An Equation Error Approach for the Identification of Elastic Parameters in Beams and Plates with H_1 Regularization

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Abstract. In this short note deals with the nonlinear inverse problem of identifying a variable parameter in fourth-order partial differential equations using an equation error approach. These equations arise in several important applications such as car windscreen modeling, deformation of plates, etc. To counter the highly ill-posed nature of the considered inverse problem, a regularization must be performed. The main contribution of this work is to show that the equation error approach permits the use of H^1 regularization whereas other optimization-based formulations commonly use H_2 regularization. We give the existence and convergence results for the equation error formulation. An illustrative numerical example is given to show the feasibility of the approach.

Keywords: Inverse problem · Equation error method · Fourth-order boundary value problem · Regularization · Parameter identification

1 Introduction

Let Ω be a bounded open domain in R^2 with a sufficiently smooth boundary Γ and let $f \in L^2(\Omega)$ be a given function. Consider the following fourth-order elliptic boundary value problem

$$\Delta(a\Delta u) = f \quad \text{in} \quad \Omega, \tag{1}$$

augmented with the clamped boundary conditions,

$$u = 0 \quad \text{on} \quad \Gamma, \tag{2a}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma. \tag{2b}$$

Dedicated to Prof. Alemdar Hasanoglu (Hasanov) on his 60th birthday

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In this work, our objective is to study the inverse problem of identifying the material parameter a from a measurement z of u. Applications of this study are in beam and plate models as well as car windshield modeling (see [16,17]). This nonlinear inverse problem has been explored using the output least squares (OLS) approach in which one attempts to find a minimizer of the functional

$$J(a) := \frac{1}{2}\|u(a) - z\|^2,$$

defined by using a suitable norm (see White [18]). Here z is the data (a measurement of u) and $u(a)$ is the unique solution of (1) that corresponds to the material parameter a,

One of the primary obstacles in a satisfactory treatment of the OLS-based optimization framework is due to the fact that the OLS, in general, is nonconvex. Our objective then is to investigate an equation error approach for solving the nonlinear inverse problem of identifying the material parameter a. In contrast to the OLS based optimization approach, the equation error approach results in solving a convex optimization problem. Some recent developments in parameter identification problems can be found in [2,4–7,7–10,13,14] and the cited references therein. Very interesting study of an identification problem in more general plate models can be found in Hasanov and Mamedov [12], see also [11].

We emphasize that the equation error approach has two advantages over the OLS approach. Firstly, it leads to a convex optimization problem and hence it only possesses global solutions. Secondly, the equation approach is computationally quite inexpensive as there is no underlying variational problem to be solved. On the other hand, a deficiency of the approach is that, due to the fact that it relies on differentiating the data, it is quite sensitive to data contamination.

The equation error approach has been studied in the context of the following simpler second-order elliptic boundary valued problem:

$$-\nabla \cdot (a\nabla u) = f \quad \text{in} \quad \Omega, \tag{3a}$$

$$u = 0 \quad \text{on} \quad \Gamma. \tag{3b}$$

For (3), the equation error approach consists of finding a minimizer of the functional

$$a \to \frac{1}{2}\|\nabla \cdot (a\nabla z) + f\|^2_{H^{-1}(\Omega)},$$

where $H^{-1}(\Omega)$ is the topological dual of $H_0^1(\Omega)$ and z is again the measured data.

In this paper, we extend the equation error approach to identify the coefficient a in the fourth-order boundary value problem (1). Our strategy is motivated by the ideas presented originally by Acar [1] and Kärkkäinen [15] for (3) (see also [3]). Besides giving an existence theorem and a convergence result for the discretized problem, we also some numerical examples.

This paper is divided into four main sections. Section 2 provides essential background material for the problem and poses the solution of the inverse problem as a solvable minimization problem. Section 3 examines the stability of the equation error method and Sect. 4 provides a brief numerical example to show the preliminary computational feasibility of the proposed method.

2 Equation Error Approach

The variational formulation of (1) will be instrumental in formulating the equation error approach. The space suitable for the variational formulation is given by

$$V := \{v \in H^2(\Omega) : \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma\}.$$

By multiplying (1) by a test function $v \in V$ and repeatedly using the well-known Green's formula we obtain the following variational formulation of (1): Find $u \in V$ such that

$$\int_\Omega a\Delta u \, \Delta v = \int_\Omega f v, \quad \text{for every } v \in V. \tag{4}$$

For a fixed pair $(a, w) \in L^\infty(\Omega) \times V$, we define the maps $E(a, w) : V \to V^*$ and $m : V \to R$ by

$$E(a, w)(v) = \int_\Omega a\Delta w \, \Delta v,$$
$$m(v) = \int_\Omega f v.$$

We note that, although the functional $E(a, w)$ was defined for fixed $a \in L^\infty(\Omega)$, $w \in V$, it remains well-defined for $a \in L^2(\Omega)$ and $w \in V \cap W^{2,\infty} := V^\infty$. In other words, we can sacrifice some regularity in a by requiring more regularity of u. This fact will play an important role below.

We first prove the following technical result for later use.

Lemma 1. *Assume that $u \in V^\infty$, $a \in L^2(\Omega)$, and $\{a_n\} \subset L^2(\Omega)$ is a sequence such that $a_n \to a$ in $L^2(\Omega)$. Then $E(a_n, u) \to E(a, u)$ in V^*.*

Proof. We begin by showing that the following inequality holds:

$$\|E(a, u)\|_{V^*} \leq \|a\|_{L^2} \|u\|_{V^\infty}. \tag{5}$$

In fact, using the definition of E, we have

$$|E(a, u)(v)| \leq \left| \int_\Omega a\Delta u \, \Delta v \right| \leq \|a\Delta u\|_{L^2} \|\Delta v\|_{L^2},$$

where

$$\|a\Delta u\|_{L^2}^2 = \int_\Omega a^2 \, (\Delta u)^2 \leq \|u\|_{V^\infty}^2 \|a\|_{L^2}^2,$$

and because $\|\Delta v\|_{L^2} \leq \|v\|_V$, we at once obtain (5).

To prove the main argument, we note that

$$(E(a_n, u) - E(a, u))(v) = \int_\Omega a_n \Delta u \Delta v - \int_\Omega a \Delta u \Delta v = \int_\Omega (a_n - a) \Delta u \Delta v,$$

which by using (5) implies that

$$|(E(a_n, u) - E(a, u))(v)| \leq \|u\|_{V^\infty} \|a_n - a\|_{L^2} \|v\|_V,$$

and consequently $\|E(a_n, u) - E(a, u)\|_{V^*} \leq \|u\|_{V^\infty} \|a_n - a\|_{L^2}$. The proof is complete. □

Since the inverse problem at hand is ill-posed, some regularization is necessary. For this, we first define $A \subset H^1(\Omega)$ to be the closed and convex set of admissible coefficients. We consider the following regularized equation error functional to estimate a^* from a measurement z of u^* by minimizing

$$J(a; z, \varepsilon) = \|E(a, z) - m\|_{V^*}^2 + \varepsilon \|a\|_{H^1}^2. \tag{6}$$

Here it is assumed that $a^* \in A$ and $u^* \in V$ satisfy (1), $\varepsilon > 0$ is a regularizing parameter, $z \in V$ is the data, and $\|\cdot\|_2^2$ is the regularization term.

Assuming that the data z is sufficiently smooth, we show that $J(\cdot; z, \varepsilon)$ has a unique minimizer in $H^1(\Omega)$ for each $\varepsilon > 0$.

Theorem 1. *Suppose $z \in W^\infty$. Then, for each $\varepsilon > 0$, there exists a unique a_ε satisfying*

$$J(a_\varepsilon; z, \varepsilon) \leq J(a; z, \varepsilon), \ \textit{for all} \ \ a \in H^1(\Omega).$$

Proof. Since the functional J is bounded below, there exists a minimizing sequence $\{a_n\}$ for J. We have $\varepsilon \|a_n\|_{H^1}^2 \leq J(a_n; z, \varepsilon)$ for all n which implies that $\{a_n\}$ is bounded in $H^1(\Omega)$. Therefore, there exists $a_\varepsilon \in H^1(\Omega)$ and a subsequence of $\{a_n\}$ (still denoted by $\{a_n\}$) such that $a_n \to a_\varepsilon$ weakly in $H^1(\Omega)$ and, by Rellich's theorem, strongly in $L^2(\Omega)$. Since $z \in V^\infty$ and $a_n \to a_\varepsilon$ in $L^2(\Omega)$, Lemma 1 confirms that $E(a_n, z) \to E(a_\varepsilon, z)$ and since the norm is weakly lower semicontinuous, it follows that

$$\begin{aligned}
\inf_{a \in H^1(\Omega)} J(a; z, \varepsilon) &= \lim_{n \to \infty} J(a_n; z, \varepsilon) \\
&= \lim_{n \to \infty} \left(\|E(a_n, z) - m\|_{V^*}^2 + \varepsilon \|a_n\|_{H^1}^2 \right) \\
&\geq \|E(a_\varepsilon, z) - m\|_{V^*}^2 + \varepsilon \|a_\varepsilon\|_{H^1}^2 \\
&= J(a_\varepsilon; z, \varepsilon),
\end{aligned}$$

confirming that a_ε is a minimizer of $J(\cdot; z, \varepsilon)$. The uniqueness of a_ε follows from the fact that the regularized equation error functional is strictly convex. The proof is complete. □

Since $J(a_\varepsilon; z, \varepsilon) \geq \inf_{a \in H^1(\Omega)} J(a; z, \varepsilon)$, the last inequality in the above proof must actually hold as an equality and hence $\lim_{n \to \infty} \|a_n\|_{H^1} = \|a_\varepsilon\|_{H^1}$ must remain valid. This, in view of the fact $a_n \to a_\varepsilon$ weakly in $H^1(\Omega)$, ensures that $\{a_n\}$ actually converges to a_ε strongly in $H^1(\Omega)$. Consequently any minimizing sequence of $J(\cdot; z, \varepsilon)$ converges in $H^1(\Omega)$ to the unique minimizer a_ε of $J(\cdot; z, \varepsilon)$.

3 Stability of the Equation Error Method

Recall that $a^* \in A$ and $u^* \in V$ are assumed to satisfy (1). However, since a^* is not unique, we define the convex set $S = \{a \in H^1(\Omega) \,:\, E(a, u^*) = m\}$.

We can now prove the following stability result for the equation error approach.

Theorem 2. *Assume that $u^* \in V^\infty$ and $a^* \in H^1(\Omega)$ satisfy (1). Let $\{z_n\} \subset V^\infty$ be a sequence of observations of u^* that satisfy, with the sequences $\{\delta_n\}$, $\{\varepsilon_n\}$, the conditions*

1. *$\delta_n^2 \le \varepsilon_n \le \delta_n$ for all $n \in \mathbb{Z}^+$;*
2. *$\delta_n^2/\varepsilon_n \to 0$ as $n \to \infty$;*
3. *$\|z_n - u^*\|_{V^\infty} \le \delta_n$ for all $n \in \mathbb{Z}^+$;*
4. *$\delta_n \to 0$ as $n \to \infty$.*

For each $n \in \mathbb{Z}^+$, let a_n be the unique solution of

$$\min_{a \in H^1(\Omega)} J(a; z_n, \varepsilon_n).$$

Then, there exists $\tilde{a} \in S$ such that $a_n \to \tilde{a}$ in $H^1(\Omega)$. Moreover, a satisfies $\|\tilde{a}\|_{H^1} \le \|a\|_{H^1}$, for all $a \in S$.

Proof. Let $a \in S$ be arbitrary. Then,

$$\begin{aligned}\varepsilon_n \|a_n\|_{H^1}^2 &\le \|E(a, z_n) - m\|_{V^*}^2 + \varepsilon_n \|a\|_{H^1}^2 \\ &= \|E(a, z_n - u^*)\|_{V^*}^2 + \varepsilon_n \|a\|_{H^1}^2 \\ &\le c\|a\|_{L^2}^2 \|z_n - u^*\|_{V^\infty}^2 + \varepsilon_n \|a\|_{H^1}^2,\end{aligned}$$

implying that

$$\|a_n\|_{H^1}^2 \le \|a\|_{L^2}^2 \frac{\delta_n^2}{\varepsilon_n} + \|a\|_{H^1}^2, \tag{7}$$

and, in particular,

$$\|a_n\|_{H^1}^2 \le \|a^*\|_{L^2}^2 \frac{\delta_n^2}{\varepsilon_n} + \|a^*\|_{H^1}^2 \le \|a^*\|_{L^2}^2 + \|a^*\|_{H^1}^2,$$

where we used the assumption $\delta_n^2 \le \varepsilon_n$. This proves that $\{a_n\}$ is bounded in $H^1(\Omega)$. Hence, by Rellich's lemma, there exists $\tilde{a} \in H^1(\Omega)$ and a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to \tilde{a}$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$.

We claim that $\tilde{a} \in S$. Indeed, for any $\hat{a} \in S$, we have

$$\begin{aligned}\|E(a_{n_k}, u^*) - m\|_{V^*}^2 &= \|E(a_{n_k}, u^*) - E(a_{n_k}, z_{n_k}) + E(a_{n_k}, z_{n_k}) - m\|_{V^*}^2 \\ &\le 2\|E(a_{n_k}, u^* - z_{n_k})\|_{V^*}^2 + 2\|E(a_{n_k}, z_{n_k}) - m\|_{V^*}^2 \\ &\le 2\|a_{n_k}\|_{L^2}^2 \|z_{n_k} - u^*\|_{V^\infty}^2 + 2\|E(\hat{a}, z_{n_k}) - m\|_{V^*}^2 + 2\varepsilon_{n_k}\|\hat{a}\|_{H^1}^2 \\ &\le 2\|a_{n_k}\|_{L^2}^2 \delta_{n_k}^2 + 2\|\hat{a}\|_{L^2}^2 \delta_{n_k}^2 + 2\varepsilon_{n_k}\|\hat{a}\|_{H^1}^2 \\ &\le 2\|a_{n_k}\|_{L^2}^2 \delta_{n_k}^2 + 4\|\hat{a}\|_{H^1}^2 \delta_{n_k},\end{aligned}$$

where we used $\delta_{n_k}^2 \leq \varepsilon_{n_k} \leq \delta_{n_k}$ and the following inequality which remains true for any $\hat{a} \in S$:

$$\|E(\hat{a}, z_{n_k}) - m\|_{V^*}^2 + \varepsilon_{n_k}\|\hat{a}\|_{H^1}^2 \leq \|\hat{a}\|_{L^2}^2 \delta_{n_k}^2 + \varepsilon_{n_k}\|\hat{a}\|_{H^1}^2.$$

Because $\{\|a_{n_k}\|_{L^2}\}$ is bounded and $\delta_{n_k} \to 0$ as $k \to \infty$, this ensures that $\|E(a_{n_k}, u^*) - m\|_{V^*} \to 0$. Since we also have $E(a_{n_k}, u^*) \to E(\tilde{a}, u^*)$ by Lemma 1, this shows that $E(\tilde{a}, u^*) = m$ and hence that $\tilde{a} \in S$.

Using the fact that $a_{n_k} \to \tilde{a}$ weakly in $H^1(\Omega)$, we have $\|\tilde{a}\|_{H^1} \leq \liminf_{k\to\infty} \|a_{n_k}\|_{H^1}$. Moreover, by (7),

$$\varepsilon_{n_k}\|a_{n_k}\|_{H^1}^2 \leq \|\tilde{a}\|_{L^2}^2 \delta_{n_k}^2 + \varepsilon_{n_k}\|\tilde{a}\|_{H^1}^2,$$

which implies that

$$\|a_{n_k}\|_{H^1}^2 \leq \|\tilde{a}\|_{L^2}^2 \frac{\delta_{n_k}^2}{\varepsilon_{n_k}} + \|\tilde{a}\|_{H^1}^2.$$

Since $\delta_{n_k}^2/\varepsilon_{n_k} \to 0$ as $k \to \infty$, this shows that $\limsup_{k\to\infty} \|a_{n_k}\|_{H^1} \leq \|\tilde{a}\|_{H^1}$. Therefore,

$$\|\tilde{a}\|_{H^1} \leq \liminf_{k\to\infty} \|a_{n_k}\|_{H^1} \leq \limsup_{k\to\infty} \|a_n\|_{H^1} \leq \|\tilde{a}\|_{H^1},$$

which shows that $\|a_{n_k}\|_{H^1} \to \|\tilde{a}\|_{H^1}$, and hence that $a_{n_k} \to a$ strongly in $H^1(\Omega)$ as $k \to \infty$.

Using (7),

$$\|\tilde{a}\|_{H^1}^2 \leq \lim_{k\to\infty} \|a_{n_k}\|_{H^1}^2 \leq \lim_{k\to\infty} \left(\|a\|_{L^2}^2 \frac{\delta_{n_k}^2}{\varepsilon_{n_k}} + \|a\|_{H^1}^2 \right) = \|a\|_{H^1}^2$$

holds for every $a \in S$.

Finally, since the set S is a convex, there is a unique minimal H^1-norm element, and we have shown that every convergent subsequence of $\{a_n\}$ converges to this unique element $\tilde{a}$. Thus the whole sequence $\{a_n\}$ must converge to $\tilde{a}$. This completes the proof. □

4 Numerical Results

To test the preliminary effectiveness of the equation error approach for this inverse problem, we consider an example boundary value problem derived from (1):

$$\begin{aligned} \Delta\left[a(x,y)\Delta u(x,y)\right] &= f(x,y) && \text{in } \Omega \\ u(x,y) = \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma \end{aligned} \tag{8}$$

where the solution u and parameter a are defined as

$$\begin{aligned} u(x,y) &= 16x^2(1-x)^2y^2(1-y)^2, \\ a(x,y) &= 4 + \sin(2\pi x)\sin(3\pi y). \end{aligned}$$

For means of this numerical experiment, we take $f(x, y)$ as subsequently defined by (8). The domain Ω is taken as the unit square, $\Omega = (0, 1) \times (0, 1)$ with the boundary Γ as the square's outside edges.

Discretization of the solution was performed using cubic Hermite finite elements on a 20×20 mesh consisting of 882 triangles and 2,048 degrees of freedom.

The discretized optimization problem was solved using a conjugate-gradient trust-region method (`cgtrust`) with a stopping criteria on $\|\nabla J\|$ of 10^{-12}. Using a value of $\varepsilon = 10^{-6}$ for the regularization parameter with the H^1-norm, `cgtrust` converged in 38 iterations. The computed solution at several iterations of the algorithm along with the output of the optimization are summarized in Fig. 1. We note that this method provides a good reconstruction of the parameter in the interior of Ω with reconstruction error concentrated mostly along the boundaries.

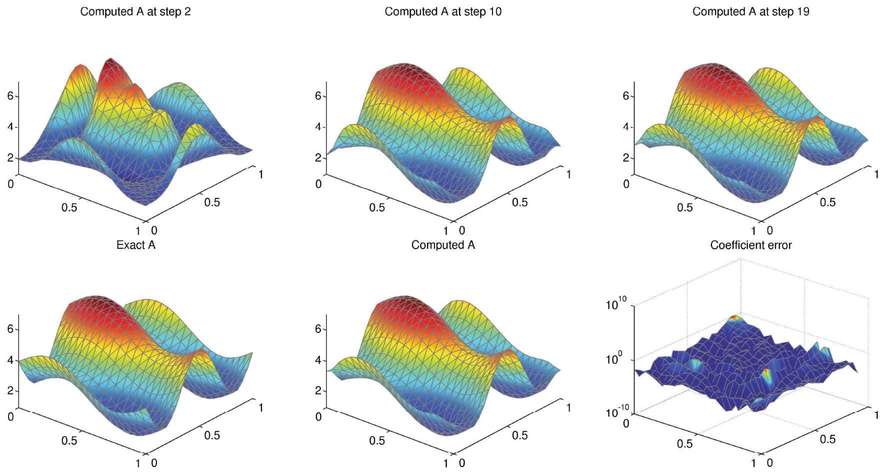

Fig. 1. Parameter recovery using the EE method and `cgtrust` with $\varepsilon = 10^{-6}$.

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