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Mathematical and Computer Modelling 43 (2006) 880–891

MATHEMATICAL AND COMPUTER MODELLING

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Random equilibrium problems on networks

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Received 26 September 2005; received in revised form 29 September 2005

Abstract

We apply the theory of random variational inequalities to study a class of random equilibrium problems on networks. By means of two classical test problems we treat the case of random demand and random cost and compute mean values and variances for two special probability distributions.

Keywords: Random variational inequalities; Traffic equilibrium; Random equilibrium; Network problems; Mosco convergence

1. Introduction

The variational inequality approach to network equilibrium problems is relatively recent. The interest in this approach is mainly due to the fact that many equilibrium problems which do not admit a direct optimization formulation can be easily cast into the framework of variational inequalities. The wide range of applications (economic models, traffic problems, mechanical equilibrium problems, etc.), as well as the methods and algorithms, are comprehensively described in, for instance, [14,6,4]. On the other hand, many problems in applied sciences have been modelled, for a long time, by equations that take into account random phenomena of various origins and, more recently, the consideration of stochastic aspects in optimization problems has led to the field of stochastic programming which has witnessed a rapid development in the last few decades [16]. The application of stochastic programming to network problems is amply investigated in [10] (see also [19]), while in [18], a generalization of the classical stochastic user equilibrium model is developed and a heuristic solution algorithm is proposed. In contrast, to our knowledge, there are few papers which deal with the introduction of random elements into the framework of variational inequalities (see for example [7] for the solution of stochastic variational inequalities on a polyhedral set via the sample-path method or [9] for an infinite dimensional formulation). Our purpose in this paper is to show how the class of random variational inequalities introduced in [9] and subsequently extended in [8] (in order to consider random constraints) can be applied to concrete network equilibrium problems in the presence of randomness. We would also like to point out that infinite dimensional variational inequalities have already been

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0895-7177/$ - see front matter
doi:10.1016/j.mcm.2005.12.007

applied to transportation networks with time-dependent data [5]. The plan of the paper is the following: in Section 2 we briefly recall the formulation in [8] and explain how the initial *infinite dimensional* variational inequality can be approximated, via a discretization procedure, by a sequence of finite dimensional problems; in Section 3 we briefly describe the classical (i.e. deterministic) traffic network problem and subsequently apply our general theory of random variational inequalities to model the traffic problem in the presence of random demand or random cost; in the last two sections we consider the random versions of two classical test problems and, for the first of them, we apply the discretization procedure described in Section 2 and perform extensive numerical computations to give approximations for the mean values and the variances of the solution, while for the second we can carry out most of the computations analytically. We stress, once more, that the purpose of this paper is not to demonstrate new theorems, but to show how the abstract formulation in [8], based on a functional analysis approach, can be applied to the *modelling* of network equilibrium problems and to their *numerical solution*.

2. The general theory

In this section we shall recall the formulation in [8], focusing on the aspects more related to the applications. Here, let (Ω, P) be the sample space, with its probability P, and consider a matrix $C \in \mathbb{R}^{m\times k}$ and a random m-vector D. We can then introduce the random set

$$M(\omega) := \{x \in \mathbb{R}^k : x \geq 0, Cx = D(\omega)\}, \quad \omega \in \Omega.$$

Moreover, let A and B be two matrices in $\mathbb{R}^{k\times k}$, $c \in \mathbb{R}^k$, and R and S two real valued random variables on Ω. With these data, consider the problem of finding a random k-vector $\hat{X} : \Omega \mapsto \mathbb{R}^k$, such that $\hat{X}(\omega) \in M(\omega)$ (P-almost surely) and the following inequality holds for P-almost every elementary event $\omega \in \Omega$ and $\forall x \in M(\omega)$:

$$S(\omega)[A\hat{X}(\omega)]^{\mathrm{T}}[x - \hat{X}(\omega)] + [B\hat{X}(\omega)]^{\mathrm{T}}[x - \hat{X}(\omega)] \geq R(\omega)c^{\mathrm{T}}[x - \hat{X}(\omega)]. \tag{1}$$

If we assume that, $\forall\omega \in \Omega$, the matrix $S(\omega)A + B$ (not necessarily symmetric) is positive definite we obtain by the classical Lions–Stampacchia Theorem [12] that (1) admits a unique solution for each ω. Moreover, one can prove (cf. [8]) that the solution map $\omega \mapsto \hat{X}(\omega)$ is a random variable (i.e. measurable). However, if we are looking for solutions with finite second-order moments we need further assumptions. Thus, we can state the following theorem which is a reformulation of Theorem 2.2 in [8] which was given there in a more abstract framework.

Theorem 2.1. *Let (Ω, P) be a probability space, S a bounded, real, random variable and R a real random variable with finite second-order moments. Let the matrix $\underline{s}A + B$ (not necessarily symmetric) be positive definite (where $\underline{s}$ is a positive constant such that $S \geq \underline{s}$ P-a.s. (almost surely)). Moreover, assume that D is an m-dimensional random vector with finite second-order moments. Furthermore, suppose that the random set $M(\omega)$ contains an element with finite second-order moments. Then, there exists a unique solution to problem (1), with finite second-order moments.*

Sketch of Proof. Since $S(\omega) \geq \underline{s} > 0$, $\forall\omega \in \Omega$ and $\underline{s}A + B > 0$ (positive definite), the existence of a unique solution $\hat{X}(\omega)$ $(\omega \in \Omega)$ follows readily from the classical Lions–Stampacchia Theorem. Further, inserting a fixed $z_0(\omega) \in M(\omega)$ in (1) we can derive an estimate for the real random variable $\|\hat{X}\|$ using some appropriate random variable with finite second-order moments that only depends on the fixed z_0 and the data of the problem. This gives that the random element $\hat{X}$ itself has finite second-order moments. □

Then, we can introduce the following closed convex nonvoid subset of $L^2_k(\Omega) := L^2(\Omega, P, \mathbb{R}^k)$:

$$M^P := \{V \in L^2_k(\Omega) : V \geq 0, CV = D, P\text{-a.s.}\}$$

and consider the following problem: Find $\hat{U} \in M^P$ such that, $\forall V \in M^P$,

$$\int_\Omega \{S(\omega)[A\hat{U}(\omega)]^{\mathrm{T}}[V(\omega) - \hat{U}(\omega)] + [B\hat{U}(\omega)]^{\mathrm{T}}[V(\omega) - \hat{U}(\omega)]\}\,\mathrm{d}P(\omega)$$
$$\geq \int_\Omega R(\omega)\,c^{\mathrm{T}}[V(\omega) - \hat{U}(\omega)]\,\mathrm{d}P(\omega). \tag{2}$$

The r.h.s. of (2) defines a continuous linear form on $L^2_k(\Omega)$, while the l.h.s. defines a continuous bilinear form on $L^2_k(\Omega) \times L^2_k(\Omega)$ which satisfies the Lions–Stampacchia Theorem. Therefore, there exists a unique solution in M^P to the problem (2) and because of uniqueness, problems (1) and (2) are equivalent.

However, in most applications, the sample space Ω is not known, while what one observes is the distribution of the random variables involved. Hence, we consider the joint distribution $\mathbb{P}$ of the random vector (R, S, D) and work with the special sample space $(\mathbb{R}^d, \mathbb{P})$, where the dimension $d := 2 + m$. To simplify our analysis we shall suppose that R, S and D are independent random vectors. Let $t = D(\omega)$, $r = R(\omega)$, $s = S(\omega)$, $y = (r, s, t)$ and consider the set

$$M(y) := \{x \in \mathbb{R}^k : x \geq 0,\ Cx = t\}$$

for $y \in \mathbb{R}^d$. Thus, the pointwise version of our problem now reads: Find $\hat{x} = \hat{x}(y)$ such that $\hat{x}(y) \in M(y)$, $\mathbb{P}$-a.s., and the following inequality holds for $\mathbb{P}$-almost every $y \in \mathbb{R}^d$ and $\forall x \in M(y)$,

$$s\,[A\hat{x}(y)]^{\mathrm{T}}[x - \hat{x}(y)] + [B\hat{x}(y)]^{\mathrm{T}}[x - \hat{x}(y)] \geq r\,c^{\mathrm{T}}[x - \hat{x}(y)]. \tag{3}$$

In order to obtain the integral formulation of (3), consider the space $L^2_k(\mathbb{R}^d)$ and introduce the closed convex nonvoid set

$$M_{\mathbb{P}} := \{v \in L^2_k(\mathbb{R}^d) : v \geq 0, Cv(r, s, t) = t, \mathbb{P}\text{-a.s.}\}.$$

This leads to the problem: Find $\hat{u} \in M_{\mathbb{P}}$ such that, $\forall v \in M_{\mathbb{P}}$,

$$\int_{\mathbb{R}^d} s\,[A\hat{u}(y)]^{\mathrm{T}}[v(y) - \hat{u}(y)] + [B\hat{u}(y)]^{\mathrm{T}}[v(y) - \hat{u}(y)]\mathrm{d}\mathbb{P}(y) \geq \int_{\mathbb{R}^d} r\,c^{\mathrm{T}}[v(y) - \hat{u}(y)]\mathrm{d}\mathbb{P}(y). \tag{4}$$

By using the same arguments as in the ω-formulation, problems (3) and (4) are equivalent.

Without loss of generality we can suppose that R, and each component of D are nonnegative, (otherwise, we can use the standard decomposition into the positive and negative parts). Moreover, we assume that the support of S is the interval $[\underline{s}, \overline{s}) \subset (0, \infty)$. Furthermore we assume that the distributions P_R, P_S, P_D have probability densities $\varphi_R, \varphi_S, \varphi_{D_i}$, respectively. Hence, $\mathbb{P} = P_R \otimes P_S \otimes P_D$, $\mathrm{d}P_R(r) = \varphi_R(r)\mathrm{d}r$, $\mathrm{d}P_S(s) = \varphi_S(s)\mathrm{d}s$ and $\mathrm{d}P_{D_i}(t_i) = \varphi_{D_i}(t_i)\mathrm{d}t_i$ for $i = 1, \ldots, m$.

In order to give a procedure for approximating the solution $\hat{u}$, let us introduce a sequence $\{\pi_n\}_n$ of partitions of the support $G := [0, \infty) \times [\underline{s}, \overline{s}] \times \mathbb{R}^m_+$ of the random variables involved. More precisely, let $\pi_n = (\pi^R_n, \pi^S_n, \pi^D_n)$, where

$$\begin{aligned}
&\pi^R_n := (r^0_n, \ldots, r^{N^R_n}_n), \quad \pi^S_n := (s^0_n, \ldots, s^{N^S_n}_n), \quad \pi^{D_i}_n := (t^0_{n,i}, \ldots, t^{N^{D_i}_n}_{n,i})\\
&0 = r^0_n < r^1_n < \cdots r^{N^R_n}_n = n\\
&\underline{s} = s^0_n < s^1_n < \cdots s^{N^S_n}_n = \overline{s}\\
&0 = t^0_{n,i} < t^1_{n,i} < \cdots t^{N^{D_i}_n}_{n,i} = n \quad (i = 1, \ldots, m)\\
&|\pi^R_n| := \max\{r^j_n - r^{j-1}_n : j = 1, \ldots, N^R_n\} \to 0 \quad (n \to \infty)\\
&|\pi^S_n| := \max\{s^k_n - s^{k-1}_n : k = 1, \ldots, N^S_n\} \to 0 \quad (n \to \infty)\\
&|\pi^{D_i}_n| := \max\{t^{h_i}_{n,i} - t^{h_i-1}_{n,i} : h_i = 1, \ldots, N^{D_i}_n\} \to 0 \quad (i = 1, \ldots, m; n \to \infty).
\end{aligned}$$

These partitions give rise to the exhausting sequence $\{G_n\}$ of subsets of G, where each G_n is given by the finite disjoint union of the intervals:

$$I^n_{jkh} := [r^{j-1}_n, r^j_n) \times [s^{k-1}_n, s^k_n) \times I^n_h,$$

where we use the multi-index $h = (h_1, \ldots, h_m)$ and

$$I^n_h := \prod_{i=1}^m [t^{h_i-1}_{n,i}, t^{h_i}_{n,i}).$$

For each $n \in \mathbb{N}$ let us now consider the space of the $\mathbb{R}^l$-valued simple vector functions ($l \in \mathbb{N}$) on G_n, extended by 0 outside of G_n:

$$X_n^l := \left\{ v_n : v_n(r,s,t) = \sum_j \sum_k \sum_h v_{jkh}^n 1_{I_{jkh}^n}(r,s,t),\, v_{jkh}^n \in \mathbb{R}^l \right\}$$

where 1_I denotes the $\{0,1\}$-valued characteristic function of a subset I.

By this discretization procedure we arrive at a finite number of finite dimensional affine variational inequalities, namely: For $\forall n \in \mathbb{N}, \forall j,k,h$ find $\hat{u}_{jkh}^n \in M_{jkh}^n$ such that, $\forall v_{jkh}^n \in M_{jkh}^n$,

$$[\tilde{A}_k^n \hat{u}_{jkh}^n]^{\mathrm{T}}[v_{jkh}^n - \hat{u}_{jkh}^n] \geq [\tilde{c}_j^n]^{\mathrm{T}}[v_{jkh}^n - \hat{u}_{jkh}^n], \tag{5}$$

where $q : (r,s,t) \in \mathbb{R}^d \mapsto t \in \mathbb{R}^m$, $\overline{q}_{jkh}^n = (\mu_{jkh}^n q) \in \mathbb{R}^m$.

$$\mu_{jkh}^n q = \frac{1}{\mathbb{P}(I_{jkh})} \int_{I_{jkh}^n} q(r,s,t)\, \mathrm{d}\mathbb{P}(r,s,t).$$

$$M_{jkh}^n := \{v_{jkh}^n \in \mathbb{R}^k : v_{jkh}^n \geq 0,\ Cv_{jkh}^n = \overline{q}_{jkh}^n\}.$$

$$\tilde{A}_k^n = s_n^{k-1} A + B, \quad \tilde{c}_j^n = r_n^{j-1} c.$$

The solutions of (5) are used to construct approximations for the infinite dimensional problem (3):

$$\hat{u}_n = \sum_j \sum_k \sum_h \hat{u}_{jkh}^n 1_{I_{jkh}} \in X_n^k. \tag{6}$$

The convergence of these approximations is established by the following theorem [8]:

Theorem 2.2. *The sequence of step functions* (6) *converge, in quadratic mean, to the solution of* (4).

Sketch of Proof. Since $\tilde{A}_k^n > 0$, there exist unique $\hat{u}_{jkh}^n$. Since $S(\omega) \geq \underline{s} > 0$, $\forall \omega \in \Omega$, the bilinear forms defined from $\tilde{A}_k^n$ are uniformly coercive in the reflexive separable space $L_k^2(\mathbb{R}^d)$. Therefore the sequence of the approximations $\hat{u}_n$ admits a weak limit point, say $\tilde{u}$. Since the cone of componentwise nonnegative functions in $L^2(\mathbb{R}^d)$ is weakly closed, C defines a linear continuous map and hence a weakly continuous map in the L^2 spaces, $\tilde{u}$ can be shown to belong to $M_{\mathbb{P}}$. To show that $\tilde{u}$ is the desired solution, consider an arbitrary $v \in M_{\mathbb{P}}$. The key argument is the construction of appropriate approximations of v, namely step functions v_n of the form (6) such that the associated v_{jkh}^n belong to M_{jkh}^n and v_n converges strongly to v. Then by a limit process using (5) one establishes that $\tilde{u}$ is a solution to (4) and by uniqueness, $\tilde{u} = \hat{u}$. Therefore the entire sequence converges weakly to $\tilde{u}$. Finally to show norm convergence, one again exploits uniform coercivity. □

The previous theorem will allow us to compute the approximate mean values and variances in the concrete problems in the sequel.

3. The random traffic equilibrium problem

The purpose of this section is to show how the formulation presented in the previous section can be applied to the modelling of random equilibrium problems. Thus, we consider a classical equilibrium problem in a random environment and show how to cast it in our framework. The traffic assignment problem has a relatively recent history. For a variational inequality formulation of this equilibrium problem we refer the reader to the influential papers by Smith [17] and Dafermos [3]. For a comprehensive treatment of models and methods we refer the reader to [15].

Let us first introduce the notation commonly used to state the standard traffic equilibrium problem from the user point of view. A traffic network consists of a triple (N, A, W) where $N = \{N_1, \ldots, N_p\}$ is the set of nodes, $A = (A_1, \ldots, A_n)$ represents the set of the directed arcs and $W = \{W_1, \ldots, W_m\}$ is the set of the origin–destination (O–D) pairs. The flow on the arc A_i is denoted by f_i, $f = (f_1, \ldots, f_n)$. We assume that each O–D pair W_j is connected by $r_j \geq 1$ paths whose set is denoted by P_j $(j = 1, \ldots, m)$. All the paths in the network are grouped in a vector $(R_1, \ldots, R_k)$. We can describe the link structure of the paths by using the arc–path incidence matrix $\Delta = \{\delta_{ir}\}_{i=1,\ldots,n;r=1,\ldots,k}$, whose entries take the value 1 if $A_i \in R_r$, 0 if $A_i \notin R_r$. To each path R_r there corresponds a flow F_r. The path flows are grouped in a vector $(F_1, \ldots, F_k)$ which is called the path (network) flow. The flow f_i on the arc A_i is equal to the sum of the path flows which contain A_i, so that $f = \Delta F$. Let us now introduce the cost of going through A_i as a function $c_i(f) \geq 0$ of the flows on the network, so that $c(f) = (c_1(f), \ldots, c_n(f))$ denotes the vector arc cost on the network. Analogously, one can define a cost on the paths as $C(F) = (C_1(F), \ldots, C_k(F))$. In most applications the cost $C_r(F)$ associated with path r is just the sum of the costs on the arcs which build that path;

$$C_r(F) = \sum_{i=1}^{n} \delta_{ir} c_i(f) \tag{7}$$

or in compact form, $C(F) = \Delta^{\mathrm{T}} C(\Delta F)$. For each O–D pair W_j there is a given traffic demand $D_j \geq 0$, so that $(D_1, \ldots, D_m)$ is the demand vector on the network. Feasible flows are flows which satisfy the demands and the capacity constraints, i.e., which belong to the set

$$\mathcal{K} := \{F \in \mathbb{R}^k \mid F \geq 0, \Phi F = D\},$$

where Φ is the well known O–D pair–path incidence matrix whose elements $\phi_{j,r}$ $(j = 1, \ldots, m; r = 1, \ldots, k)$ are set equal to 1 if the path R_r connects the pair W_j, 0 otherwise.

A path flow H is called an equilibrium flow (or *Wardrop Equilibrium*) if $H \in \mathcal{K}$ and

$$\begin{aligned} &\forall W_j \in W, \forall q, s \in P_j, \quad \text{there holds:} \\ &C_q(H) < C_s(H) \Longrightarrow H_s = 0. \end{aligned} \tag{8}$$

This statement is equivalent to:

$$H \in \mathcal{K} \quad \text{and} \quad [C(H)]^{\mathrm{T}}[F - H] \geq 0 \quad \forall F \in \mathcal{K}. \tag{9}$$

Roughly speaking, the meaning of Wardrop Equilibrium is that the road users choose minimum cost paths, and the meaning of the cost is usually that of traversal time.

We can describe the traffic equilibrium problems with respect to the arc flows and cost. In the so-called *link formulation* the feasible set of flows is now given by

$$K_A := \{f \in \mathbb{R}^n : f = \Delta F \text{ for some } F \in \mathcal{K}\}. \tag{10}$$

We say that $h \in K_A$ induces a Wardrop Equilibrium if $H \in \mathcal{K}$ is a Wardrop Equilibrium. The new variational inequality reads:

$$h \in K_A \quad \text{and} \quad [c(h)]^{\mathrm{T}}[f - h] \geq 0 \quad \forall f \in K_A. \tag{11}$$

It can be shown that if the cost is nonnegative and additive (i.e. (7) holds), the path and link formulations are equivalent. However the two formulations give rise to different algorithms (see e.g. [1,13]).

Now, let us suppose that the cost mapping is not deterministic, but is subject to random fluctuations. Thus, we are left with a *random cost*: $C : \Omega \times \mathcal{K} \mapsto \mathbb{R}^k$. We require that the random cost satisfies the hypothesis of our general theory, and in particular, it will be chosen affine for each $\omega \in \Omega$. Moreover, it is also natural to suppose that the traffic demands are also subject to random perturbations. Thus, we are left with the random convex set

$$\mathcal{K}(\omega) := \{F \in \mathbb{R}^k \mid F \geq 0, \Phi F = D(\omega)\}, \quad \omega \in \Omega.$$

We can then state the *random Wardrop conditions* as follows: $\forall \omega \in \Omega$, $H(\omega) \in \mathcal{K}(\omega)$ and

$$\begin{aligned} &\forall W_j \in W, \forall q, s \in P_j \quad \text{there holds:} \\ &C_q(\omega, (H(\omega))) < C_s(\omega, (H(\omega))) \Longrightarrow H_s(\omega) = 0. \end{aligned} \tag{12}$$

Under the natural assumption that C is nonnegative, it is evident that the *random Wardrop conditions* are equivalent to the following random variational inequality: For every $\omega \in \Omega$, find $H(\omega) \in \mathcal{K}(\omega)$ such that

$$[C(\omega, H(\omega))]^{\mathrm{T}}[F - H(\omega)] \geq 0 \qquad \forall F \in \mathcal{K}(\omega). \tag{13}$$

The equivalence between Wardrop conditions and (13) is, as we have just seen, very general. Our general theory, however, applies when the cost operator is affine (with respect to the flow variables). Moreover, we consider the case

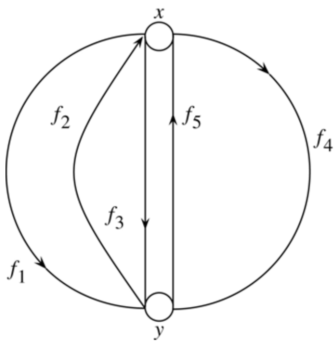

Fig. 1. Dafermos's network.

where the flow and the random variables are separable. While our theoretical results can be generalized to more general operators than the affine ones, the numerical approximation scheme sketched in Section 2 requires the separability assumption. However this assumption is very natural in many applications where the random perturbation is treated as a *modulation* of a deterministic process. Under the above-mentioned assumptions, (13) assumes the particular form:

$$S(\omega)A^{\mathrm{T}}(H(\omega))[F - H(\omega)] \geq R(\omega)b^{\mathrm{T}}[F - H(\omega)], \quad \forall F \in \mathcal{K}(\omega) \tag{14}$$

where we have split the affine cost operator into its linear and constant parts, $A \in \mathbb{R}^{k\times k}$ and $b \in \mathbb{R}^k$, respectively, while S and R are two real random variables. In Eq. (14), both the l.h.s. and the r.h.s. can be replaced with any (finite) linear combination of affine and separable terms, where each term satisfies the hypothesis of the previous section:

$$\sum_i S_i(\omega)A_i^{\mathrm{T}}(H(\omega))[F - H(\omega)] \geq \sum_j R_j(\omega)b_j^{\mathrm{T}}[F - H(\omega)], \quad \forall F \in \mathcal{K}(\omega). \tag{15}$$

Hence, in the traffic network, we could consider the case where the random perturbation has a different weight for each path. We can then transform the problem in the image space of the random variables involved and apply the approximation procedure. For the sake of clarity we shall consider two concrete networks of small dimension: the first is the classical test-problem of Dafermos, where we allow for the demand to be a random vector, while the cost is deterministic. In the second example we shall consider the celebrated Braess network, with one random parameter in the cost function.

4. Dafermos's network

The network considered by Dafermos [3] consists of two nodes x and y connected by two two-way links and by one one-way link. Thus, there are five links $\{A_1, A_2, A_3, A_4, A_5\}$ which we order according to Fig. 1, and five associated flows: $\{f_1, f_2, f_3, f_4, f_5\}$. In our random version, the travel demands are nonnegative random variables:

$$d(x, y) = \alpha, \quad d(y, x) = \beta$$

(In [3], $\alpha = 210$, $\beta = 120$.)

The link cost functions are given by:

$$c_1 = 10f_1 + 5f_2 + 1000$$
$$c_2 = 2f_1 + 20f_2 + 1000$$
$$c_3 = 10f_3 + 5f_5 + 950$$
$$c_4 = 20f_4 + 3000$$
$$c_5 = f_3 + 25f_5 + 1300.$$

The linear part of the operator defined by the previous equations is associated with a (not symmetric) positive definite matrix, so that the variational inequality that we shall consider in the sequel has a unique solution, for each value of α and β. The conservation of flows implies that: $f_1 + f_3 + f_4 = \alpha$, $f_2 + f_5 = \beta$.

Let:

$$K(\alpha,\beta) := \{f \in \mathbb{R}^5 : f_1 + f_3 + f_4 = \alpha,\ f_2 + f_5 = \beta,\ f_j \geq 0, \forall j = 1\ldots 5\}, \quad (\alpha,\beta) \in \mathbb{R}^2_+.$$

If $\mathbb{P}$ is the probability of (α,β), the pointwise variational inequality associated with our network is:
Find $h = h(\alpha,\beta)$, such that $h(\alpha,\beta) \in K(\alpha,\beta)$, $\mathbb{P}$-almost surely, and $\forall(\alpha,\beta) \in \mathbb{R}^2_+$, $\forall f \in K(\alpha,\beta)$:

$$c[h(\alpha,\beta)]^{\mathrm{T}}[f - h(\alpha,\beta)] \geq 0. \tag{16}$$

Since we want to solve (16) by the discretization method described in [9,8], we must solve the corresponding deterministic, finite dimensional, problem for each fixed α and β. We use the direct method described in [13] which is very effective for problems where the number of independent flow components is not too large. Moreover, by writing our code in MAPLE, we are able, in the affine case under examination, to get exact results (the only error being the final rounding error). In our case, because of the conservation law, the components f_1 and f_2 can be obtained from f_3, f_4, f_5, so that, following [13], we transform the original five-dimensional V.I. in an *equivalent* reduced three-dimensional V.I. where the constraints set is the polytope given by:

$$\mathbb{M}_{\mathbb{P}}(\alpha,\beta) := \{(f_3, f_4, f_5) \in \mathbb{R}^3 : f_3 + f_4 \leq \alpha,\ f_5 \leq \beta,\ f_3, f_4, f_5 \geq 0\}.$$

Thus, the new variational inequality reads: Find $h = h(\alpha,\beta) \in \mathbb{M}_{\mathbb{P}}$ such that, $\forall f \in \mathbb{M}_P$:

$$\Gamma_3(h)(f_3 - h_3) + \Gamma_4(h)(f_4 - h_4) + \Gamma_5(h)(f_5 - h_5) \geq 0 \tag{17}$$

where:

$$\begin{aligned}
\Gamma_3(h) &= 25h_3 10h_4 + 10h_5 - 10\alpha - 5\beta - 50\\
\Gamma_4(h) &= 10h_3 + 30h_4 + 5h_5 - 10\alpha - 5\beta + 2000\\
\Gamma_5(h) &= 3h_3 + 2h_4 + 45h_5 - 2\alpha - 20\beta + 300.
\end{aligned}$$

If the solution of the reduced V.I. is not in the interior of the polytope, the method, loosely speaking, is based on looking for the solution on the faces.

Thus, fix $N \in \mathbb{N}$ and assume that $\alpha \in [\alpha_0, \alpha_N]$ and $\beta \in [\beta_0, \beta_N]$, $I := [\alpha_0, \alpha_N] \times [\beta_0, \beta_N]$ and consider the decompositions:

$$\begin{aligned}
&\alpha_0 < \alpha_0 + \frac{\alpha_N - \alpha_0}{N} < \alpha_0 + 2\frac{\alpha_N - \alpha_0}{N} < \cdots < \alpha_N\\
&\beta_0 < \beta_0 + \frac{\beta_N - \beta_0}{N} < \beta_0 + 2\frac{\beta_N - \beta_0}{N} < \cdots < \beta_N.
\end{aligned}$$

Let $I^n_{jk} = [\alpha_{j-1}, \alpha_j) \times [\beta_{k-1}, \beta_k)$, $\forall j, k = 1, 2, \ldots, N$. For each value of j and k we solve the finite dimensional variational inequality corresponding to the values α_{j-1} and β_{k-1}, respectively, and denote with $h^N(\alpha_{j-1}, \beta_{k-1})$ the three-dimensional solution vector:

$$h^N(\alpha_{j-1}, \beta_{k-1}) = (h^N_3(\alpha_{j-1}, \beta_{k-1}), h^N_4(\alpha_{j-1}, \beta_{k-1}), h^N_5(\alpha_{j-1}, \beta_{k-1})).$$

Hence, $\forall(\alpha,\beta) \in I^N_{jk}$ we define the (constant) function: $h_{I^N_{jk}}(\alpha,\beta) = h^N(\alpha_{j-1}, \beta_{k-1})$. Now, we can define a simple (i.e. piecewise constant) vector function on I:

$$h^N(\alpha,\beta) = \sum_{j,k=1}^{N} h^N(\alpha_{j-1}, \beta_{k-1}) \mathbf{1}_{I^N_{jk}}(\alpha,\beta) \tag{18}$$

where $\mathbf{1}_A(x)$ denotes the characteristic function of a set A.

Let us recall [8] that the sequence of step functions in (18) approximates the solutions of (16) in quadratic mean, so that it is meaningful, for each N, to compute mean values and variances. Thus, for each component i and for each N, the mean value is given by:

$$\langle h^N_i \rangle = \int_I h^N_i(\alpha,\beta)\mathrm{d}\mathbb{P}(\alpha)\mathrm{d}\mathbb{P}(\beta) = \sum_{j,k=1}^{N} h^N(\alpha_{j-1}, \beta_{k-1}) \int_{I^N_{jk}} \mathrm{d}\mathbb{P}(\alpha)\mathrm{d}\mathbb{P}(\beta)$$

Table 1
Mean values corresponding to various approximations for $\alpha \in [9, 11]$ and $\beta = 14$

N	$\langle h_3 \rangle$	$\langle h_4 \rangle$	$\langle h_5 \rangle$
10	4.280000	0	0.008888
100	4.298000	0	0.010888
500	4.299600	0	0.011066
1 000	4.299800	0	0.011088
2 000	4.299980	0	0.011100
3 000	4.299933	0	0.011133
4 000	4.299950	0	0.011105
5 000	4.299960	0	0.011106
10 000	4.299980	0	0.011108
15 000	4.299986	0	0.011109
20 000	4.299990	0	0.011110
22 000	4.299990	0	0.011110

Table 2
Variances corresponding to various approximations for $\alpha \in [9, 11]$ and $\beta = 14$

N	σ_3^2	σ_4^2	σ_5^2
10	18.324800	0	0.000158
100	18.479468	0	0.000200
500	18.493226	0	0.000204
1 000	18.494946	0	0.000205
2 000	18.495906	0	0.000205
3 000	18.496093	0	0.000205
4 000	18.496236	0	0.000205
5 000	18.496322	0	0.000205
10 000	18.496494	0	0.000205
15 000	18.496552	0	0.000205
20 000	18.496580	0	0.000205
22 000	18.496558	0	0.000205

and the variances are given, as usual, by:

$$\sigma 2_i^N = \langle (h_i^N(\alpha, \beta))^2 \rangle - \langle h_i^N(\alpha, \beta) \rangle^2.$$

(For the sake of simplicity we are assuming that the two probabilities involved represent independent random variables.)

In the tables that follow we show several computations, in the case of *uniform distribution* of the random variables involved. Some comments are in order. At first, to show the effectiveness of our approximation technique, we fix the value of one random variable, e.g. $\beta = 14$, let $\alpha \in [9, 11]$, and test our algorithm up to $N = 22\,000$. The numerical approximations for the mean values and variances are shown in Tables 1 and 2, respectively. Let us notice that, although α varies in a relatively small interval, the solutions of the (22 000) finite dimensional variational inequalities are distributed on a rather bigger interval, as one can deduce from the analysis of the variance corresponding to h_3. This is the generic situation where the probabilistic approach is of much interest. On the other hand, for some values of the random variables we can get particularly stable solutions of the corresponding finite dimensional variational inequality. This is the case, for instance, if we let our random variables vary around the reference values in [3] (cf. Table 3). Since this time we are discretizing both random variables (and moreover on intervals bigger than the previous one), our discretization is too coarse to obtain numerical convergence as good as in the previous case, however the reference values in [3] are very special in that, as we can notice from Table 3, the mean values of the solutions approach the values 90, 0, 50 obtained when $\alpha = 210$ and $\beta = 120$, i.e., in the center of I. Another generic situation is reported in Tables 4 and 5.

Table 3
Mean values and variances corresponding to the approximations in the interval [205, 215] × [115, 125], for $N = 10, 100, 200, 220$

$\langle h_3 \rangle$	$\langle h_4 \rangle$	$\langle h_5 \rangle$
89.7922	0	49.7694
89.9794	0	49.9769
89.9896	0	49.9884
89.9905	0	49.9895
σ_3^2	σ_4^2	σ_5^2
1.2765	0	1.6212
1.2892	0	1.6374
1.2893	0	1.6375
1.2893	0	1.6375

Table 4
Mean values corresponding to various approximations in the interval [10, 20] × [15, 25]

N	$\langle h_3 \rangle$	$\langle h_4 \rangle$	$\langle h_5 \rangle$
10	9.9766	0	1.9793
30	10.7290	0	2.0921
50	10.8332	0	2.1178
70	10.8744	0	2.1290
100	10.9077	0	2.1372
120	10.9207	0	2.1404
140	10.9299	0	2.1427
160	10.9363	0	2.1445
180	10.9407	0	2.1459
200	10.9451	0	2.1469
220	10.9486	0	2.1478
250	10.9525	0	2.1489
500	10.9664	0	2.1529

Table 5
Variances corresponding to various approximations in the interval [10, 20] × [15, 25]

N	σ_3^2	σ_4^2	σ_5^2
10	11.0191	0	1.4079
30	5.0781	0	1.5536
50	4.3368	0	1.5719
70	4.0224	0	1.5792
100	3.7618	0	1.5851
120	3.6559	0	2.1427
140	3.5854	0	1.5890
160	3.5376	0	1.5901
180	3.5053	0	1.5908
200	3.4708	0	1.5916
220	3.4417	0	1.5922
250	3.4123	0	1.5929
500	3.3057	0	1.5988

5. Braess's network

Our description of Braess's network follows [2], but the equilibrium problem is formulated as a variational inequality and not as a parametric complementarity problem. We shall employ again the direct method and let the parameter be a nonnegative random variable. Thanks to the simplicity of this problem we can carry out analytically all the steps of the direct method, which for the reader's convenience are reported in detail. The network is depicted in Fig. 2: there are five links: $\{A_1, A_2, A_3, A_4, A_5\}$, and one origin–destination pair. The traffic demand is 6 and the

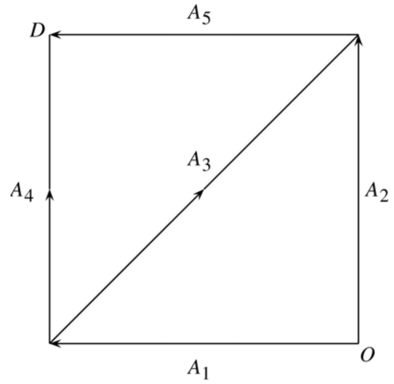

Fig. 2. Braess's network.

three paths (or routes) joining the pair (O, D) are labelled as follows:

$$R_1 = (A_1, A_4), \quad R_2 = (A_2, A_5), \quad R_3 = (A_1, A_3, A_5).$$

The link cost function is:

$$\begin{aligned} c_1(f) &= 10 f_1 \\ c_2(f) &= f_2 + 50 \\ c_3(f) &= f_3 + \lambda, \quad \lambda \geq 0 \\ c_4(f) &= f_4 + 50 \\ c_5(f) &= 10 f_5. \end{aligned}$$

The relation between the link flows, f, and the path flows, F, is:

$$\begin{aligned} f_1 &= F_1 + F_3 \\ f_2 &= F_2 \\ f_3 &= F_3 \\ f_4 &= F_1 \\ f_5 &= F_3 + F_4. \end{aligned}$$

We can then obtain the cost in the path variables:

$$\begin{aligned} C_1(F) &= c_1(f) + c_4(f) = 11F_1 + 10F_3 + 50 \\ C_2(F) &= c_2(f) + c_5(f) = 10F_3 + 11F_2 + 50 \\ C_3(F) &= c_1(f) + c_3(f) + c_5(f) = 10F_1 + 10F_2 + 21F_3 + \lambda. \end{aligned}$$

The set of feasible flows is given by:

$$K = \{(F_1, F_2, F_3) \in \mathbb{R}^3 : F_1, F_2, F_3, \geq 0, F_1 + F_2 + F_3 = 6\}.$$

The network equilibrium problem on Braess's network is equivalent to the following variational inequality problem: Find $H \in K$ such that $\forall H \in K$:

$$\sum_{r=1}^{3} C_r(H)(F_r - H_r) \geq 0. \tag{19}$$

Let us observe that because the smallest eigenvalue of the symmetric part of the matrix associated with the cost operator C is 3.77, it follows that the matrix is positive definite and (19) has a unique solution.

Since $F_1 = 6 - F_2 - F_3$, we can reduce the dimension and consider the following equivalent representation of K:

$$\tilde{K} = \{(F_2, F_3) \in \mathbb{R}^2 : F_2, F_3 \geq 0, \; F_2 + F_3 \leq 6\}.$$

Thus, the original variational inequality is equivalent to the following reduced variational inequality:
Find $H \in \tilde{K}$ such that $\forall F \in \tilde{K}$:

$$(22H_2 + 11H_3 - 66)(F_2 - H_2) + (11H_2 + 12H_3 - 56 + \lambda)(F_3 - H_3) \geq 0. \tag{20}$$

The solution of the system:

$$\begin{cases} 22H_2 + 11H_3 - 66 = 0 \\ 11H_2 + 12H_3 - 56 + \lambda = 0 \end{cases}$$

gives:

$$H_2 = \frac{16+\lambda}{13}, \quad H_3 = \frac{2(23-\lambda)}{13}. \tag{21}$$

This solution is feasible, i.e. belongs to $\tilde{K}$ provided that $\lambda \in [0, 23]$. Since $F_1 = 6 - F_2 - F_3$ we can recover, for $\lambda \in [0, 23]$, the solution of the full dimensional variational inequality:

$$H_1 = \frac{16+\lambda}{13}, \quad H_2 = \frac{16+\lambda}{13}, \quad H_3 = \frac{2(23-\lambda)}{13}. \tag{22}$$

If the system does not have feasible solutions, the solution of the variational inequality, which does exist and is unique, has to be found in one of the three (two-dimensional) faces of $\tilde{K}$. Let us consider the face defined by the equation $F_3 = 0$, and reduce once more the dimension in (20). The one-dimensional feasible set is given by:

$$F_2 \in \mathbb{R} : 0 \leq F_2 \leq 6$$

and the our problem is now:
Find $H_2 : 0 \leq H_2 \leq 6$ such that $\forall F_2 : 0 \leq H_2 \leq 6$ we have:

$$(22H_2 - 66)(F_2 - H_2) \geq 0$$

which is satisfied by the feasible point $H_2 = 3$. Moreover, we can check that $(H_2, H_3) = (3, 0)$ satisfies (20). Thus, as the last step, we can recover the solution of the original three-dimensional variational inequality, which, for $\lambda \geq 23$ is:

$$(H_1, H_2, H_3) = (3, 3, 0).$$

Let us now suppose that the parameter is a nonnegative random variable *uniformly distributed* in a given interval, for example in [0, 100]. In this case an easy computation yields for the mean values and for the variances the following results:

$$\langle H_1 \rangle = \langle H_2 \rangle = 2.7965, \quad \langle H_3 \rangle = 0.4069$$
$$\sigma^2(H_1) = \sigma^2(H_2) = 0.1980, \quad \sigma^2(H_3) = 0.8160.$$

We can also suppose that our random parameter follows the *log-normal distribution* [11], which is used for numerous applications to model nonnegative random phenomena. It is also known as the Galton–McAlister distribution and, in economics, is sometimes called the Cobb–Douglas distribution, where it has been used to model production data. Thus, let:

$$g_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

be the *normal* distribution, then the *log-normal distribution* is defined by:

$$\begin{cases} (1/x) g_{\mu,\sigma^2}(\log x), & \text{if } x > 0 \\ 0, & \text{if } x \leq 0. \end{cases}$$

If we fix, for example, $\mu = 0.5$ and $\sigma = 2$, the numerical evaluation of the mean values and variances yields:

$$\langle H_1 \rangle = \langle H_2 \rangle = 0.7408, \quad \langle H_3 \rangle = 2.2586$$
$$\sigma^2(H_1) = \sigma^2(H_2) = 2.6276, \quad \sigma^2(H_3) = 0.4207.$$

Acknowledgements

Most of this paper was written while one of the authors, F.R., was on leave at the University of the Federal Armed Forces in Munich, Germany, supported by the University of Catania (Contributo per l'aggiornamento all'estero dei docenti) and by GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le sue Applicazioni). Technical support by A.O. Caruso (in Catania) and T. Humbert (in Muenchen) is also gratefully acknowledged.

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