq \alpha(I)/\gamma(I) \geq 1$. Thus, if $m/r > \omega(I)/\gamma(I)$, then $m\gamma(I) > r\omega(I)$ and $m \geq r$. This means for any fixed constant c, we have $mt\gamma(I) > rt\omega(I) + c$ for $t \gg 0$. So, in particular, for $t \gg 0$ we have

$$\alpha(I^{(mt)}) \geq mt\gamma(I) > rt\omega(I) + c_I \geq \operatorname{reg}(I^{rt}) \geq \operatorname{satdeg}(I^{rt}).$$

Thus $(I^{(mt)})_l = 0$ when $l < \operatorname{satdeg}(I^{rt})$ (since $\operatorname{satdeg}(I^{rt}) < \alpha(I^{(mt)})$), but $I^{(mt)} \subseteq I^{(rt)}$ (since $m \geq r$) so $(I^{(mt)})_l \subseteq (I^{(rt)})_l = (I^{rt})_l$ when $l \geq \operatorname{satdeg}(I^{rt})$; i.e., we have $(I^{(mt)})_l \subseteq (I^{rt})_l$ for all l (and hence $I^{(mt)} \subseteq I^{rt}$) for all $t \gg 0$. Since we may assume m and r are relatively prime, the result follows as in (1). $\square$

Recall that $M = (x_0, \dots, x_N)$ is the irrelevant ideal in $k[\mathbb{P}^N]$.

Proof of Theorem 1.2. (1) It is clear from the definitions that $\rho_a(I) \le \rho_a'(I) \le \rho(I)$. As noted in Section 2 above, $1 \le \gamma(I) \le \alpha(I)$. Thus $\alpha(I)/\gamma(I)$ makes sense and we have $1 \le \alpha(I)/\gamma(I)$. The bound $\alpha(I)/\gamma(I) \le \rho_a(I)$ is clear from the definition of $\rho_a(I)$ using Proposition 1.1(1). We also have $I^{(h_I r)} \subseteq I^r$ for all r by [23]. But $I^{(m)} \subseteq I^{(h_I r)}$ if $m \ge h_I r$, so $h_I \ge \rho(I)$. Moreover, $h_I \le N$ except in the case that M is an associated prime for I. But in this case $I^{(m)} = I^m$ follows from the definition, so $\rho(I) = 1$. Thus $\rho(I) \le N$ and hence $\rho(I) \le h$.

(2) Let $m/r > \omega(I)/\gamma(I)$. By Proposition 1.1, we cannot have $m/r < \rho_a(I)$, so $\rho_a(I) \le m/r$ whenever $m/r > \omega(I)/\gamma(I)$; i.e., $\rho_a(I) \le \omega(I)/\gamma(I)$. As is well known and noted above, $\omega(I) \le \operatorname{reg}(I)$, which finishes the proof of (2).

(3) First note that if $m, r \ge 1$ and if there is an integer $s \ge 0$ such that $m \ge (s+1)c$ and $r \le (s+1)b$, then

$$I^{(m)} \subseteq I^{((s+1)c)} = (I^{(c)})^{s+1} \subseteq (I^b)^{s+1} = I^{(s+1)b} \subseteq I^r.$$

We now show that if $m, r \ge 1$ satisfies $r \le mb/c - b$, then $I^{(m)} \subseteq I^r$. This is because we can take s to be the largest integer such that $sb \le r$, hence $r < (s+1)b$. But now $m \ge (c/b)(r+b) \ge (c/b)(sb+b) = c(s+1)$.

In particular, if $m, r \ge t$ but $I^{(m)} \not\subseteq I^r$, then $r > mb/c - b$ so $c/b > (m/r) - (c/r)$ hence $(c/b) + (c/t) > m/r$. Therefore, $\rho(I, t) \le (c/b) + (c/t)$, so $\rho_a'(I) = \limsup_t \rho(I, t) \le \limsup_t (c/b) + (c/t) = c/b$. $\square$

Proof of Corollary 1.3. We need some facts about ideals of lines in $\mathbb{P}^N$. For the ideal $I = I(X)$ of the union X of s general lines in $\mathbb{P}^N$ for $N \ge 3$, [22] shows that $\dim(I_t) = \max(0, \binom{t+N}{N} - s(t+1))$. Thus, if s and t are such that $\binom{t+N}{N} = s(t+1)$, then $\dim(I_t) = 0$, but $\binom{t+1+N}{N} - s(t+2) = \binom{t+N}{N} + \binom{t+N}{N-1} - s(t+2) = \binom{t+N}{N} + \binom{t+N}{N}\frac{N}{t+1} - s(t+2) = s(t+1) + sN - s(t+2) = s(N-1) > 0$, so $\dim(I_{t+1}) > 0$ and hence $\alpha(I) = t+1$. We claim that $\operatorname{reg}(I) = t+1$ also. So suppose $t \ge 0$ and let $\mathcal{I}$ be the sheafification of I. We have

$$0 \to \mathcal{I}(i) \to \mathcal{O}_{\mathbb{P}^N}(i) \to \mathcal{O}_X(i) \to 0.$$

By [22], this is surjective on global sections if $i \ge t$ and $\Gamma(\mathcal{O}_{\mathbb{P}^N}(i)) \to \Gamma(\mathcal{O}_X(i))$ is injective otherwise. Now by taking the cohomology of the sheaf sequence above, it is easy to see that $h^j(\mathbb{P}^N, \mathcal{I}(i-j)) = 0$ for all $i \ge t+1$ and $j \ge 1$ but that $h^2(\mathbb{P}^N, \mathcal{I}(t-2)) > 0$ when $t = 0$, and that $h^1(\mathbb{P}^N, \mathcal{I}(t-1)) > 0$ when $t > 0$, for the latter using the fact that $\binom{t-1+N}{N} - st = \binom{t+N}{N} - \binom{t-1+N}{N-1} - s(t+1) + s = s - \binom{t-1+N}{N-1} = s - \binom{t+N}{N}\frac{N}{t+N} = s - s(t+1)\frac{N}{t+N} < 0$. Thus $\operatorname{reg}(I) = t+1$ by Eisenbud [14, Exercise 20.20]. Since $\alpha(I) = t+1 = \operatorname{reg}(I)$, the result now follows by Theorem 1.2. $\square$

Remark 3.1. Even when the bounds given in Theorem 1.2 do not directly give exact values, they can be informative when combined with computational evidence. For example, consider the ideal of $s = 4$ general lines in $\mathbb{P}^3$. In this case $\alpha(I) = 3$, $\operatorname{reg}(I) = 4$, and, by Guardo et al. [19], $\gamma(I) = 8/3$. Thus $9/8 \le \rho_a(I) \le 3/2$ by Theorem 1.2(1) and (2). But computational evidence using *Macaulay2* suggests that $I^{(3t)} = (I^{(3)})^t$ for all $t > 0$ and also that $I^{(9)} \subseteq I^8$. This would imply by Theorem 1.2(3) that $\rho_a'(I) \le 9/8$, and hence $\rho_a(I) = \rho_a'(I) = 9/8$. In fact, evidence from *Macaulay2* also suggests that $I^{(3t+i)} = (I^{(3)})^t I^i$ for $i = 1, 2$ and all $t > 0$ and that $I^{(3)} \subseteq I^2$ and $I^{(6)} \subseteq I^5$. Given that these hold, one can prove that

$I^{(m)} \subseteq I^r$ whenever $m/r \geq 9/8$, which by definition implies that $\rho(I) \leq 9/8$ and thus that $\rho_a(I) = \rho_a'(I) = \rho(I) = 9/8$. The proof is to check cases modulo 3. For example, given any $m \geq 1$, we can write $m = 3t + i$, for some t and $0 \leq i \leq 2$, and we can write $t = 3j + q$ for some $0 \leq q \leq 2$. Then $I^{(m)} = I^{(9j+3q+i)} = (I^{(3)})^{3j+q} I^i = (I^{(9)})^j (I^{(3)})^q I^i$. Also note that $m/r \geq 9/8$ is equivalent to $8j + 8q/3 + 8i/9 = 8m/9 \geq r$. If $q = 0$, then $I^{(9)} \subseteq I^8$ implies $I^{(m)} = (I^{(9)})^j I^i \subseteq I^{8j} I^i$, but $I^{8j} I^i \subseteq I^r$, since $8j + i \geq 8j + 8q/3 + 8i/9 \geq r$. If $q = 1$, then $I^{(9)} \subseteq I^8$ and $I^{(3)} \subseteq I^2$ imply $I^{(m)} = (I^{(9)})^j (I^{(3)})^q I^i \subseteq I^{8j} I^2 I^i$, but $I^{8j} I^2 I^i \subseteq I^r$ since $8j + 8/3 + 8i/9 = 8j + 8q/3 + 8i/9 \geq r$ implies $8j + 2 + i = \lfloor 8j + 8/3 + 8i/9 \rfloor \geq r$. If $q = 2$, then $I^{(9)} \subseteq I^8$ and $I^{(6)} \subseteq I^5$ imply $I^{(m)} = (I^{(9)})^j (I^{(3)})^q I^i \subseteq I^{8j} I^5 I^i$, but $I^{8j} I^5 I^i \subseteq I^r$ since $8j + 5 + i = \lfloor 8j + 8q/3 + 8i/9 \rfloor \geq r$. We expect that it is possible to push through rigorous proofs of the above statements suggested by *Macaulay*2, but the length of the resulting proofs militates against their inclusion here.

4. Proof of Theorem 1.5

Proof of Theorem 1.5. If $s = 1$, then I is a complete intersection so $\gamma(I) = \rho_a(I) = \rho_a'(I) = \rho(I) = 1 = \max(1, 2(s-1)/2)$ as noted above. So hereafter we assume $s > 1$. We begin by computing $\gamma(I)$ for $s = (N+1)/2$ general lines in $\mathbb{P}^N$ in the case that N is odd (hence $s \geq 2$). By choosing two points on each line, we obtain $N + 1$ points of $\mathbb{P}^N$ which after a change of coordinates we may assume are the coordinate vertices. The s lines, $L_1, \ldots, L_s$, now become disjoint coordinate lines. After reindexing, we may assume $I(L_i)$ is generated by the coordinate variables $\{x_j : 0 \leq j \leq N+1, j \neq 2(i-1), 2i-1\}$. Since $I^{(m)} = I(L_1)^m \cap \cdots \cap I(L_s)^m$, we see that $I^{(m)}$ is a monomial ideal, and a monomial $\mu = x_0^{m_0} \cdots x_N^{m_N}$ is in $I^{(m)}$ if and only if $\mu \in I(L_j)^m$ for all j. To get a useful criterion for $\mu \in I(L_j)^m$, let $d_i(\mu) = m_{2(i-1)} + m_{2i-1}$ for $1 \leq i \leq s$, and let $d(\mu) = \deg(\mu) = m_0 + \cdots + m_N$. Clearly $\mu \in I(L_j)^m$ holds if and only if the degree of μ in the variables generating $I(L_i)$ is at least m; i.e., if and only if $d(\mu) \geq m + d_i(\mu)$. Thus $\mu \in I^{(m)}$ if and only if $d(\mu) \geq m + d_i(\mu)$ holds for all $1 \leq i \leq s$. But $d(\mu) = d_1(\mu) + \cdots + d_s(\mu)$, so summing $d(\mu) \geq m + d_i(\mu)$ over i we conclude that $\mu \in I^{(m)}$ implies $sd(\mu) \geq sm + d(\mu)$ or $d(\mu) \geq sm/(s-1)$. Thus $\alpha(I^{(m)}) \geq sm/(s-1) = (N+1)m/(N-1)$, hence $\gamma(I) \geq (N+1)/(N-1)$.

To show in fact that $\gamma(I) = (N+1)/(N-1)$, consider the case that m is a multiple of $s-1$, so let $m = \lambda(s-1)$. We claim that $\alpha(I^{(m)}) = \lambda s$. We already know $\alpha(I^{(m)}) \geq (N+1)m/(N-1) = \lambda s$, so it suffices to find some element of degree λs in $I^{(m)}$. Consider $\mu = x_0^{m_0} \cdots x_N^{m_N}$ where $m_0 = \cdots = m_N = \lambda/2$ for any even λ. Then $d(\mu) = \lambda s = \lambda(s-1) + \lambda = m + d_i(\mu)$ holds for all i, so $\mu \in I^{(m)}$. Thus $\alpha(I^{(m)}) = \lambda s$. Taking the limit of $\alpha(I^{(m)})/m = \lambda s/(\lambda(s-1))$ as $\lambda \to \infty$ gives $\gamma(I) = s/(s-1) = (N+1)/(N-1)$, as claimed. Moreover, since $s = (N+1)/2$, we have $\binom{t+N}{N} = s(t+1)$ for $t = 1$, so, by Corollary 1.3, $\rho_a(I) = (t+1)/\gamma(I) = 2(N-1)/(N+1) = 2(s-1)/s = \max(1, 2(s-1)/s)$.

We now show that we also have $\rho(I) = 2(s-1)/s$, and hence by Theorem 1.2(1) that $\rho_a'(I) = 2(s-1)/s$. Consider the homomorphism $\phi : k[\mathbb{P}^N] = k[x_0, \ldots, x_N] \to k[y_0, \ldots, y_{s-1}] = k[\mathbb{P}^{s-1}]$ of polynomial rings where $y_0 = \phi(x_0) = \phi(x_1)$, $y_1 = \phi(x_2) = \phi(x_3)$, etc. Note that $\phi(I(L_i)) = J(p_i)$ where $p_0, \ldots, p_{s-1}$ are the coordinate vertices of $\mathbb{P}^{s-1}$. For any monomial $\mu' = y_0^{m_0} \cdots y_{s-1}^{m_{s-1}} \in k[\mathbb{P}^{s-1}]$, define $d'(\mu') = \deg(\mu') = m_0 + \cdots + m_{s-1}$ and define $d_i'(\mu') = m_i$. Thus for any monomial $\mu \in k[\mathbb{P}^N]$ we have $d'(\phi(\mu)) = d(\mu)$ and $d_i'(\phi(\mu)) = d_i(\mu)$. Therefore, if we set $J = \phi(I)$, then for any monomial $\mu \in k[\mathbb{P}^N]$, we have $d'(\phi(\mu)) \geq m + d_i'(\phi(\mu))$ if and only if $d(\mu) \geq m + d_i(\mu)$, so $\mu \in I^{(m)}$ if and only if $\phi(\mu) \in J(p_0)^m \cap \cdots \cap J(p_{s-1})^m$. In particular, $J = J(p_0) \cap \cdots \cap J(p_{s-1})$, and $\phi(I^{(m)}) = J^{(m)}$.

We now see that $I^{(m)} \subseteq I^r$ implies $J^{(m)} = \phi(I^{(m)}) \subseteq \phi(I^r) = \phi(I)^r = J^r$. For the converse, assume $J^{(m)} \subseteq J^r$ and consider a monomial $\mu \in I^{(m)}$. Then $\phi(\mu) \in J^{(m)} \subseteq J^r$ and $\phi(\mu)$ is a monomial, so we can factor $\phi(\mu)$ as $\phi(\mu) = \mu'_1 \cdots \mu'_r$ with each μ'_i being a monomial in J. For each i, there is a monomial $\mu_i \in k[x_0, \ldots, x_N]$ with $\phi(\mu_i) = \mu'_i$, and since $\phi(\mu_i) \in J^{(1)} = J$, we have $\mu_i \in I^{(1)} = I$; i.e., $\mu \in I^r$. Thus $I^{(m)} \subseteq I^r$ if and only if $J^{(m)} \subseteq J^r$, so $\rho(I) = \rho(J)$, but $\rho(J) = 2(s-1)/s$ by [5, Theorem 2.4.3(a)].

Now consider the case that $s < (N+1)/2$. Let $n = 2s-1$. A monomial $\mu = x_0^{m_0} \cdots x_N^{m_N}$ factors as $\mu = \mu_1\mu_2$ where $\mu_1 = x_0^{m_0} \cdots x_n^{m_n}$ and $\mu_2 = x_{n+1}^{m_{n+1}} \cdots x_N^{m_N}$. Let $\delta = \delta(\mu) = m_{n+1} + \cdots + m_N$. Then $\mu \in I^{(m)}$ if and only if $\mu_1 \in I^{(m-\delta)}$, where we regard nonpositive powers or symbolic powers as denoting the unit ideal. Similarly, $\mu \in I^r$ if and only if $\mu_1 \in I^{r-\delta}$. Denote by J the ideal of the s lines regarded as being in $\mathbb{P}^n$, where $k[\mathbb{P}^n] = k[x_0, \ldots, x_n] \subset k[x_0, \ldots, x_N]$. Then $\mu_1 \in I^{(m)}$ if and only if $\mu_1 \in J^{(m)}$ and $\mu_1 \in I^r$ if and only if $\mu_1 \in J^r$. Also, the s lines in $\mathbb{P}^n$ have $\rho_a(J) = \rho(J) = 2(s-1)/s$ by the previously considered case. Now, for any monomial $\mu' \in J^{(m)} \setminus J^r$, we have $\mu' \in I^{(m)} \setminus I^r$, so $\rho(I) \geq \rho(J)$ and $\rho_a(I) \geq \rho_a(J)$. On the other hand, say m and r are such that $I^{(m)} \not\subseteq I^r$. Then there is a monomial $\mu \in I^{(m)} \setminus I^r$ (hence $\delta = \delta(\mu) < r$, since $\delta(\mu) \geq r$ implies $\mu \in I^r$). If $m < r$, then $J^{(m)} \not\subseteq J^r$, so $m/r \leq \rho(J)$. Now assume $m \geq r$. Since $\mu_1 \in I^{(m-\delta)} \setminus I^{r-\delta}$, we see $\mu_1 \in J^{(m-\delta)} \setminus J^{r-\delta}$, so $m/r \leq (m-\delta)/(r-\delta) \leq \rho(J)$. Thus $m/r \leq \rho(J)$ whenever we have $I^{(m)} \not\subseteq I^r$, so we conclude $\rho(I) \leq \rho(J)$ hence $\rho_a(I) \leq \rho_a(I) \leq \rho(I) \leq \rho(J) = \rho_a(J)$. Thus $\rho_a(I) = \rho(I) = \rho(J) = \rho_a(J)$.

Finally, if $s < (N+1)/2$ (N not necessarily odd), then the s lines are contained in a hyperplane, in particular, in $x_N = 0$, so $\alpha(I^{(m)}) = m$ and hence $\gamma(I) = 1$. $\square$

For the ideal I of two disjoint lines in $\mathbb{P}^N$, Theorem 1.5 shows that $\rho(I) = 1$. An alternative way to see that $\rho(I) = \rho_a(I) = 1$ for the ideal I of two skew lines in $\mathbb{P}^N$, is to show that $I^{(m)} = I^m$ for all $m \geq 1$. To see this, let $V \subseteq \{x_0, \ldots, x_N\}$ be a non-empty subset of the coordinate variables of $k[\mathbb{P}^N]$. Let $\{V_j\}_{j=1,\ldots,r}$ be a partition of V into non-empty proper disjoint subsets. Let $I_j = (V_j)$. Since each ideal I_j is a complete intersection, we have $(I_j)^{(m)} = (I_j)^m$ for all $m \geq 1$. Then we have the following result.

Lemma 4.1. *Let $m_1, \ldots, m_r$ be nonnegative integers, not all zero. Then* $I_1^{m_1} \cdots I_r^{m_r} = \bigcap_j (I_j^{m_j})$.

Proof. Clearly we have $I_1^{m_1} \cdots I_r^{m_r} \subseteq \bigcap_j (I_j^{m_j})$. Both ideals are monomial ideals. The former is generated by the monomials of the form $\mu_1 \cdots \mu_r$ where μ_j is a monomial of degree m_j in the variables V_j. Thus it is enough for any monomial $\mu \in \bigcap_j (I_j^{m_j})$ to show that μ is divisible by such a monomial $\mu_1 \cdots \mu_r$. But $\mu \in I_j^{m_j}$ for each j and $I_j^{m_j}$ is generated by monomials of degree m_j in the variables V_j, and so there is such a monomial μ_j that divides μ. Since the elements $\mu_1, \ldots, \mu_r$ are pair-wise relatively prime, we see that $\mu_1 \cdots \mu_r$ divides μ. $\square$

Remark 4.2. We now show that $I^{(m)} = I^m$ for all $m \geq 1$ whenever I is the ideal of two skew lines in $\mathbb{P}^N$. So let $L_1, L_2 \subset \mathbb{P}^N$ be skew lines. If $N = 3$, by an appropriate choice of coordinates, we may assume $I(L_1) = (x_0, x_1)$ and $I(L_2) = (x_2, x_3)$, and $I = I(L_1) \cap I(L_2)$. By Lemma 4.1 we have $I = I(L_1)I(L_2)$ and $I^{(m)} = I(L_1)^m \cap I(L_2)^m = (I(L_1)I(L_2))^m = I^m$. If $N > 3$, we have $I(L_1) = (x_0, x_1, x_4, \ldots, x_N)$ and $I(L_2) = (x_2, \ldots, x_N)$, and so $I = (x_0x_2, x_0x_3, x_1x_2, x_1x_3, x_4, \ldots, x_N)$. Clearly, $I^m \subseteq I^{(m)}$ for all $m \geq 1$, so let $\mu \in I^{(m)}$ be a monomial. We have a factorization $\mu = \mu_1\mu_2\mu_3$ where $\mu_1 \in (x_0, x_1)$, $\mu_2 \in (x_2, x_3)$ and

$\mu_3 \in (x_4, \ldots, x_N)$. Let $\delta_i = \deg(\mu_i)$, $\delta = \deg(\mu)$. If $\delta_3 \geq m$, then $\mu_3 \in I^m$, hence $\mu \in I^m$. Assume $\delta_3 < m$. Then $\mu \in I^{(m)}$ implies $\delta_i \geq m - \delta_3$ for $i = 1, 2$. Thus $\mu_1 \in (x_0, x_1)^{m-\delta_3}$ and $\mu_2 \in (x_2, x_3)^{m-\delta_3}$. By Lemma 4.1, we have $\mu_1\mu_2 \in (x_0, x_1)^{m-\delta_3} \cap (x_2, x_3)^{m-\delta_3} = [(x_0, x_1)(x_2, x_3)]^{m-\delta_3} = (x_0x_2, x_0x_3, x_1x_2, x_1x_3)^{m-\delta_3} \subseteq I^{m-\delta_3}$, and $\mu = \mu_1\mu_2\mu_3 \in I^m$.

5. Computing $\gamma(I)$

Corollary 1.3 gives an exact answer for $\rho_a(I)$, but it is in terms of $\gamma(I)$, which is hard to determine in general. Let I be the ideal of s generic lines in $\mathbb{P}^N$. In cases such that $I^{(m)} = I^m$ for all $m \geq 1$, we have $\gamma(I) = \alpha(I)$. Thus $\gamma(I) = 1$ if $s = 1$, since I is a complete intersection, so $I^{(m)} = I^m$ for all $m \geq 1$. When $s = 2$, then again $I^{(m)} = I^m$ for all $m \geq 1$ by Remark 4.2, so $\gamma(I) = \alpha(I) = 2$ if $N = 3$ and $\gamma(I) = \alpha(I) = 1$ if $N > 3$. If $2s \leq N + 1$, then we know $\gamma(I)$ by Theorem 1.5. If $N = s = 3$, then we will show in a separate paper [19] that once again $I^{(m)} = I^m$ for all $m \geq 1$, so $\gamma(I) = \alpha(I) = 2$, and, by exploiting a connection between lines in $\mathbb{P}^3$ and points in $\mathbb{P}^1 \times \mathbb{P}^1$, that $\gamma(I) = 8/3$ if $s = 4$. (It is not hard to see at least that $\gamma(I) \leq 8/3$. If $s = 4$, then $\alpha(I) = 3$ and there is a unique quadric through any three of the four lines, so taking each combination of three of the four lines we see that the four quadrics give an element of $I^{(3)}$ of degree 8. Thus $\alpha(I^{(3)}) \leq 8$, so $\gamma(I) \leq \alpha(I^{(3)})/3 = 8/3$.)

As far as we know, $\gamma(I)$ is not known for any other cases of s generic lines in $\mathbb{P}^N$, but one can still say something. For example, say $N = 3$ and $s = 5$. Then $\alpha(I) = 4$, so using the lower bound $\alpha(I^{(m)})/(m+h_I-1) \leq \gamma(I)$ we have $2 \leq \gamma(I)$. Using the ten quadrics through combinations of three of the five lines gives $\alpha(I^{(6)}) \leq 20$, so we get $\gamma(I) \leq \alpha(I^{(6)})/6 \leq 20/6$. (Experiments with *Macaulay*2 suggest that $\alpha(I^{(6)}) = 20$. If so, using the lower bound $\alpha(I^{(m)})/(m+h_I-1) \leq \gamma(I)$, we would have $20/7 \leq \gamma(I)$. As an application, this would imply by Corollary 1.3 that $6/5 \leq \rho_a(I) \leq 7/5$. In fact, *Macaulay*2 calculations suggest that $\alpha(I^{(12)}) = 40$ which would give $40/13 \leq \gamma(I)$ and $6/5 \leq \rho_a(I) \leq 13/10$.)

There is also the question of what it is reasonable to conjecture $\gamma(I)$ to be for the ideal I of s generic lines in $\mathbb{P}^N$. The still open conjecture of Nagata discussed in the introduction addresses the question for ideals of generic points in $\mathbb{P}^2$. A generalization given by Iarrobino [24] and Evain [16] for generic points in $\mathbb{P}^N$ subsuming Nagata's conjecture can be paraphrased in the following way:

Conjecture 5.1. *Let $I \subset k[\mathbb{P}^N]$ be the ideal of $s \gg 0$ generic points of $\mathbb{P}^N$ for $N \geq 2$. Then $\gamma(I) = \sqrt[N]{s}$.*

The motivation for this conjecture is that if I is the ideal of s generic points p_i in $\mathbb{P}^N$, then it is easy to check that $\dim((I(p_i)^m)_t) = \max(0, \binom{t+N}{N} - \binom{m+N-1}{N})$. It follows that $\dim((I^{(m)})_t) \geq \max(0, \binom{t+N}{N} - s\binom{m+N-1}{N})$. Thus $\alpha(I^{(m)})$ is at most the least t such that $\binom{t+N}{N} > s\binom{m+N-1}{N}$. An asymptotic analysis now shows that $\gamma(I) \leq \sqrt[N]{s}$, and the naive hope is that the other inequality also holds. Since it is known that it can fail for certain exceptional cases, the conjecture is actually that the other inequality holds as long as $s \geq s_N$ for some constant s_N depending on N.

It is of interest to see what we get if we apply the same reasoning to the problem of determining $\gamma(I)$ for ideals I of generic lines in $\mathbb{P}^N$. Let L be a line in $\mathbb{P}^N$. We can regard $I(L)$ as being $I(L) = (x_2, \ldots, x_N) \subset k[\mathbb{P}^N]$. We can determine $\dim((I(L)^m)_t)$ for each $t \geq 0$ by counting the number of monomials in $I(L)^m$ of degree t. Of course, $\dim((I(L)^m)_t) = 0$ for $t < m$. For $t \geq m$ and $N = 3$, the result is $\dim((I(L)^m)_t) =$

$\binom{t+3}{3} - ((t+2) - (2m+1)/3)\binom{m+1}{2}$. (Here is the argument. The monomials $x_0^a x_1^b x_2^c x_3^d$ of degree t not in $I(L)^m$ are those for which $c+d < m$. There are $(t-i+1)(i+1)$ monomials of degree t such that $c+d=i$. Thus the number of monomials of degree t not in $I(L)^m$ is

$$\begin{aligned}\sum_{0\le i<m}(t-i+1)(i+1) &= \sum_{0\le i<m}(t+2-(i+1))(i+1)\\ &= (t+2)\sum_{0\le i<m}(i+1) - \sum_{0\le i<m}(i+1)^2\\ &= (t+2)\binom{m+1}{2} - (2m+1)\binom{m+1}{2}\Big/3.\end{aligned}$$

Subtracting this from $\binom{t+3}{3}$ thus gives $\dim((I(L)^m)_t)$.)

Thus for the ideal I of s distinct lines in $\mathbb{P}^3$ we have $\dim((I^{(m)})_t) = 0$ for $t < m$ and for $t \ge m$ we have

$$\dim((I^{(m)})_t) \ge \max\left(0, \binom{t+3}{3} - s\left((t+2) - (2m+1)/3\right)\binom{m+1}{2}\right). \qquad (^\circ)$$

We can use this to get an upper bound on $\gamma(I)$ when s is not too small. To do so, let $\tau = g$ be the largest real root of $\tau^3 - 3s\tau + 2s$. It is easy to see that $\sqrt{3s} - (3/4) < g < \sqrt{3s}$ as long as $s \ge 1$.

Lemma 5.2. *If $t/m > g$, then $\binom{it+3}{3} - s\left((it+2) - (2im+1)/3\right)\binom{im+1}{2} > 0$ for $i \gg 0$, while for $1 \le t/m < g$, $\binom{t+3}{3} - s\left((t+2) - (2m+1)/3\right)\binom{m+1}{2} < 0$ if $s \ge 17$.*

Proof. It is convenient to substitute $m\tau$ into $\binom{it+3}{3} - s\left((it+2) - (2im+1)/3\right)\binom{im+1}{2}$ for t. Doing so (and multiplying by 6 to clear denominators) gives the following polynomial in i:

$$i^3m^3(\tau^3 - 3s\tau + 2s) + i^2m^2(6\tau^2 - 3s\tau - 3s) + im(11\tau - 5s) + 6.$$

The condition $t/m > g$ is now equivalent to $\tau > g$, which means the leading coefficient of the polynomial is positive, hence for $i \gg 0$ the polynomial is positive.

Now assume $1 \le t/m < g$. Substituting $\tau = mt$ into $\binom{t+3}{3} - s\left((t+2) - (2m+1)/3\right)\binom{m+1}{2}$ for t gives

$$m^3(\tau^3 - 3s\tau + 2s) + m^2(6\tau^2 - 3s\tau - 3s) + m(11\tau - 5s) + 6.$$

It is easy to check that $\tau^3 - 3s\tau + 2s \le 0$ for $1 \le \tau < g$. Also, $6\tau^2 - 3s\tau - 3s < 0$ for $1 \le \tau < g$ if $s \ge 12$ (since $6\tau^2 - 3s\tau \le 0$ for $0 \le \tau \le s/2$, and $s/2 \ge \sqrt{3s} \ge g$ for $s \ge 12$). Finally, $11\tau - 5s \le 0$ for $\tau \le 5s/11$, and $5s/11 \ge \sqrt{3s} > g$ for $s \ge (11\sqrt{3}/5)^2 = 14.52$. To accommodate the constant term, 6, of the polynomial, we actually want $11\tau - 5s + 6 \le 0$, which occurs for $\tau \le (5s-6)/11$, and we have $(5s-6)/11 \ge \sqrt{3s}$ if $s \ge 17$. Thus each term of $\binom{t+3}{3} - s\left((t+2) - (2m+1)/3\right)\binom{m+1}{2}$ is nonpositive and one is negative if $1 \le t/m < g$ and $s \ge 17$. $\square$

Corollary 5.3. *Let I be the ideal of $s \ge 1$ distinct lines in $\mathbb{P}^3$. Then $\gamma(I) \le g$, where g is the largest real root of $\tau^3 - 3s\tau + 2s$. In particular, $\gamma(I) \le \sqrt{3s}$, so $\rho_a(I) \ge \alpha(I)/\sqrt{3s}$ for any s distinct disjoint lines in $\mathbb{P}^3$.*

Proof. From Lemma 5.2 we see that $\alpha(I^{(im)}) \leq it$ for $i \gg 0$ whenever $t/m > g$. Thus $\gamma(I) = \lim_{i\to\infty} \alpha(I^{(im)})/(im) \leq t/m$ whenever $t/m > g$, hence $\gamma(I) \leq g$. The bound on $\rho_a(I)$ follows by Theorem 1.2(1). □

If the s lines are generic and $s \gg 0$, we might hope that (°) is an equality for all $m, t \geq 1$. Certainly it is not an equality for all s, m and t, since for the ideal I of $s = 3$ general lines in $\mathbb{P}^3$ we know (as noted above) that $I^{(m)} = I^m$, but $\dim(I_2) = 1$ and hence $\dim((I^{(m)})_{2m}) = \dim((I^m)_{2m}) = 1$. However, $\binom{t+3}{3} - s\,((t+2) - (2m+1)/3)\binom{m+1}{2} < 0$ for $t = 2m$ and $s = 3$ when $m > 1$. But suppose for s generic lines in $\mathbb{P}^3$ for each $s \gg 0$ it were true that (°) is an equality for all $m, t \geq 1$; then, by Lemma 5.2, it would follow that $\alpha(I^{(m)}) \geq gm$ for $m \geq 1$ and hence that $\gamma(I) \geq g$. Since we know from above that $\gamma(I) \leq g$, we would have $\gamma(I) = g$.

It is therefore tantalizing to make the following conjecture:

Conjecture 5.4. *For the ideal I of $s \gg 0$ generic lines in $\mathbb{P}^3$, $\gamma(I)$ is equal to the largest real root $\tau = g$ of $\tau^3 - 3s\tau + 2s$.*

6. Concluding remarks and open questions

It is clear of course that our approach to obtaining upper bounds on $\gamma(I)$ applies much more generally than just to disjoint lines in $\mathbb{P}^3$. The main obstruction to implementing our approach to obtain explicit results for disjoint r-planes in $\mathbb{P}^N$ is combinatorial, and is thus better dealt with in a separate paper with a somewhat different focus than was the case here. In fact, while this paper was under review, just such a paper [12] has been posted to the arXiv implementing our approach, giving explicit Nagata-like conjectures for unions of $s \gg 0$ generic r-dimensional subspaces in $\mathbb{P}^N$ for $N \geq 2r + 1$, thus situating both the conjectures of Nagata, Iarrobino and Evain for generic points and the above conjecture about generic lines in $\mathbb{P}^3$ into a larger framework. These conjectures are in terms of $s \gg 0$. Nagata's conjecture is more precise because it uses $s \geq 10$ in place of $s \gg 0$ in Conjecture 5.1 when $N = 2$. It would be of interest to make both Conjecture 5.4 and the conjectures of [12] similarly more precise, but we currently do not know what would be reasonable statements; see [12] for examples addressing this issue.

Many other questions can be raised which also remain wide open. For example, for $h \leq N$, [5] shows that h is the least c such that $\rho(I) \leq c$ holds for all homogeneous ideals $I \subset k[\mathbb{P}^N]$ of big height h. The proof is to find ideals of big height h with $\alpha(I)/\gamma(I)$ arbitrarily close to h. Thus by Theorem 1.2(1), these same examples show that $\rho_a(I) \leq h$ is also optimal. If we restrict to ideals I with big height h for smooth schemes, however, it is not known whether $\rho(I) \leq h$ or $\rho_a(I) \leq h$ is still optimal.

Another wide open question is what kinds of upper bounds there are for resurgences (asymptotic or not) for ideals of singular subschemes. See [17, Theorem 4.11] for some special case results in this direction for ideals of unions of linear spaces. One can also ask how much can S_I or the resurgence vary for ideals of linear spaces with the same combinatorial data. Not much is known about S_I or the resurgence for unions of linear spaces which are not disjoint. For disjoint unions, slightly more is known. As an example we mention that we have $\rho_a(I) = 4/\gamma(I)$ by Corollary 1.3 for the ideal I of $s = 5$ general lines in $\mathbb{P}^3$. By Corollary 5.3 we have $\gamma(I) \leq 3.483$, so $\rho(I) \geq \rho_a(I) \geq 4/3.483 \approx 1.148$. But for the ideal J of five slightly more special but still disjoint lines (in particular, the five disjoint lines corresponding to five general points in $\mathbb{P}^1 \times \mathbb{P}^1$) we have $\rho(J) = \rho_a(J) = 1$ [19, Theorem 1.2].

One could also ask about the algebraic or geometric structure of S_I and its complement $S_I^* = \{(r, m) : m, r \geq 1, (r, m) \notin S_I\}$: are they defined by a finite set of linear inequalities?

Are they convex? Is S_I^* a semigroup? We know no examples where the answer to any of the questions is no, but there are currently no proofs that the answers must be yes. For an example that seems typical among those for which S_I is known, consider the ideal I of $s \geq 3$ points on a smooth conic in $\mathbb{P}^2$ with s odd. Then $(r, m) \in S_I$ if and only if $m \geq (n + 1)r/n - (1/n)$ [6]. Here we see that S_I and S_I^* are convex, defined (on the set of ordered pairs of positive integers) by a single linear inequality, and S_I^* is a semigroup (closed under component-wise addition) but S_I is not a semigroup (for example, $(1, 1) \in S_I$ but $(2, 2) \notin S_I$).

Another natural question to ask is whether it ever happens that $\rho_a(I) < \rho(I)$. We know no examples of this. The determination of $\rho_a(I)$ and $\rho(I)$ are difficult, so the cases where exact values are known are limited, and do not include any cases where the values are different. For example, one of the main situations where $\rho(I)$ is known is when I defines a zero-dimensional subscheme with $\alpha(I) = \mathrm{reg}(I)$, but in this case $\rho(I) = \alpha(I)/\gamma(I)$, and thus, by Theorem 1.2(1), $\rho_a(I) = \rho_a'(I) = \rho(I)$. For another example, S_I is known for $s \leq 9$ general points of $\mathbb{P}^2$ [6], and in these cases one can check directly that $\rho_a(I) = \rho(I)$. If it were true that S_I^* is always a semigroup, then it would follow immediately that $\rho_a(I) = \rho(I)$ always holds.

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