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Perfect blocking sets in P_3-designs

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Abstract

A well-known problem in Design Theory is the study of the possible existence of blocking sets in Steiner systems. In this paper, we introduce the concept of *perfect* blocking sets in G-designs and determine all the possible v for which there exist P_3-designs having perfect blocking sets.

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Math. Subj. Class.: Blocking sets, transversals, P_3-designs.

1 Introduction

Let K_v be the complete undirected graph defined in a vertex set X. Given a graph with n vertices, a *G-design* of *order* v (briefly a $G(v)$*-design*), for $v \geq n$, is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge set of K_v into classes generating graphs all isomorphic to G. The classes of $\mathcal{B}$ are said to be the *blocks* of Σ. A K_n-design of order v is a Steiner systems $S(2, n, v)$.

Let $\Sigma = (X, \mathcal{B})$ be a G-design of order v. Following *hypergraph* terminology, a *transversal* T of Σ is a subset of X intersecting every block of Σ. The *transversal number* of Σ is the minimum cardinality of transversals of Σ. A *blocking set* T of Σ is a transversal such that also its complementary C_T is a transversal of Σ: in other words, T is a blocking sets if and only if every block of Σ contains elements of T and elements of $C_X(T)$. For a blocking set T of Σ, the *discrepancy* is the number $\delta(\Sigma) = \big||T| - |C_X T|\big|$ (see [4, 8]). In what follows, we will indicate by $B(\Sigma)$ the set of all possible $p \in N$ for which there exist in Σ blocking sets of cardinality p. Therefore: $\delta(\Sigma) = |B(\Sigma)|$. Note that *there exist blocking sets* in a system if and only if the system is 2*-vertex-colourable*.

The problem to determine the existence of possible *blocking-sets* in Steiner systems has been studied by many authors [1, 2, 7] , expecially for $S(2, 4, v)$ [3, 9, 10, 11, 12], and for G-designs [6]. Interesting results can be found also in [5].

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In this paper, we introduce the concept of *perfect blocking-set* of a G-design and determine all possible v for which there exist P_3-designs having perfect blocking sets.

In what follows, b will always indicate the number of blocks in a G-design. Observe that, in the case of systems with $b = 1$ (Steiner systems $S(h, k, v)$ with $v = k$), the research of blocking sets is trivial. Therefore, in what follows, we will consider always systems with $b > 1$. It is known that:

Theorem 1.1. *A P_3-design of order v exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \geq 4$.*

Observe that if a path P_3 has vertices x, y, z and edges $\{x, y\}, \{y, z\}$, we will denote it by $[x, y, z]$.

2 Transversals and blocking sets in G-designs

The following results were proved in [4, 8]:

Theorem 2.1. *If $\Sigma = (X, \mathcal{B})$ is a G-design of order v and T is a blocking set of cardinality p of Σ such that $p \leq \frac{v-1}{r}$, then:*

$$\binom{p}{2} + p \cdot \left[\frac{v-1}{r} - (p-1)\right] \geq |\mathcal{B}|.$$

Theorem 2.2. *If $\Sigma = (X, \mathcal{B})$ is a P_3-design of order v and T is a transversal of cardinality p of Σ, then:*

$$\binom{p}{2} + p \cdot (v - p) \geq v(v-1)/4.$$

where the inequality is the best possible.

Proof. To see that the inequality is the best possible, consider the system $\Sigma = (X, \mathcal{B})$, defined in $X = \{1, 2, \ldots, 8\}$, and having for blocks:

$$\begin{aligned}\mathcal{B}\colon\quad & [1,2,3],\ [1,3,4],\ [1,4,2],\ [5,6,7],\ [5,7,8],\ [5,8,6],\\ & [1,5,3],\ [2,5,4],\ [1,6,3],\ [2,6,4],\\ & [1,7,3],\ [2,7,4],\ [1,8,3],\ [2,8,4].\end{aligned}$$

We can see that Σ is a P_3-design of order $v = 8$ and that $T = \{1, 2, 5\}$ is a blocking set of Σ. Further, we can verify that, from Theorem 2.2, the minimum possible value of p for $v = 8$ is $p = 3$. □

Observe that the minimum cardinality of a blocking set depends on the system and not only on its order. The following two systems $\Sigma_1 = (X, \mathcal{B}_1)$ and $\Sigma_2 = (X, \mathcal{B}_2)$, are defined both in $X = \{1, 2, \ldots, 9\}$. Therefore their order is $v = 9$. However, the minimum cardinality of a blocking set in them is different:

$$\begin{aligned}\mathcal{B}_1\colon\quad & [1,2,4],\ [1,3,4],\ [2,3,5],\\ & [1,4,7],\ [1,5,4],\ [1,6,4],\ [1,7,8],\ [1,8,4],\ [1,9,4],\\ & [2,5,8],\ [2,6,5],\ [2,7,5],\ [2,8,9],\ [2,9,5],\\ & [3,6,9],\ [3,7,6],\ [3,8,6],\ [3,9,7].\end{aligned}$$

$$\begin{array}{ll}\mathcal{B}_2: & [1,2,3],\ [1,3,4],\ [1,4,2],\ [5,6,7],\ [5,7,8],\ [5,8,6],\\ & [1,5,3],\ [2,5,4],\ [1,6,3],\ [2,6,4],\\ & [1,7,3],\ [2,7,4],\ [1,8,3],\ [2,8,4],\\ & [1,9,7],\ [2,9,6],\ [3,5,9],\ [4,9,8].\end{array}$$

We can see that:

- the minimum possible cardinality in Σ_1 is exactly $p = 3$ and that $T_1 = \{1, 2, 3\}$ is a blocking set of Σ_2;
- the minimum possible cardinality in Σ_2 is $p = 4$ and that $T_2 = \{1, 2, 5, 9\}$ is a blocking set of Σ_1.

Definition 2.3. Let $\Sigma = (X, \mathcal{B})$ be a G-design. We say that a blocking set T of Σ is *perfect* if there exists a constant $C \in N$ such that in every block $B \in \mathcal{B}$ there are exactly C edges having an extreme in T and the other extreme in $C_X T$.

Observe that the blocking set T_1 of the P_3-design Σ_1, defined above, is *perfect*; while the blocking set T_2 of Σ_2 is not *perfect*.

3 Perfect blocking sets in P_3-designs

We see the exact cardinality of any perfect blocking set in in P_3-designs.

Theorem 3.1. *If T is a perfect blocking set of any P_3-design of order v, then*

$$|T| = \frac{v \pm \sqrt{v}}{2},$$

and v must be a square.

Proof. Let $\Sigma = (X, \mathcal{B})$ be a P_3-design of order v and let T be a perfect blocking set of Σ. Observe that, from the definition of *perfect blocking set*, every block of Σ contains exactly one edge having an extreme in T and the other extreme in $C_X T$. Therefore, since $|T| = p$ and $|C_X T| = v - p$, it follows:

$$p(v - p) = \frac{v(v-1)}{4},$$

hence:

$$p = \frac{v \pm \sqrt{v}}{2}.$$

Since p is a positive integer, it follows that v must be a square. □

From Theorem 3.2, if we consider a P_3-design of order v having perfect blocking set, since $v \equiv 0$ or $1 \pmod 4$, then there is a positive integer k such that $v = (2k)^2$ or $v = (2k+1)^2$.

Theorem 3.2. *Let T be a perfect blocking set of any P_3-design of order v. If $|T| \leq |C_X T|$, then:*

(i) if $v \equiv 0 \pmod 4$, then $v = (2k)^2$ and $|T| = k(2k-1)$, for any $k \in N$;

(ii) if $v \equiv 1 \pmod 4$, then $v = (2k+1)^2$ and $|T| = k(2k+1)$, for any $k \in N$.

Proof. Let $\Sigma = (X, \mathcal{B})$ be a P_3-design of order v and let T be a perfect blocking set of Σ, such that $|T| \leq |C_X T|$.

(i): If $v \equiv 0 \pmod 4$, then there exists $k \in N$ such that $v = (2k)^2$. Further:

$$|T| = \frac{v - \sqrt{v}}{2} = \frac{4k^2 - 2k}{2} = k(2k-1).$$

(ii): If $v \equiv 1 \pmod 4$, then there exists $k \in N$ such that $v = (2k+1)^2$. Further:

$$|T| = \frac{v - \sqrt{v}}{2} = \frac{(4k^2 + 4k + 1) - (2k+1)}{2} = k(2k+1). \qquad \square$$

4 Main results

In this section we determine the spectrum of P_3-designs having perfect blocking sets.

Theorem 4.1. *There exist P_3-designs of order v having perfect blocking sets if and only if v is a square.*

Proof. Let $\Sigma = (X, \mathcal{B})$ be a P_3-design of order v and let T be a perfect blocking set of Σ. From Theorems 3.1 and 3.2, it follows that v must be a square.

Therefore, let v be an *odd* [resp. *even*] square number. This implies that $v = (2k+1)^2$ and $p = |T| = k(2k+1)$ [resp. $v = (2k)^2$ and $p = |T| = k(2k-1)$].

If X is a set of cardinality v, partition X into 3 classes X_1, X_2, X_3, defined as follows:

$$\begin{aligned}
X_1 &= \{a_1, a_2, \dots, a_{k(2k+1)}\} \qquad &&\text{[resp.: } X_1 = \{a_1, a_2, \dots, a_{k(2k-1)}\}\text{];}\\
X_2 &= \{b_1, b_2, \dots, b_{k(2k+1)}\} \qquad &&\text{[resp.: } X_2 = \{b_1, b_2, \dots, b_{k(2k-1)}\}\text{];}\\
X_3 &= \{c_1, c_2, \dots, c_{2k+1}\} \qquad &&\text{[resp.: } X_3 = \{c_1, c_2, \dots, c_{2k}\}\text{].}
\end{aligned}$$

To simplify, indicate by q the cardinality of X_3, i.e. $q = 2k+1$ [resp. $q = 2k$], and of course $p = k(2k+1)$ [resp. $p = k(2k-1)$].

Define in X the following families of paths P_3:

$$\begin{aligned}
\mathcal{F}: \quad & [a_1, a_2, b_1],\ [a_1, a_3, b_1],\ \dots,\ [a_1, a_p, b_1],\\
& [a_2, a_3, b_2],\ [a_2, a_4, b_2],\ \dots,\ [a_2, a_p, b_2],\\
& \quad \vdots\\
& [a_{p-2}, a_{p-1}, b_{p-2}],\ [a_{p-2}, a_p, b_{p-2}],\\
& [a_{p-1}, a_p, b_{p-1}];
\end{aligned}$$

$\mathcal{G}_{1,2}$: $[a_1, b_1, c_1],\ [a_2, b_2, c_1],\ \ldots,\ [a_{q-1}, b_{q-1}, c_1],$

$[a_q, b_q, c_2],\ [a_{q+1}, b_{q+1}, c_2],\ \ldots,\ [a_{2q-3}, b_{2q-3}, c_2],$

(the last index $2q-3$ is because of $2q-3 = (q-1)+(q-2)$),

$[a_{2q-2}, b_{2q-2}, c_3],\ [a_{2q-1}, b_{2q-1}, c_3],\ \ldots,\ [a_{3q-6}, b_{3q-6}, c_3],$

(the last index $3q-6$ is because of $2q-3 = (q-1)+(q-2)+(q-3)$),

$\vdots$

$[a_{p-2}, b_{p-2}, c_{q-2}],\ [a_{p-1}, b_{p-1}, c_{q-2}],$

$[a_p, b_p, c_{q-1}];$

$\mathcal{G}_{2,2}$: $[a_1, c_1, c_2],\ [a_2, c_1, c_3],\ [a_3, c_1, c_4],\ \ldots,\ [a_{q-1}, c_1, c_q],$

$[a_q, c_2, c_3],\ [a_{q+1}, c_2, c_4],\ [a_{q+2}, c_2, c_5],\ \ldots,\ [a_{2q-3}, c_2, c_{2q-3}],$

$\vdots$

$[a_{p-2}, c_{q-2}, c_{q-1}],\ [a_{p-1}, c_{q-2}, c_q],$

$[a_p, c_{q-1}, c_q];$

$\mathcal{H}_1$: $[a_1, b_2, b_1],\ [a_1, b_3, b_1],\ \ldots,\ [a_1, b_p, b_1],$

$[a_2, b_3, b_2],\ [a_2, b_4, b_2],\ \ldots,\ [a_2, b_p, b_2],$

$[a_3, b_4, b_3],\ [a_3, b_5, b_3],\ \ldots,\ [a_3, b_p, b_3],$

$\vdots$

$[a_{q-1}, b_q, b_{q-1}],\ [a_{q-1}, b_{q+1}, b_{q-1}],\ \ldots,\ [a_{q-1}, b_p, b_{q-1}],$

$[a_q, b_{q+1}, b_q],\ [a_q, b_{q+2}, b_q],\ \ldots,\ [a_q, b_p, b_q],$

$[a_{q+1}, b_{q+2}, b_{q+1}],\ [a_{q+1}, b_{q+3}, b_{q+1}],\ \ldots,\ [a_{q+1}, b_p, b_{q+1}],$

$\vdots$

$[a_{2q-3}, b_{2q-2}, b_{2q-3}],\ [a_{2q-3}, b_{2q-1}, b_{2q-3}],\ \ldots,\ [a_{2q-3}, b_p, b_{2q-3}],$

$\vdots$

$[a_{p-2}, b_{p-1}, b_{p-2}],\ [a_{p-2}, b_p, b_{p-2}],$

$[a_{p-1}, b_p, b_{p-1}];$

$\mathcal{H}_2$: $[a_1, c_2, b_1],\ [a_1, c_3, b_1],\ \ldots,\ [a_1, c_q, b_1],$

$[a_2, c_2, b_2],\ [a_2, c_3, b_2],\ \ldots,\ [a_2, c_q, b_2],$

$[a_3, c_2, b_3],\ [a_3, c_3, b_3],\ \ldots,\ [a_3, c_q, b_3],$

$\vdots$

$[a_{q-1}, c_2, b_{q-1}],\ [a_{q-1}, c_3, b_{q-1}],\ \ldots,\ [a_{q-1}, c_q, b_{q-1}],$

$[a_q, c_1, b_q],\ [a_q, c_3, b_q],\ \ldots,\ [a_q, c_q, b_q],$

$$[a_{q+1}, c_1, b_{q+1}],\ [a_{q+1}, c_3, b_{q+1}],\ \ldots,\ [a_{q+1}, c_q, b_{q+1}],$$
$$\vdots$$
$$[a_{2q-3}, c_1, b_{2q-3}],\ [a_{2q-3}, c_3, b_{2q-3}],\ \ldots,\ [a_{2q-3}, c_q, b_{2q-3}],$$
$$\vdots$$
$$[a_{p-2}, c_1, b_{p-2}],\ [a_{p-2}, c_2, b_{p-2}],\ [a_{p-2}, c_3, b_{p-2}],\ \ldots,\ [a_{p-2}, c_{q-3}, b_{p-2}],$$
$$[a_{p-2}, c_{q-1}, b_{p-2}],\ [a_{p-2}, c_q, b_{p-2}],$$
$$[a_{p-1}, c_1, b_{p-1}],\ [a_{p-1}, c_2, b_{p-1}],\ [a_{p-1}, c_3, b_{p-1}],\ \ldots,\ [a_{p-1}, c_{q-3}, b_{p-1}],$$
$$[a_{p-1}, c_{q-1}, b_{p-1}],\ [a_{p-1}, c_q, b_{p-1}],$$
$$[a_p, c_1, b_p],\ [a_p, c_2, b_p],\ [a_p, c_3, b_p],\ \ldots,\ [a_p, c_{q-3}, b_p],$$
$$[a_p, c_{q-2}, b_p],\ [a_p, c_q, b_p].$$

If $X = X_1 \cup X_2 \cup X_3$ and $\mathcal{B} = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{H}_1 \cup \mathcal{H}_2$, then it is possible to verify that $\Sigma = (X, \mathcal{B})$ is a P_3-design of order $v = (2k+1)^2$ [resp. $v = (2k)^2$], for any $k \in N$, and that X_1 having is a *perfect* blocking set of Σ of cardinality $k(2k+1)$ [resp. $v = k(2k-1)$].

Indeed, observe that:

1. the family $\mathcal{F}$ has cardinality $|\mathcal{F}| = \binom{p}{2}$ and its blocks contain exactly an edge having both extremes in X_1; no block of $\mathcal{B} - \mathcal{F}$ contains two elements of X_1; further they contain all the edges $\{a_j, b_i\}$, for every $i = 1, 2, \ldots, p-1$ and $j = i+1, i+2, \ldots, p$;
2. the family $\mathcal{G}_1$ contains all the blocks of type $[a_i, b_i, c_j]$, where:
   - $j = 1$, for $i = 1, 2, \ldots, q-1$;
   - $j = 2$, for $i = q, q+1, \ldots, 2q-3$;

     $\vdots$
   - $j = q-1$, for $i = p = \binom{q}{2} = k(2k+1)$ [resp. $= k(2k-1)$];
3. the family $\mathcal{G}_2$ contains blocks of type $\{a, c', c''\}$, where $\{a, c'\} \in X_1 \times X_3$ and $\{c', c''\} \in X_3 \times X_3$;
4. the family $\mathcal{H}_1$ contains all the blocks of type $[a_i, b_j, b_i]$, for every $i = 1, 2, \ldots, p-1$ and $j = i+1, \ldots, p$;
5. the family $\mathcal{H}_2$ contains all the blocks of type $[a_i, c_j, b_i]$, for every $i = 1, 2, \ldots, p$ and $j = 1, \ldots, q$, with exception for:
   - $j = 1$, for $i = 1, 2, \ldots, q-1$;
   - $j = 2$, for $i = q, q+1, \ldots, 2q-3$;

     $\vdots$
   - $j = q-1$, for $i = p = q$. □

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