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LAMINAR HYDROMAGNETIC FLOWS IN AN INCLINED HEATED LAYER

Paolo Falsaperla,[a] Andrea Giacobbe,[a] Sebastiano Lombardo,[a] and Giuseppe Mulone [a*]

(communicated by Paolo V. Giaquinta)

ABSTRACT. In this paper we investigate, analytically, stationary laminar flow solutions of an inclined layer filled with a hydromagnetic fluid heated from below and subject to the gravity field. In particular we describe in a systematic way the many basic solutions associated to the system. This extensive work is the basis to linear instability and nonlinear stability analysis of such motions.

1. Introduction

Laminar flows in fluid dynamics and in magneto-fluid dynamics are very important for many physical applications, for instance in geophysics and astrophysics (Alexakis *et al.* 2003; Batchelor 2000; Ferraro and Plumpton 1966). Many applications are possible in industry, e.g., in metallurgy (Branover, Mond, and Unger 1988) and in biology (Tao and Huang 2011). Usually the results related to laminar flow are applied to smooth flows of a viscous liquid through tubes or pipes (Batchelor 2000; Joseph 1976), but laminar flows in layers are also studied (see, for example, Pai 1962; Rionero and Mulone 1991). Moreover, the variation of the temperature in the layer plays an important role in convective problems (Chandrasekhar 1961; Mulone 1991b). Generally the layer is orthogonal to the gravity field (horizontal layer), but also interesting are layers inclined with respect to the horizontal plane (see, e.g., Goldstein, Ullmann, and Brauner 2015; Guo and Kaloni 1995), including the vertical case (Donaldson 1961).

In this article we investigate the analytical solutions of stationary laminar flows, that are flows in which the velocity has the form $\mathbf{U}(z) = U(z)\,\mathbf{i} + V(z)\,\mathbf{j}$ (usually, in the literature, $V(z) = 0$), of an inclined layer filled with a hydromagnetic fluid heated from below and subject to the gravity field. This is a general problem that contains, as particular cases, many very well known laminar flows like Couette, magnetic Couette, Poiseuille, and Hartmann (Alexakis *et al.* 2003; Hartmann 1937; Rogers 1992).

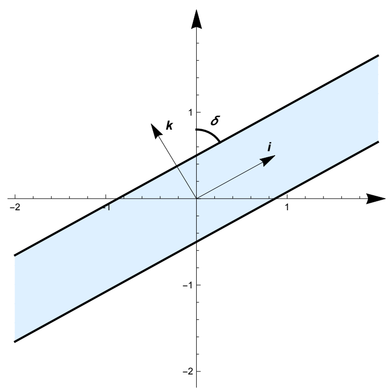
We consider many different type of boundary conditions depending on the physical nature of the bounding planes. For the velocity field $\mathbf{U}$, we suppose that the bounding planes are either rigid or stress-free; we fix the temperature T on the planes and, depending upon the electrical properties of the medium adjoining the fluid (cf. Chandrasekhar 1961, §42 p. 163), we consider either electrically non-conducting or perfect conducting boundaries for the magnetic field **H**.

We finally investigate the main limiting cases such as the coplanar case and the horizontal case (Mulone and Rionero 2003; Rionero and Mulone 1991). With *coplanar* we indicate the case in which the component of the magnetic field normal to the layer vanishes or, in other words, the Hartmann number goes to zero (Mulone and Salemi 1992; Pai 1962; Stuart 1954; Xu and Lan 2014). This case is important in medical applications (Tao and Huang 2011). With *horizontal* we indicate the case in which the layer is horizontal. In this case, also in absence of magnetic field, we expect that the linear operator associated to the perturbations equations is non symmetric and will have terms depending on the variable z (Mulone 1991a,b).

This paper is the base for a future investigation of the linear instability and the nonlinear stability of the solutions (motions) we study in the following sections.

The plan of the paper is the following: in Section 2 we give the setup of the problem. Section 3 deals with different boundary conditions. In Section 4 we consider some limiting cases, and in Section 5 we draw some conclusions.

2. Setup



The aim of this Section is to give the more general laminar solutions of the basic equations which govern the hydromagnetic dynamics of an inclined layer of fluid. Precisely, let us consider a layer $\Omega = \mathbb{R}^2 \times (-d/2, d/2)$ of width d filled with a hydromagnetic fluid and inclined of an angle $\pi/2 - \delta$ with respect to the horizontal plane Oxy. The fluid has temperature T and is subject to thermal expansion; its motion is also subjected and influences a magnetic field. The fluid has prescribed velocity $\mathbf{U}^{\pm}$ at the boundary of Ω, it has prescribed fixed temperature $T^{\pm}$ that will be assumed lower on the top and higher on the bottom of the layer $T^- > T^+$ (this conditions is referred to as *heating from below*, and its magnitude is quantified by the parameter $\beta = (T^- - T^+)/d$, that is the typical parameter associated to the onset of Rayleigh instability), and it is subjected to a fixed magnetic field $\mathbf{H}^{\pm}$ at the boundary of Ω (more general boundary conditions will be considered in Section **3**). The equations that model, in the Boussinesq approximation, such a system are (cf. Chandrasekhar 1961; Joseph 1976):

$$\begin{cases} \mathbf{U}_t + \mathbf{U}\cdot\nabla\mathbf{U} = \frac{\mu}{4\pi\rho_0}\mathbf{H}\cdot\nabla\mathbf{H} - \nabla\left(\frac{p^*}{\rho_0} + \frac{\mu}{8\pi\rho_0}|\mathbf{H}|^2\right) + (1-\alpha(T-T_0))\mathbf{g} + \nu\Delta\mathbf{U} \\ \nabla\cdot\mathbf{U} = 0 \\ \mathbf{H}_t + \mathbf{U}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla\mathbf{U} = \eta\,\Delta\mathbf{H} \\ \nabla\cdot\mathbf{H} = 0 \\ T_t + \mathbf{U}\cdot\nabla T = \kappa\Delta T\,, \end{cases} \tag{1}$$

where $(x,y,z,t) \in \Omega \times (0,\infty)$. In (1), $\mathbf{U},\mathbf{H},T,p^*$ are the unknown fields, respectively the velocity of the fluid, the magnetic field, the temperature, the pressure, ρ_0 is the reference density. The other symbols are positive physical parameters, precisely

- μ is the magnetic permeability
- σ is the electric conductivity
- κ is the thermal diffusivity
- ν is the viscosity
- $\eta = 1/(4\pi\sigma\mu)$ is the electric resistivity[1]
- α is the volume expansion coefficient
- $\beta = (T^- - T^+)/d$ is the gradient of temperature
- $\mathrm{Pr} = \nu/\kappa$ is the Prandtl number
- $\mathrm{Pm} = \nu/\eta$ is the magnetic Prandtl number
- $\mathrm{Ra} = R^2 = \dfrac{g\,\alpha\,\beta\,d^4}{\kappa\,\nu}$ is the Rayleigh number.

Under our hypotheses, choosing unit vectors

$$\mathbf{i} = \cos\delta\,\mathbf{e_3} + \sin\delta\,\mathbf{e_1}, \quad \mathbf{j} = \mathbf{e_2}, \quad \mathbf{k} = -\cos\delta\,\mathbf{e_1} + \sin\delta\,\mathbf{e_3},$$

so that $\mathbf{g} = -g\,\mathbf{e_3} = -g\cos\delta\,\mathbf{i} - g\sin\delta\,\mathbf{k}$, denoting with x,y,z the respective coordinates, and collecting all gradient-like terms in a new function $\Pi = \frac{p^*}{\rho_0} + \frac{\mu}{8\pi\rho_0}|\mathbf{H}|^2 - (1+\alpha T_0)\mathbf{g}\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$, the equations become

$$\begin{cases} \mathbf{U}_t + \mathbf{U}\cdot\nabla\mathbf{U} = \frac{\mu}{4\pi\rho_0}\,\mathbf{H}\cdot\nabla\mathbf{H} - \nabla\Pi + \alpha\,g\,T(\cos\delta\,\mathbf{i} + \sin\delta\,\mathbf{k}) + \nu\Delta\mathbf{U} \\ \nabla\cdot\mathbf{U} = 0 \\ \mathbf{H}_t + \mathbf{U}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla\mathbf{U} = \eta\,\Delta\mathbf{H} \\ \nabla\cdot\mathbf{H} = 0 \\ T_t + \mathbf{U}\cdot\nabla T = \kappa\Delta T\,. \end{cases} \tag{2}$$

Let us look for stationary and laminar solutions, in which all the fields are explicitly independent of time and the velocity field has the form

$$\mathbf{U} = U(z)\,\mathbf{i} + V(z)\,\mathbf{j} \tag{3}$$

under the simplifying hypothesis that

$$T = T(z) \quad \mathbf{H} = \mathbf{H}(z) = (H(z), K(z), H_3(z)).$$

Lemma 1. *The third component of* $\mathbf{H}$*, denoted* H_3*, is a constant real number* h_3*. The temperature is given by*

$$T(z) = -\beta z + T_0.$$

Proof. From equation $(2)_4$ we easily obtain $H_3' = 0$. Equation $(2)_5$ implies $T'' = 0$. Imposing the boundary conditions one concludes. We can assume that $T_0 = (T^- + T^+)/2$ coincides with the T_0 in (1) associated to the thermal expansion of the fluid. □

Remark 1. *The above Lemma implies that the only possible boundary conditions on the third component of* $\mathbf{H}$ *are necessarily the same constant on both boundaries. There are hence two possible cases:* $H_3 = 0$ *(the coplanar case) or* $H_3 \neq 0$*. In the first case a survey of*

[1]Here we use the definition of resistivity given by Chandrasekhar (1961, §38 p. 149), which differs from the usual definition by a factor $1/(4\pi)$.

the equations shows that equations $(2)_{1,2,5}$ *decouple from the field* $\mathbf{H}$*, and equations* $(2)_{3,4}$ *can then be solved once* $\mathbf{U}$ *is determined. We will assume for now that* $H_3 \neq 0$ *and dedicate Section **4.2** to the coplanar case.*

To nondimensionalize system (2) we introduce the following new variables:

$$z = d\,\hat{z}, \quad \Pi = \frac{\nu^2}{d^2}\widehat{\Pi}, \quad T = \sqrt{\frac{\beta\,\nu^3}{g\,\alpha\,\kappa\,d^2}}\,\widehat{T}, \quad \mathbf{U} = \frac{\nu}{d}\widehat{\mathbf{U}}, \quad \mathbf{H} = \sqrt{\frac{\rho_0\,\nu}{\mu^2\,d^2\,\sigma}}\,\widehat{\mathbf{H}},$$

$$t = \frac{d^2}{\nu}\hat{t}, \quad \partial_t = \frac{\nu}{d^2}\partial_{\hat{t}}, \qquad \nabla = \frac{1}{d}\widehat{\nabla}, \qquad \Delta = \frac{1}{d^2}\widehat{\Delta}.$$

Observe that, in this nondimensional form, the value of $\widehat{H}_3$ must be constant and equal to $\sqrt{\frac{\sigma}{\rho_0\,\nu}}\mu\,d\,h_3$. We denote such constant by γ (this constant is typically called Hartmann number, and is the square root of the Chandrasekhar number). Omitting the hats above the new variables, the equations become

$$\begin{cases} \mathbf{U}_t + \mathbf{U}\cdot\nabla\mathbf{U} = \mathrm{Pm}^{-1}\,\mathbf{H}\cdot\nabla\mathbf{H} - \nabla\Pi + R\,T(\cos\delta\,\mathbf{i} + \sin\delta\,\mathbf{k}) + \Delta\mathbf{U} \\ \nabla\cdot\mathbf{U} = 0 \\ \mathbf{H}_t + \mathbf{U}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla\mathbf{U} = \mathrm{Pm}^{-1}\Delta\mathbf{H} \\ \nabla\cdot\mathbf{H} = 0 \\ T_t + \mathbf{U}\cdot\nabla T = \mathrm{Pr}^{-1}\Delta T\,. \end{cases} \tag{4}$$

Physically relevant boundary conditions are:

- (rigid, rigid, electrically non-conducting, electrically non-conducting) up to a uniform translation, one can assume that $U(-1/2) = V(-1/2) = 0$ and $U(1/2) = u$, $V(1/2) = v$ for the velocity field, while the boundary conditions on the first two components of the magnetic field are $H(-1/2) = h_-$, $K(-1/2) = k_-$, $H(1/2) = h_+$, $K(1/2) = k_+$. This case includes Couette and Poiseuille basic solutions.
- (rigid, rigid, electrically conducting, electrically non-conducting) the conditions on the velocity field are the same, that is $U(-1/2) = V(-1/2) = 0$, $U(1/2) = u$, $V(1/2) = v$. For the magnetic field one has conditions on the first derivatives below $H'(-1/2) = h'$, $K'(-1/2) = k'$ and on the values above $H(1/2) = h$, $K(1/2) = k$.
- (rigid, stress free, electrically conducting, electrically non-conducting) $U(-1/2) = V(-1/2) = 0$, $U'(1/2) = u'$, $V'(1/2) = v'$, $H'(-1/2) = h'$, $K'(-1/2) = k'$, $H(1/2) = h$, $K(1/2) = k$

By substituting in the equations the particular solution for the temperature T, using the fact that the solutions are laminar, $\mathbf{U}$ and $\mathbf{H}$ are solenoidal vector fields, and incorporating in Π all gradient-like terms, the stationary equations for kinetic and magnetic fields become

$$\begin{cases} 0 = \mathrm{Pm}^{-1}\,\mathbf{H}\cdot\nabla\mathbf{H} - \nabla\Pi - \mathrm{Pr}^{-1}\,\mathrm{Ra}\cos\delta\,\mathbf{i}\,z + \Delta\mathbf{U} \\ \mathbf{U}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla\mathbf{U} = \mathrm{Pm}^{-1}\Delta\mathbf{H}. \end{cases} \tag{5}$$

Using the boundary conditions, the solutions of $(5)_2$ must satisfy

$$H''(z) = -\mathrm{Pm}\,\gamma U'(z), \qquad K''(z) = -\mathrm{Pm}\,\gamma V'(z), \qquad H_3 \equiv \gamma, \tag{6}$$

for all $(x,y) \in \mathbb{R}^2$, and $z \in (-1/2, 1/2)$. Integrating equations (6), we have

$$H' = -\mathrm{Pm}\,\gamma U + b_1, \qquad K' = -\mathrm{Pm}\,\gamma V + b_2$$

with b_1 and b_2 real constants.

Let us investigate the general solutions of (5) with $\gamma \neq 0$ (in Section **4.2** we will consider the case in which the magnetic field $\mathbf{H}$ is coplanar). The equations left to fulfill are equations $(5)_1$:

$$\nabla\Pi = \frac{\gamma}{\mathrm{Pm}}\begin{pmatrix} -\mathrm{Pm}\,\gamma U(z)+b_1 \\ -\mathrm{Pm}\,\gamma V(z)+b_2 \\ 0 \end{pmatrix} - \frac{\mathrm{Ra}}{\mathrm{Pr}}\begin{pmatrix} \cos\delta\, z \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} U''(z) \\ V''(z) \\ 0 \end{pmatrix}.$$

From these equations it follows that Π is independent of z and its derivatives with respect to x and y are functions of z alone. Hence, up to a constant, $\Pi = -\sigma_1 x - \sigma_2 y$. The first two equations above become

$$U''(z) - \gamma^2 U(z) = \mathrm{Pr}^{-1}\,\mathrm{Ra}\cos\delta\, z - \mathrm{Pm}^{-1}\,\gamma b_1 - \sigma_1 \qquad (7)$$

$$V''(z) - \gamma^2 V(z) = -\mathrm{Pm}^{-1}\,\gamma b_2 - \sigma_2. \qquad (8)$$

The general solutions to these equations, denoting with $\mathrm{c}(z) = \cosh(\gamma z)$ and $\mathrm{s}(z) = \sinh(\gamma z)$ are

$$U(z) = u_1\,\mathrm{c}(z) + u_2\,\mathrm{s}(z) - \frac{\mathrm{Ra}}{\mathrm{Pr}\,\gamma^2}\cos\delta\, z + \frac{b_1}{\mathrm{Pm}\,\gamma} + \frac{\sigma_1}{\gamma^2}, \qquad (9)$$

$$V(z) = v_1\,\mathrm{c}(z) + v_2\,\mathrm{s}(z) + \frac{b_2}{\mathrm{Pm}\,\gamma} + \frac{\sigma_2}{\gamma^2}, \qquad (10)$$

with u_1, u_2, v_1, v_2 integrating constants. The general solutions for the magnetic field are hence

$$H(z) = -\mathrm{Pm}\,u_1\,\mathrm{s}(z) - \mathrm{Pm}\,u_2\,\mathrm{c}(z) + \frac{\mathrm{Ra}\,\mathrm{Pm}}{\mathrm{Pr}\,\gamma}\cos\delta\,\frac{z^2}{2} - \frac{\mathrm{Pm}\,\sigma_1}{\gamma} z + c_1, \qquad (11)$$

$$K(z) = -\mathrm{Pm}\,v_1\,\mathrm{s}(z) - \mathrm{Pm}\,v_2\,\mathrm{c}(z) - \frac{\mathrm{Pm}\,\sigma_2}{\gamma} z + c_2. \qquad (12)$$

Observe that the functions must satisfy 8 boundary conditions, and the constants of integrations are 10. Among them, the two constants σ_1, σ_2 are related to an exterior force field, exerted through a non-trivial "*pressure*" function.

3. Different boundary conditions

3.1. Rigid–rigid, electrically nonconducting–electrically nonconducting. We first note that when we write *rigid–rigid* and so on, we refer the first adjective to the bottom plane and the second adjective to the top plane. As described in the previous section, particularly relevant conditions on the fields taking part in this analysis are zero velocity at the lower boundary, constant velocity $(u, v, 0)$ at the upper boundary and assigned magnetic field at the boundaries, $H(-1/2) = h_-$, $K(-1/2) = k_-$, $H(1/2) = h_+$, $K(1/2) = k_+$. Under these conditions, and denoting $\mathrm{c}_\gamma = \cosh(\gamma/2)$, $\mathrm{s}_\gamma = \sinh(\gamma/2)$, $\widehat{h} = h_+ + h_-$, $\widetilde{h} = h_+ - h_-$,

$\widehat{k} = k_+ + k_-$, $\widetilde{k} = k_+ - k_-$, the solutions are

$$U(z) = \left(\frac{u}{2} + \frac{\mathrm{Ra}\cos\delta}{2\gamma^2 Pr}\right)\frac{\mathrm{s}(z)}{\mathrm{s}_\gamma} - \frac{\mathrm{Ra}\cos\delta}{\gamma^2 Pr}z + \frac{u}{2} + \frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{2\gamma \mathrm{s}_\gamma}\left(\sigma_1 + \frac{\widetilde{h}\gamma}{\mathrm{Pm}}\right),$$

$$V(z) = \frac{v}{2}\frac{\mathrm{s}_\gamma + \mathrm{s}(z)}{\mathrm{s}_\gamma} + \frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{2\gamma \mathrm{s}_\gamma}\left(\sigma_2 + \frac{\widetilde{k}\gamma}{\mathrm{Pm}}\right).$$

Observe that these solutions depend on the boundary conditions u, v and on the two real parameters σ_1, σ_2, whose choice is related to the pressure.

Finally, the first two components of the magnetic field must satisfy the equations (6). Under the hypotheses above, the solutions are $H_3(z) = \gamma$,

$$H(z) = \mathrm{Pm}\left(\left(\frac{\mathrm{Ra}\cos\delta}{2\gamma^2\mathrm{Pr}} + \frac{u}{2}\right)\frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{\mathrm{s}_\gamma} + \frac{\mathrm{Ra}\cos\delta}{2\gamma\mathrm{Pr}}\left(z^2 - \frac{1}{4}\right) - \frac{\sigma_1 z}{\gamma} + \frac{\sigma_1 \mathrm{s}(z)}{2\gamma \mathrm{s}_\gamma}\right) + \\ + \frac{\widehat{h}}{2} + \frac{\widetilde{h}}{2}\frac{\mathrm{s}(z)}{\mathrm{s}_\gamma},$$

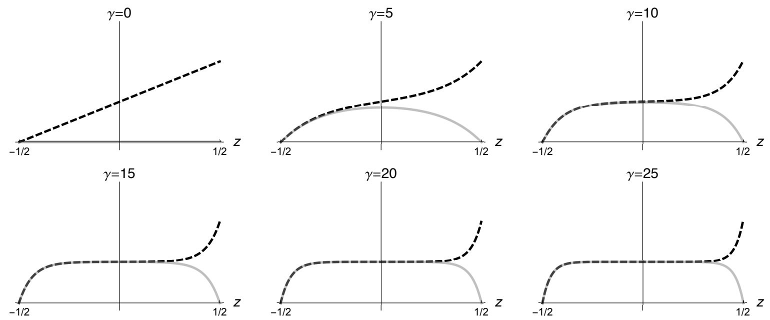
and

$$K(z) = \frac{\mathrm{Pm}}{2}\left(v\frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{\mathrm{s}_\gamma} + \left(\frac{\mathrm{s}(z)}{\mathrm{s}_\gamma} - 2z\right)\frac{\sigma_2}{\gamma}\right) + \frac{\widehat{k}}{2} + \frac{\widetilde{k}}{2}\frac{\mathrm{s}(z)}{\mathrm{s}_\gamma}.$$

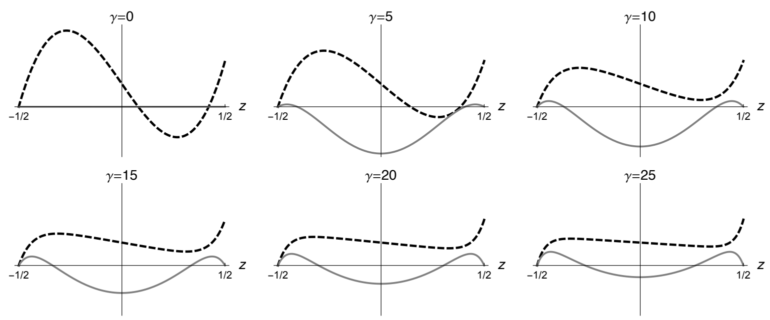
Particular care has to be taken in the *coplanar* case, which is the limit as the third component of $\mathbf{H}$ is zero (in this case H and K are affine functions of z), and in the horizontal case, when δ is $\pi/2$. We will discuss these cases in Section 4.

In the following plots we have set $\mathrm{Ra} = 1707$ (about the critical Rayleigh number for the Bénard problem in absence of magnetic field), $\mathrm{Pr} = 6.7$ (the Prandtl number of water), and $\mathrm{Pm} = 1$ (a value much higher than that of fluids such as air or water, chosen to emphasize the effects of the magnetic field, that would be qualitatively the same). The first two components of the magnetic field H, K vanish at the boundary, the velocity is zero at the lower boundary. *In all plots the dashed black line is the U component of velocity, the gray line is the H component of magnetic field.*

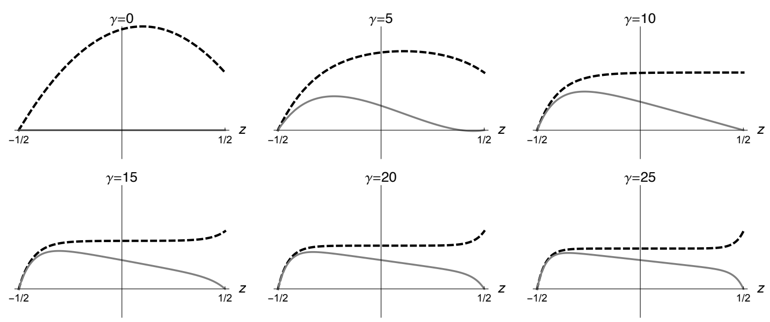
In the first block of six plots the layer is horizontal ($\delta = \pi/2$), the velocity at the upper boundary is positive ($u = 1$), the pressure is set to zero ($\sigma_1 = 0$), and γ is increased from 0 to 25. When $\gamma = 0$ one obtains a Couette flow. Incrementing slowly γ, that is proportional to the third component of the magnetic field, one obtains a family of magnetic Couette flows. We observe that the presence of a transversal component of the magnetic field makes the velocity converges to a uniform function.



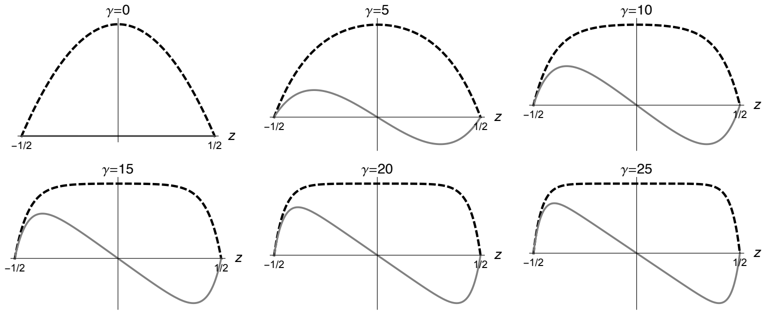
Changing the inclination of the plane to $\delta = \pi/4$ the graphs become as below. Observe that for large γ the velocity is the same as in the horizontal case, while the magnetic field changes noticeably.



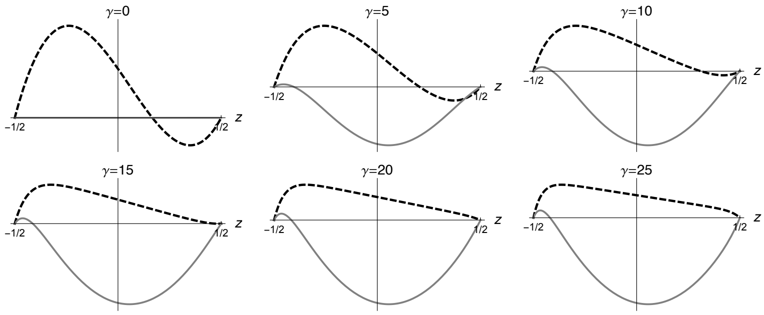
We now consider a horizontal plane but fix a positive pressure. For low γ the pressure changes the concavity of velocity, for large γ the pictures are qualitatively the same but the velocity has shifted upward because of the pressure acting.



To investigate the case which has as limit solution a Poisueille flow we consider a horizontal layer ($\delta = \pi/2$), with zero velocity at both boundaries $u = 0$, and positive pressure ($\sigma_1 = 10$). The set of pictures show the effect of incrementing γ from 0 to 25. *In the next two blocks of plots the vertical axis is rescaled with γ, in the unscaled plots both the velocity U and the magnetic field H converge to zero as γ grows.*



In what follows similar plots are drawn to investigate the effect of inclination of the layer ($\delta = \pi/4$).



3.2. Rigid–rigid, electrically conducting–electrically nonconducting. The difference between this case and the previous one is that to the magnetic field is imposed a condition on the first derivative from below. The solutions in this case are $H_3(z) = \gamma$,

$$U(z) \;=\; \frac{\mathrm{Ra}}{\gamma^2 \mathrm{Pr}} \cos\delta \left(\frac{\mathrm{s}(z)}{2\,\mathrm{s}_\gamma} - z \right) \;+\; \frac{u}{2} \left(\frac{\mathrm{c}(z)}{\mathrm{c}_\gamma} + \frac{\mathrm{s}(z)}{\mathrm{s}_\gamma} \right) \;+\; \frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{\mathrm{c}_\gamma} \left(\frac{\sigma_1}{\gamma^2} + \frac{h'}{\mathrm{Pm}\gamma} \right),$$

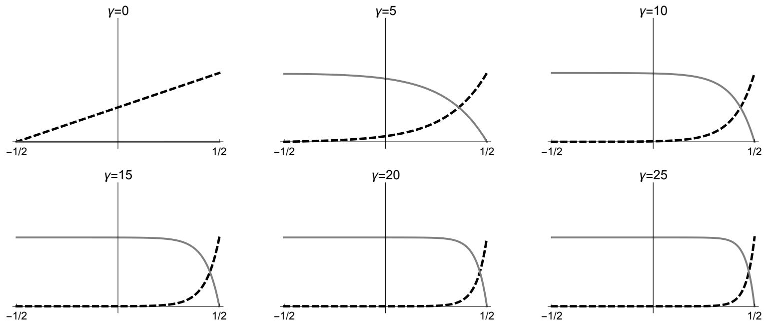
$$V(z) = \frac{v}{2} \left(\frac{\mathrm{c}(z)}{\mathrm{c}_\gamma} + \frac{\mathrm{s}(z)}{\mathrm{s}_\gamma} \right) + \frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{\mathrm{c}_\gamma} \left(\frac{\sigma_2}{\gamma^2} + \frac{k'}{\mathrm{Pm}\gamma} \right),$$

$$H(z) = h + \frac{h'}{\gamma} \frac{\mathrm{s}(z) - \mathrm{s}_\gamma}{\mathrm{c}_\gamma} + \mathrm{Pm} \left(\frac{\mathrm{Ra}\cos\delta}{2\,\mathrm{Pr}\,\gamma} \left(z^2 - \frac{1}{4} + \frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{\gamma \mathrm{s}_\gamma} \right) + \right.$$

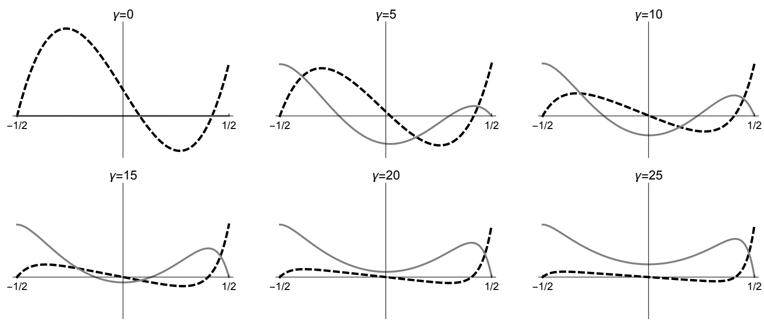
$$\left. + \frac{\sigma_1}{\gamma} \left(\frac{1}{2} - z + \frac{\mathrm{s}(z) - \mathrm{s}_\gamma}{\gamma \mathrm{c}_\gamma} \right) + \frac{u}{2} \left(\frac{\mathrm{s}_\gamma - \mathrm{s}(z)}{\mathrm{c}_\gamma} + \frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{\mathrm{s}_\gamma} \right) \right),$$

$$K(z) = k + \frac{k'}{\gamma} \frac{\mathrm{s}(z) - \mathrm{s}_\gamma}{\mathrm{c}_\gamma} + \mathrm{Pm} \left(\frac{\sigma_2}{\gamma} \left(\frac{1}{2} - z + \frac{\mathrm{s}(z) - \mathrm{s}_\gamma}{\gamma \mathrm{c}_\gamma} \right) + \frac{v}{2} \left(\frac{\mathrm{s}_\gamma - \mathrm{s}(z)}{\mathrm{c}_\gamma} + \frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{\mathrm{s}_\gamma} \right) \right).$$

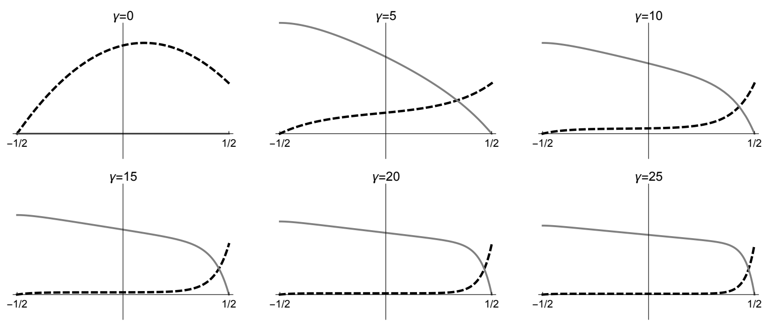
Also in this case we make a few plots to expose interesting features. The boundary conditions are set to zero in the first derivative of H at the boundary below and to zero in the value of H at the boundary above. We consider first a horizontal layer.



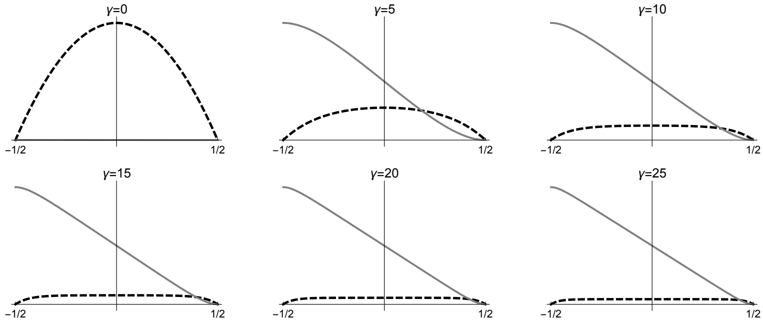
We then incline the layer posing $\delta = \pi/4$.



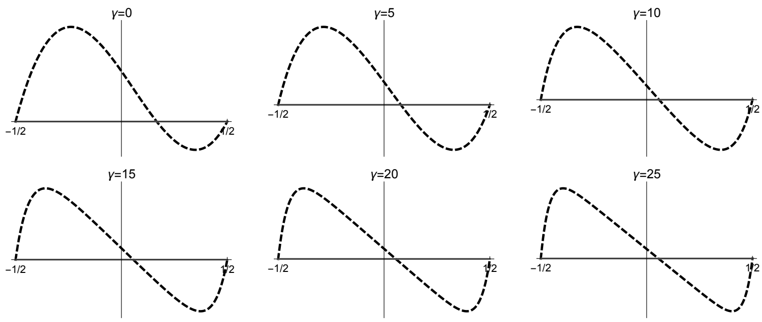
And we finally consider a horizontal layer but with a pressure term $\sigma_1 = 10$



We also consider a horizontal plane and set the velocity at the upper boundary to zero. To have a non-trivially zero velocity we pose $\sigma_1 = 10$.



We finally show the effect of inclining the layer posing $\delta = \pi/4$.



3.3. Rigid–stress free, electrically conducting–electrically nonconducting. In this final case the magnetic field must satisfy a condition on the value from above and on its first derivative from below, while the velocity must satisfy a condition on the value from below (it is not restrictive, as before, to assume it equal to zero) and a condition on the first derivative from above. In this case the general solutions are cumbersome to write, we write here only the functions U and H, the functions V, K can be easily deduced from U, H by cancelling the terms multiplied by $\cos\delta$ and making a suitable change of the boundary conditions. The first component of the velocity U has boundary value zero at the lower boundary and u' at the upper boundary, the first component of the magnetic field H has value h' at the lower boundary and h at the upper boundary. The two solutions are

$$U(z) = \frac{\left(\cosh(\gamma) - \cosh\left(\gamma(z-\frac{1}{2})\right)\right)(\gamma h' + \mathrm{Pm}\,\sigma_1) + \gamma \mathrm{Pm}\, u' \sinh\left(\gamma\left(z+\frac{1}{2}\right)\right)}{\mathrm{Pm}\,\gamma^2 \cosh(\gamma)} + \\ + \mathrm{Ra}\cos\delta\, \frac{2\sinh\left(\gamma(z+\frac{1}{2})\right) - 2\gamma z\cosh(\gamma) + \gamma\cosh\left(\gamma(z-\frac{1}{2})\right)}{2\gamma^3\,\mathrm{Pr}\cosh(\gamma)},$$

$$
\begin{aligned}
H(z) = h + h' \left(\frac{\mathrm{c}_\gamma \mathrm{s}(z) - \mathrm{s}_\gamma \mathrm{c}(z)}{\gamma \cosh(\gamma)} \right) + \mathrm{Pm} \Bigg(& \frac{\mathrm{Ra} \cos\delta}{\gamma^3 \mathrm{Pr}} - \frac{\mathrm{Ra}\cos\delta}{8\gamma \mathrm{Pr}} + \\
& + \frac{\mathrm{s}(z)}{\cosh\gamma} \left(\frac{\sigma_1 \mathrm{c}_\gamma}{\gamma^2} - \frac{\mathrm{Ra}\,\mathrm{s}_\gamma \cos\delta}{\gamma^3 \mathrm{Pr}} + \frac{\mathrm{Ra}\,\mathrm{c}_\gamma \cos\delta}{2\gamma^2 \mathrm{Pr}} - \frac{u' \mathrm{s}_\gamma}{\gamma} \right) + \\
& - \frac{\mathrm{c}(z)}{\cosh\gamma} \left(\frac{\sigma_1 \mathrm{s}_\gamma}{\gamma^2} + \frac{\mathrm{Ra}\,\mathrm{c}_\gamma \cos\delta}{\gamma^3 \mathrm{Pr}} + \frac{\mathrm{Ra}\,\mathrm{s}_\gamma \cos\delta}{2\gamma^2 \mathrm{Pr}} + \frac{u' \mathrm{c}_\gamma}{\gamma} \right) + \\
& + \frac{\mathrm{Ra}\cos\delta}{2\gamma \mathrm{Pr}} z^2 + \frac{u'}{\gamma} + \sigma_1 \left(\frac{1}{2\gamma} - \frac{z}{\gamma} \right) \Bigg).
\end{aligned}
$$

The limit as γ goes to zero can be obtained by Taylor expanding the expressions above or by computing the solutions directly from (6), (7) and (8). The limit functions are

$$
\begin{aligned}
U(z) &= \frac{\mathrm{Ra}}{\mathrm{Pr}} \cos\delta \frac{z^3}{6} - \sigma_1 \frac{z^2}{2} + \left(u' + \frac{\sigma_1}{2} - \frac{\mathrm{Ra}}{8\mathrm{Pr}} \cos\delta \right) z + \frac{\mathrm{Ra}\cos\delta}{24\,\mathrm{Pr}} - \frac{1}{8} \left(\sigma_1 + 4u' \right), \\
H(z) &= h' z + h - \frac{h'}{2}.
\end{aligned}
$$

The pictures relative to this case could be easily done and analyzed, we omit doing it here for the sake of brevity. We only observe that in this case a reasonable Rayleigh number would be $\mathrm{Ra} = 1100$.

4. Limiting cases

4.1. Isothermal or horizontal case. The coefficient $\cos\delta$, due to the inclination of the plane, always appears multiplied by the Rayleigh number Ra. In fact, in absence of thermal expansion the effect of inclination is equivalent to the effect of a driving, conservative force, that can be exerted assuming a pressure not independent of the variables x, y (i.e. non-zero constants σ_1, σ_2). This is the laminar case of electrically conducting layer of fluid. The solution in the case rigid–rigid, nonconducting–nonconducting are $H_3(z) = \gamma$ and

$$
U(z) = \frac{u}{2} \frac{\mathrm{s}(z)}{\mathrm{s}_\gamma} + \frac{u}{2} + \frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{2\gamma \mathrm{s}_\gamma} \left(\sigma_1 - \frac{\widetilde{h}\gamma}{\mathrm{Pm}} \right),
$$

$$
V(z) = \frac{v}{2} \frac{\mathrm{s}(z)}{\mathrm{s}_\gamma} + \frac{v}{2} + \frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{2\gamma \mathrm{s}_\gamma} \left(\sigma_2 - \frac{\widetilde{k}\gamma}{\mathrm{Pm}} \right),
$$

$$
H(z) = \frac{\widehat{h}}{2} + \frac{\widetilde{h}}{2} \frac{\mathrm{s}(z)}{\mathrm{s}_\gamma} + \mathrm{Pm} \left(\frac{u}{2} \frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{\mathrm{s}_\gamma} + \left(\frac{\mathrm{s}(z)}{2\mathrm{s}_\gamma} - z \right) \frac{\sigma_1}{\gamma} \right),
$$

$$
K(z) = \frac{\widehat{k}}{2} + \frac{\widetilde{k}}{2} \frac{\mathrm{s}(z)}{\mathrm{s}_\gamma} + \mathrm{Pm} \left(\frac{v}{2} \frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{\mathrm{s}_\gamma} + \left(\frac{\mathrm{s}(z)}{2\mathrm{s}_\gamma} - z \right) \frac{\sigma_2}{\gamma} \right).
$$

From these equations we can obtain, in the limiting case $\gamma \to 0$, the classical Couette, and Poiseuille flows (depending on u, v and σ_1, σ_2). We note the complete symmetry of the functions $U(z)$ and $V(z)$ and of $H(z)$ and $K(z)$. This symmetry disappears if one of the σ is zero and the other is different from zero. In this case we may have a superposition of Couette flow in one direction and Poiseuille flow in the other direction. For instance $H_3(z) = \gamma$ and

$$
\begin{aligned}
U(z) &= \frac{u}{2}\left(1+\frac{\mathrm{s}(z)}{\mathrm{s}_\gamma}\right) - \frac{\widetilde{h}}{\mathrm{Pm}}\,\frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{2\,\mathrm{s}_\gamma},\\
V(z) &= \frac{v}{2}\left(1+\frac{\mathrm{s}(z)}{\mathrm{s}_\gamma}\right) + \frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{2\gamma \mathrm{s}_\gamma}\left(\sigma_2 - \frac{\widetilde{k}\gamma}{\mathrm{Pm}}\right),\\
H(z) &= \frac{\widehat{h}}{2} + \frac{\widetilde{h}}{2}\frac{\mathrm{s}(z)}{\mathrm{s}_\gamma} + \mathrm{Pm}\left(\frac{u}{2}\frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{\mathrm{s}_\gamma}\right),\\
K(z) &= \frac{\widehat{k}}{2} + \frac{\widetilde{k}}{2}\frac{\mathrm{s}(z)}{\mathrm{s}_\gamma} + \mathrm{Pm}\left(\frac{v}{2}\frac{\mathrm{c}_\gamma - \mathrm{c}(z)}{\mathrm{s}_\gamma} + \left(\frac{\mathrm{s}(z)}{2\mathrm{s}_\gamma} - z\right)\frac{\sigma_2}{\gamma}\right).
\end{aligned}
$$

In the asymptotic case $\gamma \ll 1$ we obtain

$$
\begin{aligned}
U(z) &= \frac{u}{2} + uz + \frac{\gamma\left(z^2 - \frac{1}{4}\right)\widetilde{h}}{2\,\mathrm{Pm}} + o(\gamma),\\
V(z) &= \frac{v}{2} + vz - \frac{\sigma_2\left(z^2 - \frac{1}{4}\right)}{2} + \frac{\gamma\left(z^2 - \frac{1}{4}\right)\widetilde{k}}{2\,\mathrm{Pm}} + o(\gamma),\\
H(z) &= \frac{\widehat{h}}{2} + \widetilde{h}z - \mathrm{Pm}\,\gamma\frac{u}{2}\left(z^2 - \frac{1}{6}\right) + o(\gamma),\\
K(z) &= \frac{\widehat{k}}{2} + \widetilde{k}z - \mathrm{Pm}\,\gamma\frac{u}{2}\left(z^2 - \frac{1}{6}\right) - \mathrm{Pm}\,\gamma\frac{\sigma_2}{6}z\left(z^2 - \frac{1}{4}\right) + o(\gamma).
\end{aligned}
$$

In the limit as $\gamma \to 0$, we have

$$
\begin{aligned}
U(z) &= \frac{u}{2} + uz, \qquad & V(z) &= \frac{v}{2} + vz - \frac{\sigma_2\left(z^2 - \frac{1}{4}\right)}{2},\\
H(z) &= \frac{\widehat{h}}{2} + \widetilde{h}z, \qquad & K(z) &= \frac{\widehat{k}}{2} + \widetilde{k}z.
\end{aligned}
$$

This is, in particular, the coplanar case discussed in the next section.

4.2. Magnetic coplanar case. With *coplanar case* one indicates the case in which the third component of the magnetic field vanishes at the boundaries (the magnetic field is coplanar with the layer). This corresponds to the case $\gamma = 0$ and can be obtained with a limit process from the general solutions.

A Taylor expansion in γ of the functions U, V, H, K truncated at the second order gives

$$U(z) = (2z+1)\frac{3\mathrm{Pr}(4u+\sigma_1(1-2z))+\mathrm{Ra}z(2z-1)\cos\delta}{24\mathrm{Pr}}+ \\ +\frac{15\mathrm{Pr}(16uz+\sigma_1\left(1-4z^2\right))+\mathrm{Ra}\left(12z^2-7\right)z\cos\delta}{5760\mathrm{Pr}}\left(4z^2-1\right)\gamma^2 \quad (13)$$

$$V(z) = \frac{1}{8}(2z+1)(4v+\sigma_2(1-2z)) + \left(4z^2-1\right)\frac{16vz+\sigma_2\left(1-4z^2\right)}{384}\gamma^2 \quad (14)$$

$$H(z) = \frac{\widehat{h}}{2}+\widetilde{h}z+ \\ \mathrm{Pm}\left(4z^2-1\right)\frac{16\mathrm{Pr}(\sigma_1 z-3u)+\mathrm{Ra}\cos\delta(1-4z^2)}{384\,\mathrm{Pr}}\gamma-\frac{1}{24}\widetilde{h}z\left(4z^2-1\right)\gamma^2 \quad (15)$$

$$K(z) = \frac{\widehat{k}}{2} + \widetilde{k}z + \left(\frac{1}{24}\mathrm{Pm}\left(4z^2-1\right)(\sigma_2 z-3v)\right)\gamma + \left(\frac{1}{24}\widetilde{k}z\left(4z^2-1\right)\right)\gamma^2 \quad (16)$$

Lemma 2. *If the velocity field and the magnetic field are regular solutions of* (6) *and have the form* $\mathbf{U} = U(z)\,\mathbf{i}+V(z)\,\mathbf{j}$, $\mathbf{H} = H(z)\,\mathbf{i}+K(z)\,\mathbf{j}+H_3\mathbf{k}$, *with* $\mathbf{U}'$ *not identically zero. Then* $H_3 = 0$ *if and only if* $H(z)$ *or* $K(z)$ *are linear functions of* z*.*

Proof. From equation (6) it easily follows $H_3 = 0$ if and only if $H(z)$ or $K(z)$ are linear functions of z. □

Lemma 2 implies that the magnetic field is coplanar.

5. Conclusions

Analytical solutions of stationary laminar flows of an inclined layer filled with a hydromagnetic fluid heated from below and subject to the gravity field have been obtained for different physical boundary conditions. The effects of inclination and of heating from below have been examined. These effects give terms which are cubic functions of z on the velocity field. We have also investigated many limiting cases:

a) letting the Hartmann number $\gamma \to 0$ (the coplanar case). This case can have very important applications for example in the dynamics of blood (Tao and Huang 2011);
b) letting Ra $\cos\delta \to 0$ (isothermal or horizontal case).

Here we have studied the analytical solutions of the equations with different boundary conditions. Some figures have been drawn in order to explain how the kinetic and magnetic field change depending on the many physical parameters (the pressure σ_i, the heating Ra the inclination δ, the Hartmann or Chandrasekhar number γ and the boundary conditions). In particular the classical Hartman and Couette magnetic flow have been obtained.

In future works we plan to study linear instability and non linear stability. In particular, we will study the dependence of the stability thresholds on the inclination angle.

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[a] Università degli Studi di Catania
Dipartimento di Matematica e Informatica
Viale Andrea Doria 6, 95125 Catania, Italy

* To whom correspondence should be addressed | email: mulone@dmi.unict.it

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