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Decision algorithms for fragments of real analysis. I. Continuous functions with strict convexity and concavity predicates[☆]

Domenico Cantone[*], Gianluca Cincotti, Giovanni Gallo

Department of Mathematics and Computer Science, University of Catania, Viale Andrea Doria 6, 95125 Catania, Italy

Abstract

In this paper we address the decision problem for a fragment of unquantified formulae of real analysis, which, besides the operators of Tarski's theory of reals, includes also strict and non-strict predicates expressing comparison, monotonicity, concavity, and convexity of continuous real functions over possibly unbounded intervals.

The decision result is obtained by proving that a formula of our fragment is satisfiable if and only if it admits a parametric "canonical" model, whose existence can be tested by solving a suitable unquantified formula, expressed in the decidable language of Tarski's theory of reals and involving the numerical variables of the initial formula plus various other parameters.

This paper generalizes a previous decidability result concerning a more restrictive fragment in which predicates relative to infinite intervals or stating strict concavity and convexity were not expressible.
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1. Introduction

Formalization of mathematics in computerized environments has received increasing attention in the last few years, in part under the impulse of applications in program and hardware

verification. This, in particular, is the case for the real numbers and real analysis, due to their important applications in the verification of floating point hardware and hybrid systems. In this connection, among others, we cite the work on theorem proving with the real numbers using a version of the HOL theorem prover (Harrison, 1998), the mechanization of real analysis in Isabelle/HOL (Fleuriot, 2000), in PVS (Gottliebsen, 2001), and with the interactive proof system IMPS (Guttman and Thayer, 1993), the ongoing efforts with the Mizar system (Bonarska, 1990; Muzalewski, 1993), the attempt to formalize Cauchy's integral theorem and integral formula in the EtnaNova proof-verifier (Schwartz et al., in preparation; Cantone et al., 2003; Omodeo and Schwartz, 2002), the extension of symbolic computation tools with some theorem proving capabilities (see, for instance, Analytica (Clarke and Zhao, 1992; Bauer et al., 1998) and Mathpert (Beeson, 1990)), and so on.

To keep within reasonable limits the amount of details that a user must provide to a verification system in proof sessions, it is necessary that the verifier has a rich endowment of decision procedures, capable of formalizing "obvious" deduction steps. Thus, a proof verifier for real analysis should include in its inferential kernel a decision procedure for Tarski's elementary theory of reals (Tarski, 1951) as well as efficient decision tests for more specialized subtheories such as the existential theory of reals (Heintz et al., 1993), the theory of bounded and stable quantified constraints (Ratschan, in press), and other even more specific classes of constraints.

In some situations, one may also need to reason about real functions, represented in the language as interpreted or uninterpreted function symbols.[1] However, one must be aware that the existential theory of reals extended with the interpreted symbols $\log 2$, π, e^x, and $\sin x$ is undecidable (Richardson, 1968). On the other hand, it has been shown in Macintyre and Wilkie (1996) that the first-order theory of the real numbers extended with the exponential function e^x is decidable, provided that Schanuel's conjecture in transcendental number theory holds (Chudnovsky, 1984, Chapter 3, pp. 145–176).

The existential theory of reals has been extended in Cantone et al. (1987) with uninterpreted continuous function symbols, function sum, function point evaluation, and with predicates expressing comparison, monotonicity (strict and non-strict) and non-strict convexity of functions. More precisely, the language considered there, denoted as RMCF (theory of Reals with Monotone and Convex Functions), consists of the propositional combinations of atoms of the following form:

$$
\begin{array}{ll}
t_1 = t_2, & t_1 > t_2, \\
\mathcal{F}_1 = \mathcal{F}_2, & \mathcal{F}_1 > \mathcal{F}_2, \\
\mathrm{Up}(\mathcal{F})_{[t_1,t_2]}, & \mathrm{Strict\_Up}(\mathcal{F})_{[t_1,t_2]}, \\
\mathrm{Down}(\mathcal{F})_{[t_1,t_2]}, & \mathrm{Strict\_Down}(\mathcal{F})_{[t_1,t_2]}, \\
\mathrm{Convex}(\mathcal{F})_{[t_1,t_2]}, & \mathrm{Concave}(\mathcal{F})_{[t_1,t_2]},
\end{array}
$$

where the t's denote numerical expressions (built up from real constants and variables by means of the standard arithmetic operators) and the $\mathcal{F}$'s denote functional expressions (built up from function symbols by means of the additive arithmetic operator, where function symbols are supposed to range over continuous real functions). In particular, functional predicates of the form $\mathcal{F}_1 = \mathcal{F}_2$ and $\mathcal{F}_1 > \mathcal{F}_2$ refer to the whole real axis, whereas the remaining functional predicates are restricted to given bounded closed intervals.

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[1] Interpreted function symbols have a predefined interpretation (e.g., the exponential and the sine functions, e^x and $\sin x$, respectively), whereas uninterpreted function symbols have no predefined meaning attached to them and therefore they can be interpreted freely (e.g. the "generic" function symbols f and g).

In this paper a more expressive extension of RMCF is considered and proved to be decidable. More specifically, the extended theory, denoted by RMCF$^+$, includes the same predicates and constructs as RMCF plus the following new predicates:

$$\text{Strict\_Convex}(\mathcal{F})_{[t_1, t_2]}, \qquad \text{Strict\_Concave}(\mathcal{F})_{[t_1, t_2]},$$

with the obvious intended meaning. Also, in RMCF$^+$, most of the predicates can be restricted to either bounded or unbounded closed intervals. The only exception is the predicate $\mathcal{F}_1 > \mathcal{F}_2$, which, for technical reasons, can only be restricted to bounded intervals.

Our decidability result will be obtained by exhibiting a chain of four effective and satisfiability preserving reduction steps which, starting from an initial formula φ of RMCF$^+$, produces at the end another formula $\overline{\varphi}$ with no function symbols, expressed in the unquantified language of Tarski's theory of reals and involving the numerical variables of φ plus various other parameters. Since φ and $\overline{\varphi}$ are equisatisfiable, it is possible to establish whether φ is satisfiable by testing for satisfiability the (syntactically) simpler formula $\overline{\varphi}$, using any decision procedure for Tarski's existential theory of reals (cf. Heintz et al., 1993).

To be a little bit more specific, our result is based on the fact that an RMCF$^+$-formula φ is satisfiable if and only if it admits a parametric "canonical" model, which can be built up by suitably enriching any real model of the associated function-free formula $\overline{\varphi}$. As we will see, canonical models map function symbols into piecewise linear functions, perturbed by quadratic or exponential functions. The technique of using piecewise linear functions to decide the satisfiability problem for a fragment of real analysis was originally introduced in Cantone et al. (1987). Here we extend it by allowing perturbations with quadratic and exponential functions, so as to be able to decide also formulae involving Strict_Convex and Strict_Concave literals, over any interval.

Our work is somewhat related to the quite systematic study in Friedman and Seress (1989, 1990) of the decision problem in elementary analysis for sentences of the form $(\forall f \in \mathfrak{F})\varphi$, where $\mathfrak{F}$ is a family of functions from $\mathbb{R}$ into $\mathbb{R}$, or from $I = [0, 1]$ into I, and φ is a sentence involving the predicate symbols $>, <, =$ on $\mathbb{R}$ (or I), and the unary function symbol f. Depending on the family $\mathfrak{F}$ and the form of the sentence φ, various decidability and undecidability results are provided in Friedman and Seress (1989, 1990). In particular, when $\mathfrak{F}$ is the family of continuous, or differentiable, or infinitely many times differentiable, or analytic functions, and φ is a Σ_1- or Π_1-sentence,[2] then $(\forall f \in \mathfrak{F})\varphi$ is decidable. In addition, if $\mathfrak{F}$ is the set of continuous and strictly monotone increasing (decreasing) functions, then $(\forall f \in \mathfrak{F})\varphi$ is decidable for any first-order sentence φ involving the unary function symbol f.

In particular, our result is comparable with the class of formulae $(\forall f \in \mathfrak{F})\varphi$ studied in Friedman and Seress (1989) in the case in which φ is a Π_1-sentence and $\mathfrak{F}$ is the family of continuous functions on $\mathbb{R}$. Such a class is much less expressive than the ones studied in Cantone et al. (1987) and in the present paper. This is immediately evident on observing that its decision problem can readily be stated as the satisfiability problem for propositional combinations of atoms of the following simple forms:

$$x_1 = x_2, \quad x_1 > x_2, \quad f(x_1) = x_2,$$

where x_1, x_2 denote numerical variables and f is the *only* function symbol allowed.

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[2] We recall that Σ_1- and Π_1-sentences have respectively the forms $(\exists x_1) \cdots (\exists x_k)\varphi_0$ and $(\forall x_1) \cdots (\forall x_k)\varphi_0$, where φ_0 is quantifier-free.

The paper is organized as follows. In Section 2 we give the precise syntax and semantics of the theory RMCF$^+$. A satisfiability test for RMCF$^+$ is presented in Section 3 and its soundness is proved in Section 4. Finally, in Section 5 we draw our conclusions and hint at some possible extensions. An appendix on convex functions concludes the paper.

2. The language RMCF$^+$: Syntax and semantics

In this section we introduce the language RMCF$^+$ (augmented theory of Reals with Monotone and Convex Functions) and define its intended semantics.

2.1. Syntax

The language RMCF$^+$ has two types of symbols, namely *numerical variables*, denoted by x, y, …, and unary *function symbols*, denoted by f, g, …. Numerical variables are supposed to range over the set $\mathbb{R}$ of real numbers, whereas function symbols range over continuous real functions on $\mathbb{R}$.

In addition, RMCF$^+$ includes the interpreted *numerical constants* 0 and 1 and the interpreted function symbols **0** and **1**.

The language also includes two distinguished symbols, $-\infty$, $+\infty$, which are restricted to occur only within "range defining" parameters, as stated in the following definitions.

Definition 1. NUMERICAL TERMS are defined recursively as follows:

– every numerical variable x, y, … or constant 0,1 is a numerical term;
– if t_1, t_2 are numerical terms, then so are $(t_1 + t_2)$, $(t_1 - t_2)$, $(t_1 \cdot t_2)$, and (t_1/t_2);
– if t is a numerical term and f is a function symbol, then $f(t)$ is a numerical term.

An EXTENDED NUMERICAL VARIABLE (resp. TERM) is a numerical variable (resp. term) or one of the symbols $-\infty$ and $+\infty$.

FUNCTIONAL TERMS are defined recursively as follows:

– every uninterpreted function symbol f, g, … or interpreted function symbol **0** and **1** is a term;
– if $\mathcal{F}_1$, $\mathcal{F}_2$ are functional terms, then so are $(\mathcal{F}_1 + \mathcal{F}_2)$ and $(\mathcal{F}_1 - \mathcal{F}_2)$.

Definition 2. An ATOMIC FORMULA or ATOM of RMCF$^+$ is an expression having one of the following forms:

$$
\begin{array}{ll}
t_1 = t_2, & t_1 > t_2 \\
(\mathcal{F}_1 = \mathcal{F}_2)_{[\mathcal{T}_1, \mathcal{T}_2]}, & (\mathcal{F}_1 > \mathcal{F}_2)_{[t_1, t_2]} \\
\mathrm{Up}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]}, & \mathrm{Strict\_Up}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]}, \\
\mathrm{Down}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]}, & \mathrm{Strict\_Down}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]}, \\
\mathrm{Convex}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]}, & \mathrm{Strict\_Convex}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]}, \\
\mathrm{Concave}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]}, & \mathrm{Strict\_Concave}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]},
\end{array}
$$

where t_1, t_2 stand for numerical terms, $\mathcal{F}_1$, $\mathcal{F}_2$ stand for functional terms, and $\mathcal{T}_1$, $\mathcal{T}_2$ stand for extended numerical terms such that $\mathcal{T}_1 \neq +\infty$ and $\mathcal{T}_2 \neq -\infty$.

We will freely write $t_1 \leq t_2$ (resp. $t_1 \geq t_2$) as a shorthand for $t_1 < t_2 \vee t_1 = t_2$ (resp. $t_1 > t_2 \vee t_1 = t_2$).

Definition 3. A FORMULA of RMCF$^+$ is any propositional combination of atoms by means of logical connectives such as $\wedge$, $\vee$, $\neg$, $\longrightarrow$, and so on.

Remark 1. Notice that explicit quantification, either existential or universal, is not allowed in the language RMCF^+, and thus first-order logic over real closed fields is not captured by the fragment RMCF^+. □

For any RMCF^+-formula φ, we denote by $Num(\varphi)$ and $Fun(\varphi)$ the collections of numerical and function symbols occurring in φ, respectively.

2.2. Semantics

Now we define the intended semantics of RMCF^+.

Definition 4. A (REAL) ASSIGNMENT M for the language RMCF^+ is a map defined over terms and formulae of RMCF^+ as follows:

Definition of M over RMCF^+-terms.

- $\mathsf{M}x \in \mathbb{R}$, for every numerical variable x;
- $\mathsf{M}0 = 0$, $\mathsf{M}1 = 1$, $\mathsf{M}(+\infty) = +\infty$, and $\mathsf{M}(-\infty) = -\infty$;
- $\mathsf{M}f$ is a continuous real function over $\mathbb{R}$;
- $\mathsf{M}\mathbf{0}$ and $\mathsf{M}\mathbf{1}$ are respectively the null function and the constant function of value 1, i.e. $(\mathsf{M}\mathbf{0})(r) = 0$ and $(\mathsf{M}\mathbf{1})(r) = 1$, for every $r \in \mathbb{R}$;
- $\mathsf{M}(t_1 \otimes t_2) = \mathsf{M}t_1 \otimes \mathsf{M}t_2$, for every composite numerical term $t_1 \otimes t_2$, where $\otimes \in \{+, -, \cdot, /\}$;
- $\mathsf{M}(f(t)) = (\mathsf{M}f)(\mathsf{M}t)$, for every function symbol f and numerical term t;
- $\mathsf{M}(\mathcal{F}_1 \oplus \mathcal{F}_2)$ is the real function $(\mathsf{M}\mathcal{F}_1) \oplus (\mathsf{M}\mathcal{F}_2)$, where $\oplus \in \{+, -\}$, i.e. $(\mathsf{M}(\mathcal{F}_1 \oplus \mathcal{F}_2))(r) = (\mathsf{M}\mathcal{F}_1)(r) \oplus (\mathsf{M}\mathcal{F}_2)(r)$, for every $r \in \mathbb{R}$.

Definition of M over RMCF^+-formulae.

(In the following, t_1, t_2 will stand for numerical terms, $\mathcal{T}_1, \mathcal{T}_2$ for extended numerical terms, and $\mathcal{F}_1, \mathcal{F}_2$ for functional terms.)

- $\mathsf{M}(t_1 = t_2) = \textbf{true}$, iff $\mathsf{M}t_1 = \mathsf{M}t_2$;
- $\mathsf{M}(t_1 > t_2) = \textbf{true}$, iff $\mathsf{M}t_1 > \mathsf{M}t_2$;
- $\mathsf{M}(\mathcal{F}_1 > \mathcal{F}_2)_{[t_1, t_2]} = \textbf{true}$, iff either $\mathsf{M}t_1 > \mathsf{M}t_2$, or $\mathsf{M}t_1 \leq \mathsf{M}t_2$ and $(\mathsf{M}\mathcal{F}_1)(r) > (\mathsf{M}\mathcal{F}_2)(r)$, for every $r \in [\mathsf{M}t_1, \mathsf{M}t_2]$;
- $\mathsf{M}(\mathcal{F}_1 = \mathcal{F}_2)_{[\mathcal{T}_1, \mathcal{T}_2]} = \textbf{true}$, iff either $\mathsf{M}\mathcal{T}_1 > \mathsf{M}\mathcal{T}_2$, or $\mathsf{M}\mathcal{T}_1 \leq \mathsf{M}\mathcal{T}_2$ and $(\mathsf{M}\mathcal{F}_1)(r) = (\mathsf{M}\mathcal{F}_2)(r)$ for every $r \in [\mathsf{M}\mathcal{T}_1, \mathsf{M}\mathcal{T}_2]^3$;
- $\mathsf{MUp}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]} = \textbf{true}$ (resp. $\mathsf{MStrict\_Up}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]} = \textbf{true}$), iff either $\mathsf{M}\mathcal{T}_1 \geq \mathsf{M}\mathcal{T}_2$, or $\mathsf{M}\mathcal{T}_1 < \mathsf{M}\mathcal{T}_2$ and the function $\mathsf{M}\mathcal{F}_1$ is monotone non-decreasing (resp. strictly increasing) in the interval $[\mathsf{M}\mathcal{T}_1, \mathsf{M}\mathcal{T}_2]$;
- $\mathsf{MDown}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]} = \textbf{true}$ (resp. $\mathsf{MStrict\_Down}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]} = \textbf{true}$), iff either $\mathsf{M}\mathcal{T}_1 \geq \mathsf{M}\mathcal{T}_2$, or $\mathsf{M}\mathcal{T}_1 < \mathsf{M}\mathcal{T}_2$ and the function $\mathsf{M}\mathcal{F}_1$ is monotone non-increasing (resp. strictly decreasing) in the interval $[\mathsf{M}\mathcal{T}_1, \mathsf{M}\mathcal{T}_2]$;
- $\mathsf{MConvex}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]} = \textbf{true}$ (resp. $\mathsf{MStrict\_Convex}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]} = \textbf{true}$), iff either $\mathsf{M}\mathcal{T}_1 \geq \mathsf{M}\mathcal{T}_2$, or $\mathsf{M}\mathcal{T}_1 < \mathsf{M}\mathcal{T}_2$ and the function $\mathsf{M}\mathcal{F}_1$ is convex (resp. strictly convex) in the interval $[\mathsf{M}\mathcal{T}_1, \mathsf{M}\mathcal{T}_2]$;

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[3] For simplicity, we are using the interval notation $[\alpha, \beta]$ even for the cases in which $\alpha = -\infty$ and/or $\beta = +\infty$. Furthermore, we are assuming that $-\infty < r < +\infty$, for every $r \in \mathbb{R}$.

– $\mathsf{MConcave}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]} = \textbf{true}$ (resp. $\mathsf{MStrict\_Concave}(\mathcal{F}_1)_{[\mathcal{T}_1, \mathcal{T}_2]} = \textbf{true}$), iff either $\mathsf{M}\mathcal{T}_1 \geq \mathsf{M}\mathcal{T}_2$, or $\mathsf{M}\mathcal{T}_1 < \mathsf{M}\mathcal{T}_2$ and the function $\mathsf{M}\mathcal{F}_1$ is concave (resp. strictly concave) in the interval $[\mathsf{M}\mathcal{T}_1, \mathsf{M}\mathcal{T}_2]$;

– logical connectives are interpreted in the standard way; thus, for instance, $\mathsf{M}(\varphi_1 \wedge \varphi_2) = \mathsf{M}\varphi_1 \wedge \mathsf{M}\varphi_2$, $\mathsf{M}(\varphi_1 \vee \varphi_2) = \mathsf{M}\varphi_1 \vee \mathsf{M}\varphi_2$, and so on, for all RMCF^+-formulae φ_1 and φ_2.

Let φ be an RMCF^+-formula and let M be a real assignment for the language RMCF^+. We say that M is a (REAL) MODEL for φ, provided that $\mathsf{M}\varphi = \textbf{true}$. If φ has a model, then it is RMCF^+-SATISFIABLE (or just SATISFIABLE), otherwise it is UNSATISFIABLE. If φ is true in every RMCF^+-assignment, then φ is an RMCF^+-THEOREM (or just a THEOREM).

Remark 2. Stated in a slightly more standard way, an RMCF^+-formula φ is RMCF^+-satisfiable if its existential closure $\varphi^\exists$ is satisfiable in the standard model of $\mathbb{R}$, whereas it is RMCF^+-true if its universal closure $\varphi^\forall$ is satisfiable in the standard model of $\mathbb{R}$. For the sake of completeness, we recall that given a formula φ involving the free variables $x_1, \ldots, x_n$, the formulae $\varphi^\exists$ and $\varphi^\forall$ are respectively defined as $(\exists x_1) \cdots (\exists x_n)\varphi$ and $(\forall x_1) \cdots (\forall x_n)\varphi$. $\quad\square$

Let φ_1 and φ_2 be RMCF^+-formulae. We say that φ_1 and φ_2 are EQUISATISFIABLE if either both of them are unsatisfiable, or both of them are satisfiable. In addition, we say that φ_1 and φ_2 are EQUIVALENT if they have the same RMCF^+-models.

Finally, the SATISFIABILITY PROBLEM for RMCF^+ (abbreviated s.p.) is the problem of determining whether any given RMCF^+-formula is satisfiable or not; likewise, the THEOREMHOOD PROBLEM for RMCF^+ is the problem of determining whether any given RMCF^+-formula is a theorem of RMCF^+ or not.

Plainly the s.p. and the theoremhood problem for RMCF^+ are equivalent. Indeed, a formula φ of RMCF^+ is satisfiable if and only if its negation $\neg\varphi$ (which is an RMCF^+-formula) is *not* a theorem of RMCF^+.

We will solve the s.p. for RMCF^+ by exhibiting an algorithmic test which not only recognizes the satisfiability of RMCF^+-formulae, but also produces descriptions of RMCF^+-models in the case of satisfiable RMCF^+-formulae.

2.3. A few remarks on the expressivity of the language RMCF^+

First of all, we remark that the choice of including in the language RMCF^+ only the interpreted numerical constants 0 and 1 and the functional constants $\textbf{0}$ and $\textbf{1}$ is somewhat arbitrary. Indeed, any rational numerical constant and any rational constant function can easily be expressed in RMCF^+.

Concerning integer numerical constants, let p be any integer. If $p > 0$ then p can be expressed by the numerical term $\underbrace{1 + \cdots + 1}_{p \text{ times}}$, whereas if $p < 0$ then it can be expressed by the term $0 - (\underbrace{1 + \cdots + 1}_{|p| \text{ times}})$. Plainly, every rational number p/q, where p and $q \neq 0$ are integers, is readily expressible by a numerical term of the language RMCF^+, since RMCF^+ allows the formation of the quotient of any two numerical terms.

Rational constant functions can be expressed as follows. Let $r = p/q$ be a rational number, where p and $q \neq 0$ are integers. Then, it is an easy matter to check that a function symbol f is constrained to be the constant function $\textbf{r}$ by the following RMCF^+-formula, which for later use we denote by $\mathsf{Is\_constant}(f, r)$:

$$\mathrm{Up}(f)_{[-\infty,+\infty]} \wedge \mathrm{Down}(f)_{[-\infty,+\infty]} \wedge f(x) = r,$$

where x is any numerical variable.

We also remark that we could have left out from the language RMCF^+ even the numerical constants 0, 1, and the functional constants **0**, **1**. Indeed, these are easily expressed as z_0, z_1, f_0, and f_1, respectively, by the following RMCF^+-conjunction:

$$z_0 = z_0 + z_0 \wedge z_1 = z_1 \cdot z_1 \wedge z_1 \neq z_0 \wedge \mathrm{Is\_constant}(f_0, z_0) \wedge \mathrm{Is\_constant}(f_1, z_1).$$

Concerning uninterpreted constants, numerical ones are readily available as numerical variables, whereas constant functions are expressed by RMCF^+-formulae of the form $\mathrm{Is\_constant}(f, z)$.

Next, we give a few examples of theorems which could be proved automatically by means of a decision test for RMCF^+.

Example 1. *A strictly convex curve and a concave curve defined over the same interval can meet in at most two points.*

A formalization of the above statement is the universal closure of the following formula:

$$\left(\mathrm{Strict\_Convex}(f)_{[\mathcal{T}_1,\mathcal{T}_2]} \wedge \mathrm{Concave}(g)_{[\mathcal{T}_1,\mathcal{T}_2]} \wedge \bigwedge_{i=1}^{3} f(x_i) = g(x_i) \right.$$

$$\left. \wedge \bigwedge_{i=1}^{3} \mathcal{T}_1 \leq x_i \leq \mathcal{T}_2 \right) \longrightarrow (x_1 = x_2 \vee x_1 = x_3 \vee x_2 = x_3),$$

whose theoremhood can be tested by showing that the following RMCF^+-formula is *unsatisfiable*:

$$\mathrm{Strict\_Convex}(f)_{[\mathcal{T}_1,\mathcal{T}_2]} \wedge \mathrm{Concave}(g)_{[\mathcal{T}_1,\mathcal{T}_2]} \wedge \bigwedge_{i=1}^{3} f(x_i) = g(x_i)$$

$$\wedge \bigwedge_{i=1}^{3} \mathcal{T}_1 \leq x_i \leq \mathcal{T}_2 \wedge x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3. \quad \square$$

A second example is:

Example 2. *If g is a linear function, then a function f defined over the same domain of g is strictly convex if and only if $f + g$ is strictly convex.*

Let $\mathrm{Linear}(f)_{[\mathcal{T}_1,\mathcal{T}_2]}$ be a new predicate standing for

$$\mathrm{Convex}(f)_{[\mathcal{T}_1,\mathcal{T}_2]} \wedge \mathrm{Concave}(f)_{[\mathcal{T}_1,\mathcal{T}_2]}.$$

Notice that if M is a real assignment for RMCF^+, then $\mathsf{M}\,\mathrm{Linear}(f)_{[\mathcal{T}_1,\mathcal{T}_2]} = \mathbf{true}$ if and only if the function $\mathsf{M}f$ is linear in the interval $[\mathsf{M}\mathcal{T}_1, \mathsf{M}\mathcal{T}_2]$.[4] Therefore the statement of Example 2 is formalized by the universal closure of the following formula:

$$\mathrm{Linear}(g)_{[\mathcal{T}_1,\mathcal{T}_2]} \longrightarrow \left(\mathrm{Strict\_Convex}(f)_{[\mathcal{T}_1,\mathcal{T}_2]} \longleftrightarrow \mathrm{Strict\_Convex}(f+g)_{[\mathcal{T}_1,\mathcal{T}_2]} \right),$$

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[4] Likewise, a constant function f can be characterized by the formula $\mathrm{Up}(f)_{[\mathcal{T}_1,\mathcal{T}_2]} \wedge \mathrm{Down}(f)_{[\mathcal{T}_1,\mathcal{T}_2]}$.

whose theoremhood is equivalent to the unsatisfiability of the following RMCF^+-formula:

$$\text{Linear}(g)_{[\mathcal{T}_1, \mathcal{T}_2]} \wedge \neg \big(\text{Strict\_Convex}(f)_{[\mathcal{T}_1, \mathcal{T}_2]} \longleftrightarrow \text{Strict\_Convex}(f + g)_{[\mathcal{T}_1, \mathcal{T}_2]} \big). \quad \square$$

Another more interesting example is the following:

Example 3. *Let f and g be two real functions which take the same values at the endpoints of a closed interval $[a, b]$. Let us also assume that f and g are respectively strictly convex and linear in $[a, b]$. Then $f(c) < g(c)$ holds at each internal point c of the interval $[a, b]$.*

A possible formalization of the above statement is given by the universal closure of the formula:

$$\left(\text{Strict\_Convex}(f)_{[x_1, x_2]} \wedge \text{Linear}(g)_{[x_1, x_2]} \wedge \bigwedge_{i=1}^{2} f(x_i) = g(x_i) \right)$$
$$\longrightarrow (\forall x)(x_1 < x < x_2 \longrightarrow f(x) < g(x)),$$

which, by way of straightforward quantifier manipulations, is logically equivalent to the universal closure of the formula

$$\left(\text{Strict\_Convex}(f)_{[x_1, x_2]} \wedge \text{Linear}(g)_{[x_1, x_2]} \wedge \bigwedge_{i=1}^{2} f(x_i) = g(x_i) \right)$$
$$\longrightarrow (x_1 < x < x_2 \longrightarrow f(x) < g(x)),$$

whose theoremhood is in turn equivalent to the unsatisfiability of the following RMCF^+-formula:

$$\text{Strict\_Convex}(f)_{[x_1, x_2]} \wedge \text{Linear}(g)_{[x_1, x_2]} \wedge \bigwedge_{i=1}^{2} f(x_i) = g(x_i)$$
$$\wedge \, x_1 < x \wedge x < x_2 \wedge \neg(f(x) < g(x)). \quad \square$$

We stress the fact that one cannot expect that any deep theorem of real analysis can be directly expressed in the language RMCF^+, and therefore automatically proved. Indeed, our result is to be regarded as just one more step towards the mechanization of the "obvious", which is basic for the realization of powerful interactive proof verifiers in which the user assumes control only for the more challenging deduction steps (such as the instantiation of quantified variables), otherwise leaving the burden of the verification of small details to the system.

The rest of the paper will be devoted to the presentation of a satisfiability test for RMCF^+.

3. A satisfiability test for RMCF^+

We shall prove our main decidability result via a series of satisfiability preserving reduction steps which will reduce the s.p. for RMCF^+ to the s.p. for Tarski's existential theory of reals.[5] From the decidability of Tarski's (existential) theory of reals (cf. Tarski (1951) and Heintz et al. (1993)), the decidability of RMCF^+ then follows immediately.

We use the following reduction steps:

(1) reduction to conjunctions of basic literals (first normal form);

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[5] We recall that Tarski's theory of reals is the collection of true sentences about real numbers in the first-order language with individual variables, the constants 0 and 1, addition, multiplication, and equality.

(2) negative and dual literals removal (second normal form);
(3) explicit evaluation of functions over domain variables and guessing of equalities and inequalities among domain variables[6] (third normal form);
(4) functional literals removal (fourth normal form).

The purpose of the first two reduction steps is to simplify RMCF^+-formulae from a pure syntactical point of view, by eliminating nested terms and reducing complex formulae to flat conjunctions (first normal form), and by eliminating negative literals as well as literals of type Down, Strict_Down, Concave, and Strict_Concave, which can readily be expressed by literals of the remaining types (second normal form).

Numerical variables, which occur as arguments of function symbols or as interval bounds, play an important role in the third normal form. Such variables are called *domain variables*. In the third normal form, all function symbols need to be evaluated over all domain variables and, additionally, all possible equalities and inequalities among domain variables need to be guessed (through a very large disjunction). As will be argued later, such a reduction step may be very expensive and therefore needs particular attention, to keep unnecessary combinatorial explosions under control.

Knowledge of all equalities and inequalities which hold among the domain variables is very important for the fourth reduction step, which eliminates from a given conjunction in third normal form all functional literals, namely the ones of type

$$y = f(x), \qquad (f = g + h)_{[z_1, z_2]}, \qquad (f > g)_{[w_1, w_2]}, \qquad \mathrm{Up}(f)_{[z_1, z_2]},$$

$$\mathrm{Strict\_Up}(f)_{[z_1, z_2]}, \qquad \mathrm{Convex}(f)_{[z_1, z_2]}, \qquad \mathrm{Strict\_Convex}(f)_{[z_1, z_2]},$$

without affecting satisfiability. All such literals are eliminated by expressing them by means of elementary Tarski's relationships among the functional images of the (finitely many) domain variables. Since such relationships can be tested for satisfiability by any decision procedure for Tarski's existential theory of reals, decidability of the theory RMCF^+ follows.

3.1. First reduction step: Normalization

The first reduction step eliminates nested terms and complex boolean combinations of literals in favor of conjunctions of literals. It is based on the following general normalization process (cf. Cantone et al. (1987)).

Let T be an unquantified first-order theory, with equality =, individual variables $x_1, x_2, \ldots$, function symbols $f_1, f_2, \ldots$, and predicate symbols $P_1, P_2, \ldots$.

Definition 5. A formula φ of T is a NORMALIZED FLAT CONJUNCTION if it is a conjunction of literals of the kinds:

$$x = y, \quad x \neq y, \quad x = f(x_1, \ldots, x_n), \quad P(x_1, \ldots, x_n), \quad \neg P(x_1, \ldots, x_n),$$

where $x, y, x_1, \ldots, x_n$ are variables, f stands for a function symbol, and P stands for a predicate symbol.

The following result is elementary:

---

[6] Cf. Definition 8.

Lemma 1. *Let φ be a formula of T. Then there is an effective procedure for constructing a formula of the form $\psi_1 \vee \cdots \vee \psi_k$ such that:*

– *each ψ_i is a normalized flat conjunction, for $i = 1, \ldots, k$; and*
– *φ and $\psi_1 \vee \cdots \vee \psi_k$ are equisatisfiable.*

Proof. See Lemmas 2.2 and 2.4 in Cantone et al. (1987).   □

In view of the previous lemma, if F is the collection of all normalized flat conjunctions of T, then we have:

Corollary 1. *The s.p. for T is reducible to the s.p. for F, in the sense that a solution to the latter yields a solution to the former.*   □

We now go back to the language RMCF^+.

Definition 6 (*First Normal Form*). An RMCF^+-formula φ is in FIRST NORMAL FORM if it is a conjunction involving only literals of the following types:

$$
\begin{array}{ll}
x = y + w, & x = y \cdot w, \\
x > y, & y = f(x), \\
\pm(f = g + h)_{[z_1, z_2]}, & \pm(f > g)_{[w_1, w_2]}, \\
\pm\mathrm{Up}(f)_{[z_1, z_2]}, & \pm\mathrm{Strict\_Up}(f)_{[z_1, z_2]}, \\
\pm\mathrm{Down}(f)_{[z_1, z_2]}, & \pm\mathrm{Strict\_Down}(f)_{[z_1, z_2]}, \\
\pm\mathrm{Convex}(f)_{[z_1, z_2]}, & \pm\mathrm{Strict\_Convex}(f)_{[z_1, z_2]}, \\
\pm\mathrm{Concave}(f)_{[z_1, z_2]}, & \pm\mathrm{Strict\_Concave}(f)_{[z_1, z_2]},
\end{array}
\tag{1}
$$

where x, y, w, w_1, w_2 stand for numerical variables or constants, z_1, z_2 for extended numerical variables,[7] and f, g, h for function symbols, and where, for an atom A, the expression $\pm A$ denotes both literals A and $\neg A$.

By RMCF^+_1 we denote the collection of all RMCF^+-formulae in first normal form.

Corollary 1 and the equivalences

$$
\begin{array}{lcl}
(f_1 = f_2 - f_3)_{[z_1, z_2]} & \longleftrightarrow & (f_2 = f_1 + f_3)_{[z_1, z_2]}, \\
(f_1 = f_2)_{[z_1, z_2]} & \longleftrightarrow & (f_1 = f_2 + \mathbf{0})_{[z_1, z_2]}, \\
t_1 = t_2 - t_3 & \longleftrightarrow & t_2 = t_1 + t_3, \\
t_1 = t_2 & \longleftrightarrow & t_1 = t_2 + 0, \\
t_1 = t_2 / t_3 & \longleftrightarrow & (t_3 \neq 0) \wedge (t_2 = t_1 \cdot t_3), \\
t_1 \neq t_2 & \longleftrightarrow & (t_2 > t_1) \vee (t_1 > t_2), \\
t_1 \not> t_2 & \longleftrightarrow & (t_1 = t_2) \vee (t_2 > t_1),
\end{array}
$$

easily yield the following result, which summarizes the first reduction step:

Lemma 2. *The s.p. for RMCF^+ is reducible to the s.p. for RMCF^+_1.*   □

Hence, it will be sufficient to solve the s.p. for RMCF^+_1-conjunctions.

---

[7] We recall that from Definition 2 we have $z_1 \neq +\infty$ and $z_2 \neq -\infty$.

3.2. Second reduction step: Removal of negative literals and dual literals

The second normal form results on eliminating negative literals as well as literals of types Down, Strict_Down, Concave, and Strict_Concave from formulae in first normal form.

Definition 7 (*Second Normal Form*). An RMCF_1^+-conjunction φ is in SECOND NORMAL FORM if it is a conjunction involving only positive literals of the following types:

$$
\begin{aligned}
& x = y + w, && x = y \cdot w, \\
& \qquad x > y, && y = f(x), \\
& (f = g + h)_{[z_1, z_2]}, && (f > g)_{[w_1, w_2]}, \\
& \quad \mathrm{Up}(f)_{[z_1, z_2]}, && \mathrm{Strict\_Up}(f)_{[z_1, z_2]}, \\
& \mathrm{Convex}(f)_{[z_1, z_2]}, && \mathrm{Strict\_Convex}(f)_{[z_1, z_2]},
\end{aligned}
\tag{2}
$$

where x, y, w, w_1, w_2 stand for numerical variables or constants, z_1, z_2 for extended numerical variables, and f, g, h for function symbols.

We denote by RMCF_2^+ the collection of all RMCF^+-formulae in second normal form.

Let φ_1 be an RMCF_1^+-conjunction. By, firstly, repeatedly applying the rewrite rules in Block 1 of Table 1 to φ_1 and, secondly, the rewrite rules in Block 2, one obtains a formula φ_1' not involving either negated functional literals or literals of type Down, Strict_Down, Concave, and Strict_Concave. Hence, by using simple arithmetic manipulations and transforming to disjunctive normal form, it can easily be seen that φ_1' can be effectively transformed into a disjunction of RMCF_2^+-conjunctions. Thus, we can conclude that

Lemma 3. *The s.p. for RMCF_1^+ is reducible to the s.p. for RMCF_2^+.* □

3.3. Third reduction step: Explicit evaluation of functions over domain variables and guessing of all equalities and inequalities among domain variables

Numerical variables, which occur as arguments of function symbols or as interval bounds, play an important role in the third normal form. Such variables are called *domain variables* (see the definition below). In the third normal form, all function symbols need to be evaluated over all domain variables and, additionally, all possible equalities and inequalities among domain variables need to be "guessed" by means of a very large disjunction.

Equalities and inequalities among the domain variables will be used in the fourth reduction step to express all functional literals in terms of elementary relationships among the functional images of the domain variables.

For the sake of accuracy, we give the following definitions.

Definition 8. Let φ be an RMCF_2^+-conjunction. A DOMAIN VARIABLE for φ is any numerical variable which either occurs in φ as the argument of some function symbol (for instance, x in $y = f(x)$) or occurs as a range parameter within some literal in φ of the types $(f = g + h)_{[z_1, z_2]}$, $(f > g)_{[w_1, w_2]}$, $\mathrm{Up}(f)_{[z_1, z_2]}$, $\mathrm{Strict\_Up}(f)_{[z_1, z_2]}$, $\mathrm{Convex}(f)_{[z_1, z_2]}$, or $\mathrm{Strict\_Convex}(f)_{[z_1, z_2]}$.

The collection of the domain variables for φ is denoted by $\mathrm{Dom}(\varphi)$.

Then the third normal form is defined as follows:

Definition 9 (*Third Normal Form*). An RMCF_2^+-conjunction φ is in THIRD NORMAL FORM if

Table 1
Rewrite rules for transforming RMCF_1^+-conjunctions into second normal form formulae

Block 1

$$
\begin{aligned}
\neg(f = g + h)_{[z_1,z_2]} &\mapsto \Delta_1 \wedge (y_1 \neq y_2) \\
\neg(f > g)_{[w_1,w_2]} &\mapsto \Delta_1' \wedge y_1 \leq y_2 \\
\neg\text{Up}(f)_{[z_1,z_2]} &\mapsto \Delta_2 \wedge y_1 > y_2 \\
\neg\text{Strict\_Up}(f)_{[z_1,z_2]} &\mapsto \Delta_2 \wedge y_1 \geq y_2 \\
\neg\text{Down}(f)_{[z_1,z_2]} &\mapsto \Delta_2 \wedge y_1 < y_2 \\
\neg\text{Strict\_Down}(f)_{[z_1,z_2]} &\mapsto \Delta_2 \wedge y_1 \leq y_2 \\
\neg\text{Convex}(f)_{[z_1,z_2]} &\mapsto \Delta_3 \wedge A > B \\
\neg\text{Strict\_Convex}(f)_{[z_1,z_2]} &\mapsto \Delta_3 \wedge A \geq B \\
\neg\text{Concave}(f)_{[z_1,z_2]} &\mapsto \Delta_3 \wedge A < B \\
\neg\text{Strict\_Concave}(f)_{[z_1,z_2]} &\mapsto \Delta_3 \wedge A \leq B
\end{aligned}
$$

Block 2

$$
\begin{aligned}
\text{Down}(f)_{[z_1,z_2]} &\mapsto (f + g = \mathbf{0})_{[z_1,z_2]} \wedge \text{Up}(g)_{[z_1,z_2]} \\
\text{Strict\_Down}(f)_{[z_1,z_2]} &\mapsto (f + g = \mathbf{0})_{[z_1,z_2]} \wedge \text{Strict\_Up}(g)_{[z_1,z_2]} \\
\text{Concave}(f)_{[z_1,z_2]} &\mapsto (f + g = \mathbf{0})_{[z_1,z_2]} \wedge \text{Convex}(g)_{[z_1,z_2]} \\
\text{Strict\_Concave}(f)_{[z_1,z_2]} &\mapsto (f + g = \mathbf{0})_{[z_1,z_2]} \wedge \text{Strict\_Convex}(g)_{[z_1,z_2]}
\end{aligned}
$$

where

$$\Delta_1 \equiv (z_1 \leq x \leq z_2) \wedge y_1 = f(x) \wedge y_2 = g(x) + h(x)$$
$$\Delta_1' \equiv (w_1 \leq x \leq w_2) \wedge y_1 = f(x) \wedge y_2 = g(x)$$
$$\Delta_2 \equiv (z_1 \leq x_1 < x_2 \leq z_2) \wedge \bigwedge_{i=1}^{2} y_i = f(x_i)$$
$$\Delta_3 \equiv (z_1 \leq x_1 < x_2 < x_3 \leq z_2) \wedge \bigwedge_{i=1}^{3} y_i = f(x_i)$$
$$A \equiv \frac{y_2 - y_1}{x_2 - x_1} \qquad B \equiv \frac{y_3 - y_1}{x_3 - x_1}$$

and where the x's and y's are newly introduced variables.

(a) for every domain variable x and function symbol f occurring in φ, a literal of the form $y = f(x)$ is present in φ, for some numerical variable y;

(b) the domain variables of φ lie on a CHAIN in φ w.r.t. the relation $<$, in the sense that if $x_1, \ldots, x_n$ are the distinct domain variables of φ, then there exists a permutation π of $(1, \ldots, n)$ such that the literals $x_{\pi(i)} < x_{\pi(i+1)}$ are in φ, for $i = 1, \ldots, n - 1$.

We denote by RMCF_3^+ the collection of all RMCF^+-conjunctions in third normal form.

Let φ_2 be an RMCF_2^+-conjunction. Condition (a) of Definition 9 can easily be forced by adding to φ_2, if needed, a literal of the form $y = f(x)$, for every domain variable x and function symbol f in φ_2, with y standing for a newly introduced variable. Let φ_2' be the resulting formula. It is clear that φ_2 and φ_2' are equisatisfiable. The size of φ_2' is at most quadratic in the size of φ_2.

Next, for any finite set S, let $\text{Eq}(S)$ and $\text{Ch}(S)$ be, respectively, the collection of all equivalence relations on S and the collection of all chains on S.[8]

---

[8] Given a finite set S, by a chain on S we mean any binary relation $\prec$ of cardinality $|S| - 1$ whose transitive closure is a total ordering on S.

Let φ_2'' be the formula

$$\bigvee_{\substack{\sim \text{ in Eq}(\text{Dom}(\varphi_2')) \\ \prec \text{ in Ch}(\text{Dom}(\widetilde{\varphi}_2'))}} \left(\widetilde{\varphi}_2' \wedge \bigwedge_{\substack{x,\, y \in \text{Dom}(\widetilde{\varphi}_2') \\ \text{such that } x \prec y}} x < y \right),$$

where $\widetilde{\varphi}_2'$ denotes the formula obtained by identifying $\sim$-equivalent variables in φ_2'.

It can be checked that the conjunction

$$\widetilde{\varphi}_2' \wedge \bigwedge_{\substack{x,\, y \in \text{Dom}(\widetilde{\varphi}_2') \\ \text{such that } x \prec y}} x < y \tag{3}$$

is in third normal form, for every $\sim$ in Eq(Dom(φ_2')) and every $\prec$ in Ch(Dom($\widetilde{\varphi}_2'$)). Moreover, it can easily be verified that φ_2' and φ_2'' are equisatisfiable. Hence, we have:

Lemma 4. *The s.p. for* RMCF_2^+ *is reducible to the s.p. for* RMCF_3^+. $\quad\square$

It should be noticed that the third reduction step is particularly expensive and special care must be taken in implementing it to limit unnecessary combinatorial explosion. Indeed, for a given finite set S of size n, there are exactly B_n distinct equivalence relations over S, where B_n is the n-th Bell number. Additionally, the number of all chains on a set S of size n is $n!$. Therefore, in the worst we can expect that the above reduction of a RMCF_2^+-conjunction φ with n distinct domain variables can generate a formula involving $\mathcal{O}(n! \cdot B_n) = \mathcal{O}(n! \cdot n \cdot e^n)$ disjuncts of type (3),[9] each of which must first undergo the fourth reduction step, as described in the next section, and then subjected to a decision test for Tarski's existential theory of reals.

In practice, inequality literals present in the initial RMCF_2^+-conjunction φ can cut down considerably the number of disjuncts of type (3) that need to be generated in the reduction phase to the third normal form. This is briefly illustrated with the following simple example. Let φ be the following (unsatisfiable) RMCF_2^+-conjunction[10]:

$$\mathrm{Up}(f)_{[-\infty,+\infty]} \wedge \mathrm{Down}(f)_{[-\infty,+\infty]}$$
$$\wedge\, x_1 = f(x_1) \wedge x_3 = f(x_3) \wedge z = g(x_2) \wedge x_1 < x_2 \wedge x_1 < x_3. \tag{4}$$

Then Dom(φ) = $\{x_1, x_2, x_3\}$, so that Dom(φ) admits the following $B_3 = 5$ distinct equivalence relations:

$$\{\{x_1\}, \{x_2\}, \{x_3\}\}$$
$$\{\{x_1\}, \{x_2, x_3\}\}$$
$$\{\{x_1, x_2\}, \{x_3\}\}$$
$$\{\{x_1, x_3\}, \{x_2\}\}$$
$$\{\{x_1, x_2, x_3\}\}.$$

Only the first two equivalence relations are compatible with the inequalities $x_1 < x_2 \wedge x_1 < x_3$ in φ, so that the last three can be discarded. In addition, relatively to the first equivalence relation,

---

[9] We have used the estimate $B_n = \mathcal{O}(n \cdot e^n)$ directly implied by de Bruijn's asymptotic formula for Bell numbers (cf. de Bruijn (1981), pp. 102–109).

[10] The unsatisfiability of (4) follows immediately on observing that its first four conjuncts imply the equality $x_1 = x_3$ which, together with its last conjunct $x_1 < x_3$, yields a contradiction. Furthermore, we recall that the literal Down(f)$_{[-\infty,+\infty]}$ in φ is to be intended as a shorthand for $(f + g = \mathbf{0})_{[-\infty,+\infty]} \wedge \mathrm{Up}(g)_{[-\infty,+\infty]}$, for a fresh function variable g.

only the two chains $x_1 < x_2 < x_3$ and $x_1 < x_3 < x_2$ are compatible with the inequalities $x_1 < x_2 \wedge x_1 < x_3$, whereas relatively to the second equivalence relation we have only the compatible chain $x_1 < x_2$. Hence (4) is equisatisfiable with the following disjunction of three RMCF_3^+-conjunctions in third normal form:

$$(\psi \wedge x_1 < x_2 \wedge x_2 < x_3) \vee (\psi \wedge x_1 < x_3 \wedge x_3 < x_2) \vee (\psi\{x_3/x_2\} \wedge x_1 < x_2),$$

where ψ is the closure of φ w.r.t. Definition 9(a) and $\psi\{x_3/x_2\}$ is the formula obtained by replacing all (free) occurrences of x_3 in ψ by x_2.

We observe that if we had reduced φ to third normal form without the above described optimization, we would have obtained a disjunction of 13 distinct RMCF_3^+-conjunctions.

3.4. Fourth reduction step: Removal of functional literals

The last step will reduce the s.p. for RMCF_3^+ to the s.p. for the unquantified Tarski's theory of reals.

This is the more delicate step, since it uses a model-theoretic result needing a quite elaborate verification. Our reduction is based on the fact that any RMCF_3^+-conjunction φ_3 is satisfiable if and only if it is satisfied by a "canonical" model "$\widehat{}$" in which each function symbol g in φ_3 is modeled by a continuous real function $\widehat{g} : \mathbb{R} \to \mathbb{R}$ having the form:

$$\widehat{g}(\xi) = \begin{cases} \gamma_0^g \cdot \ell_0^g(\xi) + \alpha_0^g \cdot e_0(\xi) & \text{if } \xi \in \;] -\infty, \widehat{v}_1[\\ \ell_j^g(\xi) + \alpha_j^g \cdot q_j(\xi) & \text{if } \xi \in [\widehat{v}_j, \widehat{v}_{j+1}[, \text{ for } j = 1, \ldots, r-1 \\ \gamma_r^g \cdot \ell_r^g(\xi) + \alpha_r^g \cdot e_r(\xi) & \text{if } \xi \in [\widehat{v}_r, +\infty[, \end{cases} \tag{5}$$

where

- $v_1, v_2, \ldots, v_r$ are the domain variables of φ_3, in the order induced by the literals of type $x < y$ in φ_3;
- γ_0^g, γ_r^g and α_j^g, for $j = 0, 1, \ldots, r$, are new real parameters associated to the function symbol g;
- the functions ℓ_i^g are linear, for $i = 0, 1, \ldots, r$, and satisfy

  · $\ell_{j-1}^g(\widehat{v}_j) = l_j^g(\widehat{v}_j)$, for $1 < j < r$,
  · $\gamma_0^g \cdot \ell_0^g(\widehat{v}_1) = \ell_1^g(\widehat{v}_1)$ and $\ell_{r-1}^g(\widehat{v}_r) = \gamma_r^g \cdot \ell_r^g(\widehat{v}_r)$;

- the functions e_0 and e_r are exponential and satisfy

  · $e_0(\widehat{v}_1) = 0$, and
  · $e_r(\widehat{v}_r) = 0$;

- the functions q_j are quadratic and satisfy $q_j(\widehat{v}_j) = q_j(\widehat{v}_{j+1}) = 0$, for $j = 1, \ldots, r-1$.

In other words, the function $\widehat{g}$ is a piecewise linear function, with junction points at $\widehat{v}_1, \widehat{v}_2, \ldots, \widehat{v}_r$, exponentially perturbed at intervals $] -\infty, \widehat{v}_1[$ and $[\widehat{v}_r, +\infty[$, and quadratically perturbed at intervals $[\widehat{v}_j, \widehat{v}_{j+1}[$, for $j = 1, \ldots, r-1$.[11]

The technique of using piecewise linear functions to decide the satisfiability problem for a fragment of real analysis is drawn from Cantone et al. (1987). Here we extend it by exponential

---

[11] Observe that canonical models, as we have defined them, are closed under addition, but not under multiplication. For such a reason they are not suited for dealing with literals of the form $f = g \cdot h$, with f, g, and h function symbols.

and quadratic perturbations so as to be able to decide also formulae involving Strict_Convex and Strict_Concave literals over any interval. Other kinds of perturbations could serve the same purpose, but we think that the above choice is the most natural.

At this point, a basic question is: how can we test an RMCF_3^+-conjunction for satisfiability by a canonical model?

We will exhibit below a constructive way to associate to any given RMCF_3^+-conjunction φ_3 another formula φ_3' in the unquantified language of Tarski's theory of reals, such that

- if φ_3 is satisfiable (by any real model), then φ_3' is satisfiable; and
- if φ_3' is satisfiable, then φ_3 is satisfiable by a canonical model.

In other words, we will show how to eliminate all functional literals from φ_3.

More precisely, the fourth reduction step proceeds as follows. Again, let φ_3 be an RMCF_3^+-conjunction and let $v_1, \ldots, v_r$ be its domain variables in the order induced by the literals of type $x < y$ in φ_3. We define an INDEX MAP $ind : \mathrm{Dom}(\varphi_3) \cup \{-\infty, +\infty\} \longmapsto \{1, 2, \ldots, r\}$ as follows:

$$ind(z) = \begin{cases} 1 & \text{if } z = -\infty \\ l & \text{if } z = v_l, \text{ for some } l \in \{1, \ldots, r\} \\ r & \text{if } z = +\infty. \end{cases}$$

As the domain variables of φ_3 lie on a chain with respect to the relation $<$, the intervals appearing in all functional atoms in φ_3 of type $\mathrm{Up}(f)_{[z_1,z_2]}$, $\mathrm{Strict\_Up}(f)_{[z_1,z_2]}$, $\mathrm{Convex}(f)_{[z_1,z_2]}$, $\mathrm{Strict\_Convex}(f)_{[z_1,z_2]}$ (with $ind(z_2) \leq ind(z_1)$, $z_1 \neq -\infty$, and $z_2 \neq +\infty$), $(f > g)_{[w_1,w_2]}$ (with $ind(w_2) < ind(w_1)$), and $(f = g + h)_{[z_1,z_2]}$ (with $ind(z_2) < ind(z_1)$) are mapped either into the empty set or into a single point by any model which satisfies all literals in φ_3 of the form $x < y$. Hence they are vacuously true in any such model, so that we can drop them from φ_3, without affecting satisfiability.

For each domain variable v_i and function symbol f, let y_i^f be a numerical variable such that the literal $y_i^f = f(v_i)$ is in φ_3.[12] Let us introduce new numerical variables γ_0^f, γ_r^f, and α_j^f, with $j = 0, 1, \ldots, r$, for each function symbol f in φ_3.

Let φ_3' be the conjunction of all arithmetical literals in φ_3, namely all literals in φ_3 of type $x = y + w$, $x = y \cdot w$, and $x > y$. We now show that by further adding suitable arithmetical literals to φ_3', we can turn it into a formula equisatisfiable with φ_3. This is done as follows.

(1) For each literal $x = f(v_i)$ occurring in φ_3, with $i \in \{1, \ldots, r\}$, add the literal

$$x = y_i^f \tag{6}$$

to φ_3'.

[*Clearly, the literal $x = y_i^f$ can play the role of the literal $x = f(v_i)$.*]

(2) For each literal $(f = g + h)_{[z_1,z_2]}$ occurring in φ_3 and for $i \in \{ind(z_1), \ldots, ind(z_2)\}$ and $j \in \{ind(z_1), \ldots, ind(z_2) - 1\}$, add the following literals

$$y_i^f = y_i^g + y_i^h, \tag{7}$$
$$\alpha_j^f = \alpha_j^g + \alpha_j^h \tag{8}$$

to φ_3'.

---

[12] By the closure property in Definition 9(a), it is always possible to find such variables y_i^f.

Moreover, if z_1 is the constant $-\infty$, add the literals

$$\alpha_0^f = \alpha_0^g + \alpha_0^h, \tag{9}$$

$$\gamma_0^f = \gamma_0^g + \gamma_0^h \tag{10}$$

to φ_3'.

Also, if z_2 is the constant $+\infty$, add the literals

$$\alpha_r^f = \alpha_r^g + \alpha_r^h, \tag{11}$$

$$\gamma_r^f = \gamma_r^g + \gamma_r^h \tag{12}$$

to φ_3'.

[*Literals of types (7), (10), and (12) force the linear part of a canonical model $\widehat{f}$ of f in the interval $[\widehat{z_1}, \widehat{z_2}]$ to be equal to the sum of the linear parts of $\widehat{g}$ and $\widehat{h}$, according to (5). Likewise, literals of type (8) and literals of types (9) and (11) do the same job with the quadratic and the exponential parts, respectively, of the functions $\widehat{f}$, $\widehat{g}$, and $\widehat{h}$.*]

(3) For each literal $(f > g)_{[w_1, w_2]}$ occurring in φ_3 and for $j \in \{ind(w_1), \ldots, ind(w_2) - 1\}$, add the literals

$$(y_j^f - y_j^g) > |\alpha_j^f| + |\alpha_j^g|, \tag{13}$$

$$(y_{j+1}^f - y_{j+1}^g) > |\alpha_j^f| + |\alpha_j^g| \tag{14}$$

to φ_3'.[13]

Notice that the literals added in this case force the conditions $y_j^f > y_j^g$, for $j \in \{ind(w_1), \ldots, ind(w_2)\}$.

[*Literals of types (13) and (14) are intended to force the linear part of $\widehat{f}$ to be far enough from the linear part of $\widehat{g}$ in the interval $[\widehat{w}_1, \widehat{w}_2]$, so that even after being perturbed by quadratic functions, the function $\widehat{f}$ is greater than $\widehat{g}$ in the interval $[\widehat{w}_1, \widehat{w}_2]$.*]

(4) For each monotonicity literal of type $Up(f)_{[z_1, z_2]}$ or $Strict\_Up(f)_{[z_1, z_2]}$ occurring in φ_3 and for $j \in \{ind(z_1), \ldots, ind(z_2) - 1\}$, add the literals

$$(y_{j+1}^f - y_j^f) \ \geq/_> \ 4|\alpha_j^f| \tag{15}$$

to φ_3', where $\geq/_>$ is the relation $\geq$, in the case of literals of type $Up(f)_{[z_1, z_2]}$, but is the relation $>$, in the case of literals of type $Strict\_Up(f)_{[z_1, z_2]}$.

Moreover, if z_1 is the constant $-\infty$, then also add the literals

$$\gamma_0^f \ \geq/_> \ 0, \tag{16}$$

$$\gamma_0^f \ \geq/_> \ \alpha_0^f \tag{17}$$

to φ_3'.

Likewise, if z_2 is the constant $+\infty$, then add the literals

$$\gamma_r^f \ \geq/_> \ 0, \tag{18}$$

$$\alpha_r^f + \gamma_r^f \ \geq/_> \ 0 \tag{19}$$

to φ_3'.

---

[13] It should be clear that literals of type (13) and (14) are to be intended as shorthand for suitable numerical RMCF^+-formulae, not involving the absolute value operator. The same remark applies also to literals of type (15).

Notice that the literals added in this case force the conditions $y^f_{j+1} \geq/> y^f_j$, for $j \in \{ind(w_1), \ldots, ind(w_2) - 1\}$.

[*Literals of types (15)–(19) force the derivative of $\widehat{f}$ to be non-negative or strictly positive, whichever must be the case, in the interval $[\widehat{z}_1, \widehat{z}_2]$, according to whether we are dealing with an Up or a Strict_Up literal.*]

(5) For each convexity literal of type $Convex(f)_{[z_1,z_2]}$ or $Strict\_Convex(f)_{[z_1,z_2]}$ occurring in φ_3 and for $i \in \{ind(z_1), \ldots, ind(z_2) - 1\}$ and $j \in \{ind(z_1) + 1, \ldots, ind(z_2) - 1\}$, add the literals

$$0 \geq/> \alpha^f_i, \tag{20}$$

$$\alpha^f_j \geq \frac{1}{4}\left[y^f_j - y^f_{j+1} + (y^f_j - y^f_{j-1} - 4\alpha^f_{j-1})\frac{v_{j+1} - v_j}{v_j - v_{j-1}}\right] \tag{21}$$

to φ'_3. Here $\geq/>$ stands for the relation $\geq$, in the case of literals of type $Convex(f)_{[z_1,z_2]}$, but stands for the relation $>$, in the case of literals of type $Strict\_Convex(f)_{[z_1,z_2]}$.

Moreover, if z_1 is the constant $-\infty$, then add the literal

$$0 \geq/> \alpha^f_0 \tag{22}$$

to φ'_3 and, provided that $z_2 \neq v_1$, also add the literal

$$\frac{y^f_2 - y^f_1 + 4\alpha^f_1}{v_2 - v_1} \geq \gamma^f_0 - \alpha^f_0 \tag{23}$$

to φ'_3.

Likewise, if z_2 is the constant $+\infty$, then add the literal

$$0 \geq/> \alpha^f_r \tag{24}$$

to φ'_3 and, provided that $z_1 \neq v_r$, also add the literal

$$\alpha^f_r + \gamma^f_r \geq \frac{y^f_r - y^f_{r-1} - 4\alpha^f_{r-1}}{v_r - v_{r-1}} \tag{25}$$

to φ'_3.

[*Literals of types (20), (22), and (24) force the second derivative of $\widehat{f}$ to be piecewise non-negative or strictly positive in the interval $[\widehat{z}_1, \widehat{z}_2]$, whichever must be the case, according to whether we are dealing with a Convex or a Strict_Convex literal. Instead, the purpose of literals of types (21), (23), and (25) is to force the left derivative of $\widehat{f}$ to be (strictly) smaller than or equal to the right derivative on the junction points contained in the interval $[\widehat{z}_1, \widehat{z}_2]$.*]

Notice that the formula φ'_3 constructed by the above steps belongs to Tarski's existential theory of reals.

3.5. Decidability of RMCF^+

The soundness of the fourth reduction step is entailed by the following lemma, whose proof will be the main subject of the next section.

Lemma 5. *The RMCF^+_3-formula φ_3 is equisatisfiable with φ'_3.* $\quad\square$

An immediate consequence of Lemma 5 is the following reduction result.

Lemma 6. *The s.p. for* RMCF_3^+ *is reducible to the s.p. for Tarski's existential theory of reals.*   □

The soundness of all reduction steps, proved in Lemmas 2–4 and 6, together with the decidability of Tarski's existential theory of reals yields our main result:

Theorem 1. *The s.p. for* RMCF^+ *is solvable.*   □

4. Soundness of the fourth reduction step

In this section we prove Lemma 5, thereby proving the soundness of the fourth reduction step. Let φ_3 and φ_3' be as in Section 3.4.

4.1. From φ_3 to φ_3'

Let us first assume that φ_3 is satisfied by a real model M.

Remark 3. To enhance readability we write $\bar{x}$ and $\bar{f}$ in place of $\mathsf{M}x$ and $\mathsf{M}f$, respectively, for each numerical variable x and function symbol f in φ_3.   □

Using the model M, we shall construct another real assignment M' which satisfies φ_3'. We start by putting

$$\mathsf{M}'x =_{\mathrm{Def}} \bar{x}, \quad \text{for each numerical variable } x \text{ in } \varphi_3.$$

Observe that all literals in φ_3' involving only "old" numerical variables occurring in φ_3 are correctly modeled by M'. This is plainly true for those numerical literals also occurring in φ_3. Concerning the remaining literals, we notice that if $x = f(v_i)$ is in φ_3, then $\bar{x} = \bar{f}(\bar{v}_i)$. Likewise, $\bar{y}_i^f = \bar{f}(\bar{v}_i)$, so that $\bar{x} = \bar{y}_i^f$, implying that literals in φ_3' of type (6) are correctly modeled by M'. Moreover, the truth in M' of all literals of the form (7), introduced in φ_3' to handle literals of type $(f = g + h)_{[z_1, z_2]}$, follows easily from the closure properties of φ_3 and the fact that M satisfies φ_3.

Next we show how to extend M' over the numerical variables γ_0^f, γ_r^f, and α_j^f, for $j = 0, 1, \ldots, r$, introduced in connection with the function symbols f in φ_3.

As before, let $v_1, \ldots, v_r$ be the distinct domain variables in the order induced by φ_3. Notice that from the closure properties of φ_3, we have $\bar{v}_1 < \cdots < \bar{v}_r$.

Let $\xi_{-1} < \xi_0 < \cdots < \xi_r < \xi_{r+1}$ be fixed real numbers such that $\xi_{-1}, \xi_0 < \bar{v}_1, \xi_r, \xi_{r+1} > \bar{v}_r$, and $\bar{v}_j < \xi_j < \bar{v}_{j+1}$, for $j = 1, \ldots, r-1$.

For notational convenience, we put

$$\bar{v}_0 = \xi_{-1}, \quad \bar{v}_{r+1} = \xi_{r+1}, \quad \bar{y}_0^f = \bar{f}(\xi_{-1}), \quad \bar{y}_{r+1}^f = \bar{f}(\xi_{r+1}), \quad \text{for } f \text{ in } Fun(\varphi_3).$$

We also write $\Delta \bar{v}_i =_{\mathrm{Def}} \bar{v}_i - \bar{v}_{i-1}$ and $\Delta \bar{y}_i^f =_{\mathrm{Def}} \bar{y}_i^f - \bar{y}_{i-1}^f$, for $i = 1, \ldots, r+1$ and $f \in Fun(\varphi_3)$.

For $j = 0, 1, \ldots, r$ and $f \in Fun(\varphi_3)$, let

$$\delta_j^f =_{\mathrm{Def}} \bar{f}(\xi_j) - \left[\bar{y}_j^f + (\xi_j - \bar{v}_j) \frac{\Delta \bar{y}_{j+1}^f}{\Delta \bar{v}_{j+1}} \right].$$
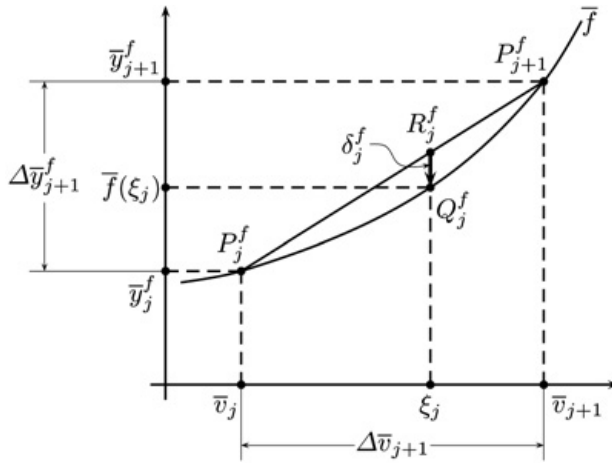
Fig. 1. Geometric interpretation of the quantities δ_j^f.

These quantities have the following geometric interpretation (see Fig. 1). If P_j^f denotes the point $(\overline{v}_j, \overline{y}_j^f)$, Q_j^f denotes the point $(\xi_j, \overline{f}(\xi_j))$, and R_j^f denotes the point with abscissa ξ_j on the straight line through P_j^f and P_{j+1}^f, then δ_j^f is the signed distance of Q_j^f from R_j^f.

Next, let $\epsilon > 0$. Define M' over the variables γ_0^f, γ_r^f, and α_j^f, for $j = 0, 1, \ldots, r$, by putting

$$\mathsf{M}'\alpha_j^f =_{\mathrm{Def}} \epsilon\delta_j^f, \quad \text{for } j = 0, 1, \ldots, r, \quad \text{and}$$

$$\mathsf{M}'\gamma_i^f =_{\mathrm{Def}} \frac{\Delta\overline{y}_{i+1}^f}{\Delta\overline{v}_{i+1}}, \quad \text{for } i \in \{0, r\}.$$

It is convenient to extend the notation introduced in Remark 3 to the newly defined variables too. Thus, we shall denote by $\overline{\alpha}_j^f$ and $\overline{\gamma}_i^f$ the values $\mathsf{M}'\alpha_j^f$ and $\mathsf{M}'\gamma_i^f$, respectively, with $j = 0, 1, \ldots, r$ and $i \in \{0, r\}$.

We show next that by a proper choice of the value of ϵ, the assignment M' can also be forced to satisfy all literals in φ_3' of types (8)–(25), introduced during the fourth reduction step.

Literals of type (8)–(12). If the literal $(f = g + h)_{[z_1, z_2]}$ occurs in φ_3 and $j \in \{ind(z_1), \ldots, ind(z_2) - 1\}$, then $\overline{f} = \overline{g} + \overline{h}$ holds in $[\overline{v}_j, \overline{v}_{j+1}]$. Thus, since $\xi_j \in [\overline{v}_j, \overline{v}_{j+1}]$, we have $\delta_j^f = \delta_j^g + \delta_j^h$, which implies $\overline{\alpha}_j^f = \overline{\alpha}_j^g + \overline{\alpha}_j^h$. One can show similarly that if $z_1 = -\infty$ then $\overline{\alpha}_0^f = \overline{\alpha}_0^g + \overline{\alpha}_0^h$ and $\overline{\gamma}_0^f = \overline{\gamma}_0^g + \overline{\gamma}_0^h$, and that if $z_2 = +\infty$ then $\overline{\alpha}_r^f = \overline{\alpha}_r^g + \overline{\alpha}_r^h$ and $\overline{\gamma}_r^f = \overline{\gamma}_r^g + \overline{\gamma}_r^h$. Hence, M' satisfies all literals in φ_3' of type (8)–(12).

Literals of type (13)–(14). If the literal $(f > g)_{[w_1, w_2]}$ occurs in φ_3 and $j \in \{ind(w_1), \ldots, ind(w_2) - 1\}$, then $\overline{f} > \overline{g}$ holds in $[\overline{v}_j, \overline{v}_{j+1}]$, so that $\overline{y}_j^f - \overline{y}_j^g > 0$ and $\overline{y}_{j+1}^f - \overline{y}_{j+1}^g > 0$. Hence, if the constant $\epsilon > 0$ is sufficiently small, then the following inequalities must hold

$$|\overline{\alpha}_j^f| + |\overline{\alpha}_j^g| = |\epsilon\delta_j^f| + |\epsilon\delta_j^g|$$
$$= \epsilon \cdot \left(|\delta_j^f| + |\delta_j^g|\right) < \min\left((\overline{y}_j^f - \overline{y}_j^g), (\overline{y}_{j+1}^f - \overline{y}_{j+1}^g)\right).$$

Therefore all literals in φ_3' of type (13)–(14) are also satisfied by M'.

Literals of type (15)–(19). If the literal $\mathrm{Up}(f)_{[z_1, z_2]}$ occurs in φ_3 and $j \in \{ind(z_1), \ldots, ind(z_2) - 1\}$, then $\overline{f}$ is increasing in $[\overline{v}_j, \overline{v}_{j+1}]$. Hence, $\Delta \overline{y}_{j+1}^f = \overline{f}(\overline{v}_{j+1}) - \overline{f}(\overline{v}_j) \geq 0$. If $\Delta \overline{y}_{j+1}^f = 0$, then $\delta_j^f = 0$, so that $|\overline{\alpha}_j^f| = |\epsilon \delta_j^f| \leq \frac{1}{4} \Delta \overline{y}_{j+1}^f$ plainly holds. On the other hand, if $\Delta \overline{y}_{j+1}^f > 0$, then, provided that $\epsilon > 0$ is sufficiently small, we have again $|\overline{\alpha}_j^f| = |\epsilon \delta_j^f| \leq \frac{1}{4} \Delta \overline{y}_{j+1}^f$. Thus, in any case, M' can be forced to satisfy literals in φ_3' of type (15).

If $z_1 = -\infty$, then $\overline{f}$ is increasing in $[\overline{v}_0, \overline{v}_1]$, so that $\overline{\gamma}_0^f = \frac{\Delta \overline{y}_1^f}{\Delta \overline{v}_1} \geq 0$, showing that literals in φ_3' of type (16) are correctly modeled by M'. If $\Delta \overline{y}_1^f = 0$, then $\overline{\alpha}_0^f = 0$, which plainly yields $\overline{\gamma}_0^f \geq \overline{\alpha}_0^f$. On the other hand, if $\Delta \overline{y}_1^f > 0$, then, provided that $\epsilon > 0$ is small enough, we have

$$\overline{\gamma}_0^f = \frac{\Delta \overline{y}_1^f}{\Delta \overline{v}_1} \geq \epsilon \delta_0^f = \overline{\alpha}_0^f.$$

Thus, in any case, provided that $\epsilon > 0$ is small enough, M' will also satisfy the literals in φ_3' of type (17).

Likewise, if $z_2 = +\infty$, we can show that M' can be forced to satisfy all literals in φ_3' of type (18) and (19), by taking $\epsilon > 0$ small enough.

The case of literals in φ_3 of type $\mathrm{Strict\_Up}(f)_{[z_1, z_2]}$ can be handled similarly.

Literals of type (20)–(25). If the literal $\mathrm{Convex}(f)_{[z_1, z_2]}$ occurs in φ_3 and $i \in \{ind(z_1), \ldots, ind(z_2) - 1\}$, then $\overline{f}$ is convex on $[\overline{v}_i, \overline{v}_{i+1}]$. Hence, since $\overline{v}_i < \xi_i < \overline{v}_{i+1}$, we have

$$\frac{\overline{f}(\xi_i) - \overline{f}(\overline{v}_i)}{\xi_i - \overline{v}_i} \leq \frac{\overline{f}(\overline{v}_{i+1}) - \overline{f}(\overline{v}_i)}{\overline{v}_{i+1} - \overline{v}_i} = \frac{\Delta \overline{y}_{i+1}^f}{\Delta \overline{v}_{i+1}},$$

which implies $\epsilon \delta_i^f \leq 0$, proving that M' satisfies all literals in φ_3' of type (20).

In addition, if the literal $\mathrm{Convex}(f)_{[z_1, z_2]}$ occurs in φ_3 and $j \in \{ind(z_1) + 1, \ldots, ind(z_2) - 1\}$, then $\overline{f}$ is convex on $[\overline{v}_{j-1}, \overline{v}_{j+1}]$. In order to prove that M' satisfies all literals in φ_3' of type (21), it is enough to show that

$$\epsilon \left(\delta_j^f + \delta_{j-1}^f \frac{\Delta \overline{v}_{j+1}}{\Delta \overline{v}_j} \right) \geq \frac{1}{4} \left(\Delta \overline{y}_j^f \frac{\Delta \overline{v}_{j+1}}{\Delta \overline{v}_j} - \Delta \overline{y}_{j+1}^f \right). \tag{26}$$

As shown above, $\delta_{j-1}^f, \delta_j^f \leq 0$. If $\delta_{j-1}^f = \delta_j^f = 0$, then (26) follows by observing that the convexity of $\overline{f}$ and Lemma 7 in the Appendix yield $\frac{\Delta \overline{y}_j^f}{\Delta \overline{v}_j} \leq \frac{\Delta \overline{y}_{j+1}^f}{\Delta \overline{v}_{j+1}}$. On the other hand, if $\delta_{j-1}^f < 0$, then, by Lemma 8 in the Appendix, $\frac{\Delta \overline{y}_j^f}{\Delta \overline{v}_j} < \frac{\Delta \overline{y}_{j+1}^f}{\Delta \overline{v}_{j+1}}$, so that $\Delta \overline{y}_j^f \frac{\Delta \overline{v}_{j+1}}{\Delta \overline{v}_j} - \Delta \overline{y}_{j+1}^f < 0$. Hence, for $\epsilon > 0$ sufficiently small, the inequality (26) holds.

The case in which $\delta_j^f < 0$ can be handled similarly.

If $z_1 = -\infty$, then $\overline{f}$ is convex on $[\overline{v}_0, \overline{v}_1]$, so that $\delta_0^f \leq 0$, showing that M' satisfies literals in φ_3' of type (22). If in addition $z_2 \neq v_1$, then much as in the previous case it

can be shown that $\epsilon(\delta_0^f + 4\frac{\delta_1^f}{\Delta \bar{v}_2}) \geq \frac{\Delta \bar{y}_1^f}{\Delta \bar{v}_1} - \frac{\Delta \bar{y}_2^f}{\Delta \bar{v}_2}$ holds for $\epsilon > 0$ sufficiently small, so that also literals in φ_3' of type (23) are correctly modeled by M'.

The case in which $z_2 = +\infty$ is completely analogous to the previous one.

Finally, literals in φ_3 of type $\text{Strict\_Convex}(f)_{[z_1,z_2]}$ can be handled in much the same way as literals of type $\text{Convex}(f)_{[z_1,z_2]}$.

The preceding discussion implies that, provided that $\epsilon > 0$ is chosen small enough, the assignment M' satisfies φ_3'.

4.2. The converse: From φ_3' to φ_3

Let us now assume that φ_3' is satisfied by a real model M'.

Using M', we shall construct a canonical model M which satisfies φ_3. We begin by putting

$$\mathsf{M}x =_{\text{Def}} \mathsf{M}'x, \quad \text{for each numerical variable } x \text{ in } \varphi_3.$$

For notational convenience, let $\bar{y}$ denote the value $\mathsf{M}'y$, for each $y \in Num(\varphi_3')$.

We next show how to define M over the function symbols of φ_3.

For each $f \in Fun(\varphi_3)$ and $i \in \{1, \ldots, r-1\}$, let $s_i^f, \epsilon_i^f : [0,1] \longrightarrow \mathbb{R}$ and $p_i : [\bar{v}_i, \bar{v}_{i+1}] \longrightarrow [0,1]$ be the real functions defined by

$$s_i^f(\xi) = \bar{y}_i^f + \xi \Delta \bar{y}_{i+1}^f, \quad \text{for } \xi \in [0,1],$$
$$\epsilon_i^f(\xi) = 4\bar{\alpha}_i^f \cdot \xi(1-\xi), \quad \text{for } \xi \in [0,1],$$
$$p_i(\eta) = \frac{\eta - \bar{v}_i}{\Delta \bar{v}_{i+1}}, \quad \text{for } \eta \in [\bar{v}_i, \bar{v}_{i+1}],$$

where, as before, $v_1, \ldots, v_r$ are the distinct domain variables of φ_3 (in the order induced by φ_3') and $\Delta \bar{y}_{i+1}^f$ and $\Delta \bar{v}_{i+1}$ denote the quantities $(\bar{y}_{i+1}^f - \bar{y}_i^f)$ and $(\bar{v}_{i+1} - \bar{v}_i)$, respectively.

Then, for each $f \in Fun(\varphi_3)$ define the following real function $\overline{f}$ over $\mathbb{R}$

$$\overline{f}(\eta) = \begin{cases} \bar{y}_1^f + \bar{\alpha}_0^f(1 - e^{\eta - \bar{v}_1}) - \bar{\gamma}_0^f(\bar{v}_1 - \eta) & \text{if } \eta \in]-\infty, \bar{v}_1[\\ s_i^f(p_i(\eta)) + \epsilon_i^f(p_i(\eta)) & \text{if } \eta \in [\bar{v}_i, \bar{v}_{i+1}[, \ i = 1, \ldots, r-1 \\ \bar{y}_r^f + \bar{\alpha}_r^f(1 - e^{\bar{v}_r - \eta}) + \bar{\gamma}_r^f(\eta - \bar{v}_r) & \text{if } \eta \in [\bar{v}_r, +\infty[, \end{cases}$$

and put

$$\mathsf{M}f =_{\text{Def}} \overline{f}.$$

It can easily be seen that each function $\overline{f}$ is continuous in $\mathbb{R}$ and differentiable in $\mathbb{R} \setminus \{\bar{v}_1, \ldots, \bar{v}_r\}$. In addition, $\overline{f}(\bar{v}_i) = \bar{y}_i^f$ holds, for $i = 1, \ldots, r$.

We next verify that M satisfies all literals in φ_3.

Literals of type $x = f(v_i)$. Let the literal $x = f(v_i)$ occur in φ_3, for some $i \in \{1, \ldots, r\}$. Then φ_3' must contain the literal $x = y_i^f$, so that $\mathsf{M}x = \bar{x} = \bar{y}_i^f = \overline{f}(\bar{v}_i) = \mathsf{M}(f(v_i))$, proving that M satisfies the literal $x = f(v_i)$.

Literals of type $(f = g + h)_{[z_1,z_2]}$. Let the literal $(f = g + h)_{[z_1,z_2]}$ occur in φ_3. We need to verify that for each $\eta \in [\mathsf{M}z_1, \mathsf{M}z_2]$ we have:

$$\overline{f}(\eta) = \overline{g}(\eta) + \overline{h}(\eta). \tag{27}$$

For each $i \in \{ind(z_1), \ldots, ind(z_2)\}$, the formula φ'_3 contains the literal $y_i^f = y_i^g + y_i^h$, so that $\overline{f}(\overline{v}_i) = \overline{y}_i^f = \overline{y}_i^g + \overline{y}_i^h = \overline{g}(\overline{v}_i) + \overline{h}(\overline{v}_i)$.

Moreover, for $j \in \{ind(z_1), \ldots, ind(z_2) - 1\}$ and $\eta \in]\overline{v}_j, \overline{v}_{j+1}[$ we have:

$$
\begin{aligned}
\overline{f}(\eta) &= s_j^f(p_j(\eta)) + \epsilon_j^f(p_j(\eta)) \\
&= \overline{y}_j^f + \Delta\overline{y}_{j+1}^f \cdot p_j(\eta) + 4\overline{\alpha}_j^f \cdot p_j(\eta) \cdot [1 - p_j(\eta)] \\
&= (\overline{y}_j^g + \overline{y}_j^h) + (\Delta\overline{y}_{j+1}^g + \Delta\overline{y}_{j+1}^h) \cdot p_j(\eta) \\
&\quad + 4(\overline{\alpha}_j^g + \overline{\alpha}_j^h) \cdot p_j(\eta) \cdot [1 - p_j(\eta)] \\
&= s_j^g(p_j(\eta)) + s_j^h(p_j(\eta)) + \epsilon_j^g(p_j(\eta)) + \epsilon_j^h(p_j(\eta)) \\
&= \overline{g}(\eta) + \overline{h}(\eta),
\end{aligned}
$$

since $\alpha_j^f = \alpha_j^g + \alpha_j^h$ is in φ'_3 and therefore $\overline{\alpha}_j^f = \overline{\alpha}_j^g + \overline{\alpha}_j^h$.

Furthermore, if $z_1 = -\infty$, then (27) holds also in the interval $]-\infty, \overline{v}_1[$. Indeed, for $\eta \in]-\infty, \overline{v}_1[$ we have

$$
\begin{aligned}
\overline{f}(\eta) &= \overline{y}_1^f + \overline{\alpha}_0^f(1 - e^{\eta - \overline{v}_1}) - \overline{\gamma}_0^f(\overline{v}_1 - \eta) \\
&= (\overline{y}_1^g + \overline{y}_1^h) + (\overline{\alpha}_0^g + \overline{\alpha}_0^h)(1 - e^{\eta - \overline{v}_1}) - (\overline{\gamma}_0^g + \overline{\gamma}_0^h)(\overline{v}_1 - \eta) \\
&= \overline{g}(\eta) + \overline{h}(\eta),
\end{aligned}
$$

since in this case $\alpha_0^f = \alpha_0^g + \alpha_0^h$ and $\gamma_0^f = \gamma_0^g + \gamma_0^h$ are in φ'_3.

Likewise, it can be shown that if $z_2 = +\infty$ then (27) holds also in the interval $]\overline{v}_r, +\infty[$.

Literals of type $(f > g)_{[w_1, w_2]}$. Let the literal $(f > g)_{[w_1, w_2]}$ occur in φ_3. We need to verify that $\overline{f}(\eta) > \overline{g}(\eta)$, for each $\eta \in [Mw_1, Mw_2]$. Notice that for each $j \in \{ind(w_1), \ldots, ind(w_2) - 1\}$, the formula φ'_3 contains the literals

$$
(y_j^f - y_j^g) > |\alpha_j^f| + |\alpha_j^g| \quad \text{and} \quad (y_{j+1}^f - y_{j+1}^g) > |\alpha_j^f| + |\alpha_j^g|,
$$

so that we have $(\overline{y}_j^f - \overline{y}_j^g) > 0$, for $j \in \{ind(w_1), \ldots, ind(w_2)\}$, and $\overline{f}(\overline{v}_j) = \overline{y}_j^f > \overline{y}_j^g = \overline{g}(\overline{v}_j)$, for $\overline{v}_j \in [Mw_1, Mw_2]$.

Moreover, for $j \in \{ind(w_1), \ldots, ind(w_2) - 1\}$ and $\eta \in]\overline{v}_j, \overline{v}_{j+1}[$, we have

$$
\begin{aligned}
s_j^f(p_j(\eta)) - s_j^g(p_j(\eta)) &= \overline{y}_j^f + \Delta\overline{y}_{j+1}^f \cdot p_j(\eta) - \overline{y}_j^g - \Delta\overline{y}_{j+1}^g \cdot p_j(\eta) \\
&= (\overline{y}_j^f - \overline{y}_j^g) \cdot (1 - p_j(\eta)) + (\overline{y}_{j+1}^f - \overline{y}_{j+1}^g) \cdot p_j(\eta) \\
&> (|\overline{\alpha}_j^f| + |\overline{\alpha}_j^g|) \cdot (1 - p_j(\eta)) + (|\overline{\alpha}_j^f| + |\overline{\alpha}_j^g|) \cdot p_j(\eta) \\
&= |\overline{\alpha}_j^f| + |\overline{\alpha}_j^g| \\
&\geq |\epsilon_j^f(p_j(\eta))| + |\epsilon_j^g(p_j(\eta))| \\
&\geq \epsilon_j^g(p_j(\eta)) - \epsilon_j^f(p_j(\eta)),
\end{aligned}
$$

where $|\epsilon_j^h(\xi)| \leq |\overline{\alpha}_j^h|$, for $\xi \in [0, 1]$, $j \in \{1, \ldots, r - 1\}$, and $h \in Fun(\varphi_3)$.

Thus, we have

$$\overline{f}(\eta) = s_j^f(p_j(\eta)) + \epsilon_j^f(p_j(\eta)) > s_j^g(p_j(\eta)) + \epsilon_j^g(p_j(\eta)) = \overline{g}(\eta),$$

for $\eta \in]\overline{v}_j, \overline{v}_{j+1}[$ and $j \in \{ind(w_1), \dots, ind(w_2) - 1\}$.

Literals of type $\mathrm{Up}(f)_{[z_1,z_2]}$ **and** $\mathrm{Strict\_Up}(f)_{[z_1,z_2]}$. Let the literal $\mathrm{Up}(f)_{[z_1,z_2]}$ occur in φ_3. We need to verify that the function f is monotone non-decreasing in the interval $[\mathsf{M}z_1, \mathsf{M}z_2]$. It is enough to show that $\overline{f}$ has a non-negative first-order derivative in each interval $]\overline{v}_i, \overline{v}_{i+1}[$ contained in $[\mathsf{M}z_1, \mathsf{M}z_2]$, with $i = 0, \dots, r$ and where we are putting $\overline{v}_0 = -\infty$ and $\overline{v}_{r+1} = +\infty$.

Let us first consider the case in which $j \in \{ind(z_1), \dots, ind(z_2) - 1\}$, namely the interval $]\overline{v}_j, \overline{v}_{j+1}[$ is contained in $[\mathsf{M}z_1, \mathsf{M}z_2]$, with $j \in \{1, \dots, r - 1\}$. Notice that φ_3' contains the literal $(y_{j+1}^f - y_j^f) \geq 4|\alpha_j^f|$.

For $\eta \in]\overline{v}_j, \overline{v}_{j+1}[$ we have:

$$
\begin{aligned}
D[\overline{f}](\eta) &= D[s_j^f(p_j)](\eta) + D[\epsilon_j^f(p_j)](\eta) \\
&= \left[D[s_j^f](\xi) + D[\epsilon_j^f](\xi) \right]_{\xi = p_j(\eta)} \cdot D[p_j](\eta) \\
&= (\Delta \overline{y}_{j+1}^f + 4\overline{\alpha}_j^f (1 - 2p_j(\eta))) \cdot \frac{1}{\Delta \overline{v}_{j+1}} \\
&\geq 0,
\end{aligned}
$$

since $\Delta \overline{y}_{j+1}^f \geq 4|\overline{\alpha}_j^f|$ (see above), $|1 - 2p_j(\eta)| \leq 1$ for $\eta \in]\overline{v}_j, \overline{v}_{j+1}[$, and $\Delta \overline{v}_{j+1} > 0$.[14]

Next, we consider the case in which the interval $]-\infty, \overline{v}_1[$ is contained in $[\mathsf{M}z_1, \mathsf{M}z_2]$, i.e. $z_1 = -\infty$. In such a case φ_3' contains the literals $\gamma_0^f \geq 0$ and $\gamma_0^f \geq \alpha_0^f$. Hence, for $\eta \in]-\infty, \overline{v}_1[$ we have:

$$
\begin{aligned}
D[\overline{f}](\eta) = \overline{\gamma}_0^f - \overline{\alpha}_0^f e^{\eta - \overline{v}_1} &\geq \overline{\gamma}_0^f + \inf_{\eta \in]-\infty, \overline{v}_1[} \{-\overline{\alpha}_0^f e^{\eta - \overline{v}_1}\} \\
&= \begin{cases} \overline{\gamma}_0^f - \overline{\alpha}_0^f & \text{if } \overline{\alpha}_0^f \geq 0 \\ \overline{\gamma}_0^f & \text{otherwise,} \end{cases}
\end{aligned}
$$

so that $D[\overline{f}](\eta) \geq 0$, for $\eta \in]-\infty, \overline{v}_1[$.

Likewise, it can be shown that if $z_2 = +\infty$, then $D[\overline{f}](\eta) \geq 0$, for $\eta \in]\overline{v}_r, +\infty[$.

Satisfiability of literals in φ_3 of type $\mathrm{Strict\_Up}(f)_{[z_1,z_2]}$ can be shown in much the same way.

Literals of type $\mathrm{Convex}(f)_{[z_1,z_2]}$ **and** $\mathrm{Strict\_Convex}(f)_{[z_1,z_2]}$. Let the literal $\mathrm{Convex}(f)_{[z_1,z_2]}$ occur in φ_3. We need to verify that the function f is convex in the interval $[\mathsf{M}z_1, \mathsf{M}z_2]$. In view of Lemma 9 in the Appendix, it is enough to verify that

(a) the second-order derivative of $\overline{f}$ is non-negative in each interval $]\overline{v}_i, \overline{v}_{i+1}[$ contained in $[\mathsf{M}z_1, \mathsf{M}z_2]$, for $i \in \{0, \dots, r\}$, where again we are putting $\overline{v}_0 = -\infty$ and $\overline{v}_{r+1} = +\infty$;

(b) $D_-[\overline{f}](\overline{v}_j) \leq D_+[\overline{f}](\overline{v}_j)$, for each $\overline{v}_j \in]\mathsf{M}z_1, \mathsf{M}z_2[$, with $j \in \{1, \dots, r\}$.

---

[14] We use the following notation: D and $D^{(2)}$ denote respectively the first and second derivative operators, whereas D_- and D_+ denote respectively the left and right first derivative operators.

Concerning (a), let us first consider the case in which an interval $]\overline{v}_j, \overline{v}_{j+1}[$ is contained in $[Mz_1, Mz_2]$, for some $j \in \{1, \ldots, r - 1\}$. Therefore $j \in \{ind(z_1), \ldots, ind(z_2) - 1\}$, which implies that φ_3' must contain the literal $\alpha_j^f \leq 0$, so that $\overline{\alpha}_j^f \leq 0$. Hence, for $\eta \in]\overline{v}_j, \overline{v}_{j+1}[$, we have

$$D^{(2)}[\overline{f}](\eta) = -8\overline{\alpha}_j^f \cdot \left(D[p_j](\eta)\right)^2 \geq 0.$$

Next, we consider the case in which the interval $] - \infty, \overline{v}_1[$ is contained in $[Mz_1, Mz_2]$, i.e. $z_1 = -\infty$. In this case, φ_3' must contain the literal $\alpha_0^f \leq 0$, so that $\overline{\alpha}_0^f \leq 0$. Hence, for $\eta \in] - \infty, \overline{v}_1[$, we have

$$D^{(2)}[\overline{f}](\eta) = -\overline{\alpha}_0^f e^{\eta - \overline{v}_1} \geq 0.$$

The case in which the interval $]\overline{v}_r, +\infty[$ is contained in $[Mz_1, Mz_2]$, i.e. $z_2 = +\infty$, is analogous to the preceding one.

Concerning (b), let $\overline{v}_j \in]Mz_1, Mz_2[$. If $j \in \{ind(z_1)+1, \ldots, ind(z_2)-1\}$, the literal

$$\alpha_j^f \geq \frac{1}{4}\left[y_j^f - y_{j+1}^f + \left(y_j^f - y_{j-1}^f - 4\alpha_{j-1}^f \right) \frac{v_{j+1} - v_j}{v_j - v_{j-1}} \right]$$

occurs in φ_3', so that

$$\overline{\alpha}_j^f \geq \frac{1}{4}\left[-\Delta\overline{y}_{j+1}^f + \left(\Delta\overline{y}_j^f - 4\overline{\alpha}_{j-1}^f \right) \frac{\Delta\overline{v}_{j+1}}{\Delta\overline{v}_j} \right]$$

holds. Hence,

$$D_-[\overline{f}](\overline{v}_j) = \frac{\Delta\overline{y}_j^f - 4\overline{\alpha}_{j-1}^f}{\Delta\overline{v}_j} \leq \frac{\Delta\overline{y}_{j+1}^f + 4\overline{\alpha}_j^f}{\Delta\overline{v}_{j+1}} = D_+[\overline{f}](\overline{v}_j).$$

If $z_1 = -\infty$, $z_2 \neq v_1$, and $j = 1$, then φ_3' contains the literal

$$\gamma_0^f - \alpha_0^f \leq \frac{y_2^f - y_1^f + 4\alpha_1^f}{v_2 - v_1},$$

so that

$$\overline{\gamma}_0^f - \overline{\alpha}_0^f \leq \frac{\Delta\overline{y}_2^f + 4\overline{\alpha}_1^f}{\Delta\overline{v}_2}$$

holds and therefore we have

$$D_-[\overline{f}](\overline{v}_1) = \overline{\gamma}_0^f - \overline{\alpha}_0^f \leq \frac{\Delta\overline{y}_2^f + 4\overline{\alpha}_1^f}{\Delta\overline{v}_2} = D_+[\overline{f}](\overline{v}_1).$$

Likewise, if $z_2 = +\infty$, $z_1 \neq v_r$, and $j = r$, it can be shown that $D_-[\overline{f}](\overline{v}_r) \leq D_+[\overline{f}](\overline{v}_r)$.

Satisfiability of literals in φ_3 of type Strict_Convex$(f)_{[z_1, z_2]}$ can be shown in much the same way.

Finally, notice that all purely arithmetic literals occurring in φ_3 also occur in φ_3' and therefore they are plainly satisfied by M.

5. Conclusions

We have proved that the satisfiability problem for the theory RMCF^+ is solvable. This result has been obtained by exhibiting a satisfiability preserving chain of four reductions that, starting from a formula φ of RMCF^+, produces at the end another formula φ', expressed in the unquantified language of Tarski's theory of reals and involving the numerical variables of the formula φ plus various other parameters. In particular, our decidability result has been based on the fact that the given formula φ is satisfiable if and only if it admits a parametric "canonical" model, which can be built up by suitably enriching any real model of the formula φ'.

As seen before, canonical models map function symbols into piecewise linear functions, perturbed by quadratic or exponential functions. We expect that by using other types of perturbations together with more sophisticated techniques to control the shape of the functions involved, other unquantified theories of continuous functions and of differentiable functions with a derivative operator can be proved decidable.

Acknowledgement

Appendix. Convex functions

In this appendix we prove some elementary properties on univariate convex real functions.

Definition 10. A real function f is said to be CONVEX on an interval I, if for any three points x, y, z in I, with $x < y < z$,

$$\frac{f(z) - f(x)}{z - x} \geq \frac{f(y) - f(x)}{y - x}.$$

If the above inequality is always strict, then f is said to be STRICTLY CONVEX on I.

Lemma 7. *Let f be a real function on an interval I. For any three distinct points $x, y, z \in I$, with $x < y < z$, the following inequalities are equivalent, in the sense that either all of them hold, or all of them are false:*

(a) $\dfrac{f(z) - f(x)}{z - x} \succ \dfrac{f(y) - f(x)}{y - x}$

(b) $\dfrac{f(z) - f(y)}{z - y} \succ \dfrac{f(z) - f(x)}{z - x}$

(c) $\dfrac{f(z) - f(y)}{z - y} \succ \dfrac{f(y) - f(x)}{y - x},$

where $\succ \in \{>, \geq\}$.

Proof. It is enough to observe that (a), (b), and (c) are all equivalent to the inequality

$$(y - x) \cdot f(z) + (z - y) \cdot f(x) \succ (z - x) \cdot f(y). \quad \square$$

Lemma 8. *Let* f *be a convex function on an interval* I. *For any* $w, x, y, z \in I$ *such that* $w < x < y < z$ *we have*

$$\text{if} \quad \frac{f(y) - f(w)}{y - w} > \frac{f(x) - f(w)}{x - w} \quad \text{then} \quad \frac{f(y) - f(w)}{y - w} < \frac{f(z) - f(y)}{z - y}.$$

Proof. By our hypothesis and by Lemma 7 above, we have $\frac{f(y)-f(w)}{y-w} < \frac{f(y)-f(x)}{y-x}$. Moreover, since f is convex, $\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x}$. Hence, we get $\frac{f(y)-f(w)}{y-w} < \frac{f(z)-f(x)}{z-x}$. □

Lemma 9. *Let* $(a, c]$ *and* $[c, b)$ *be two bounded or unbounded real intervals, and let* g : $(a, c] \rightarrow \mathbb{R}$ *and* h : $[c, b) \rightarrow \mathbb{R}$ *be two convex functions on their domains. Moreover, let us assume that*

- $g(c) = h(c)$,
- *the function* g *has left derivative* $D_-[g](c)$ *in* c, *the function* h *has right derivative* $D_+[h](c)$ *in* c, *and* $D_-[g](c) \leq D_+[h](c)$ *holds.*

Then, the function f : $(a, b) \rightarrow \mathbb{R}$ *defined as follows*

$$f(x) = \begin{cases} g(x) & \text{if } x \in (a, c] \\ h(x) & \text{if } x \in]c, b), \end{cases}$$

is convex in (a, b).

Proof. According to Definition 10, we need to verify that $\frac{f(\xi_3)-f(\xi_1)}{\xi_3-\xi_1} \geq \frac{f(\xi_2)-f(\xi_1)}{\xi_2-\xi_1}$, for any three points $\xi_1, \xi_2, \xi_3 \in (a, b)$ such that $\xi_1 < \xi_2 < \xi_3$. W.l.o.g., we can assume that $\xi_1, \xi_2 \in (a, c]$ and $\xi_3 \in]c, b)$, so that $\xi_1 < c$. By the convexity of g in $(a, c]$, we have

$$\frac{g(\xi_2) - g(\xi_1)}{\xi_2 - \xi_1} \leq \frac{g(c) - g(\xi_1)}{c - \xi_1}. \tag{1}$$

Again by the convexity of g in $(a, c]$ and by Lemma 7 in the Appendix, we have $\frac{g(c)-g(\xi_1)}{c-\xi_1} \leq \frac{g(c)-g(\eta)}{c-\eta}$, for any $\eta \in]\xi_1, c[$, so that

$$D_-[g](c) = \lim_{\eta \to c^-} \frac{g(\eta) - g(c)}{\eta - c} \geq \frac{g(c) - g(\xi_1)}{c - \xi_1}. \tag{2}$$

Likewise, by the convexity of h in $[c, b)$, we have $\frac{h(\eta)-h(c)}{\eta-c} \leq \frac{h(\xi_3)-h(c)}{\xi_3-c}$, for any $\eta \in]c, \xi_3[$, and therefore

$$D_+[h](c) = \lim_{\eta \to c^+} \frac{h(\eta) - h(c)}{\eta - c} \leq \frac{h(\xi_3) - h(c)}{\xi_3 - c}. \tag{3}$$

Thus, by (2), (3), and the hypothesis $D_-[g](c) \leq D_+[h](c)$, we get

$$\frac{g(c) - g(\xi_1)}{c - \xi_1} \leq \frac{h(\xi_3) - h(c)}{\xi_3 - c}. \tag{4}$$

Relations (1) and (4) can be rewritten as

$$\frac{f(\xi_2) - f(\xi_1)}{\xi_2 - \xi_1} \leq \frac{f(c) - f(\xi_1)}{c - \xi_1}, \tag{5}$$

$$\frac{f(c) - f(\xi_1)}{c - \xi_1} \leq \frac{f(\xi_3) - f(c)}{\xi_3 - c}, \tag{6}$$

respectively. Thus, by Lemma 7 in the Appendix, the inequality (6) implies

$$\frac{f(c) - f(\xi_1)}{c - \xi_1} \leq \frac{f(\xi_3) - f(\xi_1)}{\xi_3 - \xi_1},$$

which, together with (5), gives our thesis. □

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