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On Multifunctions of One Real Variable

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INTRODUCTION

Here and in the sequel, T is a nonempty subset of $\mathbb{R}$, Y is a topological space and F is a multifunction from T into Y, with nonempty values. We denote by $\mathrm{gr}(F)$ the set $\{(t, y) \in T \times Y: y \in F(t)\}$, i.e., the graph of F. Moreover, we denote by p the restriction to $\mathrm{gr}(F)$ of the projection from $T \times Y$ onto T. Of course, the following proposition holds:

PROPOSITION 1. *For each $t \in T$, we have*

$$\{t\} \times F(t) = p^{-1}(t).$$

The purpose of this paper is to point out various aspects of the structure of F, provided that some suitable connectedness assumption on $\mathrm{gr}(F)$ is made. To give an idea of the type of results we obtain, here are two of them. Let F be a single-valued function. If $\mathrm{gr}(F)$ is a continuous image of some connected and locally connected topological space, then F is continuous (see Theorem 2). If Y is metrizable and separable, and $\mathrm{gr}(F)$ is locally connected, then F is of first Baire class (see Theorem 20).

The key of proof of our main theorems resides in a systematic application of Proposition 1 jointly with certain recent results on continuous real functions (see [7–13]). For each result concerning F, we also state explicitly a dual version, in terms of a multifunction G from Y onto T, because of its independent interest. Finally, we want to stress that each of these dual versions, when G is single-valued, turns out to be an improvement of the corresponding well-known result on continuous real functions, applied to obtain it.

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NOTATIONS AND DEFINITIONS

Let $D, E \subseteq \mathbb{R}$, with $D \subseteq E$. We denote by $\mathrm{int}_E(D)$ the interior of D in E. Also, we put $A_0(D) = \{t \in D: \exists \rho > 0\colon\ ]t-\rho, t+\rho[\, \cap D = \{t\}\}$, $A_+(D) = \{t \in D\colon\ ]t, t+\rho[\, \cap D \neq \emptyset,\ \forall \rho > 0\}$, $A_-(D) = \{t \in D\colon\ ]t-\rho, t[\, \cap D \neq \emptyset,\ \forall \rho > 0\}$. We denote by $\mathscr{F}$ the family of all open sets in Y and, for each $y \in Y$, we put $\mathscr{F}(y) = \{\Omega \in \mathscr{F}: y \in \Omega\}$. Now, let S, X be two topological spaces and Φ be a multifunction from S into X. For any $B \subseteq S$ and $V \subseteq X$, we put $\Phi(B) = \bigcup_{s \in B} \Phi(s)$, $\Phi^-(V) = \{s \in S\colon \Phi(s) \cap V \neq \emptyset\}$ and $\Phi^+(V) = \{s \in S\colon \Phi(s) \subseteq V\}$. If $\Phi(S) = X$, we say that Φ is onto X. We recall that Φ is said to be *lower* (resp. *upper*) *semicontinuous* if $\Phi^-(V)$ (resp. $\Phi^+(V)$) is open in S for every open set $V \subseteq X$. We say that Φ is *open* (resp. *closed*) if $\Phi(B)$ is open (resp. closed) in $\Phi(S)$ for every open (resp. closed) set $B \subseteq S$. Moreover, we say that Φ is *inductively open* if there exists $S^* \subseteq S$ such that $\Phi(S^*) = \Phi(S)$ and $\Phi|_{S^*}$ is open. A multifunction (resp. a function) Ψ from S into X is said to be a *multiselection* (resp. a *selection*) of Φ if $\Psi(s) \subseteq \Phi(s)$ (resp. $\Psi(s) \in \Phi(s)$) for every $s \in S$. We denote by I_Φ the *inverse multifunction* of Φ which is defined by putting $I_\Phi(x) = \Phi^-(x)$ for every $x \in X$.

We now state a proposition which will be useful in the sequel. It follows from the proof of Theorem 2 of [9] and from Proposition 1.3 of [5].

PROPOSITION 2. *Let S be locally connected and let φ be a continuous real function on S such that* $\mathrm{int}(\varphi^{-1}(t)) = \emptyset$ *for every* $t \in \mathrm{int}(\varphi(S))$. *Then, the following are equivalent*:

(1) *φ is inductively open.*

(2) *$I_\varphi|_{\varphi(s)}$ admits a lower semicontinuous multiselection H, with nonempty values, such that* $\mathrm{card}(H(t)) = 1$ *for all* $t \in \partial(\varphi(S)) \cap \varphi(s)$.

(3) *For every $t \in \varphi(S)$, there exists a connected set $S_t \subseteq S$ such that* $t \in \mathrm{int}_{\varphi(S)}\, \varphi(S_t)$.

Now, let P be a multifunction from S into $\mathbb{R}$. We say that P is *pseudo-almost-open* if $I_P|_{P(S)}$ admits a multiselection Q, with nonempty values, such that $\sup_{t \in P(S)} \mathrm{card}(Q(t)) \leq 2$ and $Q^+(B) \subseteq \mathrm{int}_{P(S)}(P(B))$ for every open set $B \subseteq S$. Observe that, when P is single-valued, the above definition reduces to the analogous one given in [11]. Finally, we say that S is *completely connected* if it is a continuous image of some connected and locally connected topological space.

RESULTS

From now on, we will consider gr(F) as a space equipped with the relative product topology.

Our first result is the following

THEOREM 1. *Let* gr(F) *be completely connected. Then, F admits a multiselection Φ, with nonempty values, such that* $\sup_{t\in T}\operatorname{card}(\Phi(t))\leqslant 2$ *and* $\Phi^+(\Omega)\subseteq \operatorname{int}_T(F^-(\Omega))$ *for every* $\Omega\in\mathscr{F}$.

Proof. Since p is continuous, by Theorem 4.1 of [11], it is pseudo-almost-open. Hence, there exists a multiselection Q of I_p, with non-empty values, such that $\sup_{t\in T}\operatorname{card}(Q(t))\leqslant 2$ and $Q^+(B)\subseteq \operatorname{int}_T(p(B))$ for every open set B in gr(F). Therefore, by Proposition 1, there exists a multiselection Φ of F, with nonempty values, such that $\operatorname{card}(\Phi(t))\leqslant 2$ and $Q(t)=\{t\}\times\Phi(t)$ for all $t\in T$. Now, let $\Omega\in\mathscr{F}$. Then, we have $\Phi^+(\Omega)=Q^+((T\times\Omega)\cap \operatorname{gr}(F))\subseteq \operatorname{int}_T(p((T\times\Omega)\cap\operatorname{gr}(F)))=\operatorname{int}_T(F^-(\Omega))$. This completes the proof.

Thanks to Theorem 1, we have the following characterization, whose proof is left to the reader.

THEOREM 2. *Let f be a function from T into Y. Then, the following are equivalent*:

(1) *T is an interval and f is continuous.*

(2) gr(f) *is completely connected.*

Observe that there are simple examples of topological spaces X and discontinuous real functions f on X such that gr(f) is even connected and locally connected. For instance, it suffices to take as X the unit circumference of $\mathbb{R}^2$, with the usual topology, and as f the function that associates to each $x\in X$, regarded as a complex number, its argument belonging to $[0, 2\pi[$.

The dual version of Theorem 1 is

THEOREM 3. *Let G be a multifunction from Y onto T, with completely connected graph. Then, G is pseudo-almost-open.*

Proof. First, observe that the graph of I_G is completely connected, since it is homeomorphic to that of G. Next, apply Theorem 1 by taking $F=I_G$.

On the basis of the remark after Theorem 2, it is seen that Theorem 3 turns out to be an effective improvement of Theorem 4.1 of [11], in the case where G is single-valued.

The dual version of Theorem 2 is

THEOREM 4. *Let G be a multifunction from Y onto T such that $G(y) \cap G(z) = \varnothing$ for all $y, z \in Y$, with $y \neq z$. Then, the following are equivalent*:

(1) *T is an interval and G is open and closed.*

(2) *$\mathrm{gr}(G)$ is completely connected.*

Now, with the notations put in the preceding section, we state the two following propositions.

PROPOSITION 3. *For each $t \in T$, the following are equivalent*:

(1) $\mathrm{int}_{\mathrm{gr}(F)}(p^{-1}(t)) \neq \varnothing$.

(2) $t \in \bigcup_{\Omega \in \mathscr{F}} A_0(F^-(\Omega))$.

Proof. If $\mathrm{int}_{\mathrm{gr}(F)}(p^{-1}(t)) \neq \varnothing$, then, taking into account Proposition 1, there exist two open sets $D \subseteq \mathbb{R}$, $\Omega \subseteq Y$ such that $\varnothing \neq (D \times \Omega) \cap \mathrm{gr}(F) \subseteq \{t\} \times F(t)$. Thus, in particular, we have $t \in F^-(\Omega) \cap D$. Let $t^* \in (D \cap T) \setminus \{t\}$. We have $F(t^*) \cap \Omega = \varnothing$. Indeed, if there was $y^* \in F(t^*) \cap \Omega$, we would have $(t^*, y^*) \in (D \times \Omega) \cap \mathrm{gr}(F)$, and so $(t^*, y^*) \in \{t\} \times F(t)$. Hence, $t^* = t$, a contradiction. Therefore, $t \in A_0(F^-(\Omega))$. Conversely, let $t \in A_0(F^-(\Omega'))$ for some $\Omega' \in \mathscr{F}$. Then, there is an open set $D' \subseteq \mathbb{R}$ such that $D' \cap F^-(\Omega') = \{t\}$. Hence, $\varnothing \neq (D' \times \Omega') \cap \mathrm{gr}(F) \subseteq \{t\} \times F(t)$, and so, by Proposition 1, we have $\mathrm{int}_{\mathrm{gr}(F)}(p^{-1}(t)) \neq \varnothing$.

PROPOSITION 4. *Let $E = \{(t, y) \in \mathrm{gr}(F)$: (t, y) is a local extremum point for $p\}$. Then, we have*

$$p(\mathrm{gr}(F) \setminus E) \cap \mathrm{int}(T) \subseteq \bigcup_{y \in Y} \bigcap_{\Omega \in \mathscr{F}(y)} (A_-(F^-(\Omega)) \cap A_+(F^-(\Omega))).$$

Proof. Let $t_0 \in p(\mathrm{gr}(F) \setminus E) \cap \mathrm{int}(T)$. Then, there is $y_0 \in F(t_0)$ such that, for every open neighbourhood U of t_0 in T and every open neighbourhood Ω of y_0 in Y, there exist $t', t'' \in U \cap F^-(\Omega)$ satisfying the relation $t' < t_0 < t''$. Thus, $t_0 \in \bigcap_{\Omega \in \mathscr{F}(y_0)} (A_-(F^-(\Omega)) \cap A_+(F^-(\Omega)))$.

Observe that the inclusion in the statement of Proposition 4 can be strict. To see this, it suffices to take $T = [-1, 1]$, $Y = [0, 1]$, and F defined as follows:

$$F(t) = \begin{cases} \left[0, \frac{1}{2}\right] \cup \left(\bigcup_{n=3}^{\infty} \left\{ \frac{n-1}{n} \right\} \right) & \text{if } t = 0 \\ \bigcup_{n=1}^{\infty} \left\{ \frac{n-1}{n}(1-t) \right\} \cup \{1-t\} & \text{if } t \in]0, 1] \\ \left\{ \frac{1}{2} \right\} \cup \left\{ \frac{t}{2} + 1 \right\} & \text{if } t \in [-1, 0[. \end{cases}$$

In this case we have $0 \in (\bigcap_{\Omega \in \mathscr{F}(1)} (A_-(F^-(\Omega)) \cap A_+(F^-(\Omega)))) \setminus p(\mathrm{gr}(F) \setminus E)$. We also observe that the inclusion in the statement of Proposition 4 is, in fact, an equality when each value of F is closed.

Applying jointly Proposition 5.1 of [11] and Propositions 3 and 4, we obtain directly

THEOREM 5. *Let* $\mathrm{gr}(F)$ *be connected. Then, we have*

$$\mathrm{int}(T) = \left(\bigcup_{\Omega \in \mathscr{F}} A_0(F^-(\Omega) \cap \mathrm{int}(T)) \right) \cup \left(\bigcup_{y \in Y} \bigcap_{V \in \mathscr{F}(y)} (A_-(F^-(V)) \cap A_+(F^-(V))) \right).$$

The dual version of Theorem 5 is

THEOREM 6. *Let G be a multifunction from Y onto T, with connected graph. Then, we have*

$$\mathrm{int}(T) = \left(\bigcup_{\Omega \in \mathscr{F}} A_0(G(\Omega) \cap \mathrm{int}(T)) \right) \cup \left(\bigcup_{y \in Y} \bigcap_{V \in \mathscr{F}(y)} (A_-(G(V)) \cap A_+(G(V))) \right).$$

It is possible to check that Proposition 5.1 of [11] turns out to be, essentially, a particular case of Theorem 6.

The next result is a multiselection theorem.

THEOREM 7. *Let* $\mathrm{gr}(F)$ *be locally connected and let* $\bigcup_{\Omega \in \mathscr{F}} A_0(F^-(\Omega) \cap \mathrm{int}(T)) = \varnothing$. *Then, the following are equivalent*:

(1) *There exists a lower semicontinuous multiselection Φ of F, with nonempty values, such that* $\mathrm{card}(\Phi(t)) = 1$ *for all* $t \in \partial T \cap T$.

(2) *For every $t \in T$, there exists a subset J_t of T, with $t \in \mathrm{int}_T(J_t)$, and a multiselection of $F|_{J_t}$, with nonempty values and connected graph.*

Proof. Thanks to Proposition 3, by hypothesis, we have $\mathrm{int}_{\mathrm{gr}(F)}(p^{-1}(t)) = \varnothing$ for all $t \in \mathrm{int}(T)$. On the other hand, taking into account Proposition 1, it is seen that condition (1) and condition (2) are equivalent, respectively, to condition (2) and condition (3) of Proposition 2, applied by taking $S = \mathrm{gr}(F)$ and $\varphi = p$. Therefore, our conclusion is a direct consequence of Proposition 2.

A remarkable particular case of Theorem 7 is

THEOREM 8. *Let* $\mathrm{gr}(F)$ *be connected and locally connected, and let* $\bigcup_{\Omega \in \mathscr{F}} A_0(F^-(\Omega) \cap \mathrm{int}(T)) = \varnothing$. *Then, F admits a lower semicontinuous multiselection Φ, with nonempty values, such that* $\mathrm{card}(\Phi(t)) = 1$ *for all* $t \in \partial T \cap T$.

It is worth noting that Theorem 8 is no longer true, in general, if $\bigcup_{\Omega \in \mathscr{F}} A_0(F^-(\Omega) \cap \mathrm{int}(T)) \neq \varnothing$. To see this, take $T = Y = [0, 1]$ and define F by putting:

$$F(t) = \begin{cases} \{t\} & \text{if } t \in \left[0, \frac{1}{3}\right[\\ \left[\frac{1}{3}, \frac{2}{3}\right] & \text{if } t = \frac{1}{3} \\ \left\{\frac{t+1}{2}\right\} & \text{if } t \in \left]\frac{1}{3}, 1\right]. \end{cases}$$

The dual versions of Theorems 7 and 8 are the following:

THEOREM 9. *Let G be a multifunction (resp. a function) from Y onto T, with locally connected graph, such that* $\bigcup_{\Omega \in \mathscr{F}} A_0(G(\Omega) \cap \mathrm{int}(T)) = \varnothing$. *Then, the following are equivalent*:

(1) *G admits an inductively open multiselection Ψ onto T such that* $\mathrm{card}(\Psi^-(t)) = 1$ *for all $t \in \partial T \cap T$ (resp. G is inductively open).*

(2) *For every $t \in T$, there exists a set $Y_t \subseteq Y$ and a multiselection H of $G|_{Y_t}$, with nonempty values and connected graph, such that $t \in \mathrm{int}_T(H(Y_t))$.*

Proof. Take $F = I_G$. So that, $\mathrm{gr}(F)$ is locally connected and $\bigcup_{\Omega \in \mathscr{F}} A_0(F^-(\Omega) \cap \mathrm{int}(T)) = \varnothing$. On the other hand, proceeding as in the proof of Proposition 1.2 of [5], it is possible to check that conditions (1) and (2) are equivalent, respectively, to conditions (1) and (2) of Theorem 7, with F chosen as above. Hence, our conclusion is a direct consequence of Theorem 7.

THEOREM 10. *Let G be a multifunction (resp. a function) from Y onto T, with connected and locally connected graph, such that* $\bigcup_{\Omega \in \mathscr{F}} A_0(G(\Omega) \cap \mathrm{int}(T)) = \varnothing$. *Then, G admits an inductively open multiselection Ψ onto T such that* $\mathrm{card}(\Psi^-(t)) = 1$ *for all $t \in \partial T \cap T$ (resp. G is inductively open).*

Taking into account that the graph of a continuous function is homeomorphic to the domain of this, it is seen that Theorems 9 and 10 improve, respectively, Theorem 2 of [9] and Theorem 2.4 of [7].

From now on, except that in Theorems 23 and 24, we will assume that the space Y is metrizable. Of course, we will consider $\mathrm{gr}(F)$ equipped with one of the usual metrics inducing the product topology.

The next result is another consequence of Theorem 7.

THEOREM 11. *Let* $\mathrm{gr}(F)$ *be locally connected, let* $\bigcup_{\Omega\in\mathscr{F}} A_0(F^-(\Omega)\cap \mathrm{int}(T))=\varnothing$ *and let* $F(t)$ *be complete for all* $t\in\mathrm{int}(T)$. *Assume, finally, that condition* (2) *of Theorem* 7 *is satisfied. Then, there exist three multiselections* Φ_1, Φ_2, Φ_3 *of* F, *with nonempty values, such that*:

(i) Φ_1, Φ_3 *are upper semicontinuous and* Φ_2 *is lower semicontinuous*;

(ii) *for every* $t\in T$, *we have* $\mathrm{card}(\Phi_1(t))\leqslant 2$, $\Phi_1(t)\subseteq\Phi_2(t)\subseteq\Phi_3(t)$, *and* $\Phi_2(t)$, $\Phi_3(t)$ *are compact.*

Proof. By Theorem 7, F admits a lower semicontinuous multiselection Φ, with nonempty values, such that $\mathrm{card}(\Phi(t))=1$ for all $t\in\partial T\cap T$. Put $\Phi^*(t)=\overline{\Phi(t)}$ for all $t\in T$. Thus, Φ^* is a lower semicontinuous multiselection of F, with nonempty and complete values. By Theorem 1.1 of [4], there are two multiselections Φ_2, Φ_3 of Φ^*, with nonempty and compact values, such that Φ_3 is upper semicontinuous and Φ_2 is a lower semicontinuous multiselection of Φ_3. On the other hand, by Proposition 2.2 of [6], $\Phi_2(T)$ is separable, and so, by Theorem 11.4 of [1], there exists an upper semicontinuous multiselection Φ_1 of Φ_2 such that $0<\mathrm{card}(\Phi_1(t))\leqslant 2$ for all $t\in T$. This completes the proof.

The dual version of Theorem 11 is

THEOREM 12. *Let* G *be a multifunction from* Y *onto* T, *with locally connected graph, such that* $\bigcup_{\Omega\in\mathscr{F}} A_0(G(\Omega)\cap\mathrm{int}(T))=\varnothing$ *and* $G^-(t)$ *is complete for all* $t\in\mathrm{int}(T)$. *Assume, finally, that condition* (2) *of Theorem* 9 *is satisfied. Then, there exist three multiselections* Ψ_1, Ψ_2, Ψ_3 *of* G, *onto* T, *such that*:

(i) Ψ_1, Ψ_3 *are closed,* Ψ_2 *is open and* $\Psi_1(y)\subseteq\Psi_2(y)\subseteq\Psi_3(y)$ *for all* $y\in Y$;

(ii) *for every* $t\in T$, *we have* $\mathrm{card}(\Psi_1^-(t))\leqslant 2$ *and* $\Psi_2^-(t)$, $\Psi_3^-(t)$ *are compact.*

Proof. It suffices to apply Theorem 11 by taking $F=I_G$ and $\Psi_i=I_{\Phi_i}$, $i=1, 2, 3$.

In particular, we have

THEOREM 13. *Let* g *be a function from* Y *onto* T *satisfying the hypotheses of Theorem* 12. *Then, there exist three subsets* Y_1, Y_2, Y_3 *of* Y, *with* $Y_1 \subseteq Y_2 \subseteq Y_3$ *and* Y_1, Y_3 σ*-compact* (*in fact, compact if* T *is so*) *such that*:

(i) $g|_{Y_1}$, $g|_{Y_3}$ *are closed and* $g|_{Y_2}$ *is open*;

(ii) *for every* $t \in T$, *we have* $0 < \operatorname{card}(g^{-1}(t) \cap Y_1) \leqslant 2$ *and* $g^{-1}(t) \cap Y_2$, $g^{-1}(t) \cap Y_3$ *are nonempty and compact.*

Proof. Apply Theorem 12, by taking $G = g$. Let Ψ_1, Ψ_2, Ψ_3 be as in the conclusion of that theorem. For $i = 1, 2, 3$, put $Y_i = \{y \in Y: \Psi_i(y) \neq \varnothing\}$. Of course, the sets Y_i satisfy (i) and (ii). Now, we prove that, under the present assumptions, T must be σ-compact. Indeed, denote by $\mathscr{T}$ the family of all connected components of T reducing to a single point. Regard $\mathscr{T}$ as a subset of T. Let $t \in \mathscr{T}$. By hypothesis, there is a connected set $Y_t \subseteq Y$ such that $\operatorname{gr}(g|_{Y_t})$ is connected and $t \in \operatorname{int}_T(g(Y_t))$. Since $g(Y_t)$ is connected, we have $g(Y_t) = \{t\}$. Hence, t is an isolated point of T. Therefore, $\mathscr{T}$ is countable. On the other hand, by a result of Morse (see [3, p. 58]), the set $T \backslash \mathscr{T}$ is a countable union of nondegenerate intervals. Thus, T is σ-compact. Now, the fact that the sets Y_1, Y_3 are σ-compact (in fact, compact if T is so) follows from (i), (ii) and from Theorem 1 of [2].

Observe that Theorem 13 improves Theorem 2.8 of [8].

Before establishing another multiselection theorem for F, we prove

THEOREM 14. *Let* S *be a locally connected metric space and let* φ *be a continuous real function on* S *such that* $\varphi^{-1}(t) \cap \Gamma$ *is complete for every connected component* Γ *of* S *and every* $t \in \operatorname{int}(\varphi(S))$. *Moreover, assume that, for every* $t \in \varphi(S)$, *there exists a connected set* $S_t \subseteq S$ *such that* $t \in \operatorname{int}_{\varphi(S)}(\varphi(S_t))$. *Then, for every compact set* $C \subseteq \varphi(S)$ *there exists a compact set* $K \subseteq S$ *such that* $\varphi(K) = C$.

Proof. Let C be any compact subset of $\varphi(S)$. For every $t \in C$, choose a connected set $S_t \subseteq S$ and a positive real number ε_t such that $[t - \varepsilon_t, t + \varepsilon_t] \cap \varphi(S) \subseteq \varphi(S_t)$. Since C is compact, it is possible to find finitely many points $t_1, \dots, t_n$ of C in such a manner that $C \subseteq \bigcup_{i=1}^n [t_i - \varepsilon_{t_i}, t_i + \varepsilon_{t_i}]$. Now, for each $i = 1, \dots, n$, put $C_i = [t_i - \varepsilon_{t_i}, t_i + \varepsilon_{t_i}] \cap C$. Fix i. Let Γ_{t_i} be the connected component of S containing S_{t_i}. Since $C_i \subseteq \varphi(\Gamma_{t_i})$, thanks to Theorem 1 of [13], there exists a compact set $K_i \subseteq \Gamma_{t_i}$ such that $\varphi(K_i) = C_i$. Now, put $K = \bigcup_{i=1}^n K_i$. Thus, K is compact and $\varphi(K) = C$.

The above-quoted multiselection theorem is

THEOREM 15. *Let* $\operatorname{gr}(F)$ *be locally connected and let* $(\{t\} \times F(t)) \cap \Gamma$ *be complete for every connected component* Γ *of* $\operatorname{gr}(F)$ *and every* $t \in \operatorname{int}(T)$. *Moreover, assume that condition* (2) *of Theorem* 7 *is satisfied. Then,* F

admits an upper semicontinuous multiselection, with nonempty and compact values.

Proof. As we know, condition (2) of Theorem 7 is equivalent to the fact that, for every $t \in T$, there is a connected set $S_t \subseteq \mathrm{gr}(F)$ such that $t \in \mathrm{int}_T(p(S_t))$. In particular, this implies that T is locally connected. Let $\mathscr{D}$ be the family of all connected components of T nonreducing to a single point. Fix $D \in \mathscr{D}$. Let $\mathscr{E}_D$ be a family of pairwise disjoint bounded subintervals of D such that $D = \bigcup_{J \in \mathscr{E}_D} \bar{J}$. For each $J \in \mathscr{E}_D$, thanks to Theorem 14, we can find a compact set $K_J \subseteq \mathrm{gr}(F)$ such that $p(K_J) = \bar{J}$. Now, for each $t \in \bar{J}$, put $\Phi_J(t) = \{y \in Y: (t, y) \in K_J\}$. Of course, Φ_J is an upper semicontinuous multiselection of $F|_{\bar{J}}$, with nonempty and compact values. For each $t \in D \setminus \bigcup_{J \in \mathscr{E}_D} \mathrm{int}_D(J)$ there are at most two members $J_{t,1}$, $J_{t,2}$ of $\mathscr{E}_D$ such that $\bar{J}_{t,1} \cap \bar{J}_{t,2} = \{t\}$. Put

$$\Phi_D(t) = \begin{cases} \Phi_J(t) & \text{if } \ t \in \mathrm{int}_D(J),\ J \in \mathscr{E}_D \\ \Phi_{J_{t,1}}(t) \cup \Phi_{J_{t,2}}(t) & \text{if } \ t \in D \setminus \bigcup_{J \in \mathscr{E}_D} \mathrm{int}_D(J). \end{cases}$$

Thus, Φ_D is an upper semicontinuous multiselection of $F|_D$, with nonempty and compact values. Now, for each $t \in T \setminus \bigcup_{D \in \mathscr{D}} D$, choose any point $y_t \in F(t)$. Finally, put

$$\Phi(t) = \begin{cases} \Phi_D(t) & \text{if } \ t \in D,\ D \in \mathscr{D} \\ \{y_t\} & \text{if } \ t \in T \setminus \bigcup_{D \in \mathscr{D}} D. \end{cases}$$

Of course, Φ satisfies the conclusion of the theorem.

Thanks to Theorem 15, we have the following characterization, whose proof is left to the reader.

THEOREM 16. *Let f be a function from T into Y. Then, the following are equivalent*:

(1) *T is locally connected and f is continuous.*

(2) $\mathrm{gr}(f)$ *is locally connected and, for every $t \in T$, there exists a subset J_t of T, with $t \in \mathrm{int}_T(J_t)$, such that* $\mathrm{gr}(f|_{J_t})$ *is connected.*

Now, we state the dual version of Theorem 15.

THEOREM 17. *Let G be a multifunction from Y onto T, with locally connected graph, such that $(\{t\} \times G^-(t)) \cap \Gamma$ is complete for every connected component Γ of* $\mathrm{gr}(G)$ *and every $t \in \mathrm{int}(T)$. Moreover, assume that condition* (2) *of Theorem* 9 *is satisfied. Then, G admits a closed multiselection Ψ onto T such that $\Psi^-(t)$ is compact for all $t \in T$.*

It is worth noting the following particular case of Theorem 17.

THEOREM 18. *Let g be a function from Y onto T, with connected and locally connected graph, such that $g^{-1}(t)$ is complete for all $t \in \mathrm{int}(T)$. Then, for every compact set $C \subseteq T$, there exists a compact set $K \subseteq Y$ such that $g|_K$ is continuous and $g(K) = C$.*

Proof. By Theorem 17, there is a set $Y^* \subseteq Y$ such that $g|_{Y^*}$ is closed, $g(Y^*) = T$ and $g^{-1}(t) \cap Y^*$ is compact for all $t \in T$. Let C any compact subset of T. Then, by Theorems 1 and 3 of [2], the set $K = g^{-1}(C) \cap Y^*$ has the desired properties.

We now prove

THEOREM 19. *Let $\mathrm{gr}(F)$ be locally connected. Suppose that the set $T^* = \{t \in T$: there is a connected component Γ of $\mathrm{gr}(F)$ for which $p(\Gamma) = \{t\}\}$ is countable and that the set $(\{t\} \times F(t)) \cap \Gamma$ is complete for every connected component Γ of $\mathrm{gr}(F)$ and every $t \in \mathrm{int}(T)$. Then, there exist a multiselection H of F, with σ-compact graph, and a selection φ of H, of first Baire class. If, in addition, the set $p(\Gamma)$ is compact for every connected component Γ of $\mathrm{gr}(F)$ on which p is not constant, the number of such components and T^* being, furthermore, finite, then the graph of H is compact.*

Proof. Denote by $\mathscr{C}$ the family of all connected components of $\mathrm{gr}(F)$ and put $\mathscr{H} = \{\Gamma \in \mathscr{C}: p|_\Gamma \text{ is constant}\}$. For each $t \in T^*$, choose $\Gamma_t \in \mathscr{H}$ such that $p(\Gamma_t) = \{t\}$. Now, put $V = \bigcup_{\Gamma \in \mathscr{C} \setminus \mathscr{H}} \Gamma \cup (\bigcup_{t \in T^*} \Gamma_t)$. Of course, V is locally connected and the family of its connected components is $(\mathscr{C} \setminus \mathscr{H}) \cup \{\Gamma_t\}_{t \in T^*}$. Then, by Theorem 3.3 of [10], there are a σ-compact set $K \subseteq V$ and a function $h: T \to K$, of first Baire class, such that $p(h(t)) = t$ for every $t \in T$. In fact, K is compact provided that so is the set $p(\Gamma)$ for every connected component Γ of V, the number of these components being, furthermore, finite. Now, let $\varphi: T \to Y$ be the function such that $h(t) = (t, \varphi(t))$ for every $t \in T$. Also, put $H(t) = \{y \in Y: (t, y) \in K\}$ for all $t \in T$. It is easily seen that H and φ have the desired properties.

Observe that, in Theorem 19, to guarantee that the graph of H is compact, it is not sufficient to assume simply that T is compact. To see this, it suffices to take, for instance, $T = Y = [0, 1]$ and F defined by putting

$$F(t) = \begin{cases} \{0\} & \text{if } t = 0 \\ \{1\} & \text{if } t \in]0, 1]. \end{cases}$$

THEOREM 20. *Let f be a function from T onto Y, with locally connected graph. Then, the following are equivalent:*

(1) *The function f is of first Baire class and its graph is σ-compact.*

(2) *Y is separable.*

(3) *Y is σ-compact.*

Proof. The implications $(1) \Rightarrow (3) \Rightarrow (2)$ are obvious. Therefore, let us show that $(2) \Rightarrow (1)$. Since Y is separable (and metrizable), so is $\mathrm{gr}(f)$. Then, $\mathrm{gr}(f)$ has countably many connected components, since it is also locally connected. Thus, (1) follows directly from Theorem 19.

In particular, as a consequence of Theorem 20, we obtain the following

THEOREM 21. *Let Y be separable but not σ-compact. Then, there is no one-to-one real function on Y, with locally connected graph.*

The next result is the dual version of Theorem 19. Here, p' denote the projection from $Y \times T$ onto T.

THEOREM 22. *Let G be a multifunction from Y onto T, with locally connected graph. Suppose that the set $T' = \{t \in T$: there is a connected component Γ of $\mathrm{gr}(G)$ for which $p'(\Gamma) = \{t\}\}$ is countable and that the set $(\{t\} \times G^{-}(t)) \cap \Gamma$ is complete for every connected component Γ of $\mathrm{gr}(G)$ and every $t \in \mathrm{int}(T)$. Then, there exist a multiselection H of I_G, with σ-compact graph, and a selection φ of H, of first Baire class. If, in addition, the set $p'(\Gamma)$ is compact for every connected component Γ of $\mathrm{gr}(G)$ on which p' is not constant, the number of such components and T' being, furthermore, finite, then the graph of H is compact.*

It is possible to check that Theorem 3.3 of [10] turns out to be a particular case of Theorem 22.

Now, we present the final result and its dual version. In these two theorems we do not assume the metrizability of Y.

THEOREM 23. *Let $\mathrm{gr}(F)$ be locally connected and F be onto Y. Let $\bigcup_{\Omega \in \mathscr{F}} A_0(F^{-}(\Omega)) = \varnothing$. Moreover, assume that there exists a set $D \subseteq T$, dense in T, such that $F(t)$ is separable for every $t \in D$. Then, Y is separable.*

Proof. By Propositions 1 and 3, $p^{-1}(t)$ is separable for all $t \in D$ and $\mathrm{int}_{\mathrm{gr}(F)}(p^{-1}(t)) = \varnothing$ for all $t \in T$. Therefore, by Theorem 3.4 of [12], $\mathrm{gr}(F)$ is separable. Hence, so Y is.

THEOREM 24. *Let G be a multifunction from Y onto T, with locally connected graph, such that $\bigcup_{\Omega \in \mathscr{F}} A_0(G(\Omega)) = \varnothing$. Moreover, assume that there exists a set $D \subseteq T$, dense in T, such that $G^{-}(t)$ is separable for every $t \in D$. Then, Y is separable.*

Observe that Theorem 3.4 of [12] becomes now a particular case of Theorem 24.

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