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# Multiplicity results for critical $p$-Laplacian problems

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**Abstract** We prove the existence of $N$ distinct pairs of nontrivial solutions for critical $p$-Laplacian problems in $\mathbb{R}^N$, as well as in bounded domains. To overcome the difficulties arising from the lack of compactness, we use a recent global compactness result of Mercuri and Willem.

## 1 Introduction

Consider the problem

$$- \Delta_p \, u = a(x) \, |u|^{p-2} \, u + |u|^{p^*-2} \, u, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \tag{1.1}$$

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where $1 < p < N$, $p^* = Np/(N-p)$ is the critical Sobolev exponent, and $a \in L^{N/p}(\mathbb{R}^N)$ satisfies

$$\inf_{u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \, |\nabla u|_p = 1} \int_{\mathbb{R}^N} \left( |\nabla u|^p - a(x) \, |u|^p \right) dx > 0 \tag{1.2}$$

and

$$\operatorname{ess\,inf}_{x \in B_\delta(x_0)} a(x) > 0 \tag{1.3}$$

for some open ball $B_\delta(x_0) \subset \mathbb{R}^N$. Here $|\cdot|_q$ denotes the norm in $L^q(\mathbb{R}^N)$. Let

$$S = \inf_{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{\left( \displaystyle\int_{\mathbb{R}^N} |u|^{p^*} \, dx \right)^{p/p^*}} \tag{1.4}$$

be the best constant in the Sobolev inequality. We prove the following multiplicity result.

**Theorem 1.1** *Assume that $N \geq p^2$, and* (1.2) *and* (1.3) *hold. If*

$$|a^+|_{N/p} < \left( 1 - 2^{-p/N} \right) S, \tag{1.5}$$

*where $a^+$ is the positive part of $a$ defined by $a^+(x) = \max\{a(x), 0\}$, then problem* (1.1) *has at least $N$ pairs of nontrivial solutions.*

Sufficient conditions for the existence of a positive solution of problem (1.1) when $p \geq 2$ and $a(x) \leq 0$ for all $x \in \mathbb{R}^N$ were given by Alves [1]. Related multiplicity results for the subcritical scalar field equation in $R^N$ can be found in Cerami et al. [5], Clapp and Weth [6], Molle and Passaseo [13] and Perera [14].

We prove the multiplicity result in Theorem 1.1 in bounded domains also (see Theorem 3.1). In particular, we have the following corollary for the problem

$$-\Delta_p u = \lambda \, |u|^{p-2} u + |u|^{p^*-2} u, \quad u \in W_0^{1,p}(\Omega), \tag{1.6}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ and $1 < p < N$.

**Corollary 1.2** *If $N \geq p^2$ and*

$$0 < \lambda < \left( 1 - 2^{-p/N} \right) S \, |\Omega|^{-p/N}, \tag{1.7}$$

*where $|\Omega|$ is the Lebesgue measure of $\Omega$, then problem* (1.6) *has at least $N$ pairs of nontrivial solutions.*

See Devillanova and Solimini [9] for the semilinear case $p = 2$. For $N > p^2 + p$, Cao et al. [4] have recently shown that problem (1.6) has infinitely many solutions for all $\lambda > 0$ (see also Wu and Huang [19] and Perera et al. [15]). The multiplicity result in Corollary 1.2 is new when $p \neq 2$ and $p^2 \leq N \leq p^2 + p$.

From a technical point of view, we first use a recent global compactness result of Mercuri and Willen [12] to prove that suitable Palais–Smale sequences of the energy functional associated with problem (1.1) are weakly compact; see Lemma 2.2. Estimates of critical levels arising from Krasnoselskii's genus are then provided; see Lemmas 2.3–2.6. In both cases, the so-called Talenti's functions [17] play a basic role.

## 2 Proof of Theorem 1.1

Weak solutions of problem (1.1) coincide with critical points of the $C^1$-functional

$$\Phi(u) = \int_{\mathbb{R}^N} \left[ \frac{1}{p} \left( |\nabla u|^p - a(x) |u|^p \right) - \frac{1}{p^*} |u|^{p^*} \right] dx, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$$

restricted to the Nehari manifold

$$\mathcal{M} = \left\{ u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} \left( |\nabla u|^p - a(x) |u|^p \right) dx = \int_{\mathbb{R}^N} |u|^{p^*} dx \right\}.$$

We note that $\mathcal{M}$ is bounded away from the origin in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ in view of (1.2) and the continuity of the Sobolev imbedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$.

Recall that for $c \in \mathbb{R}$, $(u_j) \subset \mathcal{M}$ is a (PS)$_c$ sequence for $\Phi$ (resp. $\Phi|_\mathcal{M}$) if $\Phi'(u_j) \to 0$ (resp. $\Phi|'_\mathcal{M}(u_j) \to 0$) and $\Phi(u_j) \to c$.

**Lemma 2.1** *If $(u_j) \subset \mathcal{M}$ is a (PS)$_c$ sequence for $\Phi|_\mathcal{M}$, then $(u_j)$ is also a (PS)$_c$ sequence for $\Phi$.*

*Proof* Since $u_j \in \mathcal{M}$,

$$\Phi(u_j) = \frac{1}{N} \int_{\mathbb{R}^N} \left( |\nabla u_j|^p - a(x) |u_j|^p \right) dx,$$

and since $(\Phi(u_j))$ is bounded and (1.2) holds, this implies that $(u_j)$ is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Since $\Phi|'_\mathcal{M}(u_j) \to 0$, for some sequence of Lagrange multipliers $(\mu_j) \subset \mathbb{R}$,

$$\int_{\mathbb{R}^N} \left( |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v - a(x) |u_j|^{p-2} u_j v - |u_j|^{p^*-2} u_j v \right) dx$$

$$- \mu_j \int_{\mathbb{R}^N} \left[ p \left( |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v - a(x) |u_j|^{p-2} u_j v \right) - p^* |u_j|^{p^*-2} u_j v \right] dx$$

$$= o(\|v\|) \quad \forall v \in \mathcal{D}^{1,p}(\mathbb{R}^N). \tag{2.1}$$

Taking $v = u_j$ and using the fact that $(u_j)$ is a bounded sequence in $\mathcal{M}$ shows that

$$\mu_j \int_{\mathbb{R}^N} \left( |\nabla u_j|^p - a(x) |u_j|^p \right) dx \to 0.$$

Since (1.2) holds and $\mathcal{M}$ is bounded away from the origin, this implies that $\mu_j \to 0$, so $\Phi'(u_j) \to 0$ by (2.1). □

In the absence of a compact Sobolev imbedding, we will use the following compactness-type property of (PS)$_c$ sequences for $\Phi|_\mathcal{M}$.

**Lemma 2.2** *If*

$$\frac{1}{N} S^{N/p} < c < \frac{2}{N} S^{N/p}, \tag{2.2}$$

*then every (PS)$_c$ sequence for $\Phi|_\mathcal{M}$ has a subsequence that converges weakly to a (nontrivial) solution $v_0$ of problem (1.1) with $\Phi(v_0) = c$ or $\Phi(v_0) = c - \dfrac{1}{N} S^{N/p}$.*

*Proof* Assume (2.2) and let $(u_j) \subset \mathcal{M}$ be a (PS)$_c$ sequence for $\Phi|_{\mathcal{M}}$. Then $(u_j)$ is also a (PS)$_c$ sequence for $\Phi$ by Lemma 2.1, and by a recent global compactness result of Mercuri and Willem [12, Theorem 5.2] (see also Benci and Cerami [2] and Alves [1]), a renamed subsequence converges weakly to a solution $v_0 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ of problem (1.1) and there exist nontrivial solutions $v_1, \ldots, v_k \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, $k \geq 0$, of $-\Delta_p u = |u|^{p^*-2} u$ such that

$$\Phi(v_0) + \sum_{i=1}^{k} \Phi_\infty(v_i) = c, \tag{2.3}$$

where

$$\Phi_\infty(u) = \int_{\mathbb{R}^N} \left( \frac{1}{p} |\nabla u|^p - \frac{1}{p^*} |u|^{p^*} \right) dx, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N).$$

If $k = 0$, then $\Phi(v_0) = c$ by (2.3) and we are done, so suppose $k \geq 1$. We have

$$\Phi(v_0) = \frac{1}{N} \int_{\mathbb{R}^N} |v_0|^{p^*} dx \geq 0, \quad \Phi_\infty(v_i) = \frac{1}{N} \int_{\mathbb{R}^N} |v_i|^{p^*} dx > 0, \quad i = 1, \ldots, k. \tag{2.4}$$

For $i = 1, \ldots, k$, if $v_i$ is sign-changing, then $\Phi_\infty(v_i) \geq \frac{2}{N} S^{N/p}$ (see, e.g., Mercuri et al. [11, Lemma 2.1]), so $|v_i| > 0$ by (2.2)–(2.4) and the strong maximum principle. Hence

$$\Phi_\infty(v_i) = \frac{1}{N} S^{N/p}, \quad i = 1, \ldots, k$$

by Sciunzi [16, Theorem 1.1] (see also Caffarelli et al. [3], Damascelli et al. [7], and Vétois [18]), and then it follows from (2.2)–(2.4) again that $k = 1$ and $\Phi(v_0) = c - \frac{1}{N} S^{N/p}$. $\quad\square$

Let $\mathcal{A}$ denote the class of all nonempty compact symmetric subsets of $\mathcal{M}$, let

$$\gamma(A) = \inf \{ k \geq 1 : \exists \text{ an odd continuous map } A \to \mathbb{R}^k \backslash \{0\} \}$$

be the Krasnoselskii genus of $A \in \mathcal{A}$, let

$$\mathcal{A}_k = \{ A \in \mathcal{A} : \gamma(A) \geq k + 1 \},$$

and set

$$c_k := \inf_{A \in \mathcal{A}_k} \max_{u \in A} \Phi(u), \quad k = 0, \ldots, N.$$

We have

$$c_0 = \inf_{u \in \mathcal{M}} \Phi(u) \tag{2.5}$$

and

$$c_0 \leq c_1 \leq \cdots \leq c_N,$$

and we will show that $\frac{1}{N} S^{N/p} < c_1$ and $c_N < \frac{2}{N} S^{N/p}$ in order to apply Lemma 2.2.

**Lemma 2.3** *Every $A \in \mathcal{A}_1$ contains a point $u_0$ such that $u_0^{\pm} \in \mathcal{M}$, and hence $c_1 \geq 2c_0$.*

*Proof* Let $A \in \mathcal{A}_1$ and set $\alpha(u) = \int_{\mathbb{R}^N} \left( |\nabla u|^p - a(x) |u|^p - |u|^{p^*} \right) dx$. Then $A$ contains a point $u_0$ with $\alpha(u_0^+) = \alpha(u_0^-)$, for otherwise

$$A \to \mathbb{R}\backslash\{0\}, \quad u \mapsto \frac{\alpha(u^+) - \alpha(u^-)}{|\alpha(u^+) - \alpha(u^-)|}$$

is an odd continuous map and hence $\gamma(A) \leq 1$. We also have $\alpha(u_0^+) + \alpha(u_0^-) = \alpha(u_0) = 0$ since $u_0 \in \mathcal{M}$, so this implies that $\alpha(u_0^\pm) = 0$ and hence $u_0^\pm \in \mathcal{M}$. Then

$$\max_{u \in A} \Phi(u) \geq \Phi(u_0) = \Phi(u_0^+) + \Phi(u_0^-) \geq 2c_0$$

by (2.5), so the second assertion follows. □

We have the following lower bound for $c_0$.

**Lemma 2.4** *If $|a^+|_{N/p} < S$, then*

$$c_0 \geq \frac{1}{N} \left( S - |a^+|_{N/p} \right)^{N/p}.$$

*Proof* For all $u \in \mathcal{M}$,

$$S \left( \int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{p/p^*} \leq \int_{\mathbb{R}^N} |\nabla u|^p \, dx = \int_{\mathbb{R}^N} \left( a(x) |u|^p + |u|^{p^*} \right) dx$$

$$\leq \int_{\mathbb{R}^N} \left( a^+(x) |u|^p + |u|^{p^*} \right) dx \leq |a^+|_{N/p} \left( \int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{p/p^*} + \int_{\mathbb{R}^N} |u|^{p^*} dx$$

by (1.4) and the Hölder inequality, so

$$\int_{\mathbb{R}^N} |u|^{p^*} dx \geq \left( S - |a^+|_{N/p} \right)^{N/p}.$$

Since

$$\Phi(u) = \frac{1}{N} \int_{\mathbb{R}^N} |u|^{p^*} dx,$$

the assertion follows. □

**Lemma 2.5** $c_1 > \dfrac{1}{N} S^{N/p}$

*Proof* By Lemmas 2.3 and 2.4, and (1.5),

$$c_1 \geq \frac{2}{N} \left( S - |a^+|_{N/p} \right)^{N/p} > \frac{1}{N} S^{N/p}.$$

□

**Lemma 2.6** $c_N < \dfrac{2}{N} S^{N/p}$

*Proof* The infimum in (1.4) is attained by the family of functions

$$u_\varepsilon(x) = \frac{C_{N,p}\, \varepsilon^{-(N-p)/p}}{\left[ 1 + \left( \dfrac{|x|}{\varepsilon} \right)^{p/(p-1)} \right]^{(N-p)/p}}, \quad \varepsilon > 0,$$

where $C_{N,p} > 0$ is chosen so that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p \, dx = \int_{\mathbb{R}^N} u_\varepsilon^{p^*} \, dx = S^{N/p}.$$

Take a smooth function $\eta : [0, \infty) \to [0, 1]$ such that $\eta(s) = 1$ for $s \le 1/4$ and $\eta(s) = 0$ for $s \ge 1/2$, and set

$$\tilde{u}_\varepsilon(x) = \eta(|x|) \, u_\varepsilon(x), \quad \varepsilon > 0.$$

We have the well-known estimates

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}_\varepsilon|^p \, dx \le S^{N/p} + C\varepsilon^{(N-p)/(p-1)}, \tag{2.6}$$

$$\int_{\mathbb{R}^N} \tilde{u}_\varepsilon^{p^*} \, dx \ge S^{N/p} - C\varepsilon^{N/(p-1)}, \tag{2.7}$$

$$\int_{\mathbb{R}^N} \tilde{u}_\varepsilon^p \, dx \ge \begin{cases} \dfrac{\varepsilon^p}{C} & \text{if } N > p^2 \\ \dfrac{\varepsilon^p}{C} |\log \varepsilon| & \text{if } N = p^2, \end{cases} \tag{2.8}$$

where $C = C(N, p) > 0$ is a constant (see, e.g., Degiovanni and Lancelotti [8]).

After a translation and a dilation, we may assume that $x_0 = 0$ and $\delta = 1$ in (1.3), so we have

$$\lambda := \text{ess inf}_{x \in B_1(0)} \, a(x) > 0. \tag{2.9}$$

Let $S^{N-1}$ be the unit sphere in $\mathbb{R}^N$, let

$$S_+^N = \left\{ x = (x'\sqrt{1 - t^2}, t) : x' \in S^{N-1}, \ t \in [0, 1] \right\}$$

be the upper hemisphere in $R^{N+1}$, and consider the map $\varphi : S_+^N \to \mathcal{M}$ defined by

$$\varphi(x) = \pi \left( \tilde{u}_\varepsilon(\cdot - (1 - (2t - 1)_+) \, x'/2) - (1 - 2t)_+ \, \tilde{u}_\varepsilon(\cdot + x'/2) \right),$$

where

$$\pi : \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\} \to \mathcal{M}, \quad u \mapsto \left[ \frac{\displaystyle\int_{\mathbb{R}^N} \left( |\nabla u|^p - a(x) \, |u|^p \right) dx}{\displaystyle\int_{\mathbb{R}^N} |u|^{p^*} \, dx} \right]^{N/pp^*} u$$

is the radial projection onto $\mathcal{M}$. Clearly, $\varphi$ is continuous. Since

$$\Phi(\pi(u)) = \frac{1}{N} \left[ \frac{\displaystyle\int_{\mathbb{R}^N} \left( |\nabla u|^p - a(x) \, |u|^p \right) dx}{\left( \displaystyle\int_{\mathbb{R}^N} |u|^{p^*} \, dx \right)^{p/p^*}} \right]^{N/p}, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\},$$

and $\tilde{u}_\varepsilon(\cdot - (1 - (2t - 1)_+) \, x'/2)$ and $\tilde{u}_\varepsilon(\cdot + x'/2)$ have disjoint supports in $B_1(0)$, where $a \ge \lambda$ a.e. by (2.9),

$$\Phi(\varphi(x)) \le \frac{1}{N} \left[ \frac{\left(1 + (1 - 2t)_+^p\right) \displaystyle\int_{\mathbb{R}^N} \left( |\nabla \tilde{u}_\varepsilon|^p - \lambda \, \tilde{u}_\varepsilon^p \right) dx}{\left(1 + (1 - 2t)_+^{p^*}\right)^{p/p^*} \left( \displaystyle\int_{\mathbb{R}^N} \tilde{u}_\varepsilon^{p^*} \, dx \right)^{p/p^*}} \right]^{N/p} \quad \forall x \in S_+^N.$$

The right-hand side is nonincreasing in $t$, and (2.6)–(2.8) give

$$\frac{\displaystyle\int_{\mathbb{R}^N}\left(|\nabla\tilde{u}_\varepsilon|^p - \lambda\,\tilde{u}_\varepsilon^p\right)\mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N}\tilde{u}_\varepsilon^{p^*}\,\mathrm{d}x\right)^{p/p^*}} \leq \begin{cases} S - \dfrac{\lambda\varepsilon^p}{C} + C\varepsilon^{(N-p)/(p-1)} & \text{if } N > p^2 \\[2mm] S - \dfrac{\lambda\varepsilon^p}{C}\,|\log\varepsilon| + C\varepsilon^p & \text{if } N = p^2, \end{cases}$$

so

$$\max_{x\in S_+^N}\ \Phi(\varphi(x)) < \frac{2}{N}\,S^{N/p}$$

if $\varepsilon$ is sufficiently small. Since $\varphi$ is odd on $S^{N-1}$, it can now be extended to an odd continuous map $\tilde{\varphi} : S^N \to \mathcal{M}$ satisfying

$$\max_{u\in\tilde{\varphi}(S^N)}\ \Phi(u) < \frac{2}{N}\,S^{N/p}.$$

Then

$$\gamma(\tilde{\varphi}(S^N)) \geq \gamma(S^N) = N + 1$$

and the assertion follows. $\qquad\square$

The next lemma is due to Devillanova and Solimini [9] when $p = 2$.

**Lemma 2.7** *If $c_k = c_{k+1}$ for some $k \in \{1, \ldots, N-1\}$, then $\Phi$ has infinitely many critical points at the level $c_k$ or $c_k - \dfrac{1}{N}\,S^{N/p}$.*

*Proof* Suppose $\Phi$ has only finitely many critical points $v_1, \ldots, v_m$ at the levels $c_k$ and $c_k - \dfrac{1}{N}\,S^{N/p}$, and let $\{w_1, \ldots, w_n\}$ be a basis for their span. We have

$$v_i = \sum_{j=1}^n a_{ij}\,w_j, \quad i = 1, \ldots, m$$

for some $a_{ij} \in \mathbb{R}$. Take $(b_1, \ldots, b_n) \in \mathbb{R}^n$ such that

$$\sum_{j=1}^n a_{ij}\,b_j \neq 0, \quad i = 1, \ldots, m,$$

and let $l$ be a bounded linear functional on $D^{1,p}(\mathbb{R}^N)$ such that $l(w_j) = b_j$, $j = 1, \ldots, n$. Then

$$l(v_i) = \sum_{j=1}^n a_{ij}\,l(w_j) = \sum_{j=1}^n a_{ij}\,b_j \neq 0, \quad i = 1, \ldots, m. \tag{2.10}$$

Now take a sequence $(A_j) \subset \mathcal{A}_{k+1}$ such that $\max \Phi(A_j) \to c_{k+1}$, and let

$$\tilde{A}_j = \{u \in A_j : l(u) = 0\}.$$

By the monotonicity of the genus,

$$k + 2 \leq \gamma(A_j) \leq \gamma(\tilde{A}_j) + \gamma(A_j\backslash\tilde{A}_j),$$

and $\gamma(A_j \setminus \tilde{A}_j) \leq 1$ since $l|_{A_j \setminus \tilde{A}_j}$ is an odd continuous mapping into $\mathbb{R} \setminus \{0\}$, so $\gamma(\tilde{A}_j) \geq k+1$ and hence $\tilde{A}_j \in \mathcal{A}_k$. Then

$$c_k \leq \max_{u \in \tilde{A}_j} \Phi(u) \leq \max_{u \in A_j} \Phi(u) \to c_{k+1} = c_k,$$

so $\max \Phi(\tilde{A}_j) \to c_k$. By Ghoussoub [10, Theorem 1], then $\Phi|_{\mathcal{M}}$ has a $(PS)_{c_k}$ sequence $(u_j)$ such that

$$\operatorname{dist}(u_j, \tilde{A}_j) \to 0. \tag{2.11}$$

Since $\frac{1}{N} S^{N/p} < c_k < \frac{2}{N} S^{N/p}$ by Lemmas 2.5 and 2.6, a renamed subsequence converges weakly to some $v_i$ by Lemma 2.2. Then (2.11) implies that $l(u_j) \to 0$ and hence $l(v_i) = 0$, contradicting (2.10).  □

We are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1* By Lemmas 2.5 and 2.6,

$$\frac{1}{N} S^{N/p} < c_1 \leq \cdots \leq c_N < \frac{2}{N} S^{N/p},$$

and hence $c_k$ or $c_k - \frac{1}{N} S^{N/p}$ is a critical level of $\Phi$ for $k = 1, \ldots, N$ by Lemma 2.2. If $c_k = c_{k+1}$ for some $k \in \{1, \ldots, N-1\}$, then $\Phi$ has infinitely many critical points by Lemma 2.7 and we are done, so suppose that this is not the case. Then

$$c_1 - \frac{1}{N} S^{N/p} < \cdots < c_N - \frac{1}{N} S^{N/p} < c_1 < \cdots < c_N$$

and at least $N$ of these levels are critical for $\Phi$.  □

# 3 Bounded domains

Consider the problem

$$-\Delta_p u = a(x) |u|^{p-2} u + |u|^{p^*-2} u, \quad u \in W_0^{1,p}(\Omega), \tag{3.1}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $1 < p < N$, $a \in L^{N/p}(\Omega)$ satisfies

$$\inf_{u \in W_0^{1,p}(\Omega), \, |\nabla u|_p = 1} \int_\Omega \left( |\nabla u|^p - a(x) |u|^p \right) dx > 0 \tag{3.2}$$

and

$$\operatorname{ess\,inf}_{x \in B_\delta(x_0)} a(x) > 0 \tag{3.3}$$

for some open ball $B_\delta(x_0) \subset \Omega$, and $|\cdot|_q$ denotes the norm in $L^q(\Omega)$.

**Theorem 3.1** *Assume that $N \geq p^2$, and (3.2) and (3.3) hold. If*

$$|a^+|_{N/p} < \left( 1 - 2^{-p/N} \right) S,$$

*then problem (3.1) has at least $N$ pairs of nontrivial solutions.*

*Proof* We argue as in the proof of Theorem 1.1. Lemma 2.2 is now proved using the variant global compactness result in Proposition 3.2 below, which readily follows by arguing as in Mercuri and Willem [12] (see also Mercuri et al. [11, Remark 2.1]). The rest of the proof is unchanged. □

**Proposition 3.2** *Let* $\left(u_j\right) \subset W_0^{1,p}(\Omega)$ *be a* $(PS)_c$ *sequence for the functional*

$$\Phi(u) = \int_\Omega \left[\frac{1}{p}\left(|\nabla u|^p - a(x)\,|u|^p\right) - \frac{1}{p^*}\,|u|^{p^*}\right]dx, \quad u \in W_0^{1,p}(\Omega).$$

*Then a renamed subsequence converges weakly to a (possibly trivial) solution* $v_0 \in W_0^{1,p}(\Omega)$ *of problem* (3.1)*, and there exist nontrivial solutions* $v_1, \ldots, v_k \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ *or (up to a rotation and a translation)* $\mathcal{D}_0^{1,p}(\mathbb{R}_+^N)$*,* $k \geq 0$*, of* $-\Delta_p u = |u|^{p^*-2} u$ *and sequences* $(y_j^i) \subset \overline{\Omega}$*,* $(\lambda_j^i) \subset \mathbb{R}_+$*,* $i = 1, \ldots, k$ *such that* $\mathrm{dist}\,(y_j^i, \partial\Omega)/\lambda_j^i \to \infty$ *in the case of* $\mathbb{R}^N$ *and* $\mathrm{dist}\,(y_j^i, \partial\Omega)/\lambda_j^i$ *is bounded in the case of* $\mathbb{R}_+^N$*,*

$$\left\|u_j - v_0 - \sum_{i=1}^k (\lambda_j^i)^{-(N-p)/p}\, v_i((\cdot - y_j^i)/\lambda_j^i)\right\| \to 0,$$

$$\left\|u_j\right\|^p \to \sum_{i=0}^k \|v_i\|^p,$$

$$\Phi(v_0) + \sum_{i=1}^k \Phi_\infty(v_i) = c,$$

*where* $\Phi_\infty$ *is defined on* $\mathcal{D}^{1,p}(\mathbb{R}^N)$ *and* $\mathcal{D}_0^{1,p}(\mathbb{R}_+^N)$ *as in the proof of Lemma 2.2.*

Finally, to see that Corollary 1.2 in introduction follows from Theorem 3.1, we note that for all $u \in W_0^{1,p}(\Omega)$ with $|\nabla u|_p = 1$,

$$\int_\Omega \left(|\nabla u|^p - \lambda\,|u|^p\right) dx \geq 1 - \lambda \left(\int_\Omega |u|^{p^*} dx\right)^{p/p^*} |\Omega|^{p/N} \geq 1 - \lambda\, S^{-1}\, |\Omega|^{p/N} > 2^{-p/N}$$

by the Hölder inequality, (1.4) and (1.7).

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