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Discrete Mathematics 283 (2004) 249–253

DISCRETE MATHEMATICS

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Note

Bicolouring Steiner systems $S(2,4,v)$

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Received 17 March 2003; accepted 30 November 2003

Abstract

We discuss colourings of elements of Steiner systems $S(2,4,v)$ in which the elements of each block get precisely two colours. We show that there exist systems admitting such colourings with arbitrary large number of colours, as well as systems which are uncolourable.

Keywords: Steiner system; Bicolouring; Chromatic number

1. Introduction

In this paper, we discuss colourings of the elements of Steiner systems $S(2,4,v)$, i.e. mappings of the set of elements of an $S(2,4,v)$ into a set C whose elements are called *colours*. Here, an $S(2,4,v)$ is a pair $(V,\mathscr{B})$ where V is a finite set of *elements*, and $\mathscr{B}$ is a collection of 4-element subsets of V called *blocks* such that each 2-subset of V is contained in exactly one block. It is well known that an $S(2,4,v)$ exists if and only if $v \equiv 1$ or 4 (mod 12) (cf. [1,2]).

There are five different ways in which a block of an $S(2,4,v)$ can be coloured (cf. [8]). For known results on colourings of $S(2,4,v)$, see [1,2,8]. In this paper, we restrict ourselves to colourings which colour each block of an $S(2,4,v)$ with exactly two colours. A block is *B-coloured* if three of its elements are coloured with one colour and the remaining element is coloured with a different colour $(X,X,X,\square)$, and it is *C-coloured* if it has two elements coloured with one colour and the other two elements coloured with a different colour $(X,X,\square,\square)$.

2. Bicolourings for $S(2,4,v)$

A colouring of an $S(2,4,v)$ is a *bicolouring* if each block is coloured with precisely two colours. An $S(2,4,v)$ is *bicolourable* if it admits at least one bicolouring. A bicolouring using exactly h colours is an h-bicolouring.

Let V be the vertex set and $\mathscr{B}$ be the family of blocks of an $S(2,4,v)$. Consider a bicolouring of an $S(2,4,v)$ in which h colours are used. Let X_i be the set of elements coloured with colour i; X_i is the *colour class* whose cardinality is $|X_i| = n_i$. We associate with a bicolouring the integer vector $(n_1,n_2,\ldots,n_h)$, the *type of bicolouring* where we assume w.l.o.g. that

$$n_1 \leqslant n_2 \leqslant \cdots \leqslant n_h.$$

Let $\mathscr{S}_I$, with $|\mathscr{S}_I| = s_I$, be the union of $|I|$ colour classes, where $I \subseteq \{1,2,\ldots,h\}$ is any subset of colours used in a bicolouring.

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2003.11.016

The maximum (minimum) number of colours for which there exists a bicolouring of an $S(2,4,v)$ $(V,\mathscr{B})$, is its *upper* (*lower*) *chromatic number* and is denoted by $\bar{\chi}_{\rm bi}(V,\mathscr{B})$ $[(\chi_{\rm bi}(V,\mathscr{B})]$. We also denote $\bar{\chi}_{\rm bi}(v)$ $[\chi_{\rm bi}(v)$, respectively] the maximum of $\bar{\chi}_{\rm bi}(V,\mathscr{B})$ [the minimum of $\chi_{\rm bi}(V,\mathscr{B})$, resp.] where the maximum (minimum) is taken over all $S(2,4,v)$ $(V,\mathscr{B})$ of order v.

The numbers $\chi_{\rm bi}(v)$, $\bar{\chi}_{\rm bi}(v)$ are well defined: for every admissible order $v\equiv 1,4 \pmod{12}$ there exists an $S(2,4,v)$ with a 2-colouring ([1,2]) which is necessarily a 2-bicolouring. Moreover, this shows also that $\chi_{\rm bi}(v)=2$ for all $v\equiv 1,4 \pmod{12}$.

On the other hand, for individual systems $S(2,4,v)$ $(V,\mathscr{B})$ the numbers $\chi_{\rm bi}(V,\mathscr{B})$ and $\bar{\chi}_{\rm bi}(V,\mathscr{B})$ may not be defined. In other words, an $S(2,4,v)$ $(V,\mathscr{B})$ may be uncolourable. In fact, result of [8] can be restated for bicolourings as follows.

Theorem 2.1. *There exists a constant v^* such that for all $v>v^*$, $v\equiv 1,4 \pmod{12}$, there exists an $S(2,4,v)$ without a bicolouring.*

As all $S(2,4,v)$s with $v\leqslant 25$ are bicolourable (see proof of Proposition 3.1 in Section 3 below), it follows that $v^*\geqslant 28$.

Proposition 2.1. *If $S(2,4,v)$ has a bicolouring using h colours where x_B is the number of B-coloured blocks and x_C is the number of C-coloured blocks, then*

$$x_B=\frac{\sum_{i=1}^h n_i(n_i-1)}{2}-\frac{v(v-1)}{6},$$

$$x_C=\frac{v(v-1)}{4}-\frac{\sum_{i=1}^h n_i(n_i-1)}{2}.$$

Proof. In a B-coloured block there are three distinct pairs of elements coloured with two different colours and three distinct pairs of elements coloured with one colour, while in a C-coloured block there are four distinct pairs of elements coloured with two different colours and two distinct pairs of elements coloured with one colour, so we have

$$3x_B+2x_C=\frac{\sum_{i=1}^h n_i(n_i-1)}{2}, \tag{1}$$

and

$$3x_B+4x_C=\frac{v(v-1)}{2}-\frac{\sum_{i=1}^h n_i(n_i-1)}{2}. \tag{2}$$

The statement follows by resolving for x_B and x_C. □

For a bicolouring of type $(n_1,n_2,\dots,n_h)$ of an $S(2,4,v)$, let x_B^I and x_C^I be the numbers of B-coloured and C-coloured blocks contained in S_I respectively.

Corollary 2.1. *Given a bicolouring of an $S(2,4,v)$ of type $(n_1,n_2,\dots,n_h)$, and $I\subseteq\{1,2,\dots,h\}$, the set S_I is a subsystem $S(2,4,s_I)$ if and only if*

$$x_B^I=\frac{\sum_{j\in I} n_j(n_j-1)}{2}-\frac{s_I(s_I-1)}{6}, \tag{3}$$

$$x_C^I=\frac{s_I(s_I-1)}{4}-\frac{\sum_{j\in I} n_j(n_j-1)}{2}. \tag{4}$$

Proof. In fact (3) and (4) hold if and only if all the monochromatic pairs of elements coloured with the colour $i\in I$, belong to blocks all contained in S_I. □

Corollary 2.2. *If an $S(2,4,v)$, $v>4$ is coloured with a bicolouring which uses h colours, then*

$$\frac{v(v+2)}{3}<\sum_{i=1}^h n_i^2\leqslant\frac{v(v+1)}{2}. \tag{5}$$

Moreover, the right inequality becomes an equality if and only if $x_C=0$.

Proof. It follows from [8] that if $v > 4$, then we must have $x_B > 0$. The rest follows from the definition of a bicolouring and from the expressions for x_B and x_C, as given in Proposition 2.1. □

In particular, if a bicolouring of an $S(2,4,v)$ is of type $(1,3,3^2,\dots,3^{h-1})$, then the right inequality in (5) becomes an equality.

Lemma 2.1. *If $S(2,4,v)$ is coloured with a bicolouring of type $(n_1,n_2,\dots,n_h)$ using h colours and $I \subseteq \{1,2,\dots,h\}$, then for $2 \leqslant |I| \leqslant h$ the following inequality holds:*

$$3\sum_{j\in I} n_j(n_j-1) \geqslant s_I(s_I-1). \tag{6}$$

Proof. The generic set S_I contains at most $s_I(s_I-1)/12$ blocks. Let x_B^I and x_C^I be the number of B-coloured blocks, and the number of C-coloured blocks, respectively, contained in S_I. We have

$$\frac{s_I(s_I-1)}{12} \geqslant x_B^I + x_C^I \geqslant \frac{1}{4}(3x_B^I + 4x_C^I)) = \frac{1}{4}\left(\frac{s_I(s_I-1)}{2} - \sum_{j\in I}\frac{n_j(n_j-1)}{2}\right) \tag{7}$$

which implies

$$3\sum_{j\in I} n_j(n_j-1) \geqslant s_I(s_I-1). \qquad \square \tag{8}$$

Note that in (7) x_B^I can be equal to zero for some set I.

Proposition 2.2. *If $S(2,4,v)$ is coloured with a bicolouring using h colours, then the following implications hold:*

1. *if $n_1 = 1$, then $n_i \equiv 0 \pmod 3$ for $2 \leqslant i \leqslant h$;*
2. *if $n_1 = 2$, then there is a unique $n_j \equiv 2 \pmod 3$, $n_j = 2$ or $n_j \geqslant 11$ and $n_i \equiv 0 \pmod 3$ for $2 \leqslant i \leqslant h$ with $i \neq j$;*
3. *if $n_1 = 3$, then $n_i \geqslant 9$ for $2 \leqslant i \leqslant h$ and there exists $n_k \geqslant 10$.*

Proof.

1. If $n_1 = 1$, let x be the element in X_1; it is contained in $(v-1)/3$ B-coloured blocks, so $n_i \equiv 0 \pmod 3$ for $2 \leqslant i \leqslant h$.
2. If $n_1 = 2$, let y_1 and y_2 be two elements in X_1, y_1 is contained in $((v-1)/3-1)$ B-coloured blocks and in a C-coloured block containing y_2. Then there exists a unique $n_j \equiv 2 \pmod 3$ and $n_i \equiv 0 \pmod 3$ such that $2 \leqslant i \leqslant h$ and $i \neq j$; clearly n_j cannot equal 2, but further analysis shows that it cannot equal 5 or 8, either.
3. If $n_1 = 3$, let z_1, z_2 and z_3 be three elements in X_1. If $b = \{z_1,z_2,z_3,x\}$ is a block of $S(2,4,v)$, then all the blocks different from b and containing z_1 or z_2 or z_3 are B-coloured, and from the definition of $S(2,4,v)$ we have that $n_i \geqslant 9$ for $2 \leqslant i \leqslant h$. In particular, since $n_1 \neq 1$, we have that X_k, the colour class which contains x, has cardinality $n_k \geqslant 10$. If b is not a block of $S(2,4,v)$, then z_i, $1 \leqslant i \leqslant 3$, is contained in $((v-1)/3-2)$ B-coloured blocks and in two C-coloured blocks. It follows that $n_i \geqslant 9$ for $2 \leqslant i \leqslant h$ and since $n_1 \neq 1$ we have that there is a colour class with cardinality at least 10. □

Proposition 2.3. *In a bicolouring of $S(2,4,v)$ of type $(n_1,n_2,\dots,n_h)$, we have $n_i \geqslant 2^{i-1}$ for $1 \leqslant i \leqslant h$, and there is at least one n_j with $n_j > 2^{j-1}$.*

Proof. By Lemma 2.1 and Lemma 2 in [7] we have that $n_i \geqslant 2^{i-1}$ for $1 \leqslant i \leqslant h$. If $n_1 = 1$, then by Proposition 2.2 $n_i \equiv 0 \pmod 3$ and $n_i > 2^{i-1}$ for $2 \leqslant i \leqslant h$. □

3. An upper bound for $\bar{\chi}_{\mathrm{bi}}$

Theorem 3.1. *If $S(2,4,v)$ admits a bicolouring and $v \leqslant ((3^d-1)/2)(d \in N)$, then*

$$\bar{\chi}_{\mathrm{bi}} \leqslant \left\lfloor \frac{d}{\log_3 2} \right\rfloor - 1.$$

Proof. If $\mathscr{P}$ is a bicolouring of $S(2,4,v)$ which uses h colours, then by Proposition 2.3 we have the sequence of inequalities

$$\frac{3^d-1}{2} \geqslant v = \sum_{i=1}^{h} n_i > \sum_{i=1}^{h} 2^{i-1} = 2^h - 1 \tag{9}$$

which implies

$$3^d \geqslant 2^{h+1}. \tag{10}$$

Inequality (10) is true for bicolourings which use $h = \bar{\chi}_{\rm bi}$ colours, hence the theorem follows. □

In the theorem above, $d \in N$ is usually taken to be the smallest integer for which the inequality $v \leqslant (3^d-1)/2$ holds.

3.1. Results for $S(2,4,v)$ of small orders

In this subsection we prove some results about the upper and the lower chromatic numbers for small $S(2,4,v)$.

Proposition 3.1. *An $S(2,4,v)$, with $v \in \{13,16,25,28\}$ can only be coloured with a bicolouring which uses either two or three colours. Thus $\bar{\chi}_{\rm bi}(v) = 3$ for these orders.*

Proof. By Theorem 3.1 we have that $\bar{\chi}_{\rm bi}(13) \leqslant 3$ and $\bar{\chi}_{\rm bi}(v) \leqslant 5$ for $v \in \{16,25,28\}$. Propositions 2.2 and 2.3 imply that the unique S(2,4,13) admits only bicolourings of type (6,7), (1,3,9), (2,2,9) and (1,6,6). Similarly, the unique S(2,4,16) admits only bicolourings of type (8,8) and (1,6,9).

There are 18 nonisomorphic systems S(2,4,25) ([3,9]). Only two of these admit a 2-bicolouring which is then, by a result of [5], necessarily of type (12,13). The remaining 16 systems do not admit a 2-colouring. All 18 Steiner systems S(2,4,25) admit a 3-bicolouring of type (1,12,12). Systems No. 2, No. 6 and No. 8 in [3] or [6] (with automorphism group of order 63, 150, and 6, respectively) admit also a 3-bicolouring of type (1,9,15). Whether there exists a bicolouring of type (2,11,12) (the only remaining possibility) is an open problem. By Corollary 2.1 and Proposition 2.2, an S(2,4,25) has no 4-bicolouring.

There exist exactly 4466 nonisomorphic systems S(2,4,28) with nontrivial automorphism group [4]. Several of these admit a bicolouring of type (14,14). Whether a 2-bicolouring of type (13, 15) is possible is an open problem [5].

To show that there exists an S(2, 4, 28) with a 3-bicolouring, consider the following system (cf. [4]). The set of elements is $\{0,1,\dots,27\}$, and the set of blocks is obtained by developing the base blocks $\{0,1,7,9\}$, $\{0,2,12,14\}$, $\{0,3,18,21\}$, $\{0,11,26,27\}$, $\{0,16,17,23\}$, $\{0,20,22,24\}$, $\{7,8,21,25\}$, $\{7,10,15,20\}$, $\{7,14,18,26\}$ under the automorphism $(0\ 1\ \dots 6)\ (7\ 8 \dots 13)\ (14\ 15\ \dots 20)\ (21\ 22\ \dots 27)$ of order 7. It is straightforward to verify that $\{0\}$, $\{4,6,8,11,13,15,20,22,24,25,26,27\}$ and $\{1,2,3,5,7,9,10,12,14,16,17,18,19,21,23\}$ are colour classes of a 3-bicolouring of type $(1,12,15)$. No S(2,4,28) can have a 4-bicolouring by Corollary 2.1 and Proposition 2.2. □

4. Bicolourings for a class of $S(2,4,v)$

Theorem 4.1. *If an $S(2,4,v)$ is bicolourable with a colouring of type $(n_1,n_2,\dots,n_k,n_{k+1},\dots,n_h)$ and the sets $\bigcup_{j=1}^{k} X_j$, for $l \leqslant k \leqslant h$ with $l \geqslant 2$, induce a sequence of subsystems $S_k(2,4,s_k)$ of $S(2,4,v)$, then $n_i \geqslant 2^{i-1}$ for $1 \leqslant i \leqslant l$ and $n_{k+1} = 2s_k + 1$ for $l \leqslant k \leqslant h$, and moreover, all the C-coloured blocks belong to the subsystem $S_l(2,4,s_l)$.*

Proof. For the induced bicolouring on $S_l(2,4,s_l)$ of type $(n_1,n_2,\dots,n_l)$, by Proposition 2.2 we have $n_i \geqslant 2^{i-1}$ for $1 \leqslant i \leqslant l$. Let x be an element of $S_k(2,4,s_k)$, $l \leqslant k < h$. It follows that $x \in X_i$, with $1 \leqslant i \leqslant k$. Let y be an element of X_{k+1}; the pair $\{x,y\}$ is contained in a B-coloured block because all pairs of elements coloured with the colour i belong to blocks of $S_k(2,4,s_k)$. By the arbitrary choice of x and by the fact that any pair of elements of X_{k+1} belongs to a block of $S_{k+1}(2,4,s_{k+1})$, we have that all elements of $S_k(2,4,s_k)$ induce a resolvable STS$(2s_k+1)$ with s_k parallel classes on X_{k+1} and $n_{k+1} = 2s_k + 1$ for $l \leqslant k \leqslant h$. Clearly, all the blocks outside S_l are B-coloured. □

Corollary 4.1. *If an $S(2,4,v)$ is bicolourable with a colouring of type $(n_1,n_2,\dots,n_l,\dots,n_h)$ and the set $\bigcup_{j=1}^{l} X_j$ induces a subsystem $S_l(2,4,s_l)$ of $S(2,4,v)$, then $n_{l+1} \geqslant 2s_l + 1$.*

Theorem 4.1 is based on the "$3v+1$ construction" (cf. [8]) which allows to obtain an $(h+1)$-bicolourable $S(2,4,3v+1)$ if there exists an (h)-bicolourable $S(2,4,v)$. In particular, this construction allows one to construct h-bicolourable systems with $h \geqslant 3$.

It has already been observed in Section 2 that for every admissible order v, there exists a 2-bicolourable $S(2,4,v)$. Applying the "$3v+1$ construction" yields:

Theorem 4.2. *For every* $v \equiv 4,\ 13 \pmod{36}$, $v \geqslant 13$, *there exists a* 3-*bicolourable* $S(2,4,v)$. *For every* $v \equiv 13,\ 40 \pmod{108}$, $v \geqslant 40$, *there exist a* 4-*bicolourable* $S(2,4,v)$. *In general, for every* $h \geqslant 2$ *and for every*

$$v \equiv \frac{3^{h-1}-1}{2} \quad or \quad \frac{3^h-1}{2} \pmod{4 \cdot 3^{h-1}}, \qquad v \geqslant \frac{3^h-1}{2},$$

there exists an h-bicolourable $S(2,4,v)$, *and the number of such systems goes to infinity with* v.

Proof. Only the last statement requires proof; it follows easily from the fact that the number of nonisomorphic Kirkman triple systems of order v goes to infinity with v [10]. □

References

[1] F. Franek, T.S. Griggs, C.C. Lindner, A. Rosa, Completing the spectrum of 2-chromatic $S(2,4,v)$, Discrete Math. 247 (2002) 225–228.

[2] D. Hoffman, C. Lindner, K. Phelps, Blocking sets in designs with block size four, Europ. J. Combin. 1 (1990) 451–457.

[3] E.S. Kramer, S.S. Magliveras, R. Mathon, The Steiner systems $S(2,4,25)$ with nontrivial automorphism group, Discrete Math. 77 (1989) 137–157.

[4] V. Krčadinac, Steiner 2-designs $S(2,4,28)$ with nontrivial automorphisms group, Glasnik Mat. 37 (57) (2002) 259–268.

[5] G. Lo Faro, Sui blocking sets nei sistemi di Steiner, Boll. U.M.I. (7) 4-A (1990) 71–76.

[6] R. Mathon, A. Rosa, 2-(v,k,λ) designs of small order, in: C.J. Colbourn, J.H. Dinitz (Eds.), CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, 1996, pp. 3–41.

[7] L. Milazzo, Zs. Tuza, Upper chromatic number of Steiner triple and quadruple systems, Discrete Math. 174 (1997) 247–259.

[8] S. Milici, A. Rosa, V. Voloshin, Colouring Steiner systems with specified block colours patterns, Discrete Math. 240 (2001) 145–160.

[9] E. Spence, The complete classification of Steiner systems $S(2,4,25)$, J. Combin. Designs 4 (1996) 295–300.

[10] D.R. Stinson, S.A. Vanstone, A note on nonisomorphic Kirkman triple systems, J. Combin. Inform. Syst. Sci. 9 (1984) 113–116.