

MULTIPOLAR HARDY INEQUALITIES ON RIEMANNIAN MANIFOLDS

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Abstract. We prove multipolar Hardy inequalities on complete Riemannian manifolds, providing various curved counterparts of some Euclidean multipolar inequalities due to Cazacu and Zuazua [Improved multipolar Hardy inequalities, 2013]. We notice that our inequalities deeply depend on the curvature, providing (quantitative) information about the deflection from the flat case. By using these inequalities together with variational methods and group-theoretical arguments, we also establish non-existence, existence and multiplicity results for certain Schrödinger-type problems involving the Laplace-Beltrami operator and bipolar potentials on Cartan-Hadamard manifolds and on the open upper hemisphere, respectively.

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1. INTRODUCTION

The classical *unipolar Hardy inequality* (or, uncertainty principle) states that if $n \geq 3$, then

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx, \quad \forall u \in C_0^\infty(\mathbb{R}^n);$$

here, the constant $\frac{(n-2)^2}{4}$ is sharp and not achieved. Many efforts have been made over the last two decades to improve/extend Hardy inequalities in various directions. One of the most challenging research topics in this direction is the so-called *multipolar Hardy inequality*. Such kind of extension is motivated by molecular physics and quantum chemistry/cosmology. Indeed, by describing the behavior of electrons and atomic nuclei in a molecule within the theory of Born-Oppenheimer approximation or Thomas–Fermi theory, particles can be modeled as certain singularities/poles $x_1, \dots, x_m \in \mathbb{R}^n$, producing their effect within the form $x \mapsto |x - x_i|^{-1}$, $i \in \{1, \dots, m\}$. Having such mathematical models, several authors studied the behavior of the operator with

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1 inverse square potentials with multiple poles, namely

$$2 \quad \mathcal{L} := -\Delta - \sum_{i=1}^m \frac{\mu_i^+}{|x - x_i|^2},$$

3 see Bosi, Dolbeaut and Esteban [5], Cao and Han [7], Felli, Marchini and Terracini [16], Guo, Han and Niu [18],
4 Lieb [29], Adimurthi [1], and references therein. Very recently, Cazacu and Zuazua [10] proved an optimal
5 multipolar counterpart of the above (unipolar) Hardy inequality, *i.e.*,

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_{\mathbb{R}^n} \frac{|x_i - x_j|^2}{|x - x_i|^2 |x - x_j|^2} u^2 dx, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \quad (1.1)$$

6 where $n \geq 3$, and $x_1, \dots, x_m \in \mathbb{R}^n$ are different poles; moreover, the constant $\frac{(n-2)^2}{m^2}$ is optimal. By using the
7 parallelogram law, (1.1) turns to be equivalent to

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_{\mathbb{R}^n} \left| \frac{x - x_i}{|x - x_i|^2} - \frac{x - x_j}{|x - x_j|^2} \right|^2 u^2 dx, \quad \forall u \in C_0^\infty(\mathbb{R}^n). \quad (1.2)$$

8 All of the aforementioned works considered the flat/isotropic setting where no external force is present. Once
9 the ambient space structure is perturbed, coming for instance by a magnetic or gravitational field, the above
10 results do not provide a full description of the physical phenomenon due to the presence of the *curvature*.

11 In order to discuss such a curved setting, we put ourselves into the Riemannian realm, *i.e.*, we consider an
12 $n (\geq 3)$ -dimensional complete Riemannian manifold (M, g) , $d_g : M \times M \rightarrow [0, \infty)$ is its usual distance function
13 associated to the Riemannian metric g , dv_g is its canonical volume element, $\exp_x : T_x M \rightarrow M$ is its standard
14 exponential map, and $\nabla_g u(x)$ is the gradient of a function $u : M \rightarrow \mathbb{R}$ at $x \in M$, respectively. Clearly, in the
15 curved setting of (M, g) , the vector $x - x_i$ and distance $|x - x_i|$ should be reformulated into a geometric context
16 by considering $\exp_{x_i}^{-1}(x)$ and $d_g(x, x_i)$, respectively. Note that

$$17 \quad \nabla_g d_g(\cdot, y)(x) = -\frac{\exp_x^{-1}(y)}{d_g(x, y)} \text{ for every } y \in M, x \in M \setminus (\{y\} \cup \text{cut}(y)),$$

18 where $\text{cut}(y)$ denotes the cut-locus of y on (M, g) . In this setting, a natural question arises: if $\Omega \subseteq M$ is an
19 open domain and $S = \{x_1, \dots, x_m\} \subset \Omega$ is the set of distinct poles, can we prove

$$\int_{\Omega} |\nabla_g u|^2 dv_g \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_{\Omega} V_{ij}(x) u^2 dx, \quad \forall u \in C_0^\infty(\Omega), \quad (1.3)$$

20 where

$$21 \quad V_{ij}(x) = \frac{d_g(x_i, x_j)^2}{d_g(x, x_i)^2 d_g(x, x_j)^2} \text{ or } V_{ij}(x) = \left| \frac{\nabla_g d_g(x, x_i)}{d_g(x, x_i)} - \frac{\nabla_g d_g(x, x_j)}{d_g(x, x_j)} \right|^2?$$

22 Clearly, in the Euclidean space \mathbb{R}^n , inequality (1.3) corresponds to (1.1) and (1.2), for the above choices of V_{ij} ,
23 respectively. It turns out that the answer deeply depends on the curvature of the Riemannian manifold (M, g) .
24 Indeed, if the Ricci curvature verifies $\text{Ric}(M, g) \geq c_0(n-1)g$ for some $c_0 > 0$ (as in the case of the n -
25 dimensional unit sphere \mathbb{S}^n), we know by the theorem of Bonnet-Myers that (M, g) is compact; thus, we may
26 use the constant functions $u \equiv c \in \mathbb{R}$ as test-functions in (1.3), and we get a contradiction. However, when
27 (M, g) is a Cartan-Hadamard manifold (*i.e.*, complete, simply connected Riemannian manifold with non-
28 positive sectional curvature), we can expect the validity of (1.3), see Theorems 1.1 and 1.3 and suitable Laplace
29 comparison theorems, respectively.

Accordingly, the primary aim of the present paper is to investigate *multipolar Hardy inequalities on complete Riemannian manifolds*. We emphasize that such a study requires new technical and theoretical approaches. In fact, we need to explore those geometric and analytic properties which are behind of the theory of multipolar Hardy inequalities in the flat context, formulated now in terms of curvature, geodesics, exponential map, *etc.* We notice that striking results were also achieved recently in the theory of unipolar Hardy-type inequalities on curved spaces. The pioneering work of Carron [9], who studied Hardy inequalities on complete non-compact Riemannian manifolds, opened new perspectives in the study of functional inequalities with singular terms on curved spaces. Further contributions have been provided by D'Ambrosio and Dipierro [11], Kombe and Özaydin [22,23], Xia [39], and Yang, Su and Kong [40], where various improvements of the usual Hardy inequality is presented on complete, non-compact Riemannian manifolds. Moreover, certain unipolar Hardy and Rellich type inequalities were obtained on non-reversible Finsler manifolds by Farkas, Kristály and Varga [15], and Kristály and Repovš [27].

In the sequel we shall present our results; for further use, let Δ_g be the Laplace-Beltrami operator on (M, g) . Let $m \geq 2$, $S = \{x_1, \dots, x_m\} \subset M$ be the set of poles with $x_i \neq x_j$ if $i \neq j$, and for simplicity of notation, let $d_i = d_g(\cdot, x_i)$ for every $i \in \{1, \dots, m\}$. Our first result reads as follows.

Theorem 1.1 (Multipolar Hardy inequality I). *Let (M, g) be an n -dimensional complete Riemannian manifold and $S = \{x_1, \dots, x_m\} \subset M$ be the set of distinct poles, where $n \geq 3$ and $m \geq 2$. Then*

$$\int_M |\nabla_g u|^2 dv_g \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_M \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 dv_g + \frac{n-2}{m} \sum_{i=1}^m \int_M \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 dv_g, \quad \forall u \in C_0^\infty(M). \tag{1.4}$$

Moreover, in the bipolar case (*i.e.*, $m = 2$), the constant $\frac{(n-2)^2}{m^2} = \frac{(n-2)^2}{4}$ is optimal in (1.4).

Remark 1.2.

- (a) The proof of inequality (1.4) is based on a direct calculation. If $m = 2$, the local behavior of geodesic balls implies the optimality of the constant $\frac{(n-2)^2}{m^2} = \frac{(n-2)^2}{4}$; in particular, the second term is a lower order perturbation of the first one of the RHS (independently of the curvature).
- (b) The optimality of $\frac{(n-2)^2}{m^2}$ seems to be a hard nut to crack. A possible approach could be a fine Agmon-Allegretto-Piepenbrink-type spectral estimate developed by Devyver [12] and Devyver, Fraas and Pinchover [13] whenever (M, g) has asymptotically non-negative Ricci curvature (see Pigola, Rigoli and Setti [32], Cor. 2.17, p. 44). Indeed, under this curvature assumption one can prove that the operator $-\Delta_g - W$ is critical (see [13], Def. 4.3), where

$$W = \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 + \frac{n-2}{m} \sum_{i=1}^m \frac{d_i \Delta_g d_i - (n-1)}{d_i^2}.$$

Although expected, we have no full control on the second summand with respect to the first one in W , *i.e.*, the latter term could compete with the 'leading' one; clearly, in the Euclidean setting no such competition is present, thus the optimality of $\frac{(n-2)^2}{m^2}$ immediately follows by the criticality of W . It remains to investigate this issue in a forthcoming study.

- (c) We emphasize that the second term in the RHS of (1.4) has a crucial role. Indeed, on one hand, when the Ricci curvature verifies $\text{Ric}(M, g) \geq c_0(n-1)g$ for some $c_0 > 0$, one has that $d_i(x) = g_d(x, x_i) \leq \pi/\sqrt{c_0}$ for every $x \in M$ and by the Laplace comparison theorem, we have that $d_i \Delta_g d_i - (n-1) \leq$

1 $(n - 1)(\sqrt{c_0}d_i \cot(\sqrt{c_0}d_i) - 1) < 0$ for $d_i > 0$, *i.e.* for every $x \neq x_i$. Thus, this term modifies the original
 2 problem (1.3) by filling the gap in a suitable way. On the other hand, when (M, g) is a Cartan–Hadamard
 3 manifold, one has $d_i \Delta_g d_i - (n - 1) \geq 0$, and inequality (1.4) implies (1.3). This result will be resumed
 4 in Corollary 4.1 (i). In particular, when $M = \mathbb{R}^n$ is the Euclidean space, then $\exp_x(y) = x + y$ for every
 5 $x, y \in \mathbb{R}^n$ and $|x|\Delta|x| = n - 1$ for every $x \neq 0$; therefore, Theorem 1.1 and the criticality of $-\Delta - W$
 6 immediately yield the main result of Cazacu and Zuazua [10], *i.e.*, inequality (1.2) (and equivalently (1.1)).

7 For further use, we notice that $\mathbf{K} \geq c$ (resp. $\mathbf{K} \leq c$) means that the sectional curvature on (M, g) is bounded
 8 from below (resp. above) by $c \in \mathbb{R}$ at any point and direction.

9 For every $c \in \mathbb{R}$, let $\mathbf{s}_c, \mathbf{ct}_c : [0, \infty) \rightarrow \mathbb{R}$ be defined by

$$\mathbf{s}_c(r) = \begin{cases} \frac{\sin(\sqrt{c}r)}{\sqrt{c}} & \text{if } c > 0, \\ r & \text{if } c = 0, \\ \frac{\sinh(\sqrt{-c}r)}{\sqrt{-c}} & \text{if } c < 0, \end{cases} \quad \text{and} \quad \mathbf{ct}_c(r) = \begin{cases} \sqrt{c} \cot(\sqrt{c}r) & \text{if } c > 0, \\ \frac{1}{r} & \text{if } c = 0, \\ \sqrt{-c} \coth(\sqrt{-c}r) & \text{if } c < 0. \end{cases} \quad (1.5)$$

10 Although the paralelogrammoid law in the Euclidean setting provides the equivalence between (1.1) and (1.2),
 11 this property is no longer valid on generic manifolds. However, a curvature-based quantitative paralelogrammoid
 12 law and a Toponogov-type comparison result provide a suitable counterpart of inequality (1.1):

13 **Theorem 1.3 (Multipolar Hardy inequality II).** *Let (M, g) be an n -dimensional complete Riemannian*
 14 *manifold with $\mathbf{K} \geq k_0$ for some $k_0 \in \mathbb{R}$ and let $S = \{x_1, \dots, x_m\} \subset M$ be the set of distinct poles belonging to*
 15 *a strictly convex open set $\tilde{S} \subset M$, where $n \geq 3$ and $m \geq 2$. Then we have the following inequality:*

$$\begin{aligned} \int_{\tilde{S}} |\nabla_g u|^2 dv_g &\geq \frac{4(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_{\tilde{S}} \frac{\mathbf{s}_{k_0}^2\left(\frac{d_{ij}}{2}\right)}{d_i d_j \mathbf{s}_{k_0}(d_i) \mathbf{s}_{k_0}(d_j)} u^2 dv_g + \sum_{1 \leq i < j \leq m} \int_{\tilde{S}} R_{ij}(k_0) u^2 dv_g \\ &+ \frac{n-2}{m} \sum_{i=1}^m \int_{\tilde{S}} \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 dv_g, \quad \forall u \in C_0^\infty(\tilde{S}), \end{aligned} \quad (1.6)$$

16 where $d_{ij} = d_g(x_i, x_j)$ and

$$17 \quad R_{ij}(k_0) = \begin{cases} \frac{1}{d_i^2} + \frac{1}{d_j^2} - \frac{2}{k_0 d_i d_j} \left(\frac{1}{\mathbf{s}_{k_0}(d_i) \mathbf{s}_{k_0}(d_j)} - \mathbf{ct}_{k_0}(d_i) \mathbf{ct}_{k_0}(d_j) \right), & \text{if } k_0 \neq 0, \\ 0, & \text{if } k_0 = 0. \end{cases}$$

18 **Remark 1.4.** When (M, g) is a Cartan–Hadamard manifold and $k_0 \leq 0$, one has that $R_{ij}(k_0) \geq 0$; thus we
 19 obtain a similar result as in (1.3); the precise statement will be given in Corollary 4.1 (ii).

20 *Applications.* As we already noticed, multipolar Hardy inequalities have been applied in the flat case to guaranty
 21 existence and uniqueness of solutions for various elliptic PDEs. If the particles (*e.g.* the fermions appearing in
 22 the Thomas–Fermi theory, see Lieb [29]) are distributed in a curved space, the aforementioned works cannot
 23 be applied. For instance, if some external forces perturb the flat model (present as a magnetic or gravitational
 24 field), the curvature will appear. Such a typical case occurs in the study of classical particles in the Lobachevsky
 25 hyperbolic model or spherical Riemannian model, described recently by Kudryashov, Kurochkin, Ovsyuk and
 26 Red'kov [28], and Cariñena, Rañada and Santander [8].

Motivated by the latter investigations on curved frameworks, we consider two model Schrödinger-type equations involving bipolar potentials in two different geometrical settings, namely, in the negatively and positively curved case, where our multipolar Hardy inequalities can be successfully applied:

A. *Non-positively curved case.* Let (M, g) be an $n (\geq 3)$ -dimensional Cartan–Hadamard manifold with $\mathbf{K} \geq k_0$ for some $k_0 \leq 0$, and $S = \{x_1, x_2\} \subset M$ be the set of poles. By keeping the previous notations, we consider the problem

$$-\Delta_g u + V(x)u = \lambda \frac{\mathbf{s}_{k_0}^2 \left(\frac{d_{12}}{2}\right)}{d_1 d_2 \mathbf{s}_{k_0}(d_1) \mathbf{s}_{k_0}(d_2)} u + \mu W(x) f(u) \quad \text{in } M, \quad (\mathcal{P}_M^\mu) \quad 7$$

where $V, W : M \rightarrow \mathbb{R}$ are positive potentials, $\lambda \in [0, (n-2)^2]$ is fixed, $\mu \geq 0$ is a parameter, and the continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is sublinear at infinity. In Theorem 4.3 we prove that problem (\mathcal{P}_M^μ) has only the zero solution for small values of μ , while it exists $\mu_0 > 0$ such that (\mathcal{P}_M^μ) has two distinct weak solutions in a suitable functional space whenever $\mu \geq \mu_0$.

B. *Positively curved case.* If \mathbb{S}_+^n denotes the open upper hemisphere and $S = \{x_1, x_2\} \subset \mathbb{S}_+^n$ is the set of poles, we study the Dirichlet problem

$$\begin{cases} -\Delta_g u + C(n, \beta)u = \lambda \left| \frac{\nabla_g d_1}{d_1} - \frac{\nabla_g d_2}{d_2} \right|^2 u + |u|^{p-2} u, & \text{in } \mathbb{S}_+^n \\ u = 0, & \text{on } \partial \mathbb{S}_+^n, \end{cases} \quad (\mathcal{P}_{\mathbb{S}_+^n}^\mu) \quad 15$$

where g is the usual Riemannian structure on the unit sphere \mathbb{S}^n inherited by \mathbb{R}^{n+1} , $\lambda \in [0, \frac{(n-2)^2}{4}]$ is fixed, $C(n, \beta) > 0$ is given in Corollary 4.5 and $p \in (2, 2^*)$; hereafter, $2^* = 2n/(n-2)$ is the critical Sobolev exponent. In Theorem 4.6 we prove the existence of infinitely many solutions for $(\mathcal{P}_{\mathbb{S}_+^n}^\mu)$; moreover, by using group-theoretical arguments, we provide qualitative results on the solutions concerning their symmetries whenever the poles x_1 and x_2 are in specific positions.

The plan of the paper is as follows. In Section 2 we present a series of preparatory definitions and results which are used throughout the paper. In Section 3 we prove the multipolar Hardy inequalities, *i.e.*, Theorems 1.1 & 1.3. In Section 4 we study problems (\mathcal{P}_M^μ) and $(\mathcal{P}_{\mathbb{S}_+^n}^\mu)$, while in Section 5 we formulate some remarks concerning further questions/perspectives.

2. PRELIMINARIES

Let (M, g) be an n -dimensional complete Riemannian manifold ($n \geq 3$). As usual, $T_x M$ denotes the tangent space at $x \in M$ and $TM = \bigcup_{x \in M} T_x M$ is the tangent bundle. Let $d_g : M \times M \rightarrow [0, \infty)$ be the distance function associated to the Riemannian metric g , and $B_r(x) = \{y \in M : d_g(x, y) < r\}$ be the open geodesic ball with center $x \in M$ and radius $r > 0$. If dv_g is the canonical volume element on (M, g) , the volume of a bounded open set $S \subset M$ is $\text{Vol}_g(S) = \int_S dv_g$. The behaviour of the volume of small geodesic balls can be expressed as follows, see Gallot, Hulin and Lafontaine [17]; for every $x \in M$ we have

$$\text{Vol}_g(B_r(x)) = \omega_n r^n (1 + o(r)) \quad \text{as } r \rightarrow 0. \quad (2.1)$$

Let $u : M \rightarrow \mathbb{R}$ be a function of class C^1 . If (x^i) denotes the local coordinate system on a coordinate neighbourhood of $x \in M$, and the local components of the differential of u are denoted by $u_i = \frac{\partial u}{\partial x_i}$, then the local components of the gradient $\nabla_g u$ are $u^i = g^{ij} u_j$. Here, g^{ij} are the local components of $g^{-1} = (g_{ij})^{-1}$. In particular, for every $x_0 \in M$ one has the eikonal equation

$$|\nabla_g d_g(x_0, \cdot)| = 1 \quad \text{a.e. on } M. \quad (2.2)$$

1 In fact, relation (2.2) is valid for every point $x \in M$ outside of the cut-locus of x_0 (which is a null measure set).

2 When no confusion arises, if $X, Y \in T_x M$, we simply write $|X|$ and $\langle X, Y \rangle$ instead of the norm $|X|_x$ and
 3 inner product $g_x(X, Y) = \langle X, Y \rangle_x$, respectively. The $L^p(M)$ norm of $\nabla_g u(x) \in T_x M$ is given by

$$4 \quad \|\nabla_g u\|_{L^p(M)} = \left(\int_M |\nabla_g u|^p dv_g \right)^{1/p}.$$

5 The space $H_g^1(M)$ is the completion of $C_0^\infty(M)$ with respect to the norm

$$6 \quad \|u\|_{H_g^1(M)} = \sqrt{\|u\|_{L^2(M)}^2 + \|\nabla_g u\|_{L^2(M)}^2}.$$

7 The Laplace-Beltrami operator is given by $\Delta_g u = \operatorname{div}(\nabla_g u)$ whose expression in a local chart of associated
 8 coordinates (x^i) is $\Delta_g u = g^{ij} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial u}{\partial x_k} \right)$, where Γ_{ij}^k are the coefficients of the Levi-Civita connection.

9 In the sequel, we shall explore the following comparison results (see Shen [35], Wu and Xin [38], Thms. 6.1
 10 and 6.3, Pigola, Rigoli and Setti [32], Thm. 2.4):

11 • *Laplace comparison theorem I*: if $\mathbf{K} \leq c$ for some $c \in \mathbb{R}$, then

$$\Delta_g d_g(x_0, x) \geq (n-1) \mathbf{ct}_c(d_g(x_0, x)); \quad (2.3)$$

12 • *Laplace comparison theorem II*: if $\mathbf{K} \geq k_0$ for some $k_0 \in \mathbb{R}$, then

$$\Delta_g d_g(x_0, x) \leq (n-1) \mathbf{ct}_{k_0}(d_g(x_0, x)), \quad (2.4)$$

13 where these relations are understood in the distributional sense. Note that in (2.4) it is enough to have the
 14 lower bound $(n-1)k_0$ for the Ricci curvature.

15 3. MULTIPOLAR HARDY INEQUALITIES: PROOF OF THEOREMS 1.1 AND 1.3

16 *Proof of Theorem 1.1.* Let $E = \prod_{i=1}^m d_i^{2-n}$ and fix $u \in C_0^\infty(M)$ arbitrarily. A direct calculation on the set $M \setminus$
 17 $\bigcup_{i=1}^m (\{x_i\} \cup \operatorname{cut}(x_i))$ yields that

$$18 \quad \nabla_g \left(u E^{-\frac{1}{m}} \right) = E^{-\frac{1}{m}} \nabla_g u + \frac{n-2}{m} u E^{-\frac{1}{m}} \sum_{i=1}^m \frac{\nabla_g d_i}{d_i}.$$

19 Integrating the latter relation, the divergence theorem and eikonal equation (2.2) give that

$$\begin{aligned} 20 \quad \int_M \left| \nabla_g \left(u E^{-\frac{1}{m}} \right) \right|^2 E^{\frac{2}{m}} dv_g &= \int_M |\nabla_g u|^2 dv_g + \frac{(n-2)^2}{m^2} \int_M \left| \sum_{i=1}^m \frac{\nabla_g d_i}{d_i} \right|^2 u^2 dv_g \\ 21 \quad &+ \frac{n-2}{m} \sum_{i=1}^m \int_M \left\langle \nabla_g u^2, \frac{\nabla_g d_i}{d_i} \right\rangle dv_g \\ 22 \quad &= \int_M |\nabla_g u|^2 dv_g + \frac{(n-2)^2}{m^2} \int_M \left| \sum_{i=1}^m \frac{\nabla_g d_i}{d_i} \right|^2 u^2 dv_g \\ 23 \quad &- \frac{n-2}{m} \sum_{i=1}^m \int_M \operatorname{div} \left(\frac{\nabla_g d_i}{d_i} \right) u^2 dv_g. \end{aligned}$$

24

Due to (2.2), we have

$$\operatorname{div} \left(\frac{\nabla_g d_i}{d_i} \right) = \frac{d_i \Delta_g d_i - 1}{d_i^2}, \quad i \in \{1, \dots, m\}.$$

Thus, an algebraic reorganization of the latter relation provides an Agmon-Allegretto-Piepenbrink-type multipolar representation

$$\int_M |\nabla_g u|^2 dv_g - \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_M \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 dv_g = \int_M \left| \nabla_g \left(u E^{-1/m} \right) \right|^2 E^{2/m} dv_g + \frac{n-2}{m} \sum_{i=1}^m \mathcal{K}_i(u), \quad (3.1)$$

where $\mathcal{K}_i(u) = \int_M \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 dv_g$. Inequality (1.4) directly follows by (3.1).

In the sequel, we deal with the optimality of the constant $\frac{(n-2)^2}{m^2}$ in (1.4) when $m = 2$. In this case the right hand side of (1.4) behaves as $\frac{(n-2)^2}{4} d_g(x, x_i)^{-2}$ whenever $x \rightarrow x_i$ and by the local behavior of the geodesic balls (see (2.1)) we may expect the optimality of $\frac{(n-2)^2}{4}$. In order to be more explicit, let $A_i[r, R] = \{x \in M : r \leq d_i(x) \leq R\}$ for $r < R$ and $i \in \{1, \dots, m\}$. If $0 < r \ll R$ are within the range of (2.1), a layer cake representation yields for every $i \in \{1, \dots, m\}$ that

$$\begin{aligned} \int_{A_i[r, R]} d_i^{-n} dv_g &= \frac{\operatorname{Vol}_g(B_R(x_i))}{R^n} - \frac{\operatorname{Vol}_g(B_r(x_i))}{r^n} + n \int_r^R \operatorname{Vol}_g(B_\rho(x_i)) \rho^{-1-n} d\rho \\ &= o(R) + n\omega_n \log \frac{R}{r}. \end{aligned} \quad (3.2)$$

Let $S = \{x_1, x_2\}$ be the set of poles, $x_1 \neq x_2$. Let $\varepsilon \in (0, 1)$ be small enough such that it belongs to the range of (2.1), and $B_{2\sqrt{\varepsilon}}(x_1) \cap B_{2\sqrt{\varepsilon}}(x_2) = \emptyset$. Let

$$u_\varepsilon(x) = \begin{cases} \frac{\log \left(\frac{d_i(x)}{\varepsilon^2} \right)}{\log \left(\frac{1}{\varepsilon} \right)} d_i(x)^{\frac{2-n}{2}}, & \text{if } x \in A_i[\varepsilon^2, \varepsilon]; \\ \frac{2 \log \left(\frac{\sqrt{\varepsilon}}{d_i(x)} \right)}{\log \left(\frac{1}{\varepsilon} \right)} d_i(x)^{\frac{2-n}{2}}, & \text{if } x \in A_i[\varepsilon, \sqrt{\varepsilon}]; \\ 0, & \text{otherwise,} \end{cases}$$

with $i \in \{1, 2\}$. Note that $u_\varepsilon \in C^0(M)$, having compact support $\bigcup_{i=1}^2 A_i[\varepsilon^2, \sqrt{\varepsilon}] \subset M$; in fact, u_ε can be used as a test function in (1.4). For later use let us denote by $\varepsilon^* = \frac{1}{\log \left(\frac{1}{\varepsilon} \right)^2}$,

$$\mathcal{I}_\varepsilon = \int_M |\nabla_g u_\varepsilon|^2 dv_g, \quad \mathcal{L}_\varepsilon = \int_M \frac{\langle \nabla_g d_1, \nabla_g d_2 \rangle}{d_1 d_2} u_\varepsilon^2 dv_g, \quad \mathcal{K}_\varepsilon = \sum_{i=1}^2 \int_M \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u_\varepsilon^2 dv_g$$

and

$$\mathcal{J}_\varepsilon = \int_M \left[\frac{1}{d_1^2} + \frac{1}{d_2^2} \right] u_\varepsilon^2 dv_g.$$

The proof is based on the following claims:

$$\mathcal{I}_\varepsilon - \mu_H \mathcal{J}_\varepsilon = \mathcal{O}(1), \quad \mathcal{L}_\varepsilon = \mathcal{O}(\sqrt[4]{\varepsilon}) \quad \text{and} \quad \mathcal{K}_\varepsilon = \mathcal{O}(\sqrt[4]{\varepsilon}) \quad \text{as } \varepsilon \rightarrow 0, \quad (3.3)$$

and

$$\lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon = +\infty. \quad (3.4)$$

1 The above properties can be obtained by direct computations, based on the estimates (2.1), (3.2) and

$$2 \quad \left| \Delta_g d_i - \frac{n-1}{d_i} \right| \leq 1 \quad \text{a.e. in } B_{\sqrt{\varepsilon}}(x_i),$$

3 (for $\varepsilon > 0$ small enough), see Kristály and Repovš [27]. Combining relations (3.3) and (3.4) with inequality (1.4),
4 we have that

$$5 \quad \mu_H \leq \frac{\mathcal{I}_\varepsilon - \frac{n-2}{2}\mathcal{K}_\varepsilon}{\mathcal{J}_\varepsilon - 2\mathcal{L}_\varepsilon} \leq \frac{\mathcal{I}_\varepsilon + \frac{n-2}{2}|\mathcal{K}_\varepsilon|}{\mathcal{J}_\varepsilon - 2|\mathcal{L}_\varepsilon|} = \frac{\mu_H \mathcal{J}_\varepsilon + \mathcal{O}(1)}{\mathcal{J}_\varepsilon + \mathcal{O}(\sqrt[4]{\varepsilon})} \rightarrow \mu_H \quad \text{as } \varepsilon \rightarrow 0,$$

6 which concludes the proof. \square

7 **Remark 3.1.** Let us assume that in Theorem 1.1, (M, g) is a Riemannian manifold with sectional curvature
8 verifying $\mathbf{K} \leq c$. By the Laplace comparison theorem I (see (2.3)) we have:

$$9 \quad \int_M |\nabla_g u|^2 dv_g \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_M \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 dv_g$$

$$10 \quad + \frac{(n-2)(n-1)}{m} \sum_{i=1}^m \int_M \frac{\mathbf{D}_c(d_i)}{d_i^2} u^2 dv_g, \quad \forall u \in C_0^\infty(M), \quad (3.5)$$

11 where $\mathbf{D}_c(r) = r \mathbf{ct}_c(r) - 1$, $r \geq 0$. In addition, if (M, g) is a Cartan–Hadamard manifold with $\mathbf{K} \leq c \leq 0$,
12 then $\mathbf{D}_c(r) \geq \frac{3|c|r^2}{\pi^2 + |c|r^2}$ for all $r \geq 0$. Accordingly, stronger curvature of the Cartan–Hadamard manifold implies
13 improvement in the multipolar Hardy inequality (3.5).

14 *Proof of Theorem 1.3.* It is clear that

$$\left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 = \frac{1}{d_i^2} + \frac{1}{d_j^2} - 2 \frac{\langle \nabla_g d_i, \nabla_g d_j \rangle}{d_i d_j}. \quad (3.6)$$

15 Let us fix two arbitrary poles x_i and x_j ($i \neq j$), and a point $x \in \tilde{S}$. We consider the Alexandrov comparison
16 triangle with vertexes \tilde{x}_i , \tilde{x}_j and \tilde{x} in the space M_0 of constant sectional curvature k_0 , associated to the points
17 x_i , x_j and x , respectively. More precisely, M_0 is the n -dimensional hyperbolic space of curvature k_0 when $k_0 < 0$,
18 the Euclidean space when $k_0 = 0$, and the sphere with curvature k_0 when $k_0 > 0$.

19 We first prove that the perimeter $L(x_i x_j x)$ of the geodesic triangle $x_i x_j x$ is strictly less than $\frac{2\pi}{\sqrt{k_0}}$; clearly,
20 when $k_0 \leq 0$ we have nothing to prove. Due to the strict convexity of \tilde{S} , the unique geodesic segments joining
21 pairwise the points x_i , x_j and x belong entirely to \tilde{S} and as such, these points are not conjugate to each
22 other. Thus, due to do Carmo ([14], Prop. 2.4, p. 218), every side of the geodesic triangle has length $\leq \frac{\pi}{\sqrt{k_0}}$.
23 By Klingenberg ([21], Thm. 2.7.12, p. 226) we have that $L(x_i x_j x) \leq \frac{2\pi}{\sqrt{k_0}}$. Moreover, by the same result of
24 Klingenberg, if $L(x_i x_j x) = \frac{2\pi}{\sqrt{k_0}}$, it follows that either $x_i x_j x$ forms a closed geodesic, or $x_i x_j x$ is a geodesic
25 biangle (one of the sides has length $\frac{\pi}{\sqrt{k_0}}$ and the two remaining sides form together a minimizing geodesic of
26 length $\frac{\pi}{\sqrt{k_0}}$). In both cases we find points on the sides of the geodesic triangle $x_i x_j x$ which can be joined by two
27 minimizing geodesics, contradicting the strict convexity of \tilde{S} .

28 We are now in the position to apply a Toponogov-type comparison result, see Klingenberg ([21], Prop. 2.7.7,
29 p. 220); namely, we have the comparison of angles

$$30 \quad \gamma_{M_0} = m(\widehat{\tilde{x}_i \tilde{x}_j \tilde{x}}) \leq \gamma_M = m(\widehat{x_i x_j x}).$$

31 Therefore, $\langle \nabla_g d_i, \nabla_g d_j \rangle = \cos(\gamma_M) \leq \cos(\gamma_{M_0})$.

On the other hand, by the cosine-law on the space form M_0 , see Bridson and Haefliger ([6], p. 24), we have

$$\begin{cases} \cosh(\sqrt{-k_0}d_{ij}) = \cosh(\sqrt{-k_0}d_i) \cosh(\sqrt{-k_0}d_j) - \sinh(\sqrt{-k_0}d_i) \sinh(\sqrt{-k_0}d_j) \cos(\gamma_{M_0}), & \text{if } k_0 < 0; \\ \cos(\sqrt{k_0}d_{ij}) = \cos(\sqrt{k_0}d_i) \cos(\sqrt{k_0}d_j) + \sin(\sqrt{k_0}d_i) \sin(\sqrt{k_0}d_j) \cos(\gamma_{M_0}), & \text{if } k_0 > 0; \\ d_{ij}^2 = d_i^2 + d_j^2 - 2d_i d_j \cos(\gamma_{M_0}), & \text{if } k_0 = 0. \end{cases}$$

Consequently,

$$\begin{cases} \cos(\gamma_M) \leq \frac{\cosh(\sqrt{-k_0}d_i) \cosh(\sqrt{-k_0}d_j) - \cosh(\sqrt{-k_0}d_{ij})}{\sinh(\sqrt{-k_0}d_i) \sinh(\sqrt{-k_0}d_j)}, & \text{if } k_0 < 0; \\ \cos(\gamma_M) \leq \frac{\cos(\sqrt{k_0}d_{ij}) - \cos(\sqrt{k_0}d_i) \cos(\sqrt{k_0}d_j)}{\sin(\sqrt{k_0}d_i) \sin(\sqrt{k_0}d_j)}, & \text{if } k_0 > 0; \\ \cos(\gamma_M) \leq \frac{d_i^2 + d_j^2 - d_{ij}^2}{2d_i d_j}, & \text{if } k_0 = 0, \end{cases}$$

which implies

$$\frac{1}{d_i^2} + \frac{1}{d_j^2} - \frac{2 \cos(\gamma_M)}{d_i d_j} \geq \begin{cases} \frac{4}{d_i d_j} \frac{\mathbf{s}_{k_0}^2\left(\frac{d_{ij}}{2}\right)}{\mathbf{s}_{k_0}(d_i) \mathbf{s}_{k_0}(d_j)} + R_{ij}(k_0), & \text{if } k_0 \neq 0; \\ \frac{d_{ij}^2}{d_i^2 d_j^2}, & \text{if } k_0 = 0, \end{cases}$$

where the expression $R_{ij}(k_0)$ is given in the statement of the theorem. Relation (3.6), the above inequality and (1.4) imply together (1.6). \square

4. APPLICATIONS: BIPOLAR SCHRÖDINGER-TYPE EQUATIONS ON CURVED SETTINGS

In this section we present two applications in different geometric frameworks. In order to avoid technicalities, we shall restrict our attention to problems with only two poles; the interested reader may extend these results to multiple poles with suitable modifications.

4.1. A bipolar Schrödinger-type equation on Cartan–Hadamard manifolds

First of all, by using inequalities (1.4) and (1.6), we obtain the following non-positively curved versions of Cazacu and Zuazua's inequalities (1.2) and (1.1) for multiple poles, respectively:

Corollary 4.1. *Let (M, g) be an n -dimensional Cartan–Hadamard manifold and let $S = \{x_1, \dots, x_m\} \subset M$ be the set of distinct poles, with $n \geq 3$ and $m \geq 2$. Then we have the following inequality:*

$$\int_M |\nabla_g u|^2 dv_g \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_M \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 dv_g, \quad \forall u \in H_g^1(M). \quad (4.1)$$

Moreover, if $\mathbf{K} \geq k_0$ for some $k_0 \in \mathbb{R}$, then

$$\int_M |\nabla_g u|^2 dv_g \geq \frac{4(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_M \frac{\mathbf{s}_{k_0}^2\left(\frac{d_{ij}}{2}\right)}{d_i d_j \mathbf{s}_{k_0}(d_i) \mathbf{s}_{k_0}(d_j)} u^2 dv_g, \quad \forall u \in H_g^1(M). \quad (4.2)$$

Proof. Since (M, g) is a Cartan–Hadamard manifold, by using inequality (1.4) and the Laplace comparison theorem I (i.e., inequality (2.3) for $c = 0$), standard approximation procedure based on the density of $C_0^\infty(M)$ in $H_g^1(M)$ and Fatou's lemma immediately imply (4.1). Moreover, elementary properties of hyperbolic functions show that $R_{ij}(k_0) \geq 0$ (since $k_0 \leq 0$). Thus, the latter inequality and (1.6) yield (4.2). \square

1 In the sequel, let (M, g) be an n -dimensional Cartan–Hadamard manifold ($n \geq 3$) with $\mathbf{K} \geq k_0$ for some
 2 $k_0 \leq 0$, and $S = \{x_1, x_2\} \subset M$ be the set of poles. In this subsection we deal with the Schrödinger-type equation

$$3 \quad -\Delta_g u + V(x)u = \lambda \frac{\mathbf{s}_{k_0}^2 \left(\frac{d_{12}}{2}\right)}{d_1 d_2 \mathbf{s}_{k_0}(d_1) \mathbf{s}_{k_0}(d_2)} u + \mu W(x) f(u) \quad \text{in } M, \quad (\mathcal{P}_M^\mu)$$

4 where $\lambda \in [0, (n-2)^2]$ is fixed, $\mu \geq 0$ is a parameter, and the continuous function $f : [0, \infty) \rightarrow \mathbb{R}$ verifies

5 (f_1) $f(s) = o(s)$ as $s \rightarrow 0^+$ and $s \rightarrow \infty$;

6 (f_2) $F(s_0) > 0$ for some $s_0 > 0$, where $F(s) = \int_0^s f(t) dt$.

7 According to (f_1) and (f_2) , the number $c_f = \max_{s>0} \frac{f(s)}{s}$ is well defined and positive.

8 On the potential $V : M \rightarrow \mathbb{R}$ we require that

9 (V_1) $V_0 = \inf_{x \in M} V(x) > 0$;

10 (V_2) $\lim_{d_g(x_0, x) \rightarrow \infty} V(x) = +\infty$ for some $x_0 \in M$,

11 and $W : M \rightarrow \mathbb{R}$ is assumed to be positive. Elliptic problems with similar assumptions on V have been
 12 studied on Euclidean spaces, see *e.g.* Bartsch, Pankov and Wang [3], Bartsch and Wang [4], Rabinowitz [34]
 13 and Willem [37].

14 Before to state our result, let us consider the functional space

$$15 \quad H_V^1(M) = \left\{ u \in H_g^1(M) : \int_M (|\nabla_g u|^2 + V(x)u^2) dv_g < +\infty \right\}$$

16 endowed with the norm

$$17 \quad \|u\|_V = \left(\int_M |\nabla_g u|^2 dv_g + \int_M V(x)u^2 dv_g \right)^{1/2}.$$

18 The next Rabinowitz-type compactness result (see Rabinowitz [34]) is crucial in the study of weak solutions of
 19 problem (\mathcal{P}_M^μ) :

20 **Lemma 4.2.** *If V satisfies (V_1) and (V_2) , the embedding $H_V^1(M) \hookrightarrow L^p(M)$ is compact, $p \in [2, 2^*)$.*

21 *Proof.* Let $\{u_k\}_k \subset H_V^1(M)$ be a bounded sequence in $H_V^1(M)$, *i.e.*, $\|u_k\|_V \leq \eta$ for some $\eta > 0$. Let $q > 0$ be
 22 arbitrarily fixed; by (V_2) , there exists $R > 0$ such that $V(x) \geq q$ for every $x \in M \setminus B_R(x_0)$. Thus,

$$23 \quad \int_{M \setminus B_R(x_0)} (u_k - u)^2 dv_g \leq \frac{1}{q} \int_{M \setminus B_R(x_0)} V(x) |u_k - u|^2 \leq \frac{(\eta + \|u\|_V)^2}{q}.$$

24 On the other hand, by (V_1) , we have that $H_V^1(M) \hookrightarrow H_g^1(M) \hookrightarrow L_{\text{loc}}^2(M)$; thus, up to a subsequence we have
 25 that $u_k \rightarrow u$ in $L_{\text{loc}}^2(M)$. Combining the above two facts and taking into account that $q > 0$ can be arbitrary
 26 large, we deduce that $u_k \rightarrow u$ in $L^2(M)$; thus the embedding follows for $p = 2$. Now, if $p \in (2, 2^*)$, by using an
 27 interpolation inequality and the Sobolev inequality on Cartan–Hadamard manifolds (see Hebey [19], Chapt. 8),
 28 one has

$$29 \quad \begin{aligned} \|u_k - u\|_{L^p(M)}^p &\leq \|u_k - u\|_{L^{2^*}(M)}^{n(p-2)/2} \|u_k - u\|_{L^2(M)}^{n(1-p/2^*)} \\ &\leq \mathcal{C}_n \|\nabla_g(u_k - u)\|_{L^2(M)}^{n(p-2)/2} \|u_k - u\|_{L^2(M)}^{n(1-p/2^*)}, \end{aligned}$$

31 where $\mathcal{C}_n > 0$ depends on n . Therefore, $u_k \rightarrow u$ in $L^p(M)$ for every $p \in (2, 2^*)$. □

32 The main result of this subsection is as follows.

Theorem 4.3. *Let (M, g) be an n -dimensional Cartan–Hadamard manifold ($n \geq 3$) with $\mathbf{K} \geq k_0$ for some $k_0 \leq 0$ and let $S = \{x_1, x_2\} \subset M$ be the set of distinct poles. Let $V, W : M \rightarrow \mathbb{R}$ be positive potentials verifying (V_1) , (V_2) and $W \in L^1(M) \cap L^\infty(M) \setminus \{0\}$, respectively. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function verifying (f_1) and (f_2) , and $\lambda \in [0, (n-2)^2]$ be fixed. Then the following statements hold:*

- (i) *Problem (\mathcal{P}_M^μ) has only the zero solution whenever $0 \leq \mu < V_0 \|W\|_{L^\infty(M)}^{-1} c_f^{-1}$;*
- (ii) *There exists $\mu_0 > 0$ such that problem (\mathcal{P}_M^μ) has at least two distinct non-zero, non-negative weak solutions in $H_V^1(M)$ whenever $\mu > \mu_0$.*

Proof. According to (f_1) , one has $f(0) = 0$. Thus, we may extend the function f to the whole \mathbb{R} by $f(s) = 0$ for $s \leq 0$, which will be considered throughout the proof. Fix $\lambda \in [0, (n-2)^2]$.

- (i) Assume that $u \in H_V^1(M)$ is a non-zero weak solution of problem (\mathcal{P}_M^μ) . Multiplying (\mathcal{P}_M^μ) by u , an integration on M gives that

$$\int_M |\nabla_g u|^2 dv_g + \int_M V(x)u^2 dv_g = \lambda \int_M \frac{\mathbf{s}_{k_0}^2\left(\frac{d_{12}}{2}\right)}{d_1 d_2 \mathbf{s}_{k_0}(d_1) \mathbf{s}_{k_0}(d_2)} u^2 dv_g + \mu \int_M W(x)f(u)u dv_g. \quad (12)$$

By the latter relation, Corollary 4.1 (see relation (4.2)) and the definition of c_f , it yields that

$$\begin{aligned} \int_M |\nabla_g u|^2 dv_g + V_0 \int_M u^2 dv_g &\leq \int_M |\nabla_g u|^2 dv_g + \int_M V(x)u^2 dv_g \\ &= \lambda \int_M \frac{\mathbf{s}_{k_0}^2\left(\frac{d_{12}}{2}\right)}{d_1 d_2 \mathbf{s}_{k_0}(d_1) \mathbf{s}_{k_0}(d_2)} u^2 dv_g + \mu \int_M W(x)f(u)u dv_g \\ &\leq \int_M |\nabla_g u|^2 dv_g + \mu \|W\|_{L^\infty(M)} c_f \int_M u^2 dv_g. \end{aligned} \quad (14)$$

Consequently, if $0 \leq \mu < V_0 \|W\|_{L^\infty(M)}^{-1} c_f^{-1}$, then u is necessarily 0, a contradiction.

- (ii) Let us consider the energy functional associated with problem (\mathcal{P}_M^μ) , i.e., $\mathcal{E}_\mu : H_V^1(M) \rightarrow \mathbb{R}$ defined by

$$\mathcal{E}_\mu(u) = \frac{1}{2} \int_M (|\nabla_g u|^2 + V(x)u^2) dv_g - \frac{\lambda}{2} \int_M \frac{\mathbf{s}_{k_0}^2\left(\frac{d_{12}}{2}\right)}{d_1 d_2 \mathbf{s}_{k_0}(d_1) \mathbf{s}_{k_0}(d_2)} u^2 dv_g - \mu \int_M W(x)F(u) dv_g. \quad (19)$$

One can show that $\mathcal{E}_\mu \in C^1(H_V^1(M), \mathbb{R})$ and for all $u, w \in H_V^1(M)$ we have

$$\mathcal{E}'_\mu(u)(w) = \int_M (\langle \nabla_g u, \nabla_g w \rangle + V(x)uw) dv_g - \lambda \int_M \frac{\mathbf{s}_{k_0}^2\left(\frac{d_{12}}{2}\right)}{d_1 d_2 \mathbf{s}_{k_0}(d_1) \mathbf{s}_{k_0}(d_2)} uw dv_g - \mu \int_M W(x)f(u)w dv_g. \quad (21)$$

Therefore, the critical points of \mathcal{E}_μ are precisely the weak solutions of problem (\mathcal{P}_M^μ) in $H_V^1(M)$. By exploring the sublinear character of f at infinity, Corollary 4.1 and Lemma 4.2, one can see that \mathcal{E}_μ is bounded from below, coercive and satisfies the usual Palais-Smale condition for every $\mu \geq 0$. Moreover, by an elementary computation one can see that assumption (f_1) is inherited as a sub-quadratic property in the sense that

$$\lim_{\|u\|_V \rightarrow 0} \frac{\int_M W(x)F(u) dv_g}{\|u\|_V^2} = \lim_{\|u\|_V \rightarrow \infty} \frac{\int_M W(x)F(u) dv_g}{\|u\|_V^2} = 0. \quad (4.3)$$

Due to (f_2) and $W \neq 0$, we can construct a non-zero truncation function $u_0 \in H_V^1(M)$ such that $\int_M W(x)F(u_0) dv_g > 0$. Thus, we may define

$$\mu_0 = \frac{1}{2} \inf \left\{ \frac{\|u\|_V^2}{\int_M W(x)F(u) dv_g} : u \in H_V^1(M), \int_M W(x)F(u) dv_g > 0 \right\}. \quad (29)$$

By the relations in (4.3), we clearly have that $0 < \mu_0 < \infty$.

Let us fix $\mu > \mu_0$. Then there exists $\tilde{u}_\mu \in H_V^1(M)$ with $\int_M W(x)F(\tilde{u}_\mu)dv_g > 0$ such that $\mu > \frac{\|\tilde{u}_\mu\|_V^2}{2 \int_M W(x)F(\tilde{u}_\mu)dv_g} \geq \mu_0$. Consequently,

$$c_\mu^1 := \inf_{H_V^1(M)} \mathcal{E}_\mu \leq \mathcal{E}_\mu(\tilde{u}_\mu) \leq \frac{1}{2} \|\tilde{u}_\mu\|_V^2 - \mu \int_M W(x)F(\tilde{u}_\mu) < 0.$$

Since \mathcal{E}_μ is bounded from below and satisfies the Palais-Smale condition, the number c_μ^1 is a critical value of \mathcal{E}_μ , i.e., there exists $u_\mu^1 \in H_V^1(M)$ such that $\mathcal{E}_\mu(u_\mu^1) = c_\mu^1 < 0$ and $\mathcal{E}'_\mu(u_\mu^1) = 0$. In particular, $u_\mu^1 \neq 0$ is a weak solution of problem (\mathcal{P}_M^μ) .

Standard computations based on Corollary 4.1 and the embedding $H_V^1(M) \hookrightarrow L^p(M)$ for $p \in (2, 2^*)$ show that there exists a sufficiently small $\rho_\mu \in (0, \|\tilde{u}_\mu\|_V)$ such that

$$\inf_{\|u\|_V = \rho_\mu} \mathcal{E}_\mu(u) = \eta_\mu > 0 = \mathcal{E}_\mu(0) > \mathcal{E}_\mu(\tilde{u}_\mu),$$

which means that the functional \mathcal{E}_μ has the mountain pass geometry. Therefore, we may apply the mountain pass theorem, see Rabinowitz [34], showing that there exists $u_\mu^2 \in H_V^1(M)$ such that $\mathcal{E}'_\mu(u_\mu^2) = 0$ and $\mathcal{E}_\mu(u_\mu^2) = c_\mu^2$, where $c_\mu^2 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{E}_\mu(\gamma(t))$, and $\Gamma = \{\gamma \in C([0,1]; H_V^1(M)) : \gamma(0) = 0, \gamma(1) = \tilde{u}_\mu\}$. Due to the fact that $c_\mu^2 \geq \inf_{\|u\|_V = \rho_\mu} \mathcal{E}_\mu(u) > 0$, it is clear that $0 \neq u_\mu^2 \neq u_\mu^1$. Moreover, since $f(s) = 0$ for every $s \leq 0$, the solutions u_μ^1 and u_μ^2 are non-negative. \square

Remark 4.4. Theorem 4.3 can be applied on the hyperbolic space $\mathbb{H}^n = \{y = (y_1, \dots, y_n) : y_n > 0\}$ endowed with the metric $g_{ij}(y_1, \dots, y_n) = \frac{\delta_{ij}}{y_n^2}$; it is new even on the Euclidean space \mathbb{R}^n , $n \geq 3$.

4.2. A bipolar Schrödinger-type equation on the upper hemisphere

A positively curved counterpart of (4.1) can be stated as follows by using (1.4) and a Mittag-Leffler expansion (the interested reader can establish a similar inequality to (4.2) as well):

Corollary 4.5. *Let \mathbb{S}_+^n be the open upper hemisphere and let $S = \{x_1, \dots, x_m\} \subset \mathbb{S}_+^n$ be the set of distinct poles, with $n \geq 3$ and $m \geq 2$. Let $\beta = \max_{i=1,m} d_g(x_0, x_i)$, where $x_0 = (0, \dots, 0, 1)$ is the north pole of the sphere*

\mathbb{S}^n and g is the natural Riemannian metric of \mathbb{S}^n inherited by \mathbb{R}^{n+1} . Then we have the following inequality:

$$\|u\|_{\mathcal{C}(n,\beta)}^2 \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_{\mathbb{S}_+^n} \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 dv_g, \quad \forall u \in H_g^1(\mathbb{S}_+^n), \quad (4.4)$$

where $\|u\|_{\mathcal{C}(n,\beta)}^2 = \int_{\mathbb{S}_+^n} |\nabla_g u|^2 dv_g + \mathcal{C}(n, \beta) \int_{\mathbb{S}_+^n} u^2 dv_g$ and $\mathcal{C}(n, \beta) = (n-1)(n-2) \frac{7\pi^2 - 3(\beta + \frac{\pi}{2})^2}{2\pi^2(\pi^2 - (\beta + \frac{\pi}{2})^2)}$.

Proof. Let $M = \mathbb{S}^n$ be the standard unit sphere in \mathbb{R}^{n+1} and the open upper hemisphere $\mathbb{S}_+^n = \{y = (y_1, \dots, y_{n+1}) \in \mathbb{S}^n : y_{n+1} > 0\}$. By Theorem 1.1 we have

$$\begin{aligned} \int_{\mathbb{S}_+^n} |\nabla_g u|^2 dv_g &\geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_{\mathbb{S}_+^n} \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 dv_{g_0} \\ &+ \frac{n-2}{m} \sum_{i=1}^m \int_{\mathbb{S}_+^n} \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 dv_g, \quad \forall u \in C_0^\infty(\mathbb{S}_+^n). \end{aligned}$$

Since $\mathbf{K} \equiv 1$, the two-sided Laplace comparison theorem (or a direct computation) shows that $\Delta_g d_i = (n - 1) \cot(d_i)$.

Fix $u \in C_0^\infty(\mathbb{S}_+^n)$. By using the Mittag–Leffler expansion of the cotangent function, *i.e.*,

$$\cot t = \frac{1}{t} + 2t \sum_{k=1}^{\infty} \frac{1}{t^2 - \pi^2 k^2}, \quad t \in (0, \pi),$$

and the fact that $0 < d_i < \pi$, $i \in \{1, \dots, m\}$ (up to the poles, which has null measure), one has

$$\int_{\mathbb{S}_+^n} \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 dv_g = -2(n-1) \int_{\mathbb{S}_+^n} \sum_{k=1}^{\infty} \frac{u^2}{\pi^2 k^2 - d_i^2} dv_g.$$

Since $d_i < \pi$, we get that

$$\int_{\mathbb{S}_+^n} \sum_{k=2}^{\infty} \frac{u^2}{\pi^2 k^2 - d_i^2} dv_g \leq \int_{\mathbb{S}_+^n} \sum_{k=2}^{\infty} \frac{u^2}{\pi^2 k^2 - \pi^2} dv_g = \frac{3}{4\pi^2} \int_{\mathbb{S}_+^n} u^2 dv_g.$$

Moreover, since $\beta = \max_{i=1,m} d_g(x_0, x_i) < \frac{\pi}{2}$, one can see that for every $x \in \mathbb{S}_+^n$, $d_i(x) = d_g(x, x_i) \leq d_g(x, x_0) + d_g(x_0, x_i) < \frac{\pi}{2} + \beta$. Thus, $\pi^2 - d_i^2 > \pi^2 - (\beta + \frac{\pi}{2})^2 > 0$, which implies

$$\int_{\mathbb{S}_+^n} \frac{u^2}{\pi^2 - d_i^2} dv_g \leq \frac{1}{\pi^2 - (\beta + \frac{\pi}{2})^2} \int_{\mathbb{S}_+^n} u^2 dv_g.$$

Combining the above two estimates, we have that

$$\int_{\mathbb{S}_+^n} |\nabla_g u|^2 dv_g + \mathbf{C}(n, \beta) \int_{\mathbb{S}_+^n} u^2 dv_g \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_{\mathbb{S}_+^n} \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 dv_g,$$

where $\mathbf{C}(n, \beta) = (n-1)(n-2) \frac{7\pi^2 - 3(\beta + \frac{\pi}{2})^2}{2\pi^2(\pi^2 - (\beta + \frac{\pi}{2})^2)}$. The latter inequality can be extended to $H_g^1(\mathbb{S}_+^n)$ by standard approximation argument. \square

For simplicity, let $S = \{x_1, x_2\} \in \mathbb{S}_+^n$ be the set of poles. We consider the Dirichlet problem

$$\begin{cases} -\Delta_g u + \mathbf{C}(n, \beta)u = \lambda u \left| \frac{\nabla_g d_1}{d_1} - \frac{\nabla_g d_2}{d_2} \right|^2 + |u|^{p-2}u, & \text{in } \mathbb{S}_+^n \\ u = 0, & \text{on } \partial\mathbb{S}_+^n, \end{cases} \quad (\mathcal{P}_{\mathbb{S}_+^n})$$

where g is the natural Riemannian structure on the standard unit sphere \mathbb{S}^n inherited by \mathbb{R}^{n+1} , $p \in (2, 2^*)$, $\lambda \in \left[0, \frac{(n-2)^2}{4}\right)$ is fixed and $\mathbf{C}(n, \beta) = (n-1)(n-2) \frac{7\pi^2 - 3(\beta + \frac{\pi}{2})^2}{2\pi^2(\pi^2 - (\beta + \frac{\pi}{2})^2)}$; hereafter, $x_0 = (0, \dots, 0, 1)$ is the north pole of \mathbb{S}^n and $\beta = \max\{d_g(x_0, x_1), d_g(x_0, x_2)\}$.

Theorem 4.6. *Let \mathbb{S}_+^n be the open upper hemisphere ($n \geq 3$), $S = \{x_1, x_2\} \subset \mathbb{S}_+^n$ be the set of poles and $p \in (2, 2^*)$. The following statements hold:*

- (i) *Problem $(\mathcal{P}_{\mathbb{S}_+^n})$ has infinitely many weak solutions in $H_g^1(\mathbb{S}_+^n)$. In addition, if $x_1 = (a, 0, \dots, 0, b)$ and $x_2 = (-a, 0, \dots, 0, b)$ for some $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and $b > 0$, then problem $(\mathcal{P}_{\mathbb{S}_+^n})$ has a sequence $\{u_k\}_{k \in \mathbb{N}}$ of distinct weak solutions in $H_g^1(\mathbb{S}_+^n)$ of the form*

$$u_k := u_k \left(y_1, \sqrt{y_2^2 + \dots + y_n^2}, y_{n+1} \right) = u_k \left(y_1, \sqrt{1 - y_1^2 - y_{n+1}^2}, y_{n+1} \right).$$

1 (ii) If $n = 5$ or $n \geq 7$, and $x_1 = (a, 0, \dots, 0, b)$, $x_2 = (-a, 0, \dots, 0, b)$ for some $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and
 2 $b > 0$, then there exists at least $s_n = \lfloor \frac{n}{2} \rfloor + (-1)^{n-1} - 2$ sequences of sign-changing weak solutions of $(\mathcal{P}_{\mathbb{S}_+^n})$
 3 in $H_g^1(\mathbb{S}_+^n)$ whose elements mutually differ by their symmetries.

4 *Proof.* Fix $\lambda \in [0, \frac{(n-2)^2}{4})$ arbitrarily. The energy functional $\mathcal{E} : H_g^1(\mathbb{S}_+^n) \rightarrow \mathbb{R}$ associated with problem $(\mathcal{P}_{\mathbb{S}_+^n})$ is
 5 given by

$$6 \quad \mathcal{E}(u) = \frac{1}{2} \|u\|_{\mathcal{C}(n,\beta)}^2 - \frac{\lambda}{2} \int_{\mathbb{S}_+^n} u^2 \left| \frac{\nabla_g d_1}{d_1} - \frac{\nabla_g d_2}{d_2} \right|^2 dv_g - \frac{1}{p} \int_{\mathbb{S}_+^n} |u|^p dv_g.$$

7 It is clear that $\mathcal{E} \in C^1(H_g^1(\mathbb{S}_+^n), \mathbb{R})$ and its critical points are precisely the weak solutions of $(\mathcal{P}_{\mathbb{S}_+^n})$.

8 (i) We notice that the embedding $H_g^1(\mathbb{S}_+^n) \hookrightarrow L^p(\mathbb{S}_+^n)$ is compact for every $p \in (2, 2^*)$, see *e.g.* Hebey [19].
 9 By means of Corollary 4.5, one can prove that the functional \mathcal{E} satisfies the assumptions of the symmetric
 10 version of the mountain pass theorem, see *e.g.* Jabri ([20], Thm. 11.5) or Rabinowitz ([33], Thm. 9.12), thus
 11 there exists a sequence of distinct critical points of \mathcal{E} which are weak solutions of problem $(\mathcal{P}_{\mathbb{S}_+^n})$ in $H_g^1(\mathbb{S}_+^n)$.
 12 In particular, let $x_1 = (a, 0, \dots, 0, b)$ and $x_2 = (-a, 0, \dots, 0, b)$ for some $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and $b > 0$.
 13 We notice that in this case $\beta = d_g(x_0, x_1) = d_g(x_0, x_2) = \arccos b$. We shall prove that the energy functional
 14 \mathcal{E} is invariant w.r.t. the group $G_0 = \text{id}_{\mathbb{R}} \times O(n-1) \times \text{id}_{\mathbb{R}}$ via the action

$$15 \quad \zeta u(x) = u(\zeta^{-1}x)$$

16 for every $u \in H_g^1(\mathbb{S}_+^n)$, $\zeta \in G_0$ and $x \in \mathbb{S}_+^n$. First, since $\zeta \in G_0$ is an isometry on \mathbb{R}^{n+1} , a change of variables
 17 easily implies that

$$18 \quad u \mapsto \frac{1}{2} \|u\|_{\mathcal{C}(n,\beta)}^2 - \frac{1}{p} \int_{\mathbb{S}_+^n} |u|^p dv_g$$

19 is G_0 -invariant. Thus, it remains to focus on the G_0 -invariance of the functional

$$20 \quad u \mapsto \int_{\mathbb{S}_+^n} u^2 \left| \frac{\nabla_g d_1}{d_1} - \frac{\nabla_g d_2}{d_2} \right|^2 dv_g.$$

21 To do this, we recall that

$$22 \quad \left| \frac{\nabla_g d_1}{d_1} - \frac{\nabla_g d_2}{d_2} \right|^2 = \frac{1}{d_1^2} + \frac{1}{d_2^2} - 2 \frac{\langle \nabla_g d_1, \nabla_g d_2 \rangle}{d_1 d_2}.$$

23 and $\nabla_g d_g(\cdot, y)(x) = -\frac{\exp_x^{-1}(y)}{d_g(x,y)}$ for every $x, y \in \mathbb{S}_+^n$, $x \neq y$. According to Udriște ([36], p. 19), one has

$$24 \quad \exp_x^{-1} x_i = \frac{d_i(x_i - x \cos d_i)}{\sin d_i}, \quad i \in \{1, 2\}, x \in \mathbb{S}_+^n \setminus \{x_i\}.$$

25 Therefore,

$$26 \quad \nabla_g d_i(x) = \nabla_g d_g(x, x_i) = -\frac{\exp_x^{-1}(x_i)}{d_i} = \frac{x \cos d_i - x_i}{\sin d_i}, \quad i \in \{1, 2\}, x \in \mathbb{S}_+^n \setminus \{x_i\}. \quad (4.5)$$

26 Let $\zeta \in G_0$, $i \in \{1, 2\}$ and $x \in \mathbb{S}_+^n \setminus \{x_i\}$ be fixed. Since $\zeta x_i = x_i$, it follows that

$$27 \quad d_i(\zeta x) = d_g(\zeta x, x_i) = d_g(\zeta x, \zeta x_i) = d_g(x, x_i) = d_i(x),$$

28 and by (4.5),

$$29 \quad \langle \nabla_g d_g(\zeta x, x_1), \nabla_g d_g(\zeta x, x_2) \rangle = \langle \nabla_g d_g(x, x_1), \nabla_g d_g(x, x_2) \rangle.$$

Summing up the above properties (combined with a trivial change of variable), it follows that the energy functional \mathcal{E} is G_0 -invariant, *i.e.*, $\mathcal{E}(\zeta u) = \mathcal{E}(u)$ for every $u \in H_g^1(\mathbb{S}_+^n)$ and $\zeta \in G_0$.

We now can apply the same variational argument as above for the functional $\mathcal{E}_0 = \mathcal{E}|_{H_{G_0}(\mathbb{S}_+^n)}$ where $H_{G_0}(\mathbb{S}_+^n) = \{u \in H_g^1(\mathbb{S}_+^n) : \zeta u = u \text{ for every } \zeta \in G_0\}$. Accordingly, one can find a sequence $\{u_k\}_{k \in \mathbb{N}} \subset H_{G_0}(\mathbb{S}_+^n)$ of distinct critical points of \mathcal{E}_0 . Moreover, due to the principle of symmetric criticality of Palais [31], the critical points of \mathcal{E}_0 are also critical points for the original energy functional \mathcal{E} , thus weak solutions of problem $(\mathcal{P}_{\mathbb{S}_+^n})$. Since u_k are G_0 -invariant functions, they have the form $u_k := u_k(y_1, \sqrt{y_2^2 + \dots + y_n^2}, y_{n+1}) = u_k(y_1, \sqrt{1 - y_1^2 - y_{n+1}^2}, y_{n+1})$, $k \in \mathbb{N}$.

(ii) Let $n = 5$ or $n \geq 7$, and denote by $s_n = \lfloor \frac{n}{2} \rfloor + (-1)^{n-1} - 2$. (Note that $s_6 = 0$.) For every $j \in \{1, \dots, s_n\}$ we define

$$G_j^n = \begin{cases} O(j+1) \times O(n-2j-3) \times O(j+1), & \text{if } j \neq \frac{n-3}{2}; \\ O\left(\frac{n-1}{2}\right) \times O\left(\frac{n-1}{2}\right), & \text{if } j = \frac{n-3}{2}, \end{cases}$$

where $O(k)$ is the orthogonal group in \mathbb{R}^k . For a fixed G_j^n , we define the function τ_j associated to G_j^n as

$$\tau_j(\sigma) = \begin{cases} (\sigma_3, \sigma_2, \sigma_1), & \text{if } j \neq \frac{n-3}{2} \text{ and } \sigma = (\sigma_1, \sigma_2, \sigma_3) \text{ with } \sigma_1, \sigma_2 \in \mathbb{R}^{j+1}, \sigma_3 \in \mathbb{R}^{n-2j-3}; \\ (\sigma_3, \sigma_1), & \text{if } j = \frac{n-3}{2} \text{ and } \sigma = (\sigma_1, \sigma_3) \text{ with } \sigma_1, \sigma_3 \in \mathbb{R}^{\frac{n-1}{2}}. \end{cases}$$

Note that $\tau_j \notin G_j^n$, $\tau_j G_j^n \tau_j^{-1} = G_j^n$ and $\tau_j^2 = \text{id}_{\mathbb{R}^{n-1}}$. Similarly as in Kristály [26], we introduce the action of the group

$$G_{j, \tau_j}^n = \text{id}_{\mathbb{R}} \times \langle G_j^n, \tau_j \rangle \times \text{id}_{\mathbb{R}} \subset O(n+1)$$

on the space $H_g^1(\mathbb{S}_+^n)$ by

$$\zeta u(x) = u(\zeta^{-1}x), \quad (\tilde{\tau}_j \zeta)u(x) = -u(\zeta^{-1} \tilde{\tau}_j^{-1}x), \quad (4.6)$$

for every $\zeta \in \tilde{G}_j^n = \text{id}_{\mathbb{R}} \times G_j^n \times \text{id}_{\mathbb{R}}$, $\tilde{\tau}_j = \text{id}_{\mathbb{R}} \times \tau_j \times \text{id}_{\mathbb{R}}$, $u \in H_g^1(\mathbb{S}_+^n)$ and $x \in \mathbb{S}_+^n$. We define the subspace of $H_g^1(\mathbb{S}_+^n)$ containing all the symmetric points w.r.t. the compact group G_{j, τ_j}^n , *i.e.*,

$$H_{G_{j, \tau_j}^n}(\mathbb{S}_+^n) = \left\{ u \in H_g^1(\mathbb{S}_+^n) : \tilde{\zeta}u = u \text{ for every } \tilde{\zeta} \in G_{j, \tau_j}^n \right\}.$$

Note that (see Kristály [26], Thm. 3.1) for every $j \neq k \in \{1, 2, \dots, s_n\}$ one has

$$H_{G_{j, \tau_j}^n}(\mathbb{S}_+^n) \cap H_{G_{k, \tau_k}^n}(\mathbb{S}_+^n) = \{0\}. \quad (4.7)$$

In a similar way as above, we can prove that the energy functional \mathcal{E} is G_{j, τ_j}^n -invariant for every $j \in \{1, \dots, s_n\}$ (note that \mathcal{E} is an even functional), where the group action on $H_g^1(\mathbb{S}_+^n)$ is given by (4.6). Therefore, for every $j \in \{1, \dots, s_n\}$ there exists a sequence $\{u_k^j\}_{k \in \mathbb{N}} \subset H_{G_{j, \tau_j}^n}(\mathbb{S}_+^n)$ of distinct critical points of $\mathcal{E}_j = \mathcal{E}|_{H_{G_{j, \tau_j}^n}(\mathbb{S}_+^n)}$.

Again by Palais [31], $\{u_k^j\}_{k \in \mathbb{N}} \subset H_{G_{j, \tau_j}^n}(\mathbb{S}_+^n)$ are distinct critical points also for \mathcal{E} , thus weak solutions for problem $(\mathcal{P}_{\mathbb{S}_+^n})$. It is clear that every u_k^j is sign-changing (see (4.6)) and according to (4.7), elements in different sequences have mutually different symmetry properties. \square

Remark 4.7. For $n = 6$ in Theorem 4.6 (ii), one has $s_6 = 0$; therefore, in this case we cannot apply the above group-theoretical argument to guarantee the existence of sign-changing solutions for problem $(\mathcal{P}_{\mathbb{S}_+^n})$.

5. CONCLUDING REMARKS

In the present paper we presented some multipolar Hardy inequalities on complete Riemannian manifolds by exploring the presence of the curvature and giving some applications in the theory of elliptic equations involving bipolar potentials; as far as we know, this is the first study in such a geometrical setting. During the preparation of the manuscript we faced several problems which, - in our opinion, - are worth to be tackled in forthcoming investigations. In the sequel, we shall formulate some of them:

- (a) As we already pointed out in Remark 1.2 (b), the optimality of $\frac{(n-2)^2}{m^2}$ in (1.4) for generic Riemannian manifolds is not yet understood for $m \geq 3$ which requires further studies. We notice that multipolar inequalities involving non-uniform weights on complete Riemannian manifolds can also be obtained, following Devyver, Fraas and Pinchover [13].
- (b) For simplicity reasons, in Section 4 we considered only some model elliptic problems with familiar growth assumptions, *i.e.*, sublinear and subcritical pure power term. However, multipolar Hardy inequalities (*cf.* Thms. 1.1 and 1.3) allow to study other classes of elliptic problems involving other type of nonlinear terms (critical, concave-convex, *etc.*).
- (c) A challenging problem is to study the heat equation involving multiple poles on strip-like domains or curved tubes (embedded into appropriate Riemannian manifolds). We notice that in the Euclidean setting such equations have been investigated by Baras and Goldstein [2], Krejčířík and Zuazua [24, 25] *via* Hardy-type inequalities; see also references therein. We notice that deep studies already exist concerning linear heat equations on Riemannian manifolds having non-negative Ricci curvature which is related to the Perelman's volume non-collapsing result, see Ni [30].

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