The type of a good semigroup and the almost symmetric condition

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Abstract

We study the type and the almost symmetric condition for good subsemigoups of $\mathbb{N}^2$, a class of semigroups containing the value semigroups of curve singularities with two branches. We define the type in term of a partition of a specific set associated to the semigroup and we show that this definition generalizes the well known notion of type of a numerical semigroup and has a good behaviour with respect to the corresponding concept for algebroid curves. Then we study almost symmetric good semigroups, their connections with maximal embedding dimension good semigroups and their Apéry set, generalizing to this context several existent known results.

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1 Introduction

The concept of *good semigroup* was formally stated in [1], in order to study value semigroups of noetherian, analytically unramified, one-dimensional, semilocal, reduced rings, e.g. the local rings arising from curve singularities (and from their blowups), with more than one branch; the properties of these semigroups were already considered in [3], [5], [6], [8], [11], [12], [13], but it was in [1] that their structure was systematically studied. Similarly to the one branch case, when the value semigroup is a numerical semigroup, the properties of the rings can be translated and studied at semigroup level. For example, the celebrated result by Kunz (see [19]) that a one-dimensional, analytically irreducible, local domain is Gorenstein if and only if its value semigroup is symmetric, can be generalized to analytically unramified rings (see [12] and also [6]) and also the numerical characterization of the canonical module in the analytically irreducible case (see [17]) can be given in the more general case (see [8]).

However good semigroups present some problems that make difficult their study; first of all, they are not finitely generated as monoids (even if they can be completely determined by

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a finite set of elements (see [13], [7] and [9])); moreover the class of good semigroups is not closed under finite intersections. Secondly, the class of good ideals (e.g. the ideals arising as values of ideals of the corresponding ring) is not closed under sums and differences (see e.g. [1] and [18]).

Hence, unlike what happens for numerical semigroups (in analogy to analytically irreducible domains), it is not clear how to define some important concepts like embedding dimension, type and good semigroups.

Moreover, the class of good semigroups is larger than the class of value semigroups (see [1]), and no characterization of value good semigroups is known (notice that, for the numerical semigroup case, it is easily seen that any such semigroup is the value semigroup of the ring of the corresponding monomial curve). This means that, to prove a property for good semigroups, it is not possible to take advantage of the nature of value semigroups and it is necessary to work only with semigroup techniques.

Despite this bad facts, we showed in [10] that, even if we work with infinite sets, sometimes it is possible to produce partitions of them in a finite number of subsets, that we call levels, and the number and the nature of the levels give key informations on the semigroup we are dealing with. More precisely, in [10], working in the case of good subsemigoups of $\mathbb{N}^2$, we applied this idea to the concept of *Apéry Set* and we proved that it can be divided in e levels $\mathbf{Ap}(S) = \bigcup_{i=1}^{e} A_i$, where $e = e_1 + e_2$ and (e_1, e_2) is the minimal nonzero element in S and, in case S is a value semigroup, it represents also the multiplicity of the corresponding ring. Moreover, if S is the value semigroup of a ring $(R, \mathfrak{m}, k)$, it is possible to choose e elements $\boldsymbol{\alpha}_i$ in the Apéry Set, one for each A_i, so that, taking any element $f_i \in R$ of valuation $v(f_i) = \boldsymbol{\alpha}_i$, the classes $\bar{f}_i$ are a basis of the e-dimensional k-vector space $R/(x)$ (where x is a minimal reduction of $\mathfrak{m}$). In the same paper [10], it has been also shown that, for good symmetric semigroups S, the partition of $\mathbf{Ap}(S, \boldsymbol{e})$ satisfies a duality property similar to the duality that holds for the Apéry set in the numerical case.

The partition in levels of the Apéry set seems to be very useful also to study other properties of good semigroups. This fact is confirmed by a recent work, [20], where the authors define the concept of embedding dimension for a good semigroup and they use the levels of the Apéry set to prove that the embedding dimension is bounded above by the multiplicity (as it happens for numerical semigroups and for one-dimensional rings).

In this paper we want to use the partition in levels for particular infinite sets, in order to define and study the type of a good semigroup and to study almost symmetric good semigroups and the symmetry of their Apéry set. To do this we show that the partition in levels can be defined for any subset of $\mathbb{N}^2$, whose complement is a proper good ideal E, and that the number of levels is exactly the number $d(S \setminus E)$, where $d(\_ \setminus \_)$ is the analogue at semigroup level of the length function (see Theorem 2.5). As in the previous paper [10], we restrict to the case of good subsemigroups of $\mathbb{N}^2$, to avoid too many technicalities arising in $\mathbb{N}^n$.

Using Theorem 2.5, we are able to define, in Section 3, the type of a good subsemigroup of $\mathbb{N}^2$. We have to deal with the infinite set $(S - M) \setminus S = \{\boldsymbol{\alpha} \in (\mathbb{N}^2 \setminus S) \mid \boldsymbol{\alpha} + M \subseteq S\}$. In case $S - M$ is a good semigroup, the type can be defined and computed as $d((S - M) \setminus S)$ (see [1]); but, unfortunately, the ideal $S - M$ it is not always good, hence the function d cannot be computed. So, after noticing that we can divide $(S - M) \setminus S$ in levels, we define the type

$t(S)$ as the number of these levels. Then we show that, in case $S - M$ is a good semigroup, the type coincides with $d((S - M) \setminus S)$ (see Proposition 3.5). Finally, we prove that our definition of type is coherent with the corresponding definitions for numerical semigroups and one dimensional Cohen Macaulay rings: in fact, $t(S)$ is bounded above by the multiplicity minus 1 (see Proposition 3.7) and, if S is a value semigroup $S = v(R)$, then $t(R) \leq t(S)$, as it happens in the numerical case (see Proposition 3.8). Moreover, in Theorem 3.11, we show that S is symmetric if and only if its type is 1.

In the second part of the paper we concentrate our attention to a very important class of semigroups: almost symmetric good semigroups. This definition was given for numerical semigroups by Barucci and Froberg in [4], together with the corresponding definition of almost Gorenstein one-dimensional analytically unramified rings and was generalized in [1] for good semigroups. Recently the concept of almost Gorenstein ring has been generalized for any dimension and extensively studied by many authors (see e.g. [14] and [15]). In particular Nari proved in [21] a characterization of almost symmetric numerical semigroups via a symmetry property of its Apéry set.

More precisely, in Section 4 we give some general results for good semigroups such that $S - M = M - \boldsymbol{e}$; this condition, for numerical semigroups, and the corrsponding condition for one-dimensional rings is equivalent to maximal embeddig dimension. In particular, we show that $S - M$ is a symmetric good semigroup if and only if S is an almost symmetric good semigroup with $S - M = M - \boldsymbol{e}$ (Theorem 4.5). The analogue of this result has been proved in many different contexts (see [4], [14], [1]), but there is no proof for the good semigroups case. Then we characterize in terms of Apéry set those good semigroups such that $S - M = M - \boldsymbol{e}$ (Theorem 4.6) and we use this fact to compute their type (Corollary 4.7) and to give a procedure to construct almost symmetric good semigroups (Corollary 4.9). Finally, in Section 5, we study the Apéry set of an almost symmetric good semigroup and we generalize Nari's duality in Theorem 5.6.

2 Complementary sets of good ideals

Let $\mathbb{N}$ be the set of non-negative integers. Given $\boldsymbol{\alpha} = (\alpha_1, \alpha_2), \boldsymbol{\beta} = (\beta_1, \beta_2) \in \mathbb{N}^2$, we set $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = (\min(\alpha_1, \beta_1), \min(\alpha_2, \beta_2))$. Observe that the element $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ is the infimum of the set $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ with respect to the usual partial ordering $\leq$ in $\mathbb{N}^2$.

Let S be a submonoid of $(\mathbb{N}^2, +)$. We say that S is a *good semigroup* if

(G1) for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S$, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \in S$;

(G2) if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S$ and $\alpha_i = \beta_i$ for some $i \in \{1, 2\}$, then there exists $\boldsymbol{\delta} \in S$ such that $\delta_i > \alpha_i = \beta_i$, $\delta_j \geq \min\{\alpha_j, \beta_j\}$ for $j \in \{1, 2\} \setminus \{i\}$ and $\delta_j = \min\{\alpha_j, \beta_j\}$ if $\alpha_j \neq \beta_j$;

(G3) there exists $\boldsymbol{c} \in S$ such that $\boldsymbol{c} + \mathbb{N}^2 \subseteq S$.

Notice that, by condition (G1), there exists a minimum $\boldsymbol{c} \in \mathbb{N}^2$ for which condition (G3) holds. Therefore we will say that

$$\boldsymbol{c} := \min\{\boldsymbol{\alpha} \in \mathbb{Z}^2 \mid \boldsymbol{\alpha} + \mathbb{N}^2 \subseteq S\}$$

is the *conductor* of S. We set $\boldsymbol{\gamma} := \boldsymbol{c} - \mathbf{1}$.

Let $\boldsymbol{\alpha} \in \mathbb{Z}^2$. The following definitions are commonly used in the literature about good semigroups:

(1) $\Delta_i(\boldsymbol{\alpha}) := \{\boldsymbol{\beta} \in \mathbb{Z}^2 \mid \alpha_i = \beta_i \text{ and } \alpha_j < \beta_j \text{ for } j \neq i\}$,

(2) $\Delta_i^S(\boldsymbol{\alpha}) := \Delta_i(\boldsymbol{\alpha}) \cap S$,

(3) $\Delta(\boldsymbol{\alpha}) := \Delta_1(\boldsymbol{\alpha}) \cup \Delta_2(\boldsymbol{\alpha})$,

(4) $\Delta^S(\boldsymbol{\alpha}) := \Delta(\boldsymbol{\alpha}) \cap S$.

An element $\boldsymbol{\alpha} \in S$ is said to be *absolute* if $\Delta^S(\boldsymbol{\alpha}) = \emptyset$. By definition of conductor we immediately get $\Delta^S(\boldsymbol{\gamma}) = \emptyset$. Given $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^2$, we say that $\boldsymbol{\beta}$ is *above* $\boldsymbol{\alpha}$ if $\boldsymbol{\beta} \in \Delta_1(\boldsymbol{\alpha})$ and that $\boldsymbol{\beta}$ is *on the right* of $\boldsymbol{\alpha}$ if $\boldsymbol{\beta} \in \Delta_2(\boldsymbol{\alpha})$.

If $\boldsymbol{\alpha} \in S$ and the conductor $\boldsymbol{c} \in \Delta_i^S(\boldsymbol{\alpha})$ (for $i \in \{1, 2\}$), then, by properties (G1) and (G2), $\Delta_i(\boldsymbol{\alpha}) = \Delta_i^S(\boldsymbol{\alpha})$, meaning that each element above or, respectively, on the right of $\boldsymbol{\alpha}$ is in S.

A good semigroup is called *local* if $\mathbf{0} = (0, 0)$ is an absolute element. In the following, unless when specified, we will work only with local good semigroups hence we will omit the word local.

The value semigroups of Noetherian, analytically unramified, residually rational, one-dimensional, reduced semilocal rings with two minimal primes are good subsemigroups of $\mathbb{N}^2$ and one of such rings R is local if and only its value semigroup is local [1, Proposition 2.1]. In this article we will always assume these hypotheses on the rings R unless differently stated.

The concept of ideal has been extended from the theory of rings to semigroups. A *relative ideal* of a good semigroup S is a subset $\emptyset \neq E \subseteq \mathbb{Z}^2$ such that $E + S \subseteq E$ and $\boldsymbol{\alpha} + E \subseteq S$ for some $\boldsymbol{\alpha} \in S$. A relative ideal E contained in S is simply called an *ideal*. A relative ideal E satisfying properties (G1) and (G2), is called a *good ideal* of S (by definition, condition (G3) is always satisfied by a relative ideal). The maximal proper good ideal of a good semigroup is the set of all the nonzero elements of S and we denote it, as usual, by M. Given two good ideals E, F we define the set

$$E - F = \{\boldsymbol{\alpha} \in \mathbb{Z}^2 \mid \boldsymbol{\alpha} + F \subseteq E\}.$$

This set is not necessarily a good ideal.

A tool frequently used in the study of good semigroups is the distance function between two (relative) ideals.

Let A be a subset of $\mathbb{N}^2$ and let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A$. We say that $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are consecutive in A if, for every $\boldsymbol{\mu} \in \mathbb{N}^2$ with $\boldsymbol{\alpha} < \boldsymbol{\mu} < \boldsymbol{\beta}$, then $\boldsymbol{\mu} \notin A$.

A chain of elements of A, $\{\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 < \cdots < \boldsymbol{\alpha}_h < \cdots < \boldsymbol{\alpha}_n = \boldsymbol{\beta}\}$, with $\boldsymbol{\alpha}_{h+1}$ consecutive of $\boldsymbol{\alpha}_h$, is called *saturated* of length n. In this case $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are called, respectively, the initial and the final point of the chain.

In [8] and [18], it has been proved that all the saturated chains in a good ideal E of S with fixed initial and final points have the same length.

If $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are elements of a good ideal E, we denote by $d_E(\boldsymbol{\alpha}, \boldsymbol{\beta})$ the common length of all the saturated chains in the good ideal E with initial point $\boldsymbol{\alpha}$ and final point $\boldsymbol{\beta}$. Furthermore, if $E \supseteq F$ are two good ideals, we call $\boldsymbol{m}_E$ and $\boldsymbol{m}_F$ the minimal elements, respectively, of E and F and, taking $\boldsymbol{\alpha} \geq \boldsymbol{c}_F$, where $\boldsymbol{c}_F$ is the conductor of F, we define the distance function $d(E \setminus F) := d_E(\boldsymbol{m}_E, \boldsymbol{\alpha}) - d_F(\boldsymbol{m}_F, \boldsymbol{\alpha})$ (in [8] and [18] is proved that it is a well defined distance).

Another main concept that we recall is that of Apéry Set. The Apéry Set of a good semigroup S with respect to an element $\boldsymbol{\beta} \in S$ is defined as the set

$$\mathbf{Ap}(S, \boldsymbol{\beta}) = \{\boldsymbol{\alpha} \in S | \boldsymbol{\alpha} - \boldsymbol{\beta} \notin S\}.$$

By Property (G1), a local good semigroup always has a smallest non zero element that we will denote by $\boldsymbol{e} = (e_1, e_2)$. In literature, authors usually consider the Apéry Set of S with respect to $\boldsymbol{e} = (e_1, e_2)$. In this case, we will simply write $\mathbf{Ap}(S)$. By definitions of conductor of S and of $\mathbf{Ap}(S)$, it is clear that

$$\{\boldsymbol{\alpha} \in \mathbb{N}^2 \mid \boldsymbol{\alpha} \geq \boldsymbol{\gamma} + \boldsymbol{e} + \mathbf{1}\} \cap \mathbf{Ap}(S) = \emptyset.$$

The Apéry Set of a good semigroup is clearly infinite, but, inspired by the fact that the Apéry Set of a numerical semigroup has as many elements as the multiplicity of the semigroup, in [10] it has been defined a partition of the Apéry Set of S in $e := e_1 + e_2$ subsets, called *levels* (a similar partition was defined also in [2] but only for value semigroups of plane curves). This partition $\mathbf{Ap}(S) = \bigcup_{i=1}^{e} A_i$ has several useful consequences.

For instance, if S is the value semigroup of a ring $(R, \mathfrak{m}, k)$, it is possible to choose e elements $\boldsymbol{\alpha}_i$ in the Apéry set, one for each A_i, so that, taking any element $f_i \in R$ of valuation $v(f_i) = \boldsymbol{\alpha}_i$, the classes $\bar{f}_i$ are a basis of the e-dimensional k-vector space $R/(x)$ (where x is a minimal reduction of $\mathfrak{m}$)[10, Theorem 3.9]. Moreover, in the case S is a good symmetric semigroup, this partition of $\mathbf{Ap}(S)$ satisfies a duality property similar to the duality that holds in the numerical case [10, Theorem 5.3].

Many properties discussed in [10] about the partition of $\mathbf{Ap}(S)$ hold more in general for a subset $A \subseteq S$ such that $E := S \setminus A$ is a proper good ideal of S (observe that $\mathbf{Ap}(S, \boldsymbol{\beta}) = S \setminus (\boldsymbol{\beta} + S)$ and $\boldsymbol{\beta} + S$ is a good ideal).

We recall the definition of the partition and some properties in this more general context. In order to define the partition, we only need subsets of S whose complementary set satisfies property (G3). We are going to make use of the following partial order relation on $\mathbb{N}^2$: we say that $(\alpha_1, \alpha_2) \leq\leq (\beta_1, \beta_2)$ if an only if $(\alpha_1, \alpha_2) = (\beta_1, \beta_2)$ or $(\alpha_1, \alpha_2) \neq (\beta_1, \beta_2)$ and $(\alpha_1, \alpha_2) \ll (\beta_1, \beta_2)$, where the last means $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$.

Definition 2.1. Let $A \subseteq S$ be a subset for which there exists $\boldsymbol{c} \in S$ such that $\boldsymbol{c} + \mathbb{N}^2 \subseteq S \setminus A$. We define the following subsets of A:

$$B^{(1)} = \{\boldsymbol{\alpha} \in A : \boldsymbol{\alpha} \text{ is maximal with respect to } \leq\leq\},$$

$$C^{(1)} := \{\boldsymbol{\alpha} \in B^{(1)} : \boldsymbol{\alpha} = \boldsymbol{\beta}_1 \wedge \boldsymbol{\beta}_2 \text{ for some } \boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in B^{(1)} \setminus \{\boldsymbol{\alpha}\}\} \text{ and } D^{(1)} = B^{(1)} \setminus C^{(1)}.$$

Assume $i > 1$ and that $D^{(1)}, \ldots, D^{(i-1)}$ have been defined; we set

$$B^{(i)} = \{\boldsymbol{\alpha} \in A \setminus (\bigcup_{j<i} D^{(j)}) : \boldsymbol{\alpha} \text{ is maximal with respect to } \leq\leq\},$$

$$C^{(i)} := \{\boldsymbol{\alpha} \in B^{(i)} : \boldsymbol{\alpha} = \boldsymbol{\beta}_1 \wedge \boldsymbol{\beta}_2 \text{ for some } \boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in B^{(i)} \setminus \{\boldsymbol{\alpha}\}\} \quad \text{and} \quad D^{(i)} = B^{(i)} \setminus C^{(i)}.$$

By the assumption on $\boldsymbol{c}$, for some $N \in \mathbb{N}_+$, we have $A = \bigcup_{i=1}^{N} D^{(i)}$ and $D^{(i)} \cap D^{(j)} = \emptyset$, for any $i \neq j$. For simplicity, we prefer to number the set of the partition in increasing order, setting $A_i = D^{(N+1-i)}$ and we get

$$A = \bigcup_{i=1}^{N} A_i$$

We call the sets A_i *levels* of A.

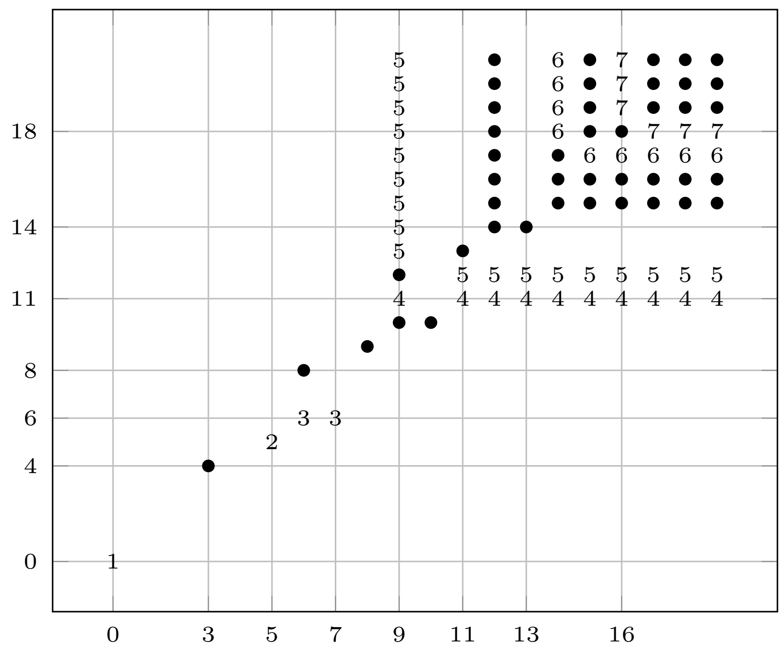

Figure 1: The value semigroup of the ring $k[[X,Y,Z]]/(X^4 - YZ, Y^2 - XZ, X^3Y - Z^2) \cap (X^3 - Z^2, Y^2 - XZ)$ and the partition of its Apéry Set $\mathbf{Ap}(S) = \bigcup_{i=1}^{7} A_i$. We mark the elements of the set A_i with the number i.

Assuming that the complementary set $E := S \setminus A$ is a proper good ideal of S, we can prove for A, with the same arguments, all the properties listed and proved in Lemma 3.2, Lemma 3.3, Theorem 3.4, Proposition 3.5 of [10] in the particular case $A = \mathbf{Ap}(S)$.

Lemma 2.2. *Let $S \subseteq \mathbb{N}^2$ be a good semigroup. Let $A \subseteq S$ such that $E := S \setminus A$ is a proper good ideal of S and let $A = \bigcup_{i=1}^{N} A_i$ be the partition of A. Let $\boldsymbol{c}_E = (c_1, c_2)$ be the conductor of E (i.e. the minimal element $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} + \mathbb{N}^2 \subseteq E$). The sets A_i satisfy the following properties:*

(1) For any $\boldsymbol{\alpha} \in A_i$ there exists $\boldsymbol{\beta} \in A_{i+1}$ such that $\boldsymbol{\alpha} \ll \boldsymbol{\beta}$ or $\boldsymbol{\alpha} = \boldsymbol{\beta}_1 \wedge \boldsymbol{\beta}_2$ with $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in A_{i+1}$ (both cases can happen at the same time).

(2) For every $\boldsymbol{\alpha} \in A_i$ and $\boldsymbol{\beta} \in A_j$, with $j \geq i$, $\boldsymbol{\beta} \not\ll \boldsymbol{\alpha}$.

(3) If $\boldsymbol{\alpha} \in A_i$, $\boldsymbol{\beta} \in A$ and $\boldsymbol{\beta} \geq \boldsymbol{\alpha}$, then $\boldsymbol{\beta} \in A_i \cup \cdots \cup A_N$.

(4) *If* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2), \boldsymbol{\beta} = (\alpha_1, \beta_2) \in A_i$*, with* $\alpha_2 < \beta_2$*, then for any* $\boldsymbol{\delta} = (\delta_1, \delta_2) \in A$ *such that* $\delta_1 > \alpha_1$ *and* $\delta_2 \geq \alpha_2$*, we get* $\boldsymbol{\delta} \in A_{i+1} \cup \cdots \cup A_N$*; an analogous statement holds switching the components.*

(5) *If* $\boldsymbol{\alpha} \ll \boldsymbol{\beta} \in A$ *and they are consecutive in* S*, then there exists* $i > 0$ *such that* $\boldsymbol{\alpha} \in A_i$ *and* $\boldsymbol{\beta} \in A_{i+1}$*; if* $\boldsymbol{\alpha} < \boldsymbol{\beta} \in A$*, they share a component and they are consecutive in* S*, then there exists* $i > 0$ *such that either* $\boldsymbol{\alpha} \in A_i$ *and* $\boldsymbol{\beta} \in A_{i+1}$ *or* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A_i$*.*

(6) *Let* $\boldsymbol{\alpha} \in A_i$ *and let be* $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_j$ *all the elements of* A*,* $\boldsymbol{\alpha} < \boldsymbol{\beta}_r$ *and consecutive to* $\boldsymbol{\alpha}$ *in* A*. Then at least one of them belongs to* A_{i+1}*.*

(7) $\boldsymbol{\alpha} = (\alpha_1, c_2) \in A_i \Leftrightarrow (\alpha_1, n) \in A_i$ *for some* $n \geq c_2 \Leftrightarrow (\alpha_1, n) \in A_i$ *for all* $n \geq c_2$*; an analogous statement holds switching the components.*

(8) *If* $\boldsymbol{\alpha} = (\alpha_1, c_2) \in A_i$ *and* $\boldsymbol{\beta} = (\beta_1, c_2) \in A$*, with* $\beta_1 < \alpha_1$ *and such that for every* a*,* $\beta_1 < a < \alpha_1$*, the element* $(a, c_2) \notin A$*, then* $\boldsymbol{\beta} \in A_{i-1}$*; an analogous statement holds switching the components.*

Proof. The proof is analogous to that of [10, Lemma 3.2]. In the proof of (7), to see that, if $\boldsymbol{\alpha} = (\alpha_1, c_2) \in A$ then $(\alpha_1, n) \in A$ for all $n \geq c_2$, just assume by way of contradiction $(\alpha_1, n) \in E$, for some $n \geq c_2$. Hence, since $\boldsymbol{c} = (c_1, c_2) \in E$, by property (G1) also $\boldsymbol{\alpha} = (\alpha_1, n) \wedge \boldsymbol{c} \in E$, contradicting the assumption. The fact that they belong to the same level A_i follows as in the proof of [10, Lemma 1]. □

Lemma 2.3. *Let* $S \subseteq \mathbb{N}^2$ *be a good semigroup. Let* $A \subseteq S$ *such that* $E := S \setminus A$ *is a proper good ideal of* S *and let* $A = \bigcup_{i=1}^{N} A_i$ *be the partition of* A*. The following assertions hold:*

(1) *Let* $\boldsymbol{\alpha} \in \mathbb{N}^2$ *and assume there is a finite positive number of elements in* $\Delta_1^S(\boldsymbol{\alpha}) \cap E$*. Call* $\boldsymbol{\delta}$ *the maximum of them. Hence* $\Delta^S(\boldsymbol{\delta}) \subseteq A$*;*

(2) *Let* $\boldsymbol{\alpha} \in A$*. If there exists* $\boldsymbol{\beta} \in E \cap \Delta_1(\boldsymbol{\alpha})$*, then* $\Delta_2^S(\boldsymbol{\alpha}) \subseteq A$*;*

(3) *Let* $\boldsymbol{\alpha} = (a_1, a_2) \in A_i$ *and suppose there exists* $b_2 < a_2$ *such that* $\boldsymbol{\delta} = (a_1, b_2) \in S$ *and* $\Delta_2^S(\boldsymbol{\delta}) \subseteq A$*. Then the minimal element* $\boldsymbol{\beta} = (b_1, b_2)$ *of* $\Delta_2^S(\boldsymbol{\delta})$ *is in* A_j *for some* $j \leq i$*. In particular, if* $\Delta^S(\boldsymbol{\delta}) \subseteq A$ *and* $\boldsymbol{\alpha}$ *is the minimal element of* $\Delta_1^S(\boldsymbol{\delta})$*,* $\boldsymbol{\beta} \in A_i$*.*

(4) *Let* $\boldsymbol{\alpha} = (a_1, a_2) \in A_i$ *and suppose there exists* $\boldsymbol{\delta} \in E \cap \Delta_1(\boldsymbol{\alpha})$*. Then* $\Delta_2^S(\boldsymbol{\alpha}) \subseteq A$ *and the minimal element* $\boldsymbol{\beta} = (b_1, a_2)$ *of* $\Delta_2^S(\boldsymbol{\alpha})$ *is also in* A_i*.*

(5) *Let* $\boldsymbol{\alpha} \in A_i$ *and assume* $\Delta_1^S(\boldsymbol{\alpha}) \subseteq A$*. Assume also that there exists* $\boldsymbol{\beta} \in \Delta_1^S(\boldsymbol{\alpha}) \cap A_{i+1}$*. Then there exists* $\boldsymbol{\theta} \in (\Delta_1^S(\boldsymbol{\alpha}) \cap A_i) \cup \{\boldsymbol{\alpha}\}$ *such that* $\boldsymbol{\theta} < \boldsymbol{\beta}$ *and* $\Delta^S(\boldsymbol{\theta}) \subseteq A$*.*

The analogous assertions hold switching the components.

Proof. The proof is analogous to that of [10, Lemma 3.3]. □

Lemma 2.4. *Let* S *be a good semigroups and* $A = \bigcup_{i=1}^{N} A_i \subseteq S$ *be such that* $E := S \setminus A$ *is a proper good ideal of* S*. Then, for every* $2 \leq i \leq N$ *there exists* $\boldsymbol{\beta} \in A_{i-1}$ *such that* $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$*.*

Proof. The proof is the same of [10, Proposition 3.5]. □

Theorem 2.5. *Let $S \subseteq \mathbb{N}^2$ be a good semigroup. Let $A \subseteq S$ such that $E := S \setminus A$ is a proper good ideal of S and let $A = \bigcup_{i=1}^{N} A_i$ be the partition of A. Then*

$$N = d(S \setminus E).$$

Proof. This result can be proved following the same proof of [10, Theorem 3.4], in which it is proved that the number of levels of $\mathbf{Ap}(S)$ is $d(S \setminus (\boldsymbol{e} + S)) = e_1 + e_2$. □

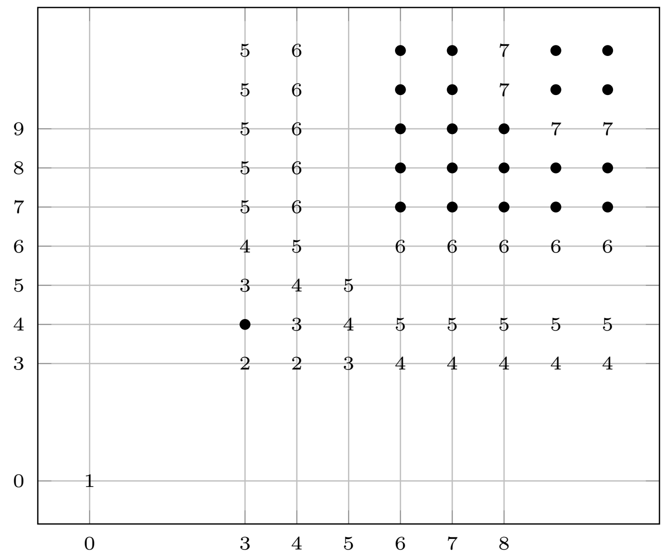

Figure 2: In this good semigroup S, we consider the Apéry set $\mathbf{Ap}(S, \boldsymbol{\beta}) = \bigcup_{i=1}^{7} A_i$, with respect to the element $\boldsymbol{\beta} = (3, 4)$. We indicate with black marks the elements of $\boldsymbol{\beta} + S$ and by the number i the elements of the set A_i.

3 The type of a good semigroup

The *type* of a ring is a classical invariant studied in commutative algebra. For a local one-dimensional ring $(R, \mathfrak{m})$ the type $t(R)$ is equal to the length of the R-module

$$\frac{\mathfrak{m}^{-1}}{R}$$

(see for instance [16, Proposition 2.16]). Analogously, for a numerical semigroup S having maximal ideal M, the type is defined as the number of elements of the set $(S - M) \setminus S$ (cf. [16]). In [1], the same notion has been defined for a good semigroup S, extending that one given for numerical semigroups, but only in the case $S - M$ is a good relative ideal of S.

Indeed there, the type of S is defined as the distance $d((S - M) \setminus S)$ which is a well defined quantity only if $S - M$ is a good relative ideal. Unfortunately $S - M$ is not necessarily a good relative ideal, even if S is the value semigroup of a ring, as shown in [1, Examples 2.10, 3.3] and recalled here in Figure 3. Anyway, in [1] are also considered classes of rings R for which it is well defined the type of $S = v(R)$.

In order to define the type of any good semigroup (even for those that are not value semigroups of rings), we make use of the partition described in Definition 2.1.

Definition 3.1. The set of *pseudo-frobenius elements* of a good semigroup S is defined as

$$\mathbf{PF}(S) = \{\boldsymbol{\alpha} \in (\mathbb{N}^2 \setminus S) \mid \boldsymbol{\alpha} + M \subseteq S\}.$$

It is easy to observe the following fact:

Proposition 3.2. *The following are equivalent for $\boldsymbol{\alpha} \in \mathbb{N}^2$:*

1. $\boldsymbol{\alpha} \in \mathbf{PF}(S)$.
2. $\boldsymbol{\alpha} + \boldsymbol{e} \in \mathbf{Ap}(S)$ *and* $\boldsymbol{\alpha} + \boldsymbol{e} + \boldsymbol{\beta} \notin \mathbf{Ap}(S)$ *for every* $\boldsymbol{\beta} \in M$.

Moreover we observe also that $\Delta(\boldsymbol{\gamma}) \subseteq \mathbf{PF}(S)$ and, since $\boldsymbol{\gamma} + (1,1) + \mathbb{N}^2 \subseteq S$, the complementary set of $\mathbf{PF}(S)$ in the good semigroup $\mathbb{N}^2$ satisfies property (G3). Hence, applying Definition 2.1, we write its partition

$$\mathbf{PF}(S) = \bigcup_{h=1}^{t} P_h.$$

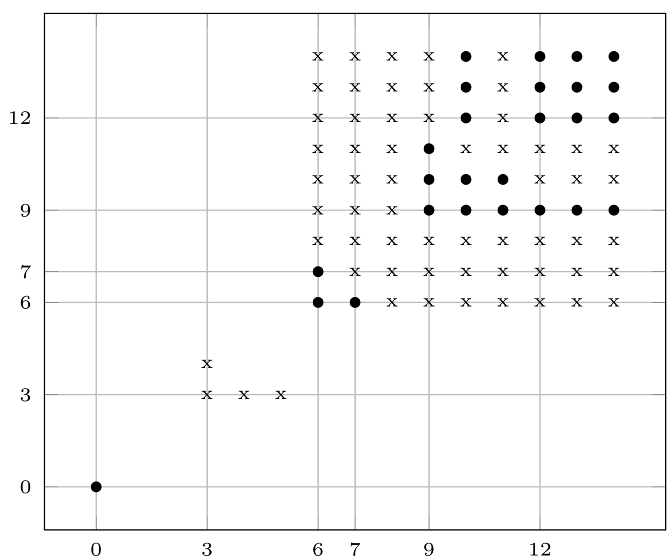

Figure 3: This good semigroup is a value semigroup (see[1, Example 3.3]). Here we indicate with black marks the elements of S and by x the pseudo-frobenius elements of S (elements of $S - M$ but not of S). We observe that $S - M$ is not good since it does not satisfy property (G2).

Definition 3.3. Let, as above $\mathbf{PF}(S) = \bigcup_{h=1}^{t} P_h$. The *type* of the good semigroup S is defined as the number of levels of the pseudo-frobenius elements, $t(S) := t$.

Following this definition, one may check that the type of the good semigroup in Figure 3 is 8. Now, we need to verify that this definition of type coincide with that given in [1] (in the case $S - M$ good) and that common properties of the type of numerical semigroups and of their associated semigroup rings are satisfied also in this more general context.

Lemma 3.4. *Let $T \subseteq \mathbb{N}^2$ be a good semigroup (not necessarily local) and let $A \subseteq T$ be a subset for which there exists $\boldsymbol{c} \in T$ such that $\boldsymbol{c} + \mathbb{N}^2 \subseteq T \setminus A$. Consider the partition $A = \bigcup_{i=1}^{N} A_i$ as in Definition 2.1. Assume $\mathbf{0} \in A$ and $\Delta^T(\mathbf{0}) \subseteq A$. Then $A_1 = \{\mathbf{0}\}$.*

Proof. By way of contradiction assume that exists $\boldsymbol{\beta} \neq \mathbf{0}$ such that $\boldsymbol{\beta} \in A_1$. By Lemma 2.2(2), $\boldsymbol{\beta} \not\gg \mathbf{0}$ and thus $\boldsymbol{\beta} \in \Delta^T(\mathbf{0})$. By property (G2), there exists a minimal element $\boldsymbol{\delta} \in \Delta^T(\mathbf{0})$ such that $\mathbf{0} = \boldsymbol{\beta} \wedge \boldsymbol{\delta}$, and, by Lemma 2.2(3), we may assume also $\boldsymbol{\beta}$ to be minimal in $\Delta^T(\mathbf{0})$. Applying Lemma 2.3(3), we are forced to have $\boldsymbol{\delta} \in A_1$, but this is a contradiction by Lemma 2.2(4). It follows that $A_1 = \{\mathbf{0}\}$. □

Proposition 3.5. *Let S be a good semigroup such that $S - M$ is a good relative ideal of S. Then $t(S) = d((S - M) \setminus S)$.*

Proof. Being a good relative ideal of S, $S - M$ is also a good semigroup (not necessarily local, as one can see in Figure 5). We observe that the complementary set of the good ideal M in $S - M$ is $\mathbf{PF}(S) \cup \{\mathbf{0}\}$. By Lemma 3.4, the partition in levels of $\mathbf{PF}(S) \cup \{\mathbf{0}\}$ it is equal to

$$\{\mathbf{0}\} \cup \bigcup_{h=1}^{t(S)} P_h$$

and thus the number of levels of this set is $t(S) + 1$. Now, by Theorem 2.5, the number of levels of $\mathbf{PF}(S) \cup \{\mathbf{0}\}$ is also equal to the distance $d((S - M) \setminus M)$, hence

$$t(S) + 1 = d((S - M) \setminus M) = d((S - M) \setminus S) + 1.$$

This concludes the proof. □

The following lemma, stating an inequality between the numbers of levels of two sets, one contained in the other, is needed to prove the next two results.

Lemma 3.6. *Let $A \subseteq S$ be a subset for which there exists $\boldsymbol{c} \in S$ such that $\boldsymbol{c} + \mathbb{N}^2 \subseteq S \setminus A$ and let $F \subseteq A$ with the same property. Write the partitions of the two sets $A = \bigcup_{i=1}^{N} A_i$ and $F = \bigcup_{i=1}^{M} F_i$ as in Definition 2.1. Then $M \leq N$.*

Proof. Clearly we have $F_M \subseteq A$. For any element $\boldsymbol{\alpha} \in F_{M-1}$ we can find $\boldsymbol{\beta} \in F_M$ such that either $\boldsymbol{\alpha} \ll \boldsymbol{\beta}$ or $\boldsymbol{\alpha} = \boldsymbol{\beta} \wedge \boldsymbol{\delta}$ for some other element $\boldsymbol{\delta} \in F_M$. But now since $\boldsymbol{\beta}, \boldsymbol{\delta} \in A$, by Lemma 2.2 we must have at least $\boldsymbol{\alpha} \notin A_N$ and therefore $F_{M-1} \subseteq \bigcup_{i=1}^{N-1} A_i$. Similarly, if $\boldsymbol{\alpha} \in F_{M-2}$, we find $\boldsymbol{\beta}, \boldsymbol{\delta} \in F_{M-1}$ such that either $\boldsymbol{\alpha} \ll \boldsymbol{\beta}$ or $\boldsymbol{\alpha} = \boldsymbol{\beta} \wedge \boldsymbol{\delta}$ and since now $F_{M-1} \subseteq \bigcup_{i=1}^{N-1} A_i$, we must have $\boldsymbol{\alpha} \notin A_{N-1}$ and therefore $F_{M-2} \subseteq \bigcup_{i=1}^{N-2} A_i$. Assuming $M \geq N$ and iterating this process we get $F_{M-N+1} \subseteq A_1$ and this implies, applying the same argument of above, that we cannot have elements in the level F_{M-N}. Hence, it must be $M \leq N$. □

Next result extends to the type of good semigroups the upper bound given in term of the multiplicity and well known for the type of numerical semigroups/semigroup rings. Successively, we also extend to this context the known inequality between the type of a ring and the type of its value semigroup.

Proposition 3.7. *Let S be a good semigroup having minimal nonzero element $\boldsymbol{e} = (e_1, e_2)$. Then*

$$t(S) \le e_1 + e_2 - 1.$$

Proof. The type of a good semigroup S is defined as the number of levels of the set $\mathbf{PF}(S)$. The number of levels does not change under the translation of the set by summing an element. Since the set $\boldsymbol{e} + \mathbf{PF}(S)$ is a subset of $\mathbf{Ap}(S)$ not containing $\mathbf{0}$, we get the thesis by Lemma 3.6 and Theorem 2.5, observing that $\{\mathbf{0}\}$ is the first set of the partition of $\mathbf{Ap}(S)$ and that the number of the levels of $\mathbf{Ap}(S)$ is $e_1 + e_2$. □

Proposition 3.8. *Let $(R, \mathfrak{m}, k)$ be an analytically unramified one dimensional local reduced ring, having value semigroup $S = v(R)$. Then*

$$t(R) \le t(S).$$

Proof. It is well known that the set $v(\mathfrak{m}^{-1})$ is a good relative ideal of S and the type of R is equal to the distance $d(v(\mathfrak{m}^{-1}) \setminus S)$. Moreover, $v(\mathfrak{m}^{-1})$ is a good semigroup and M is a good ideal of it. Let n be the number of levels of the set $v(\mathfrak{m}^{-1}) \setminus M$. Applying Theorem 2.5, we get that n is equal to the distance $d(v(\mathfrak{m}^{-1}) \setminus M) = t(R) + 1$. Moreover, by Lemma 3.4 and since

$$v(\mathfrak{m}^{-1}) \setminus M = (v(\mathfrak{m}^{-1}) \setminus S) \cup \{\mathbf{0}\},$$

the number of levels of $v(\mathfrak{m}^{-1}) \setminus S$ is equal to $n - 1 = t(R)$.

Now, we know that $t(S)$ is the number of levels of $\mathbf{PF}(S)$ and, since $v(\mathfrak{m}^{-1}) \subseteq (S - M)$, we get $(v(\mathfrak{m}^{-1}) \setminus S) \subseteq \mathbf{PF}(S)$. We conclude applying Lemma 3.6 to see that $t(R) \le t(S)$. □

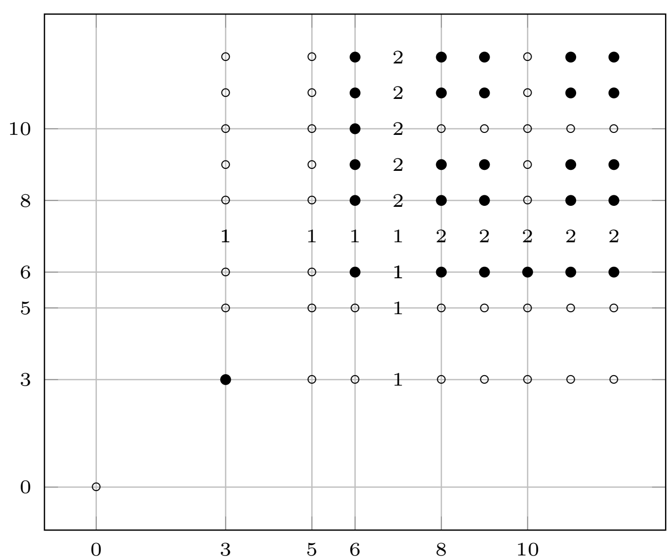

Figure 4: A good semigroup of type 2. We indicate with black marks the elements of $\boldsymbol{e} + S$, with white marks the element of the Apéry Set and by 1 and 2 the elements of the two levels of the pseudo-frobenius elements.

It is well known that Gorenstein rings are characterized by having type equal to one. Similarly symmetric semigroups (i.e. value semigroups of Gorenstein analytically unramified one dimensional local reduced rings) are characterized by having type equal to one, following the definition given in [1]. We want to show that this condition characterizes symmetry for good semigroups also with our more general definition of type.

Definition 3.9. A good semigroup S is *symmetric* if, for every $\boldsymbol{\alpha} \in \mathbb{N}^2$, $\boldsymbol{\alpha} \in S$ if and only if $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset$.

Properties of symmetric good semigroup are surveyed in [10, Section 4 and 5]. We recall that if $\boldsymbol{\alpha} \in S$, then it is always true that $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset$, while the converse fails for some element in the case of a non-symmetric good semigroup. Pseudo-frobenius elements not belonging to $\Delta(\boldsymbol{\gamma})$ are elements for which this second implication fails.

Lemma 3.10. *Let S be a good semigroup. Then*

$$\Delta(\boldsymbol{\gamma}) \subseteq \mathbf{PF}(S) \subseteq \Delta(\boldsymbol{\gamma}) \cup \{\boldsymbol{\alpha} \in (\mathbb{N}^2 \setminus S) \mid \Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset\}.$$

Proof. The first inclusion follows easily from the definitions. For the second, take $\boldsymbol{\alpha} \in \mathbf{PF}(S) \setminus \Delta(\boldsymbol{\gamma})$ and observe that, since $\boldsymbol{\alpha} + \boldsymbol{\beta} \in S$ for every $\boldsymbol{\beta} \in M$, we get $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha} - \boldsymbol{\beta}) = \emptyset$ for every $\boldsymbol{\beta} \in M$. It follows

$$\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) \subseteq \bigcap_{\boldsymbol{\beta} \in M} \mathbf{Ap}(S, \boldsymbol{\beta}) = \{\mathbf{0}\}.$$

Since $\boldsymbol{\alpha} \notin \Delta(\boldsymbol{\gamma})$, we must necessarily have $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset$. □

Theorem 3.11. *A good semigroup S is symmetric if and only if $t(S) = 1$.*

Proof. S is symmetric if and only if the set $\{\boldsymbol{\alpha} \in (\mathbb{N}^2 \setminus S) \mid \Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset\}$ is empty. Lemma 3.10 implies that this condition is equivalent to have $\mathbf{PF}(S) = \Delta(\boldsymbol{\gamma})$. By Definition 3.3 this is equivalent to say $t(S) = 1$. □

4 Almost symmetric good semigroups

In [4, Definition-Proposition 20], a local one dimensional analitically unramified ring $(R, \mathfrak{m})$ is called *almost Gorenstein* if $\mathfrak{m}\omega = \mathfrak{m}$, where ω is a canonical ideal of R lying between R and $\overline{R}$. Gorenstein rings are exactly the almost Gorenstein rings of type one.

For a good semigroup S, the *canonical ideal* is defined as

$$K := \{\boldsymbol{\alpha} \in \mathbb{Z}^2 \mid \Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset\}.$$

By property (G3), $K \subseteq \mathbb{N}^2$ and it proved in [8] that K is a good relative ideal of S. Moreover, if $S = v(R)$, K is the set of values of the canonical ideal ω.

Since $S \subseteq K$, S is symmetric if and only if $K = S$. In [1] a good semigroup S is called *almost symmetric* if $M = K + M$. Clearly any symmetric semigroup is almost symmetric. In general $S - M \subseteq K \cup \Delta(\boldsymbol{\gamma})$ (see Lemma 3.10) and in [1, Lemma 3.5] it is proved that S is almost symmetric if and only if the equality holds.

Thus, if S is almost symmetric, since $K \cup \Delta(\boldsymbol{\gamma})$ is a good relative ideal, so it is $S - M$. From these facts, it easily follows another useful equivalent condition for almost symmetry that we will often use in this work. We state it in the next lemma:

Lemma 4.1. *A good semigroup S is almost symmetric if and only if*

$$\mathbf{PF}(S) = \Delta(\boldsymbol{\gamma}) \cup \{\boldsymbol{\alpha} \in (\mathbb{N}^2 \setminus S) \mid \Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset\}.$$

Remark 4.2. By what said in the previous paragraph together with Proposition 3.5, it follows that if a good semigroup S is almost symmetric, then $d(K \setminus S) = t(S) - 1$. The last property is a well-known characterization of almost symmetric numerical semigroups but is still unclear if the good semigroups for which $d(K \setminus S) = t(S) - 1$ have to be necessarily almost symmetric.

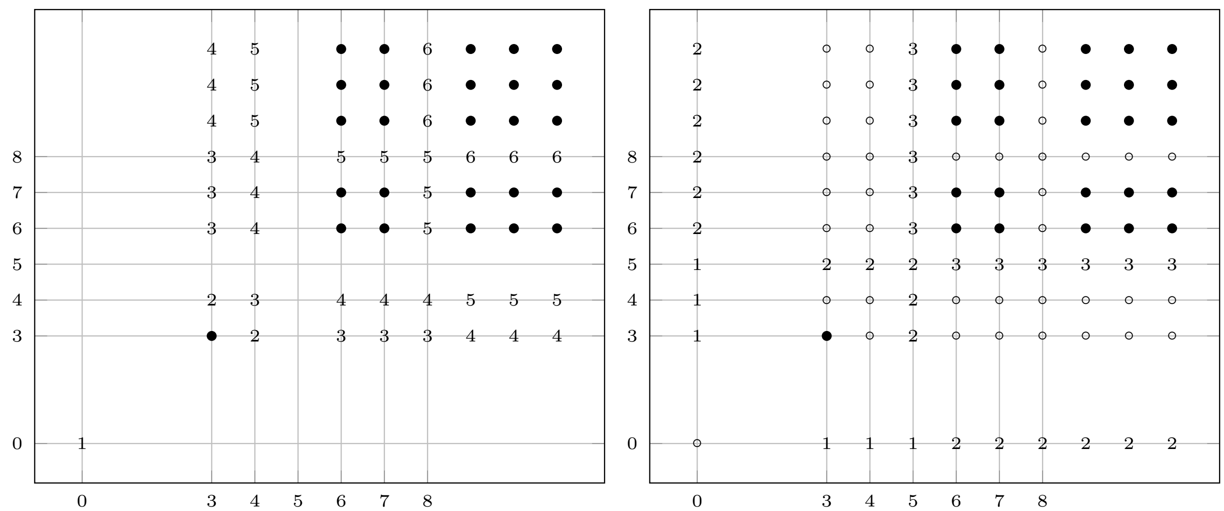

Figure 5: An almost symmetric good semigroup of type 3. In the picture on the left elements in different levels of $\mathbf{Ap}(S)$ are indicated by different numbers, while in the picture on the right the elements of $\mathbf{Ap}(S)$ are indicated by white marks and the numbers indicate the levels of the pseudo-frobenius elements.

A classical result for numerical semigroups states that, if S is a numerical semigroup, $S - M$ is symmetric if and only if S is almost symmetric and $M - M = M - e$ where e is the minimal nonzero element of S (this last condition it is equivalent to having S of maximal embedding dimension). We prove that the same result holds for good semigroups. First we observe that, by property (G3), $\boldsymbol{\gamma} + (1, 1) + \mathbb{N}^2 \subseteq S$, $\mathbf{Ap}(S)$ contains $\Delta^S(\boldsymbol{\gamma} + \boldsymbol{e})$, and therefore $S - M \subseteq \mathbb{N}^2$, implying $S - M = M - M$.

Lemma 4.3. *Let S be a good semigroup and assume $S - M$ to be a relative good ideal of S. Then $S - M$ is a good semigroup and its conductor is $\boldsymbol{\gamma} - \boldsymbol{e} + (1, 1)$.*

Proof. The fact that $S - M$ is a good semigroup is trivial since it contains $\mathbf{0}$. Since $\Delta^S(\boldsymbol{\gamma}) = \emptyset$, clearly $\Delta^{S-M}(\boldsymbol{\gamma} - \boldsymbol{e}) = \emptyset$. Thus, since $\boldsymbol{e}$ is the minimal element of M and $\boldsymbol{\gamma} + (1, 1)$ is the conductor of S, we get that $\boldsymbol{\gamma} - \boldsymbol{e} + (1, 1)$ is the conductor of $S - M$. □

In [4, Proposition 2.5], Barucci and Fröberg prove that a one dimensional local Noetherian ring $(R, \mathfrak{m})$ is at the same time almost Gorenstein and of maximal embedding dimension (or, equivalently, $\mathfrak{m}$ is stable) if and only if the fractional ideal $\mathfrak{m}^{-1}$ is Gorenstein as a ring. This classical statement can be translated to the value semigroup $S = v(R)$, obtaining that S is almost symmetric and $S - M = M - \boldsymbol{e}$ if and only if $S - M$ is symmetric, but this has been proved in [1, Proposition 3.7, Corollary 3.16] only with the extra assumption that $t(S) = t(R)$. Here, we give a proof of a more general result for good semigroups without assuming that they are value semigroups of some ring and without assumptions on their type.

Before starting with the proof, we need to observe that the semigroup $S - M$ is not necessarily local, and therefore we need to justify that our notion of symmetric good semigroup introduced in Definition 3.9 is compatible with the definition of symmetric good semigroup in the non-local case. According to [1, Theorem 2.5, Remark 2.6], every good semigroup S is expressible in a unique way as a direct product of local good semigroups and in the case $S \subseteq \mathbb{N}^2$ is not local, then $S = S_1 \times S_2$ is the direct product of its two numerical projections. By [1, Lemma 3.10] a non-local good semigroup $S = S_1 \times S_2 \subseteq \mathbb{N}^2$ is symmetric if and only if S_1 and S_2 are symmetric numerical semigroups. We prove that this definition coincides with Definition 3.9.

Proposition 4.4. *Let $S = S_1 \times S_2 \subseteq \mathbb{N}^2$ be a non-local good semigroup. The following conditions are equivalent:*

1. *$\boldsymbol{\alpha} \in S$ if and only if $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset$.*

2. *S_1 and S_2 are symmetric numerical semigroups.*

Proof. (1) $\Rightarrow$ (2): First we observe that, since $S = S_1 \times S_2$, then the components of the conductor of S are the conductors of the projections and hence $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ where γ_i is the Frobenius number of S_i. The numerical semigroup S_i is symmetric if for every positive integer a, the condition $\gamma_i - a \notin S_i$ implies $a \in S_i$. Hence for some integer $a \geq 0$, assume $\gamma_1 - a \notin S_1$. It follows that $(\gamma_1 - a, b) \notin S$ for every $b \in \mathbb{N}$, and, by assumption (1), $\Delta^S(a, \gamma_2 - b) \neq \emptyset$ for every $b \in \mathbb{N}$. This necessarily implies $a \in S_1$, and hence S_1 is symmetric. With the same argument, one can show that also S_2 is symmetric.
(2) $\Rightarrow$ (1): As observed previously in this article, $\boldsymbol{\alpha} \in S$ always implies $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset$. Hence assume $\boldsymbol{\alpha} = (a, b) \notin S$ and, without loss of generality, $a \notin S_1$. But, being S_1 symmetric, this means $\gamma_1 - a \in S_1$. Hence for every $c \in S_2$, $(\gamma_1 - a, c) \in S$, implying $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) \neq \emptyset$. □

Now we are ready to prove the anticipated result:

Theorem 4.5. *Let S be a good semigroup. The following conditions are equivalent:*

1. *$S - M$ is a symmetric good semigroup.*

2. *S is almost symmetric and $S - M = M - \boldsymbol{e}$.*

Proof. (1) $\Rightarrow$ (2): Assume $S - M$ a symmetric good semigroup and let $\boldsymbol{\alpha} \in \mathbb{N}^2 \setminus S$ such that $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset$. Hence $\Delta^{S-M}(\boldsymbol{\gamma} - \boldsymbol{e} - \boldsymbol{\alpha}) = \emptyset$ and, since $S - M$ is symmetric, this implies $\boldsymbol{\alpha} \in S - M$. Since $(S - M) \setminus S = \mathbf{PF}(S)$, we conclude that S is almost symmetric by Lemma 4.1 and Lemma 3.10.

For the other condition, we observe that $S - M \subseteq M - \boldsymbol{e}$ and prove the opposite inclusion. Let $\boldsymbol{\alpha} \in M - \boldsymbol{e}$ and assume by way of contradiction $\boldsymbol{\alpha} \notin S - M$. By symmetry of $S - M$, we can find $\boldsymbol{\beta} \in \Delta^{S-M}(\boldsymbol{\gamma} - \boldsymbol{e} - \boldsymbol{\alpha}) \neq \emptyset$, but since $\boldsymbol{\alpha} + \boldsymbol{e} \in S$, $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{e} - \boldsymbol{\alpha}) = \emptyset$ and therefore $\boldsymbol{\beta} \in \mathbf{PF}(S)$. Now, $\boldsymbol{\alpha} + \boldsymbol{e} \in \Delta^S(\boldsymbol{\gamma} - \boldsymbol{\beta})$ and, if $\boldsymbol{\beta} \in \Delta(\boldsymbol{\gamma})$, this means $\boldsymbol{\alpha} + \boldsymbol{e} = \mathbf{0}$ that is a

contradiction since $\mathbf{0} \notin M$. By Lemma 3.10, this implies that $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\beta}) = \emptyset$ which is a contradiction too. Thus we must have $\boldsymbol{\alpha} \in S - M$.
(2) ⇒ (1): First recall that if S is almost symmetric, then $S - M$ is a good semigroup. To prove that $S - M$ is symmetric we take $\boldsymbol{\alpha} \in \mathbb{N}^2$ such that $\Delta^{S-M}(\boldsymbol{\gamma} - \boldsymbol{e} - \boldsymbol{\alpha}) = \emptyset$ and prove $\boldsymbol{\alpha} \in S - M$. If $\boldsymbol{\alpha} \in S$, we are done, therefore we assume $\boldsymbol{\alpha} \notin S$ and by Lemma 4.1, we get that either $\boldsymbol{\alpha} \in \mathbf{PF}(S) \subseteq S - M$ or $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) \neq \emptyset$. But now, if $\boldsymbol{\beta} \in \Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha})$, we get $\boldsymbol{\beta} - \boldsymbol{e} \in (M - \boldsymbol{e}) \cap \Delta(\boldsymbol{\gamma} - \boldsymbol{e} - \boldsymbol{\alpha}) = \Delta^{S-M}(\boldsymbol{\gamma} - \boldsymbol{e} - \boldsymbol{\alpha}) = \emptyset$ and this is a contradiction. □

The last theorem allows to easily construct examples of almost symmetric non-symmetric good semigroup, using the next characterization of the good semigroups for which $S - M = M - M = M - \boldsymbol{e}$.

Theorem 4.6. *Let S be a good semigroup and denote $\mathbf{Ap}(S)^\star := \mathbf{Ap}(S) \setminus \{\mathbf{0}\}$. The following conditions are equivalent:*

1. *$S - M = M - \boldsymbol{e}$.*
2. *There exists a good semigroup T (not necessarily local) and $\boldsymbol{\omega} \in T$, such that $S = (\boldsymbol{\omega} + T) \cup \{\mathbf{0}\}$.*
3. *For every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{Ap}(S)^\star$, the sum $\boldsymbol{\alpha} + \boldsymbol{\beta} \notin \mathbf{Ap}(S)^\star$.*
4. *$\mathbf{PF}(S) = \{\boldsymbol{\alpha} - \boldsymbol{e} \,|\, \boldsymbol{\alpha} \in \mathbf{Ap}(S)^\star\}$.*

Proof. (1) ⇒ (2): Since $S - M = M - \boldsymbol{e}$, then it is a good semigroup. Hence take $T = S - M$ and $\boldsymbol{\omega} = \boldsymbol{e}$ to get $(\boldsymbol{\omega} + T) \cup \{\mathbf{0}\} = S$.
(2) ⇒ (3): Let $S = (\boldsymbol{\omega} + T) \cup \{\mathbf{0}\}$ and observe that in this case $\boldsymbol{\omega} = \boldsymbol{e}$ is the minimal nonzero element of S. Take $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{Ap}(S)^\star$ and write $\boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{e}$ and $\boldsymbol{\beta} = \boldsymbol{\beta}' + \boldsymbol{e}$ for $\boldsymbol{\alpha}', \boldsymbol{\beta}' \in T$. Hence

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = \boldsymbol{\alpha}' + \boldsymbol{e} + \boldsymbol{\beta}' + \boldsymbol{e} \in \boldsymbol{e} + (\boldsymbol{e} + T) \subseteq \boldsymbol{e} + S.$$

(3) ⟺ (4): Follows by Proposition 3.2.
(3) ⇒ (1): Let $\boldsymbol{\theta} \in M - \boldsymbol{e}$ and $\boldsymbol{\delta} \in M$. Assume $\boldsymbol{\theta} \notin S$ and hence $\boldsymbol{\theta} + \boldsymbol{e} \in \mathbf{Ap}(S)^\star$, otherwise clearly $\boldsymbol{\theta} \in S - M$. In the case $\boldsymbol{\delta} \in \boldsymbol{e} + S$ we may write

$$\boldsymbol{\theta} + \boldsymbol{\delta} = (\boldsymbol{\theta} + \boldsymbol{e}) + (\boldsymbol{\delta} - \boldsymbol{e}) \in M + S \subseteq M.$$

Otherwise, if $\boldsymbol{\delta} \in \mathbf{Ap}(S)$, we have by assumption (3),

$$\boldsymbol{\theta} + \boldsymbol{\delta} = (\boldsymbol{\theta} + \boldsymbol{e}) + \boldsymbol{\delta} - \boldsymbol{e} \in (\mathbf{Ap}(S)^\star + \mathbf{Ap}(S)^\star) - \boldsymbol{e} \subseteq (\boldsymbol{e} + M) - \boldsymbol{e} = M.$$

In every case, we get $\boldsymbol{\theta} \in M - M = S - M$. □

In the numerical semigroup case, all the conditions listed in the preceding theorem are equivalent to have S of maximal embedding dimension (i.e. the number of generators of the semigroup S is equal to the minimal nonzero element of S). The embedding dimension of good semigroups has been recently defined in [20] in a way that makes this notion compatible with the standard notion of embedding dimension of a ring. In [20], the authors define the good semigroups of maximal embedding dimension as those having emebedding dimension equal to $e_1 + e_2$ and show that Arf good semigroups are of maximal embedding dimension (as the Arf numerical semigroups and the Arf rings are). Furthermore, they conjecture that good semigroups fulfilling the conditions of Theorem 4.6 are also equivalent to have S of maximal embedding dimension.

At the same time, in the numerical case, maximal embedding dimension is also equivalent to have maximal type (i.e $t(S) = e - 1$ where e is the minimal nonzero element of S). Next corollary, easily following by Definition 3.3, shows that the equivalent conditions given in Theorem 4.6 are sufficient to imply maximal type, but it is still unclear if they are also necessary condition for it. We leave this fact as a question.

Corollary 4.7. *The type of a good semigroup S such that $S - M = M - \boldsymbol{e}$ is $e_1 + e_2 - 1$.*

Question 4.8. Let S be a good semigroup such that $t(S) = e_1 + e_2 - 1$. Is true that $S - M = M - \boldsymbol{e}$?

Theorem 4.6 can be used in order to describe an easy way to produce many examples of almost symmetric good semigroups, starting from symmetric ones. One may construct examples of symmetric good semigroups using for instance the theory developed in [10, Section 4].

Corollary 4.9. *Let T be a symmetric good semigroup having conductor $\mathbf{c}(T) = \boldsymbol{\gamma}(T)+(1,1)$. Then for every element $\boldsymbol{\omega} > (1,1)$ in T, the semigroup $S = (\boldsymbol{\omega}+T)\cup\{\mathbf{0}\}$ is almost symmetric non-symmetric.*

Proof. S is almost symmetric by Theorem 4.5 and Theorem 4.6. To show that it is non-symmetric, observe that $\boldsymbol{\omega}$ is the minimal nonzero element of S and that $\boldsymbol{\gamma}(T) + \boldsymbol{\omega} + (1,1)$ is the conductor of S. Since $\boldsymbol{\omega} > (1,1)$, we can find $\boldsymbol{\alpha} \in \mathbf{Ap}(T, \boldsymbol{\omega}) \setminus (\{\mathbf{0}\} \cup \Delta(\boldsymbol{\gamma}(T) + \boldsymbol{\omega}))$. It follows that $\boldsymbol{\alpha} \notin S$, $\boldsymbol{\alpha} - \boldsymbol{\omega} \notin T$ and hence $\Delta^T(\boldsymbol{\gamma}(T) + \boldsymbol{\omega} - \boldsymbol{\alpha}) \subseteq \mathbf{Ap}(T, \boldsymbol{\omega})$. Since $\mathbf{Ap}(T, \boldsymbol{\omega}) \cap S = \{\mathbf{0}\}$, this means $\Delta^S(\boldsymbol{\gamma}(T)+\boldsymbol{\omega}-\boldsymbol{\alpha}) = \emptyset$ and therefore S is not symmetric. □

Notice that the example of almost symmetric semigroup given in Figure 5 does not satisfy the equivalent conditions of Theorem 4.6. Hence not all the almost symmetric good semigroups are constructed by translating symmetric good semigroups as those described in this corollary.

5 Duality of almost symmetric good semigroups

For a numerical semigroup S with smallest non zero element (multiplicity) e, it is well known that, writing $\mathbf{Ap}(S) = \{w_1, \dots, w_e\}$ where $w_i < w_{i+1}$, then S is symmetric if and only if $w_i + w_{e-i+1} = w_e$.

In the case of a symmetric good semigroup a correspondent, but less intuitive, duality relation does exist for the levels of the partition $\mathbf{Ap}(S) = \bigcup_{i=1}^{e} A_i$, where $e = e_1 + e_2 = d(S/\boldsymbol{e} + S)$. For $\boldsymbol{\omega} \in \mathbb{N}^2$, define $\boldsymbol{\omega}' := \boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\omega}$. Denoting

$$A_i' := \left(\bigcup_{\boldsymbol{\omega} \in A_i} \Delta^S(\boldsymbol{\omega}') \right) \setminus \left(\bigcup_{\boldsymbol{\omega} \in A_j,\ j<i} \Delta^S(\boldsymbol{\omega}') \right),$$

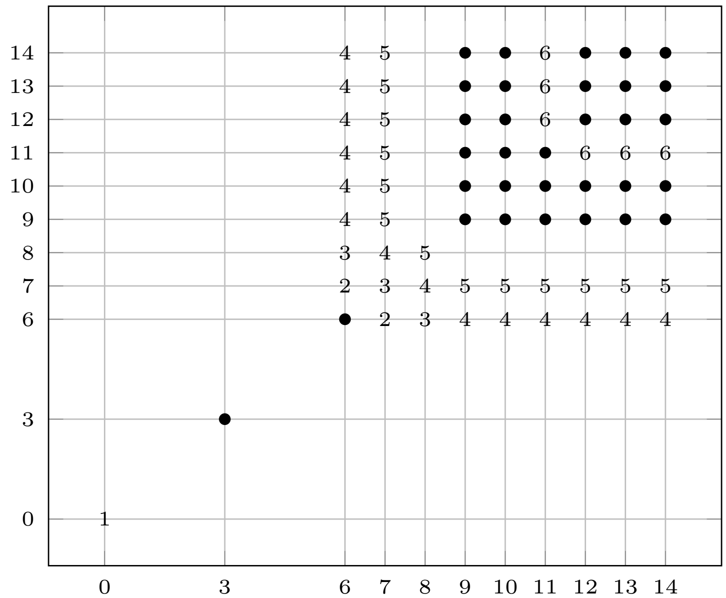

Figure 6: This is the almost symmetric good semigroup $S = (\boldsymbol{\omega} + T) \cap \{\mathbf{0}\}$ where T is the symmetric semigroup in Figure 2 and $\boldsymbol{\omega} = (3, 3)$. We indicate by the number i the elements of the level A_i of $\mathbf{Ap}(S)$.

in [10, Theorem 5.3] it is proved that S is symmetric if and only if $A'_i = A_{e-i+1}$ for every $i = 1, \ldots, e$, and hence, following this idea, the level A_{e-i+1} is dual of the level A_i.

We show here how a similar property holds for almost symmetric good semigroup, inspiring our investigation to the duality found by Nari in [21] for almost symmetric numerical semigroups. Indeed, if S is a numerical semigroup with minimal nonzero element e and Frobenius number $f(S)$, we can divide its Apéry Set in two distinct subsets:

$$\mathbf{Ap}(S) = \{0 = a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_{t(S)-1}\}$$

where $a_m - e = f(S)$ and the set of pseudo-frobenius numbers of S is

$$\mathbf{PF}(S) = \{b_1 - e, \ldots, b_{t(S)-1} - e\} \cup \{f(S)\}.$$

In [21, Theorem 2.4] it is proved that S is almost symmetric if and only if $a_i + a_{m-i+1} = a_m$ and $b_j + b_{t(S)-j} = a_m + e$ and also if and only if $f_i + f_{t(s)-i} = f(S)$, where the f_i are the pseudo-frobenius numbers of S listed in increasing order.

In order to prove a similar result for almost symmetric good semigroups, we generalize Theorem 5.3 of [10] to any subset of a good semigroup S which is the complementary set of a good ideal of S and fulfilling some properties of symmetry. Successively, we are going to consider two subsets, one of $\mathbf{Ap}(S)$ and the other of $S - M$ which, in the case S is almost symmetric satisfy a duality generalizing Nari's duality to good semigroups. Moreover, we show that the duality for one of this sets holds as a characterization of almost symmetric good semigroups.

We need now to state a preliminary lemma whose proof is the same obtained in [10] in the case $A = \mathbf{Ap}(S)$ and S symmetric. To make this proof work in general, we need two technical assumptions playing the role that symmetry plays in the case of $\mathbf{Ap}(S)$ (see [10, Proposition 4.3]).

Our setting is the following: we consider a good semigroup S and a subset $A \subseteq S$ for which $E := S \setminus A$ is a proper good ideal of S. We write the partition $A = \bigcup_{i=1}^{N} A_i$ as made in Definition 2.1. We denote the conductor of the good ideal E by $\boldsymbol{c}_E = \boldsymbol{\gamma}_E + (1,1)$ and, for $\boldsymbol{\alpha} \in \mathbb{N}^2$, we set $\boldsymbol{\alpha}' := \boldsymbol{\gamma}_E - \boldsymbol{\alpha}$. The following technical conditions are assumed on A:

(a) For every $\boldsymbol{\alpha} \in A$, $\Delta^S(\boldsymbol{\alpha}') \subseteq A$ and it is not empty.

(b) For every $\boldsymbol{\alpha} \in A$ and for $i = 1, 2$, $\Delta_i^S(\boldsymbol{\alpha}') = \emptyset$ if and only if $\Delta_i^S(\boldsymbol{\alpha}) \nsubseteq A$.

Lemma 5.1. *Let S be a good semigroup and $A = \bigcup_{i=1}^{N} A_i \subseteq S$ be such that $E := S \setminus A$ is a proper good ideal of S and assume conditions (a) and (b) hold for A. Let $\boldsymbol{\alpha} \in A_{e-i+1}$. Then for every $j < i$, $\Delta^S(\boldsymbol{\alpha}') \cap A_j = \emptyset$ and $\Delta^S(\boldsymbol{\alpha}') \cap A_i \neq \emptyset$.*

Proof. Here it is possible to use the same argument of Lemma 5.1 and Lemma 5.2 of [10] using some of the properties listed in Lemma 2.2, Lemma 2.3 and Lemma 2.4. The conditions (a) and (b) assumed here, replace the results taken from Proposition 4.3 of [10] and needed in the proof of the cited lemmas in the case $A = \mathbf{Ap}(S)$. □

Theorem 5.2. (Generalized duality)
Let S be a good semigroup and $A = \bigcup_{i=1}^{N} A_i \subseteq S$ be such that $E := S \setminus A$ is a proper good ideal of S and assume conditions (a) and (b) hold for A. Let

$$A_i' := \left(\bigcup_{\boldsymbol{\omega} \in A_i} \Delta^S(\boldsymbol{\omega}') \right) \setminus \left(\bigcup_{\boldsymbol{\omega} \in A_j,\ j<i} \Delta^S(\boldsymbol{\omega}') \right).$$

Hence, $A_i' = A_{N-i+1}$ for every $i = 1, \ldots, N$.

Proof. This result can be proved applying Lemma 5.1 and following the same argument used for the implication (1)⇒ (2) of Theorem 5.3 of [10]. □

Our aim is to apply Theorem 5.2 to almost symmetric good semigroups, defining opportune subsets which are complementary of good relative ideals. For a good semigroup S having Apéry Set $\mathbf{Ap}(S) = \bigcup_{i=1}^{e} A_i$, we define:

$$Z := \mathbf{PF}(S) \cup \{\mathbf{0}\},$$

and:

$$W := \{\mathbf{0}\} \cup \Delta(\boldsymbol{\gamma} + \boldsymbol{e}) \cup \{\boldsymbol{\alpha} \in \bigcup_{i=2}^{e-1} A_i \mid \boldsymbol{\alpha} - \boldsymbol{e} \notin \mathbf{PF}(S)\}.$$

Recall that when S is almost symmetric, $S - M$ is a good relative ideal of S. Also notice that when S is symmetric, $Z = \{\mathbf{0}\} \cup \Delta(\boldsymbol{\gamma})$ and $W = \mathbf{Ap}(S)$.

Proposition 5.3. *Let S be an almost symmetric good semigroup. The following assertions hold:*

1. *The set $Z \subseteq S - M$ is the complementary of a good ideal of $S - M$.*

2. *The set $W \subseteq S$ is the complementary of a good ideal of S.*

Proof. To prove (1) we observe that $Z = (S - M) \setminus M$ and M is clearly a good ideal of $S - M$.
To prove (2), we observe that $S \setminus W = (\boldsymbol{e} + (S - M)) \setminus \Delta(\boldsymbol{\gamma} + \boldsymbol{e})$ and that $\boldsymbol{e} + (S - M)$ is a proper good ideal of S. Removing $\Delta(\boldsymbol{\gamma} + \boldsymbol{e})$ from a good ideal still keeps properties (G1),(G2) and (G3) for the set $S \setminus W$, but we need to show that this set is still an ideal of S. This is equivalent to show that for every $\boldsymbol{\alpha} \in S \setminus W$ and $\boldsymbol{\beta} \in S$, $\boldsymbol{\alpha} + \boldsymbol{\beta} \notin \Delta(\boldsymbol{\gamma} + \boldsymbol{e})$. But this is true, since $\boldsymbol{\alpha} \in \boldsymbol{e} + (S - M)$ and, by Proposition 3.2, $\boldsymbol{\alpha} + \boldsymbol{\beta} \notin \mathbf{Ap}(S)$ for every $\boldsymbol{\beta} \in M$. □

Now, we show in Proposition 5.5 that when S is almost symmetric, both Z and W fulfill conditions (a) and (b) stated at the beginning of this section.

Lemma 5.4. *Let S be a good semigroup and let $\boldsymbol{\alpha} \in \mathbb{N}^2$. Assume that $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\beta}) \neq \emptyset$ for every $\boldsymbol{\beta} \in \Delta(\boldsymbol{\gamma} - \boldsymbol{\alpha})$. Then $\boldsymbol{\alpha} \in S$.*

Proof. Assume $\boldsymbol{\alpha} \notin S$. Considering $\boldsymbol{\beta} \in \Delta(\boldsymbol{\gamma} - \boldsymbol{\alpha})$ large enough such that $\Delta_1^S(\boldsymbol{\gamma} - \boldsymbol{\beta}) = \emptyset$, we get that $\Delta_2^S(\boldsymbol{\gamma} - \boldsymbol{\beta}) \neq \emptyset$ and hence either there exists $\boldsymbol{\theta} \in \Delta_2^S(\boldsymbol{\alpha})$ or $\boldsymbol{\alpha} \in \Delta_2^S(\boldsymbol{\delta})$ for some $\boldsymbol{\delta} \in S$. In the second case we may assume $\boldsymbol{\delta}$ to be the closest element of S on the left of $\boldsymbol{\alpha}$ and, since $\boldsymbol{\gamma} - \boldsymbol{\delta} \in \Delta(\boldsymbol{\gamma} - \boldsymbol{\alpha})$, by the hypothesis we get $\Delta^S(\boldsymbol{\delta}) \neq \emptyset$. Hence also in this case, by property (G2), there must exist $\boldsymbol{\theta} \in \Delta_2^S(\boldsymbol{\alpha})$. Doing the same process, we can find $\boldsymbol{\omega} \in \Delta_1^S(\boldsymbol{\alpha})$. Hence $\boldsymbol{\alpha} = \boldsymbol{\theta} \wedge \boldsymbol{\omega} \in S$ by property (G1). □

Proposition 5.5. *Let S be an almost symmetric good semigroup. The following assertions hold:*

1. *For every $\boldsymbol{\alpha} \in W$, $\Delta^S(\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\alpha}) \subseteq W$ and it is not empty.*
2. *For every $\boldsymbol{\theta} \in Z$, $\Delta^{S-M}(\boldsymbol{\gamma} - \boldsymbol{\theta}) \subseteq Z$ and it is not empty.*
3. *For every $\boldsymbol{\alpha} \in W$ and for $i = 1, 2$, $\Delta_i^S(\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\alpha}) = \emptyset$ if and only if $\Delta_i^S(\boldsymbol{\alpha}) \nsubseteq W$.*
4. *For every $\boldsymbol{\theta} \in Z$ and for $i = 1, 2$, $\Delta_i^{S-M}(\boldsymbol{\gamma} - \boldsymbol{\theta}) = \emptyset$ if and only if $\Delta_i^{S-M}(\boldsymbol{\theta}) \nsubseteq Z$.*

Proof. (1) Let $\boldsymbol{\alpha} \in W$. For $\boldsymbol{\alpha} \in \{\mathbf{0}\} \cup \Delta(\boldsymbol{\gamma} + \boldsymbol{e})$ the thesis is clear, hence we may assume $\boldsymbol{\alpha} - \boldsymbol{e} \in \mathbb{N}^2 \setminus (\mathbf{PF}(S) \cup S)$. By Lemma 4.1, $\Delta^S(\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\alpha}) \neq \emptyset$ and hence we take $\boldsymbol{\beta} \in \Delta^S(\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\alpha})$. Since $\boldsymbol{\alpha} \in S$, $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset$ and hence $\Delta^S(\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\alpha}) \subseteq \mathbf{Ap}(S)$ implying $\boldsymbol{\beta} \in \mathbf{Ap}(S)$. Since $\boldsymbol{\alpha} \in \Delta^S(\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\beta})$, it follows $\Delta^S(\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\beta}) \neq \emptyset$ and therefore, again by Lemma 4.1, $\boldsymbol{\beta} - \boldsymbol{e} \notin \mathbf{PF}(S)$ implying $\boldsymbol{\beta} \in W$.
(2) Let $\boldsymbol{\theta} \in Z$, and as before assume $\boldsymbol{\theta} \notin \{\mathbf{0}\} \cup \Delta(\boldsymbol{\gamma})$ otherwise the thesis is clear. By Lemma 4.1, $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\theta}) = \emptyset$, hence $\Delta^{S-M}(\boldsymbol{\gamma} - \boldsymbol{\theta}) \subseteq Z$. Now, we still need to prove that $\Delta^{S-M}(\boldsymbol{\gamma} - \boldsymbol{\theta})$ is not empty. We use the following argument: assume by way of contradiction that every $\boldsymbol{\beta} \in \Delta(\boldsymbol{\gamma} - \boldsymbol{\theta})$ is not in Z. Thus by Lemma 4.1, $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\beta}) \neq \emptyset$ for every $\boldsymbol{\beta} \in \Delta(\boldsymbol{\gamma} - \boldsymbol{\theta})$ and this implies, by Lemma 5.4, $\boldsymbol{\theta} \in S$ that is a contradiction.
(3) Let $i = 1$ (for $i = 2$ the proof is identical) and take $\boldsymbol{\alpha} \in W$. Assume $\Delta_1^S(\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\alpha}) = \emptyset$. Hence by (1), $\Delta_2^S(\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\alpha}) \neq \emptyset$ and therefore $\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\alpha} \notin S$. By Lemma 4.1, there are now two possibilities: either $\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\alpha} \in \mathbf{PF}(S)$ or $\Delta^S(\boldsymbol{\gamma} - (\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\alpha})) = \Delta^S(\boldsymbol{\alpha} - \boldsymbol{e}) \neq \emptyset$.

In the first case by (2), $\Delta^{S-M}(\boldsymbol{\alpha} - \boldsymbol{e}) \subseteq \mathbf{PF}(S)$ and it is not empty, implying $\Delta^S(\boldsymbol{\alpha}) \nsubseteq W$. In the second case, again we get $\Delta^S(\boldsymbol{\alpha}) \nsubseteq W$, since $\Delta^S(\boldsymbol{\alpha}) \nsubseteq \mathbf{Ap}(S)$. Now, let

$\boldsymbol{\beta} \in \Delta_2^S(\boldsymbol{\gamma}+\boldsymbol{e}-\boldsymbol{\alpha})$ and observe that by (1), $\Delta_2^S(\boldsymbol{\alpha}) \subseteq \Delta_2^S(\boldsymbol{\gamma}+\boldsymbol{e}-\boldsymbol{\beta}) \subseteq W$. Thus we must have $\Delta_1^S(\boldsymbol{\alpha}) \nsubseteq W$.

Conversely, if $\Delta_1^S(\boldsymbol{\alpha}) \nsubseteq W$, take $\boldsymbol{\beta} \in \Delta_1^S(\boldsymbol{\alpha}) \setminus W$. Hence $\boldsymbol{\beta}-\boldsymbol{e} \in S-M$, implying $\Delta^S(\boldsymbol{\gamma}+\boldsymbol{e}-\boldsymbol{\beta}) = \emptyset$. We conclude observing that $\Delta_1^S(\boldsymbol{\gamma}+\boldsymbol{e}-\boldsymbol{\alpha}) \subseteq \Delta_1^S(\boldsymbol{\gamma}+\boldsymbol{e}-\boldsymbol{\beta})$.
(4) Again let $i=1$ (for $i=2$ the proof is identical) and take $\boldsymbol{\theta} \in Z$. As in the proof of (3), assuming $\Delta_1^{S-M}(\boldsymbol{\gamma}-\boldsymbol{\theta}) = \emptyset$ and using (2), we imply that $\Delta_2^{S-M}(\boldsymbol{\gamma}-\boldsymbol{\theta}) \subseteq Z$ and it is non empty, and therefore $\boldsymbol{\gamma}-\boldsymbol{\theta} \notin S-M$. We can find $\boldsymbol{\omega} \in M$ such that $\boldsymbol{\gamma}-\boldsymbol{\theta}+\boldsymbol{\omega} \in \mathbf{PF}(S)$, and hence, again by (2), $\Delta^{S-M}(\boldsymbol{\theta}-\boldsymbol{\omega}) \subseteq Z$ and it is non empty. It follows that $\Delta^S(\boldsymbol{\theta}) \neq \emptyset$ and necessarily that $\Delta_1^{S-M}(\boldsymbol{\theta}) \nsubseteq Z$, since, for some $\boldsymbol{\delta} \in \Delta_2^{S-M}(\boldsymbol{\gamma}-\boldsymbol{\theta}) \subseteq Z$, we have $\Delta_2^{S-M}(\boldsymbol{\theta}) \subseteq \Delta_2^{S-M}(\boldsymbol{\gamma}-\boldsymbol{\delta}) \subseteq Z$.

Conversely, if $\Delta_1^{S-M}(\boldsymbol{\theta}) \nsubseteq Z$, take $\boldsymbol{\delta} \in \Delta_1^S(\boldsymbol{\theta})$. Now, since $\boldsymbol{\delta} \in S$, we get $\Delta^S(\boldsymbol{\gamma}-\boldsymbol{\delta}) = \emptyset$, and, if by way of contradiction there exists $\boldsymbol{\omega} \in \mathbf{PF}(S) \cap \Delta(\boldsymbol{\gamma}-\boldsymbol{\delta})$, we would get $\boldsymbol{\delta} \in \Delta^S(\boldsymbol{\gamma}-\boldsymbol{\omega}) = \emptyset$. Hence $\Delta_1^{S-M}(\boldsymbol{\gamma}-\boldsymbol{\theta}) \subseteq \Delta_1^{S-M}(\boldsymbol{\gamma}-\boldsymbol{\delta}) = \emptyset$. □

We can state and prove the main theorem of this section, extending Nari's duality to almost symmetric good semigroups.

Theorem 5.6. *Let $S \subseteq \mathbb{N}^2$ be a good semigroup. Define the sets Z and W as above in this section and let $Z = \bigcup_{h=1}^{n} Z_h$ and $W = \bigcup_{i=1}^{m} W_i$ be their partitions obtained as in Definition 2.1. Set*

$$Z'_h = \left(\bigcup_{\boldsymbol{\delta} \in Z_h} \Delta^{S-M}(\boldsymbol{\gamma}-\boldsymbol{\delta}) \right) \setminus \left(\bigcup_{\boldsymbol{\delta} \in Z_j,\ j<h} \Delta^{S-M}(\boldsymbol{\gamma}-\boldsymbol{\delta}) \right)$$

and

$$W'_i = \left(\bigcup_{\boldsymbol{\omega} \in W_i} \Delta^{S}(\boldsymbol{\gamma}+\boldsymbol{e}-\boldsymbol{\omega}) \right) \setminus \left(\bigcup_{\boldsymbol{\omega} \in W_j,\ j<i} \Delta^{S}(\boldsymbol{\gamma}+\boldsymbol{e}-\boldsymbol{\omega}) \right).$$

The following assertions are equivalent:

1. *S is almost symmetric.*
2. *$Z'_h = Z_{n-h+1}$ for every $h = 1, \ldots, n$ and $W'_i = W_{m-i+1}$ for every $i = 1, \ldots, m$.*

Proof. (1) $\Rightarrow$ (2): Follows combining Theorem 5.2 with Propositions 5.3 and 5.5.
(2) $\Rightarrow$ (1): We argue by way of contradiction. Assuming S not almost symmetric, we can find $\boldsymbol{\alpha} \notin S-M$ such that $\Delta^S(\boldsymbol{\gamma}-\boldsymbol{\alpha}) = \emptyset$. Thus, there exists $\boldsymbol{\beta} \in M$, such that $\boldsymbol{\theta} := \boldsymbol{\alpha}+\boldsymbol{\beta} \in \mathbf{PF}(S)$. It follows that $\Delta^{S-M}(\boldsymbol{\gamma}-\boldsymbol{\theta}) = \emptyset$, since, if there exists some $\boldsymbol{\delta} \in \Delta^{S-M}(\boldsymbol{\gamma}-\boldsymbol{\theta})$, then $\boldsymbol{\delta}+\boldsymbol{\beta} \in \Delta^S(\boldsymbol{\gamma}-\boldsymbol{\alpha})$, that is empty. This shows $\boldsymbol{\theta} \notin Z'_h$ for every $h = 1, \ldots, n$, negating the assumption of (2). Indeed, if $\boldsymbol{\theta} \in \Delta^{S-M}(\boldsymbol{\gamma}-\boldsymbol{\omega})$ for some $\boldsymbol{\omega} \in Z_h$, then $\boldsymbol{\omega} \in \Delta^{S-M}(\boldsymbol{\gamma}-\boldsymbol{\theta}) = \emptyset$ and this is a contradiction. □

Remark 5.7. Notice that the duality on the set W does not imply that S is almost symmetric. Indeed, for instance, when $S-M = M-\boldsymbol{e}$, we get by Theorem 4.6,

$$Z = \{\mathbf{0}\} \cup \{\boldsymbol{\alpha}-\boldsymbol{e} \,|\, \boldsymbol{\alpha} \in \mathbf{Ap}(S)^\star\}$$

and $W = \{\mathbf{0}\} \cup \Delta(\boldsymbol{\gamma}+\boldsymbol{e})$. Hence the duality holds for the levels of W but S may fail to be almost symmetric (as seen in Theorem 4.5).

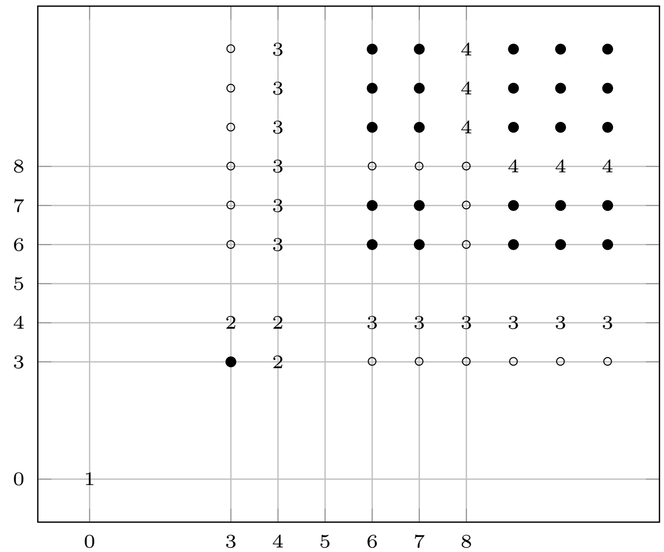

Figure 7: In the almost symmetric good semigroup of Figure 5 we indicate by white marks the elements of $\mathbf{Ap}(S) \setminus W$ and by the numbers the different levels of W. One may check using this figure together with Figure 5 that the duality for Z and W holds for this semigroup.

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