HARNACK INEQUALITY FOR STRONGLY DEGENERATE ELLIPTIC OPERATORS WITH NATURAL GROWTH

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ABSTRACT. We prove that positive and bounded weak solutions of a strongly degenerate elliptic equation satisfy the Harnack inequality. The structure of the differential operator includes a nonlinear term in the gradient with quadratic growth. Moreover, the lower order terms belong to some Stummel classes defined in term of sum operators introduced in [13].

1. INTRODUCTION

Recently the regularity of weak solutions of the equation

$$- X_j^*(a_{ij}X_i u + d_j u) + b_i X_i u + cu = f - X_i^* h_i \tag{1.1}$$

was studied in [6]. Here $X = (X_1, X_2, \ldots, X_m)$ is a system of first-order locally Lipschitz vector fields in $\mathbb{R}^n$, and the lower order terms belong to suitable Stummel classes modeled on a special geometry introduced in [13]. Namely, local boundedness and continuity of the weak solutions have been proved. Later, in [7], the results in [6] were generalized to a class of operators satisfying a weighted degeneracy condition. There, the principal part of (1.1) is controlled by a A_2 Muckenhoupt weight, and the operators considered in [6, 7] have a controlled growth in the gradient. As it is well known, this implies that any weak solution is locally bounded.

In this note we study an operator that is similar to those in [6, 7] but very different in the growth. In fact, the operator considered here satisfies a quadratic growth with respect to the gradient. It is worth to remark that adding such a term destroy the local boundedness property of the weak solutions. This phenomenon forces us to assume that the weak solutions are locally bounded and then we can show regularity only for the bounded solutions.

To state our result, let us consider the equation

$$- X_j^*(a_{ij}X_i u + d_j u) + \frac{b_0}{\lambda} w |Xu|^2 + b_i X_i u + cu = f - X_i^* h_i, \tag{1.2}$$

where X is as before and the coefficients a_{ij} satisfy weighted ellipticity condition with respect to a Muckenhoupt A_2 weight. Assuming the lower order terms in appropriate weighted Stummel classes (see section 2 for definitions) we prove that the *bounded* positive solutions of equation (1.2) satisfy a Harnack inequality. As a consequence, this will imply that the *bounded* weak solutions are continuous.

Some comments are now in order. We assume that X satisfies the following (1-2) weighted Poincaré inequality

$$\frac{1}{w(B)} \int_B |u - u_B| w \, dy \le C_P r \Big(\frac{1}{w(B)} \int_B |Xu|^2 w \, dy \Big)^{1/2} \quad \forall u \in C^\infty \,.$$

As showed by Franchi Perez and Wheeden [13], the above Poincaré inequality implies the following subrepresentation formula

$$|u(x) - u_{B_0}| \le C \sum_{j=0}^{\infty} r(B_j(x)) \Big(\frac{1}{w(B_j(x))} \int_{B_j(x)} |Xu|^2 w(y) dy \Big)^{1/2}$$

where $\{B_j(x)\}_{j=1}^{\infty}$ is special chain of balls related to a fixed ball B_0. The subrepresentation formula allowed us to prove a Fefferman type inequality and to define suitable Stummel classes modeled on the geometry introduced in [13] (for more details see [5]).

The Fefferman inequality is a fundamental tool in our proof. Indeed, following the classical pattern in Trudinger paper [18] (analogous results in different settings are shown in [3, 4, 8, 9]) we are faced with products between lower order terms and test functions. Due to the low integrability of the lower order terms we use the Fefferman inequality to complete the iteration process and obtain our results.

2. Stummel type classes and Fefferman inequality

Let $X = (X_1, X_2, \ldots, X_m)$ be a system of locally Lipschitz vector fields in $\mathbb{R}^n$ and d the associated Carnot-Carathéodory distance. We assume that d is finite for any $x, y \in \mathbb{R}^n$ and denote by $B = B_r = B(x, r)$ the Carnot-Carathéodory ball centered at x of radius r. Let us recall the definition of Muckenhoupt weight A_p.

Definition 2.1 (A_p Muckenhoupt weights). Let w be a non negative and locally integrable function in $\mathbb{R}^n$ and $1 < p < +\infty$. We say that w is an A_p weight if

$$[w]_p \equiv \sup_B \Big(\frac{1}{|B|} \int_B w(x) \, dx \Big) \Big(\frac{1}{|B|} \int_B [w(x)]^{\frac{-1}{p-1}} \, dx \Big)^{p-1} < +\infty$$

where the supremum is taken over all metric balls B in $\mathbb{R}^n$. The number $[w]_p$ is called the A_p constant of w.

Throughout this article we assume the following.
(A1) The distance d is continuous with respect to the Euclidean distance in $\mathbb{R}^n$.
(A2) There exists a positive constant C_D such that

$$|B(x, 2r)| \le C_D |B(x, r)| \quad \forall x \in \mathbb{R}^n, r > 0 \,.$$

(A3) If B_0 is a given ball in $\mathbb{R}^n$ and $w \in A_2$, there exists a positive constant C_P such that

$$\frac{1}{w(B)} \int_B |u - u_B| w \, dy \le C_P r \Big(\frac{1}{w(B)} \int_B |Xu|^2 w \, dy \Big)^{1/2}$$

for all $B \subset B_0$ and all $u \in C^\infty(\overline{B}_0)$. Here $u_B = \frac{1}{w(B)} \int_B u \, w \, dy$, $w(B) = \int_B w \, dy$ and r is the radius of B.

The number $Q = \log_2 C_D$ will be called homogeneous dimension of $\mathbb{R}^n$.

To state and prove our results we need to define the Sobolev classes with respect to the measure $w \, dx$ where $w \in A_2$.

Definition 2.2 (Sobolev spaces). Let $w \in A_2$ and Ω be a bounded domain in $\mathbb{R}^n$. We say that u belongs to $W^{1,2}(\Omega, w)$ if u, $X_i u \in L^2(\Omega, w)$ for any $i = 1, \ldots m$. Moreover, we denote by $W_0^{1,2}(\Omega, w)$ the closure of the smooth and compactly supported functions in $W^{1,2}(\Omega, w)$ with respect to the norm

$$\|u\|_{W^{1,2}(\Omega,w)} = \|u\|_{L^2(\Omega,w)} + \sum_{i=1}^m \|X_i u\|_{L^2(\Omega,w)}.$$

and we say that u belongs to $W_{\mathrm{loc}}^{1,2}(\Omega, w)$ if $u \in W^{1,2}(\Omega', w)$ for any $\Omega' \subset\subset \Omega$.

We recall the useful embedding Theorem for Sobolev spaces (see [12, 15, 14]).

Theorem 2.3. Let $w \in A_2$ and K be a compact subset of Ω. Then there exist $r_0 > 0$, $q_0 > 2$ and C depending on K, Ω and $\{X_j\}$ such that for any metric ball $B = B(x, r)$, $x \in K$, we have

$$\Big(\frac{1}{w(B)} \int_B |u - u_B|^q w \, dy\Big)^{1/q} \leq Cr\Big(\frac{1}{w(B)} \int_B |Xu|^2 w \, dy\Big)^{1/2}, \quad \forall u \in C^\infty(\overline{B})$$

provided $0 < r < r_0$ and $2 < q < q_0$.

The following definition is useful for stating the subrepresentation formula that we will use later (see [13]).

Definition 2.4. Given $B_0 = B(x_0, r)$ and $x \in B_0$, let us denote by $\{B_i\} = \{B_i(x)\}_{i=1}^\infty$ a chain of balls, of radius $r(B_i)$, such that

(H1) $B_i \subset B_0$ for all $i \geq 0$
(H2) $r(B_i) \sim 2^{-i} r(B_0)$ for all $i \geq 0$
(H3) $\rho(B_i, x) \leq Cr(B_i)$ for all $i \geq 0$
(H4) for all $i \geq 0$, $B_i \cap B_{i+1}$ contains a ball S_i with $r(S_i) \sim r(B_i)$.

Theorem 2.5. Given a weight $w \in A_2$ and a ball B let $\{B_j(x)\}_{j=1}^\infty$ be a chain of balls as in Definition 2.4. Let $u \in W^{1,2}(B_0, w)$ be such that for any ball $B \subset B_0$

$$\frac{1}{w(B)} \int_B |u - u_B| w \, dx \leq Cs\Big(\frac{1}{w(B)} \int_B |Xu|^2 w \, dy\Big)^{1/2} \tag{2.1}$$

where s is the radius of B. Then there exists $C' > 0$ such that

$$|u(x) - u_{B_0}| \leq C' \sum_{j=0}^\infty r(B_j(x)) \Big(\frac{1}{w(B_j(x))} \int_{B_j(x)} |Xu|^2 w(y) dy\Big)^{1/2}$$

where C' is a geometric constant which also depends on C.

Since we are interested to prove our result assuming low integrability properties on the lower order term we introduce the Stummel and Morrey classes adapted to our setting.

Definition 2.6 (Stummel and Morrey classes). Let $w \in A_2$, B_0 be a ball and $\{B_j(x)\}_{j=1}^\infty$ be a chain of balls as in Definition 2.4. We say that $V \in L_{\mathrm{loc}}^1(\mathbb{R}^n, w)$ belongs to the class $\tilde{S}(\mathbb{R}^n, w)$ if

$$\eta_V(r) \equiv \sup_{x_0 \in \mathbb{R}^n} \sup_{y \in B_0} \int_{B_0} \sum_{j=0}^\infty \frac{r^2(B_j(x))|V(x)|}{w(B_j(x))} \chi_{B_j(x)}(y) w(x) dx$$

is finite for all $r > 0$. We say that V belongs to $S(\mathbb{R}^n, w)$ if, in addition, we have $\lim_{r \to 0} \eta_V(r) = 0$. We say that $V \in S'(\mathbb{R}^n, w)$ if there exists $\delta > 0$ such that

$$\int_0^\delta \frac{\eta_V(t)}{t} dt < +\infty.$$

We say that V belongs to the Morrey space $M_\sigma(\mathbb{R}^n, w)$ if there exist $C > 0$ such that $\eta_V(r) \leq Cr^\sigma$.

We close this section giving the proof of the weighted embedding result. As we have already noted, it will allow us to get our main results. The unweighted result and some corollaries has been proven in [5] (see also [2], [10, 11, 16, 19, 20, 21]). Here we extend the embedding to the weighted case.

Theorem 2.7. *Let $w \in A_2$, B_0 be a ball and V a function in $\tilde{S}(\mathbb{R}^n, w)$. Then, there exists a positive constant C such that*

$$\int_{B_0} |V(x)|\, |u(x) - u_{B_0}|^2 w\, dx \leq C\eta_V(r) \int_{B_0} |Xu(x)|^2 w\, dx$$

for any $u \in C^\infty(B_0)$.

Proof. Let u be a smooth function in B_0. Theorem 2.5 yields the following subrepresentation formula for u

$$|u(x) - u_{B_0}| \leq C \sum_{j=0}^\infty r(B_j(x)) \Big(\frac{1}{w(B_j(x))} \int_{B_j(x)} |Xu|^2 w(y) dy \Big)^{1/2} \tag{2.2}$$

for a.e. $x \in B_0$. Now from (2.2) and Hölder inequality

$$\int_{B_0} |V(x)||u(x) - u_{B_0}|^2 w(x) dx$$

$$\leq \int_{B_0} |V(x)||u(x) - u_{B_0}| \sum_{j=0}^\infty r(B_j(x))$$

$$\times \Big[\frac{1}{w(B_j(x))} \int_{B_j(x)} |Xu(y)|^2 w(y) dy \Big]^{1/2} w(x) dx$$

$$\leq \Big[\int_{B_0} |V(x)||u(x) - u_{B_0}|^2 w(x) dx \Big]^{1/2}$$

$$\times \Big[\int_{B_0} \sum_{j=0}^\infty |V(x)| \frac{r^2(B_j(x))}{w(B_j(x))} \int_{B_j(x)} |Xu(y)|^2 w(y) dy w(x) dx \Big]^{1/2}$$

$$\leq \Big[\int_{B_0} |V(x)||u(x) - u_{B_0}|^2 w(x) dx \Big]^{1/2}$$

$$\times \Big[\int_{B_0} \sum_{j=0}^\infty |V(x)| \frac{r^2(B_j(x))}{w(B_j(x))} \int_{B_0} |Xu(y)|^2 \chi_{B_j(x)}(y) w(y) dy w(x) dx \Big]^{1/2}$$

$$\leq \Big[\int_{B_0} |V(x)||u(x) - u_{B_0}|^2 w(x) dx \Big]^{1/2}$$

$$\times \Big[\int_{B_0} |Xu(y)|^2 \int_{B_0} \sum_{j=0}^\infty |V(x)| \frac{r^2(B_j(x))}{w(B_j(x))} \chi_{B_j(x)}(y) w(x) dx w(y) dy \Big]^{1/2}$$

$$\leq \Big[\int_{B_0} |V(x)||u(x) - u_{B_0}|^2 w(x) dx \Big]^{1/2} \eta_V^{1/2}(r)$$

$$\times \Big[\int_{B_0} |Xu(y)|^2 w(y) dy \Big]^{1/2}$$

from which

$$\int_{B_0} |V(x)||u(x) - u_{B_0}|^2 w(x) dx \leq C\eta_V(r) \int_{B_0} |Xu(x)|^2 w(x) dx \,.$$

$\square$

From Theorem 2.7 we obtain the following corollaries.

Corollary 2.8. *Let V be a function in $\tilde{S}(\mathbb{R}^n, w)$. Then, there exists a positive constant C such that*

$$\int_{\mathbb{R}^n} |V(x)| \, |u(x)|^2 w \, dx \leq C\eta_V(r) \int_{\mathbb{R}^n} |Xu(x)|^2 w \, dx$$

for any compactly supported smooth function u in $\mathbb{R}^n$.

Corollary 2.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and V in $S(\Omega, w)$. Then, for any $\varepsilon > 0$ there exists a positive function $K(\varepsilon) \sim \frac{\varepsilon}{[\eta_V^{-1}(\varepsilon)]^{Q+2}}$ (where η_V^{-1} is the inverse function of η_V), such that*

$$\int_{\Omega} |V(x)| \, |u(x)|^2 w \, dx \leq \varepsilon \int_{\Omega} |Xu(x)|^2 \, dx + K(\varepsilon) \int_{\Omega} |u(x)|^2 \, dx \qquad (2.3)$$

for any compactly supported smooth function u in Ω.

3. Harnack inequality for strongly degenerate equations

Let Ω be a bounded domain in $\mathbb{R}^n$. Let $X = (X_1, X_2, \ldots, X_m)$ be a system of locally Lipschitz vector fields in $\mathbb{R}^n$. For $i = 1, 2, \ldots, m$ we denote by X_i^* the formal adjoint of the vector fields X_i. Let $\{a_{ij}(x)\}$ be a symmetric matrix of measurable functions in Ω satisfying the weighted ellipticity condition: There exists $\lambda > 0$ such that

$$\lambda^{-1} w(x)|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda w(x)|\xi|^2 \quad \text{a.e. } x \in \Omega \text{ and } \forall \xi \in \mathbb{R}^m \qquad (3.1)$$

for some Muckenhoupt weight $w \in A_2$.

Let us consider the strongly degenerate elliptic equation in divergence form

$$- X_j^*(a_{ij} X_i u + d_j u) + \frac{b_0}{\lambda} w|Xu|^2 + b_i X_i u + cu = f - X_i^* h_i \,, \qquad (3.2)$$

where

$$b_0 \in \mathbb{R} \setminus \{0\}, \quad \Big(\frac{b_i}{w}\Big)^2, \quad \frac{c}{w}, \quad \Big(\frac{d_i}{w}\Big)^2, \quad \frac{f}{w}, \quad \Big(\frac{h_i}{w}\Big)^2 \in S'(\Omega, w) \,. \qquad (3.3)$$

To begin with we give the definition of weak super, sub solutions and solutions.

Definition 3.1 (Weak supersolutions, subsolutions, solutions)**.** Let $w \in A_2$ and $u \in W_{\text{loc}}^{1,2}(\Omega, w)$. We say that u is a local weak supersolution (subsolution) of (3.2) if for any $\varphi \in W_0^{1,2}(\Omega, w)$, $\varphi \geq 0$

$$\int_{\Omega} \Big[(a_{ij} X_i u + d_j u) X_j \varphi + \Big(\frac{b_0}{\lambda} w|Xu|^2 + b_i X_i u + cu\Big)\varphi \Big] dx$$

$$\geq (\leq) \int_{\Omega} (f\varphi + h_i X_i \varphi) dx \,.$$

We say that $u \in W^{1,2}_{\text{loc}}(\Omega, w)$ is a local weak solution of (3.2) if it is both a super-solution and a subsolution.

Our first result is the weak Harnack inequality for supersolutions of (3.2). We follow the pattern drawn in [18].

Theorem 3.2. *Let us assume conditions (3.1) and (3.3) are satisfied, $w \in A_2$, and let u be a weak nonnegative supersolution of equation (3.2) in a ball $B_{3r} \subset\subset \Omega$. Let $M > 0$ be a constant such that $u \leq M$ in B_{3r}. Then, there exists C depending on Q, M, λ and the A_2 constant of w, such that*

$$w^{-1}(B_{2r}) \int_{B_{2r}} u\, w dx$$

$$\leq C\Big\{\min_{B_r} u + r^\sigma \|\frac{f}{w}\|_{\sigma, B_{3r}} + \Big(r^\sigma \sum_{i=1}^n \|\big(\frac{h_i}{w}\big)^2\|_{\sigma, B_{3r}}\Big)^{1/2}\Big\}.$$

Proof. Let

$$k = \|\frac{f}{w}\|_{\sigma, B_{3r}} + \Big(\sum_{i=1}^n \|\big(\frac{h_i}{w}\big)^2\|_{\sigma, B_{3r}}\Big)^{1/2}$$

and $v = u + k$. For $\eta \in C_0^1(B_{3r})$, $\eta \geq 0$, we set $\varphi(x) = \eta^2(x)v^\beta(x)e^{-|b_0|v(x)}$, $\beta < 0$, as a test function in (3.2). Since u is a supersolution in B_{3r} of (3.2) we have

$$\int_{B_{3r}} \Big[2\eta(a_{ij}X_i u + d_j u - h_j)X_j\eta v^\beta e^{-|b_0|v}$$

$$+ (-|\beta|v^{\beta-1} - |b_0|v^\beta)\eta^2 e^{-|b_0|v}(a_{ij}X_i u + d_j u - h_j)X_j v$$

$$+ \frac{b_0}{\lambda}w|Xu|^2\eta^2 v^\beta e^{-|b_0|v} + (b_i X_i u + cu - f)\eta^2 v^\beta e^{-|b_0|v}\Big]dx \geq 0$$

and

$$\int_{B_{3r}} \eta^2 e^{-|b_0|v}(b_0 v^\beta + |\beta|v^{\beta-1})|Xv|^2 w dx$$

$$\leq \int_{B_{3r}} \eta^2 e^{-|b_0|v}(|b_0|v^\beta + |\beta|v^{\beta-1})|Xv|^2 w dx$$

$$\leq \lambda \int_{B_{3r}} \eta^2 e^{-|b_0|v}(|b_0|v^\beta + |\beta|v^{\beta-1})a_{ij}X_i v X_j v\, dx$$

$$\leq \lambda \int_{B_{3r}} \eta^2 e^{-|b_0|v}(|\beta|v^{\beta-1} + |b_0|v^\beta)(h_j - d_j u)X_j v\, dx$$

$$+ 2\lambda \int_{B_{3r}} \eta(a_{ij}X_i v + d_j u - h_j)X_j\eta v^\beta e^{-|b_0|v}dx$$

$$+ \int_{B_{3r}} b_0 w|Xv|^2\eta^2 v^\beta e^{-|b_0|v}dx$$

$$+ \lambda \int_{B_{3r}} (b_i X_i v + cu - f)\eta^2 v^\beta e^{-|b_0|v}dx\,.$$

From this inequality it follows that

$$\int_{B_{3r}} \eta^2 e^{-|b_0|v}|\beta|v^{\beta-1}|Xv|^2 w dx$$

$$\leq \lambda \int_{B_{3r}} \eta^2 e^{-|b_0|v}(|\beta|v^{\beta-1} + |b_0|v^\beta)(h_j - d_j u)X_j v\, dx$$

$$+ 2\lambda \int_{B_{3r}} \eta(a_{ij}X_i v + d_j u - h_j)X_j \eta v^\beta e^{-|b_0|v} dx$$

$$+ \lambda \int_{B_{3r}} (b_i X_i v + cu - f)\eta^2 v^\beta e^{-|b_0|v} dx \,.$$

Since v is bounded, we may drop the exponential to obtain

$$\int_{B_{3r}} \eta^2 |\beta||v^{\beta-1}||Xv|^2 w dx$$

$$\leq C(M, b_0) \Big[2\lambda \int_{B_{3r}} \eta a_{ij} X_i v X_j \eta v^\beta dx + \lambda|\beta| \int_{B_{3r}} |d_j||X_j v||v^\beta \eta^2 dx$$

$$+ 2\lambda \int_{B_{3r}} |d_j||v^{\beta+1} X_j \eta \eta dx + 2\lambda \int_{B_{3r}} |h_j||v^\beta X_j \eta \eta dx + \lambda \int_{B_{3r}} |b_i||X_i v \eta^2 v^\beta$$

$$+ \lambda \int_{B_{3r}} |c|\eta^2 v^{\beta+1} dx + \lambda \int_{B_{3r}} |f|\eta^2 v^\beta dx$$

$$+ \lambda|\beta| \int_{B_{3r}} h_j X_j v v^{\beta-1} \eta^2 dx + \lambda \int_{B_{3r}} |d_j||v_{x_i}|\eta^2 v^\beta dx \Big] \,.$$

Now, set

$$V = \sum_{i=1}^n \frac{|b_i|^2}{w} + |c| + \sum_{j=1}^n \frac{|d_j|^2}{w} + k^{-1}|f| + k^{-2} \sum_{i=1}^n \frac{|h_i|^2}{w} \,.$$

Using Young's inequality yields

$$\int_{B_{3r}} \eta^2 v^{\beta-1}|Xv|^2 w dx$$

$$\leq C(M, b_0, \lambda) \Big[\frac{|\beta|+1}{\beta^2} \int_{B_{3r}} v^{\beta+1}|X\eta|^2 w dx + \Big(\frac{|\beta|+1}{\beta}\Big)^2 \int_{B_{3r}} V\eta^2 v^{\beta+1} dx \Big] \quad (3.4)$$

$$\leq C(M, b_0, \lambda)\Big(\frac{|\beta|+1}{\beta}\Big)^2 \Big[\int_{B_{3r}} v^{\beta+1}|X\eta|^2 w dx + \int_{B_{3r}} V\eta^2 v^{\beta+1} dx \Big] \,.$$

Now we set

$$\mathcal{U}(x) = \begin{cases} v^{\frac{\beta+1}{2}}(x) & \text{if } \beta \neq -1 \\ \log v(x) & \text{if } \beta = -1 \end{cases}$$

and by (3.4) we have

$$\int_{B_{3r}} \eta^2 |X\mathcal{U}|^2 w \, dx$$
$$\leq C(\beta+1)^2 \Big(\frac{|\beta|+1}{\beta}\Big)^2 \Big\{ \int_{B_{3r}} |X\eta|^2 \mathcal{U}^2 w \, dx + \int_{B_{3r}} V\eta^2 \mathcal{U}^2 \, dx \Big\}, \quad \beta \neq -1 \qquad (3.5)$$

while

$$\int_{B_{3r}} \eta^2 |X\mathcal{U}|^2 w \, dx \leq C\Big\{ \int_{B_{3r}} |X\eta|^2 w \, dx + \int_{B_{3r}} V\eta^2 \, dx \Big\} \qquad (3.6)$$

if $\beta = -1$.

Let us start with the case $\beta = -1$. By Corollary 2.8 we have

$$\int_{B_{3r}} \eta^2 |X\mathcal{U}|^2 w \, dx \leq C\Big(\int_{B_{3r}} |X\eta|^2 w \, dx + \int_{B_{3r}} \eta^2 w dx \Big) \,.$$

Let B_h be a ball contained in B_{2r}. Choosing $\eta(x)$ so that $\eta(x) = 1$ in B_h, $0 \le \eta \le 1$ in $B_{3r} \setminus B_h$ and $|X\eta| \le \frac{3}{h}$, we obtain

$$\|X\mathcal{U}\|_{L^2(B_h,w)} \le C \frac{w(B_h)^{1/2}}{h}.$$

By Theorem 2.3 and John-Nirenberg lemma (see [1]) we have $\mathcal{U}(x) = \log v(x) \in BMO$. Then there exist two positive constants p_0 and C, such that

$$\Big(\fint_{B_{2r}} e^{p_0 \mathcal{U}} w \, dx \Big)^{1/p_0} \Big(\fint_{B_{2r}} e^{-p_0 \mathcal{U}} w \, dx \Big)^{1/p_0} \le C. \tag{3.7}$$

Let us consider the family of seminorms

$$\Phi(p,h) = \Big(\fint_{B_h} |v|^p w \, dx \Big)^{1/p}, \quad p \ne 0.$$

By (3.7) we have

$$\frac{1}{w(B_{2r})^{1/p_0}} \Phi(p_0, 2r) \le C w(B_{2r})^{1/p_0} \Phi(-p_0, 2r).$$

Now we consider $\beta \ne 1$ (see inequality (3.5)). By Corollary 2.9 we obtain

$$\int_{B_{3r}} |X\mathcal{U}|^2 \eta^2 w \, dx$$
$$\le C \Big\{ \Big[\big(\frac{\beta+1}{2} \big)^2 + 1 \Big] \big(1 + \frac{1}{|\beta|} \big)^2 \int_{B_{3r}} |X\eta|^2 \mathcal{U}^2 w \, dx \tag{3.8}$$
$$+ \Big[\frac{1}{\phi^{-1} \big(\frac{V}{w}; \big(\frac{\beta+1}{2} \big)^{-2} \big(1 + \frac{1}{|\beta|} \big)^{-2} \big)} \Big]^{Q+2} \int_{B_{3r}} \eta^2 \mathcal{U}^2 w \, dx \Big\}.$$

By Theorem 2.3 we have

$$\Big(\fint_{B_{3r}} |\eta \mathcal{U}|^{\tau p} w \, dx \Big)^{1/\tau}$$
$$\le c w(B_{3r})^{\frac{1}{\tau}-1} \Big\{ \Big[\big(\frac{\beta+1}{2} \big)^2 + 2 \Big] \big(1 + \frac{1}{|\beta|} \big)^2 \int_{B_{3r}} |X\eta|^2 \mathcal{U}^2 w \, dx \tag{3.9}$$
$$+ \Big[\frac{1}{\phi^{-1} \big(\frac{V}{w}; \big(\frac{\beta+1}{2} \big)^{-2} \big(1 + \frac{1}{|\beta|} \big)^{-2} \big)} \Big]^{Q+2} \int_{B_{3r}} \eta^2 \mathcal{U}^2 w \, dx \Big\}$$

where c is a positive constant independent of w.

Now we choose the function η. For r_1 and r_2 such that $r \le r_1 < r_2 \le 2r$ we choose η such that $\eta(x) = 1$ in B_{r_1}, $0 \le \eta(x) \le 1$ in B_{r_2}, $\eta(x) = 0$ outside B_{r_2}, $|X\eta| \le \frac{c}{r_2 - r_1}$ for some fixed constant c. Then we have

$$\Big(\fint_{B_{r_1}} \mathcal{U}^{2\tau} w \, dx \Big)^{1/\tau}$$
$$\le c w(B_{3r})^{\frac{1}{\tau}-1} \frac{1}{(r_2 - r_1)^2} \Big[\big(\frac{\beta+1}{2} \big)^2 + 2 \Big]$$
$$\times \big(1 + \frac{1}{|\beta|} \big)^2 \Big[\frac{1}{\phi^{-1} \big(\frac{V}{w}; \big(\frac{\beta+1}{2} \big)^{-2} \big(1 + \frac{1}{|\beta|} \big)^{-2} \big)} \Big]^{Q+2} \int_{B_{r_2}} \mathcal{U}^2 w \, dx.$$

Setting $\gamma = \beta + 1$ and recalling that $\mathcal{U}(x) = v^{\frac{\beta+1}{2}}(x)$, we obtain

$$\Phi(\tau\gamma, r_1) \geq c^{1/\gamma} w(B_{3r})^{\frac{1}{\gamma}(\frac{1}{\tau}-1)} \Big[\big(\frac{\beta+1}{2}\big)^2 + 2\Big]^{1/\gamma}$$
$$\times \Big[\frac{1}{\phi^{-1}\big(\frac{V}{w}; \big(\frac{\beta+1}{2}\big)^{-2}\big)}\Big]^{\frac{Q+2}{\gamma}} \frac{1}{(r_2 - r_1)^{2/\gamma}} \Phi(\gamma, r_2) , \qquad (3.10)$$

for negative γ.

We are going to iterate the inequality just obtained. Setting $\gamma_i = \tau^i p_0$ and $r_i = r + \frac{r}{2^i}$, $i = 1, 2, \ldots$ iteration of (3.10) and use of [17, Lemma 3.4] yield

$$\Phi(-\infty, r) \geq c\big(\phi_{\frac{V}{w}}, \operatorname{diam}\Omega\big) w(B_{3r})^{1/p_0} \Phi(-p_0, 2r) .$$

Therefore, by Hölder inequality,

$$\Phi(p_0', 2r) \leq \Phi(p_0, 2r) w(B_{3r})^{\frac{1}{p_0'} - \frac{1}{p_0}} , \quad p_0' \leq p_0$$

so we obtain

$$w^{-1}(B_{2r}) \Phi(1, 2r) \leq c \Phi(-\infty, r)$$

and the result follows. $\qquad\qquad\Box$

The following weak Harnack inequality for subsolutions can be obtained in a similar way.

Theorem 3.3. *Let u be a weak nonnegative subsolution of (3.2) in $B_{3r} \subset\subset \Omega$. Assume (3.1) and (3.3). Let $M > 0$ be a constant such that $u \leq M$ in B_{3r}. Then there exists C depending on Q, M, λ and the A_2 constant of w, such that*

$$\max_{B_r} u \leq C\Big\{w^{-1}(B_{2r}) \int_{B_{2r}} u\, w dx + r^\sigma \|\frac{f}{w}\|_{\sigma, B_{3r}} \Big(r^\sigma \sum_{i=1}^n \|\big(\frac{h_i}{w}\big)^2\|_{\sigma, B_{3r}}\Big)^{1/2}\Big\}.$$

Now, from our previous results, we obtain the Harnack inequality for solutions.

Theorem 3.4. *Let us assume conditions (3.1) and (3.3) are satisfied, $w \in A_2$, and let u be a weak nonnegative supersolution of (3.2) in a ball $B_{3r} \subset\subset \Omega$. Let $M > 0$ be a constant such that $u \leq M$ in B_{3r}. Then, there exists C depending on Q, M, λ and the A_2 constant of w such that*

$$\max_{B_r} u \leq C\Big\{\min_{B_r} u + r^\sigma \|\frac{f}{w}\|_{\sigma, B_{3r}} + \Big(r^\sigma \sum_{i=1}^n \|\big(\frac{h_i}{w}\big)^2\|_{\sigma, B_{3r}}\Big)^{1/2}\Big\}.$$

As a consequence of Harnack inequality we can show that the weak solutions of (3.2) are continuous with respect to the Carnot-Carathéodory metric.

Theorem 3.5. *Let us assume conditions (3.1) and (3.3) are satisfied, $w \in A_2$. Let u be a weak solution of (3.2) in Ω and let $\sup_\Omega |u| = L < +\infty$. Then u is continuous in Ω.*

The next result is a natural consequence of the previous one if we assume the lower order terms to belong to the Morrey classes M_σ.

Theorem 3.6. *Let us assume condition (3.1) is satisfied, $w \in A_2$. Let u be a weak solution of (3.2) in Ω, let $\sup_\Omega |u| = L < +\infty$ and moreover*

$$\big(\frac{b_i}{w}\big)^2, \quad \frac{c}{w}, \quad \big(\frac{d_i}{w}\big)^2, \quad \frac{f}{w}, \quad \big(\frac{h_i}{w}\big)^2 \in M_\sigma(\Omega, w) .$$

Then u is locally Hölder continuous in Ω.

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