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Regularity for minimizers for functionals of double phase with variable exponents

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Abstract: The functionals of double phase type

$$\mathcal{H}(u) := \int \left(|Du|^p + a(x)|Du|^q\right) dx, \quad (q > p > 1, \ a(x) \geq 0)$$

are introduced in the epoch-making paper by Colombo-Mingione [1] for constants p and q, and investigated by them and Baroni. They obtained sharp regularity results for minimizers of such functionals. In this paper we treat the case that the exponents are functions of x and partly generalize their regularity results.

1 Introduction and main theorem

The main goal of this paper is to provide a regularity theorem for minimizers of a class of integral functionals of the calculus of variations called *of double phase type* with variable exponents defined for $u \in W^{1,1}(\Omega;\mathbb{R}^N)$ $(\Omega \in \mathbb{R}^n, \ n, N \geq 2)$ as

$$\mathcal{F}(u,\Omega) := \int_\Omega \left(|Du|^{p(x)} + a(x)|Du|^{q(x)}\right) dx, \quad q(x) \geq p(x) > 1, \ a(x) \geq 0,$$

where $p(x)$, $q(x)$ and $a(x)$ are assumed to be Hölder continuous. They do not only have strongly non-uniform ellipticity but also discontinuity of growth order at points where $a(x) = 0$. The above functional is provided by the following type of functionals with variable exponent growth

$$u \mapsto \int g(x,Du)dx, \quad \lambda|z|^{p(x)} \leq g(x,z) \leq \Lambda(1+|z|)^{p(x)}, \quad \Lambda \geq \lambda > 0,$$

which are called *of $p(x)$-growth*. These $p(x)$-growth functionals have been introduced by Zhikov [2] (in this article $\alpha(x)$ is used as variable exponents) in the setting of Homogenization theory. He showed higher integrability for minimizers and, on the other hand, he gave an example of discontinuous exponent $p(x)$ for which the Lavrentiev phenomenon occurs ([3, 4]).

Such functionals provide a useful prototype for describing the behaviour of strongly inhomogeneous materials whose strengthening properties, connected to the exponent dominating the growth of the gradient variable, significantly change with the point. In [3], Zhikov pointed out the relationship between $p(x)$-growth functionals and some physical problems including thermistor. As another application, the theory of electrorheological materials and fluids is known. About these objects see, for example, [5–8].

These kind of functionals have been the object of intensive investigation over the last years, starting with the inspiring papers by Marcellini [9–11], where he introduced so-called (p,q)- or *nonstandard* growth

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functionals:

$$u \mapsto \int f(x, u, Du)dx, \quad \lambda|z|^p \le f(x, u, z) \le \Lambda(1 + |z|)^q, \quad q \ge p \ge 1, \quad \Lambda \ge \lambda > 0.$$

About general (p, q)-growth functionals, see for example [3, 4, 12–19] and the survey [20].

For the continuous variable exponent case, nowadays many results on the regularity for minimizer are known, see [21–24]. Further results in this direction can be, for instance, found in [25–41] for partial regularity results for $p(x)$-energy type functionals:

$$u \mapsto \int \left(A_{ij}^{\alpha\beta}(x, u) D_\alpha u^i(x) D_\beta u^j(x) \right)^{p(x)} dx, \quad A_{ij}^{\alpha\beta}(x, u) z_\alpha^i z_\beta^j \ge \lambda |z|^2$$

In 2015 a new class of functional so-called *functionals of double phase* are introduced by Colombo-Mingione [1]. In the primary model they have in mind are

$$u \mapsto \mathcal{H}(u; \Omega) := \int H(x, Du)dx, \quad H(x, z) := |z|^p + a(x)|z|^q,$$

where p and q are constants with $q \ge p > 1$ and $a(\cdot)$ is a Hölder continuous non-negative function. By Colombo-Mingione [1, 42, 43] and Baroni-Colombo-Mingione [44–46] many sharp results are given about the regularity of local minimizers of the functional defined as

$$u \mapsto \mathcal{G}(u; \Omega) := \int_\Omega G(x, u, Du)dx, \tag{1.1}$$

where $G(x, u, z) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to R$ is a Carathéodory function satisfying the following growth condition for some constants $\Lambda \ge \lambda > 0$ besides several natural assumptions:

$$\lambda H(x, z) \le G(x, u, z) \le \Lambda H(x, z).$$

For the scalar valued case, in [46] regularity results are given comprehensively. Under the conditions

$$a(\cdot) \in C^{0,\alpha}(\Omega), \quad \alpha \in (0, 1] \quad \text{and} \quad \frac{q}{p} \le 1 + \frac{\alpha}{n}, \tag{1.2}$$

or

$$u \in L^\infty(\Omega), \quad a(\cdot) \in C^{0,\alpha}(\Omega), \quad \alpha \in (0, 1] \quad \text{and} \quad \frac{q}{p} \le 1 + \frac{\alpha}{p}, \tag{1.3}$$

they showed that a local minimizer of $\mathcal{G}$ defined as (1.1) is in the class $C^{1,\beta}$ for some $\beta \in (0, 1)$.

For the scaler valued case, see also [47]. They proved Harnack's inequality and the Hölde continuity for quasiminimizer of the functional fo type

$$\int \varphi(x, |Du|)dx,$$

where φ is the so-called *Φ-function*. We mention that Harnack's inequality is not valid in the vector valued cases which we are considering in the present paper.

On the other hand, for vector valued case, in [1], under the condition

$$a(\cdot) \in C^{0,\alpha}(\Omega), \quad \alpha \in (0, 1] \quad \text{and} \quad \frac{q}{p} < 1 + \frac{\alpha}{n}, \tag{1.4}$$

$C^{1,\beta}$-regularity, for some $\beta \in (0, 1)$, of local minimizers is given.

Zhikov has given in [3, 4] examples of functionals with discontinuous growth order for which Lavrentiev phenomenon occurs. So, in general settings, we can not expect regularity of minimizers for such functionals which change their growth order discontinuously. So, conditions (1.2), (1.3) and (1.4), which guarantee the regularity of minimizers, are very significant.

In this paper we deal with a typical type of functionals of double phase with variable exponents and show a regularity result for minimizers.

In our opinion these results present new and interesting features from the point of view of regularity theory.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p(x)$, $q(x)$ and $a(x)$ functions on Ω satisfying

$$p, q \in C^{0,\sigma}(\Omega), \quad q(x) \geq p(x) \geq p_0 > 1, \quad \text{forall} \quad x \in \Omega \tag{1.5}$$

where p_0 is a fixed constant strictly larger than one and

$$a \in C^{0,\alpha}(\Omega), \quad a(x) \geq 0, \tag{1.6}$$

for $\alpha,\ \sigma \in (0, 1]$. Moreover, we assume that $p(x)$ and $q(x)$ satisfy

$$\sup_{x\in\Omega} \frac{q(x)}{p(x)} < 1 + \frac{\beta}{n}, \quad \beta = \min\{\alpha, \sigma\}, \tag{1.7}$$

at every $x \in \Omega$ (compare these conditions with (1.2)). Let $F : \Omega \times \mathbb{R}^{nN} \to [0, \infty)$ be a function defined by

$$F(x, z) := |z|^{p(x)} + a(x)|z|^{q(x)}. \tag{1.8}$$

We consider the functional with double phase and variable exponents defined for $u : \Omega \to \mathbb{R}^N$ and $D \Subset \Omega$ as

$$\mathcal{F}(u, D) = \int_D F(x, Du)dx. \tag{1.9}$$

For a bounded open set $\Omega \subset \mathbb{R}^n$ and a function $p : \Omega \to [1, +\infty)$, we define $L^{p(x)}(\Omega; \mathbb{R}^N)$ and $W^{1,p(x)}(\Omega; \mathbb{R}^N)$ as follows:

$$L^{p(x)}(\Omega; \mathbb{R}^N) := \{u \in L^1(\Omega; \mathbb{R}^N)\, ;\ \int_\Omega |u|^{p(x)}dx < +\infty\}.$$

$$W^{1,p(x)}(\Omega; \mathbb{R}^N) := \{u \in L^{p(x)} \cap W^{1,1}(\Omega; \mathbb{R}^N)\, ;\ Du \in L^{p(x)}(\Omega; \mathbb{R}^{nN})\}.$$

In what follows we omit the target space $\mathbb{R}^N$. We also define $L^{p(x)}_{\text{loc}}(\Omega)$ and $W^{1,p(x)}_{\text{loc}}(\Omega)$ similarly. As mentioned in [48], if $p(x)$ is uniformly continuous and $\partial\Omega$ satisfies uniform cone property, then

$$W^{1,p(x)}(\Omega) = \{u \in W^{1,1}(\Omega)\, ; Du \in L^{p(x)}(\Omega)\}.$$

Let us define *local minimizers of* $\mathcal{F}$ as follows:

Definition 1.1. *A function $u \in W^{1,1}\Omega)$ is called to be a local minimizer of $\mathcal{F}$ if $F(x, Du) \in L^1(\Omega)$ and satisfies*

$$\mathcal{F}(u; \text{supp}\varphi) \leq \mathcal{F}(u + \varphi; \text{supp}\varphi),$$

for any $\varphi \in W^{1,p(x)}_{\text{loc}}(\Omega)$ with compact support in Ω.

The main result of this paper is the following:

Theorem 1.2. *Assume that the conditions (1.5), (1.6) and (1.7) are fulfilled. Let $u \in W^{1,1}(\Omega)$ be a local minimizer of $\mathcal{F}$. Then $u \in C^{1,\gamma}_{\text{loc}}(\Omega)$ for some $\gamma \in (0, 1)$.*

Remark 1.3 (About the symbols for Hölder spaces)**.** *If we follow the standard textbooks, Dacorogna [49], Evans [50], Gilberg-Trudinger [51], etc., for $k \in \mathbb{N}, 0 < \alpha \leq 1$, $C^{k,\alpha}(\Omega)$ mean the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous. However, recently many authors (especially ones who study regularity problems) write them as $C^{k,\alpha}_{\text{loc}}(\Omega)$, and they use $C^{k,\alpha}(\Omega)$ for $C^{k,\alpha}(\bar{\Omega})$ (namely, for uniformly Hölder continuous cases). Anyway, with "loc" there is no doubt of misunderstanding. So, in this paper we follow their usage for Hölder spaces.*

In order to prove the above theorem, we employ a freezing argument; namely we consider a frozen functional which is given by freezing the exponents, and compare a minimizer of the original functional under consideration with that of frozen one.

2 Preliminary results

In what follows, we use C as generic constants, which may change from line to line, but does not depend on the crucial quantities. When we need to specify a constant, we use small letter c with index.

For double phase functional with constant exponents, namely for

$$\mathcal{H}(u, D) := \int_D H(x, Du)dx, \quad H(x, z) = |z|^p + a(x)|z|^q, \tag{2.1}$$

we prepare the following Sobolev-Poincaré inequality which is a slightly generalised version of [1, Theorem 1.6] due to Colombo-Mingione.

Theorem 2.1. *Let $a(x) \in C^{0,\beta}(\Omega)$ for some $\beta \in (0, 1)$ and $1 < p < q$ constants satisfying*

$$\frac{q}{p} < 1 + \frac{\beta}{n},$$

and let $\omega \in L^\infty(\mathbb{R}^n)$ with $\omega \geq 0$ and $\int_{B_R} \omega dx = 1$ for $B_R \subset \Omega$ with $R \in (0, 1)$. Then, there exists a constant C depending only on $n, p, q, [a]_{0,\beta}, R^n\|\omega\|_{L^\infty}$ and $\|Dw\|_{L^p(B_R)}$ and exponents $d_1 > 1 > d_2$ depending only on n, p, q, β such that

$$\left(\fint_{B_R} \left[H\left(x, \frac{u - \langle u\rangle_\omega}{R}\right)\right]^{d_1} dx\right)^{\frac{1}{d_1}} \leq C\left(\fint_{B_R} [H(x, Du)]^{d_2} dx\right)^{\frac{1}{d_2}} \tag{2.2}$$

holds whenever $u \in W^{1,p}(B_R)$, where

$$\langle u\rangle_\omega := \int_{B_R} u(x)\omega(x)dx.$$

Note that for the special choice $\omega = |B_R|^{-1}\chi_{B_R}$ we have

$$\langle u\rangle_\omega = \fint_{B_R} u(x)dx.$$

Proof. We can proceed exactly as in the proof of [1, Theorem 1.6] only replacing (3.11) of [1] by

$$\frac{|u(x) - \langle u\rangle_\omega|}{R} \leq \frac{C}{R}\int_{B_R} \frac{|Du(y)|}{|x-y|^{n-1}} dy,$$

which is shown by [52, Lemma 1.50] (see also the proof of [53, Theorem 7]). □

From the above theorem, we have the following corollary.

Corollary 2.2. *Assume that all conditions of Theorem 2.1 are satisfied, and let D be a subset of B_R with positive measure. Then, there exists a constant C depending only on $n, p, q, [a]_{0,\beta}, R^n/|D|$ and $\|Du\|_{L^p(B_R)}$ and exponents $d_1 > 1 > d_2$ depending only on n, p, q, β such that the following inequality holds whenever $u \in W^{1,p(x)}(B_R)$ satisfies $u \equiv 0$ on D:*

$$\left(\fint_{B_R} \left[H\left(x, \frac{u}{R}\right)\right]^{d_1} dx\right)^{\frac{1}{d_1}} \leq C\left(\fint_{B_R} [H(x, Du)]^{d_2} dx\right)^{\frac{1}{d_2}}. \tag{2.3}$$

Proof. Choosing ω so that

$$\omega(x) = \begin{cases} 0 & x \in B_R \setminus D \\ \frac{1}{|D|} & x \in D \end{cases}$$

and applying Theorem 2.1, we get the assertion. □

Remark 2.3. *In [1, Theorem 6.1], and therefore also in the above theorem and corollary, the exponent $d_2 \in (0,1)$ is chosen so that the following conditions hold:*

$$\frac{q}{p} < 1 + \frac{\beta d_2}{n} \tag{2.4}$$

$$\frac{p}{q(n-1)} + 1 > \frac{1}{d_2}. \tag{2.5}$$

In fact, in [1], they choose a constant $y \in (1,p)$ so that

$$\frac{q}{p} < 1 + \frac{\alpha}{yn} \quad \text{and} \quad \frac{p+q(n-1)}{yq(n-1)} > 1,$$

(see [1, (3.6), (3.14)]), and put $d_2 = 1/y$. Let us mention the that if d_2 satisfies (2.4) and (2.5) for some $q = q_0$ and $p = p_0$, then the same d_2 satisfies these inequalities for any q and p with $q/p \leq q_0/p_0$.

For any $y \in \Omega$ and $R > 0$ with $B_R(x) \subset \Omega$ let us put

$$p_2(y,R) := \sup_{B_R(y)} p(x), \;\; p_1(y,R) := \inf_{B_R(y)} p(x), \tag{2.6}$$

$$q_2(y,R) := \sup_{B_R(y)} q(x), \;\; q_1(y,R) := \inf_{B_R(y)} q(x). \tag{2.7}$$

We prove interior higher integrability of the gradient of a minimizer, similar results are contained in [54].

Proposition 2.4. *Let $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ be a local minimizer of $\mathcal{F}$. Then, for any compact subset $K \subset \Omega$, $F(x,Du) \in L^{1+\delta_0}(K)$ and there exists a positive constant δ_0 and C depending only on the given data and K such that*

$$\left(\fint_{B_{R/2}(y)} F(x,Du)^{1+\delta_0} dx \right)^{\frac{1}{1+\delta_0}} \leq C + C \fint_{B_R(y)} F(x,Du) dx \tag{2.8}$$

holds for any $B_R(y) \Subset K$.

Proof. Let $K \subset \Omega$ be a compact subset and $R_0 \in (0, \mathrm{dist}(K, \partial\Omega))$ a constant such that

$$0 < R_0^\sigma \leq \frac{p_0}{2^{1+\sigma}[q]_{0,\sigma}} \left(1 + \frac{\beta}{n} - \sup_{x\in\Omega} \frac{q(x)}{p(x)} \right). \tag{2.9}$$

For any $x_0 \in \overset{\circ}{K}$, put

$$\kappa_0 := \frac{1}{4} \left(1 + \frac{\beta}{n} - \sup_{x \in B_R(x_0)} \frac{q(x)}{p(x)} \right) > 0. \tag{2.10}$$

Then, letting $x_- \in \bar{B}_{R_0}(x_0)$ be a such that $p(x_-) = p_1(x_0, R_0)$, we have

$$\begin{aligned} \frac{q_2(x_0,R_0)}{p_1(x_0,R_0)} &= \frac{q(x_-) + (q_2(x_0,R_0) - q(x_-))}{p_1(x_0,R_0)} \\ &\leq \sup_{x\in B_{R_0}(x_0)} \frac{q(x)}{p(x)} + \frac{2^\sigma [q]_{0,\sigma} R_0^\sigma}{p_0} \\ &\leq \sup_{x\in B_{R_0}(x_0)} \frac{q(x)}{p(x)} + \frac{1}{2}\left(1 + \frac{\beta}{n} - \sup_{x\in B_{R_0}(x_0)} \frac{q(x)}{p(x)} \right) \\ &= \frac{1}{2}\left(1 + \frac{\beta}{n} + \sup_{x\in B_{R_0}(x_0)} \frac{q(x)}{p(x)} \right) \leq 1 + \frac{\beta}{n} - 2\kappa_0 \end{aligned} \tag{2.11}$$

The above estimate (2.11) implies that

$$q_2(x_0,R_0) < (p_1(x_0,R_0))^* = \frac{np_1(x_0,R_0)}{n - p_1(x_0,R_0)}. \tag{2.12}$$

For any $B_R(y) \subset B_{R_0}(x_0)$ with $0 < R < 1$, and $0 < t \le s \le R$, let η be a cut-off function such that $\eta \equiv 1$ on $B_t(y)$, $\eta \equiv 0$ outside $B_s(y)$ and $|D\eta| \le \frac{2}{s-t}$. Put $w := u - \eta(u - u_R)$, where $u_R = \fint_{B_R(y)} u dx$. Since

$$Dw = (1-\eta)Du + (u - u_R)D\eta,$$

we have

$$F(x, Dw) \le c_0\Big[\big((1-\eta)|Du|\big)^{p(x)} + (|u-u_R||D\eta|)^{p(x)} + a(x)\big((1-\eta)|Du|\big)^{q(x)} + (|u-u_R||D\eta|)^{q(x)}\Big],$$

where c_0 is a constant depending only on $\max_K q(x)$. On the other hand, since $F(x, Du) \in L^1$, we have

$$u \in W^{1,p(x)} \subset W^{1,p_1(x_0,R_0)} \subset L^{p_1(x_0,R_0)^*} \subset L^{p_2(x_0,R_0)} \subset L^{q(x)},$$

on $B_{R_0}(x_0)$. Thus, mentioning also that $w = u$ outside $B_s(y)$, we see that $F(x, Dw) \in L^1(K)$, namely w is an admissible function. In the following part of the proof, let us abbreviate

$$p_i := p_i(y, R), \quad q_i := q_i(y, R) \quad (i = 1, 2).$$

Then, we have

$$\begin{aligned}
\int_{B_s(y)} F(x, Du)dx \le \int_{B_s(y)} F(x, Dw)dx &\le c_0 \int_{B_s(y)} (1-\eta)^{p(x)}\big(|Du|^{p(x)} + a(x)|Du|^{q(x)}\big)dx \\
&\quad + c_0 \int_{B_s(y)} \left[\left|\frac{u-u_R}{s-t}\right|^{p(x)} + a(x)\left|\frac{u-u_R}{s-t}\right|^{q(x)}\right]dx \\
&\le c_0 \int_{B_s(y)\setminus B_t(y)} F(x, Du)dx + \frac{c_0}{(s-t)^{p_2}} \int_{B_s(y)} |u-u_R|^{p(x)} \\
&\quad + \frac{c_0}{(s-t)^{q_2}} \int_{B_s(y)} a(x)|u-u_R|^{q(x)}dx
\end{aligned} \tag{2.13}$$

We can use *hole-filling method*. Add $c_0 \int_{B_s(y)\setminus B_t(y)} F(x, Du)dx$ to the both side and divide them by $c_0 + 1$, then we get

$$\int_{B_t(y)} F(x, Du)dx \le \frac{c_0}{c_0+1}\left(\int_{B_s(y)} F(x, Du)dx + \frac{1}{(s-t)^{p_2}}\int_{B_s(y)} |u-u_R|^{p(x)}dx + \frac{1}{(s-t)^{q_2}}\int_{B_s(y)} a(x)|u-u_R|^{q(x)}dx\right). \tag{2.14}$$

Using an iteration lemma [55, Lemma 6.1], we see, for some constant $C = C(c_0, p_2, q_2)$, that

$$\int_{B_t(y)} F(x, Du)dx \le \frac{C}{(s-t)^{p_2}}\int_{B_s(y)} |u-u_R|^{p(x)} + \frac{C}{(s-t)^{q_2}}\int_{B_s(y)} a(x)|u-u_R|^{q(x)}dx.$$

Putting $s = R$ and $t = R/2$, we have

$$\begin{aligned}
\int_{B_{\frac{R}{2}}(y)} F(x, Du)dx &\le \frac{C}{R^{p_2}}\int_{B_R(y)} |u-u_R|^{p(x)} + \frac{C}{R^{q_2}}\int_{B_R(y)} a(x)|u-u_R|^{q(x)}dx \\
&\le CR^{p_1-p_2}\int_{B_R(y)} \left|\frac{u-u_R}{R}\right|^{p(x)}dx + CR^{q_1-q_2}\int_{B_R(y)} a(x)\left|\frac{u-u_R}{R}\right|^{q(x)}dx \\
&\le CR^{p_1-p_2}\int_{B_R(y)} \left(1 + \left|\frac{u-u_R}{R}\right|\right)^{p_2}dx + CR^{q_1-q_2}\int_{B_R(y)} \left(1 + a(x)^{\frac{1}{q(x)}}\left|\frac{u-u_R}{R}\right|\right)^{q_2}dx.
\end{aligned} \tag{2.15}$$

Since $R^{p_1-p_2}$ and $R^{q_1-q_2}$ are bounded because of the Hölder continuity of exponents $p(x)$ and $q(x)$, putting

$$\tilde{a}(x) := \left(a(x)\right)^{\frac{q_2}{q(x)}},$$

from (2.15), we obtain the estimate

$$\begin{aligned}\int_{B_{\frac{R}{2}}(y)} F(x,Du)dx &\le CR^n + CR^n \fint_{B_R(y)} \left(\left|\frac{u-u_R}{R}\right|^{p_2} dx + \tilde{a}(x)\left|\frac{u-u_R}{R}\right|^{q_2}\right) dx \\ &=: I + II. \end{aligned} \tag{2.16}$$

In order to get the boundedness of $R^{p_1-p_2}$ and $R^{q_1-q_2}$ the so-called "log-*Hölder continuity*" (see [56, section 4.1]) is sufficient. On the other hand by virtue of the Hölder continuity of $q(\cdot)$, we have that $\tilde{a} \in C^{0,\beta}$ $(\beta = \min\{\alpha,\sigma\})$. Let $d_2 \in (0,1)$ be a constant satisfying (2.4) and (2.5) for $\beta = \min\{\alpha,\sigma\}$, $q = q_2(x_0,R_0)$ and $p = p_1(x_0,R_0)$. Then, for any $B_R(y) \subset B_{R_0}(x_0)$, this d_2 satisfy (2.4) and (2.5) with $q = q_2(y,R)$ and $p = p_2(y,R)$.

By Theorem 2.1, we can estimate II as follows.

$$\begin{aligned} II &\le CR^n \left(\fint_{B_R(y)} \left(|Du|^{p_2} + \tilde{a}(x)|Du|^{q_2}\right)^{d_2} dx\right)^{\frac{1}{d_2}} \\ &\le CR^n \left(\fint_{B_R(y)} |Du|^{d_2p_2}dx\right)^{\frac{1}{d_2}} + CR^n \left(\fint_{B_R(y)} \left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{d_2q_2} dx\right)^{\frac{1}{d_2}}. \end{aligned} \tag{2.17}$$

As mentioned above, (2.17) holds for for any $B_R(y) \subset B_{R_0}(x_0)$ with same d_2. Now, take $R > 0$ sufficiently small so that

$$d_2p_2(y,R) < p_1(y,R) \ \text{ and } \ d_2q_2(y,R) < q_1(y,R),$$

and let $\theta \in (d_2, 1)$ be a constant satisfying

$$d_2p_2(y,R) < \theta p_1(y,R) \ \text{ and } \ d_2q_2(y,R) < \theta q_1(y,R). \tag{2.18}$$

Then, using Hölder inequality, we can estimate the first term of the right hand side of (2.17) as follows.

$$\begin{aligned} \left(\fint_{B_R(y)} |Du|^{d_2p_2}dx\right)^{\frac{1}{d_2}} &\le \left(\fint_{B_R(y)} |Du|^{\theta p_1}dx\right)^{\frac{p_2}{\theta p_1}} = \left(\fint_{B_R(y)} |Du|^{\theta p_1}dx\right)^{\frac{p_2-p_1}{\theta p_1}} \cdot \left(\fint_{B_R(y)} |Du|^{\theta p_1}dx\right)^{\frac{1}{\theta}} \\ &\le \left(\fint_{B_R(y)} (1+|Du|^{p(x)})dx\right)^{\frac{p_2-p_1}{\theta p_1}} \cdot \left(\fint_{B_R(y)} \left(1+|Du|^{\theta p_1}\right)dx\right)^{\frac{1}{\theta}}. \end{aligned} \tag{2.19}$$

Since,

$$\int_{B_R(y)} |Du|^{p(x)}dx \le \mathcal{F}(u, B_R(y)) \le \mathcal{F}(u,K)$$

and u locally minimizes $\mathcal{F}$, $\int_{B_R(y)} |Du|^{p(x)}dx$ is bounded. On the other hand, as mentioned after (2.15), $R^{-(p_2-p_1)}$ is bounded. So, there exists a constant $c_1 = c_1(\mathcal{F}(u,K), p(x), d_2, n, \theta)$

$$\begin{aligned}\left(\fint_{B_R(y)} |Du|^{p(x)}dx\right)^{\frac{p_2-p_1}{\theta p_1}} &\le (\omega_n R^n)^{\frac{-(p_2-p_1)}{\theta p_1}} \mathcal{F}(u,K)^{\frac{p_2-p_1}{\theta p_1}} \\ &\le c_1(\mathcal{F}(u,K), p(x), d_2, n, \theta),\end{aligned}$$

where ω_n denotes the volume of a n-dimensional unit ball. Thus, from (2.19) we obtain for some positive constant $c_2 = c_2(c_1, \theta)$

$$\left(\fint_{B_R(y)} |Du|^{d_2p_2}dx\right)^{\frac{1}{d_2}} \le c_2 + c_2 \left(\fint_{B_R(y)} |Du|^{\theta p(x)}dx\right)^{\frac{1}{\theta}}. \tag{2.20}$$

Similarly, we can estimate the second term of the left hand side of (2.17) as follows.

$$\begin{aligned}\left(\fint_{B_R(y)}\left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{d_2q_2}dx\right)^{\frac{1}{d_2}} &\le \left(\fint_{B_R(y)}\left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{\theta q_1}dx\right)^{\frac{q_2}{\theta q_1}}\\
&\le \left(\fint_{B_R(y)}\left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{\theta q_1}dx\right)^{\frac{q_2-q_1}{\theta q_1}}\left(\fint_{B_R(y)}\left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{\theta q_1}dx\right)^{\frac{1}{\theta}}\\
&\le \left(\fint_{B_R(y)}\left(1+\left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{q(x)}\right)dx\right)^{\frac{q_2-q_1}{\theta q_1}}\left(\fint_{B_R(y)}\left(1+\left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{\theta q(x)}\right)dx\right)^{\frac{1}{\theta}}. \end{aligned} \tag{2.21}$$

As above, using local minimality of u and the fact that $R^{-(q_2-q_1)}$ is bounded, we have for a positive constant $c_3 = c_3(\mathcal{F}(u,K), q(x), d_2, n, \theta)$

$$\left(\fint_{B_R(y)}\left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{d_2q_2}dx\right)^{\frac{q_2-q_1}{\theta q_1}} \le c_3(\mathcal{F}(u,K), q(x), d_2, n, \theta). \tag{2.22}$$

Thus, we obtain for some positive constant $c_4 = c_4(c_3, \theta)$

$$\left(\fint_{B_R(y)}\left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{d_2q_2}dx\right)^{\frac{1}{d_2}} \le c_4 + c_4\left(\fint_{B_R(y)}\left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{\theta q(x)}dx\right)^{\frac{1}{\theta}}. \tag{2.23}$$

Combining (2.16), (2.17), (2.20) and (2.23), we see that there exists a constant C depending on the given data and $\mathcal{F}(u,K)$ such that

$$\fint_{B_{\frac{R}{2}}(y)} F(x,Du)dx \le C + C\left(\fint_{B_R(y)} F(x,Du)^{\theta}dx\right)^{\frac{1}{\theta}} \tag{2.24}$$

for any $B_R(y) \subset B_{R_0} \subset K \Subset \Omega$. Now, by virtue of the *reverse Hölder inequality with increasing domain* due to Giaquinta-Modica [57], we get the assertion. □

For δ_0 determined in Proposition 2.4, in what follows, we always take $R > 0$ sufficiently small so that

$$\left(1+\frac{\delta_0}{2}\right)p_2(y,R) \le (1+\delta_0)p_1(y,R) \text{ and } \left(1+\frac{\delta_0}{2}\right)q_2(y,R) \le (1+\delta_0)q_1(y,R). \tag{2.25}$$

We need also higher integrability results on the neighborhood of the boundary. Let us use the following notation: for $T > 0$ we put

$$\begin{aligned} &B_T := B_T(0),\ \ B_T^+ := \{x \in \mathbb{R}^n\,;\ |x| < T,\ x^n > 0\},\\ &\Gamma_T := \{x \in \mathbb{R}^n\,;\ |x| < T,\ x^n = 0\}, \end{aligned}$$

We say "$f = g$ on Γ_T" when for any $\eta \in C_0^\infty(B_T)$ we have $(f-g)\eta \in W_0^{1,1}(B_T^+)$. For $y \in B_T$, we write

$$\Omega_r := B_r(y) \cap B_T^+.$$

Then, we have the following proposition on the higher integrability near the boundary, independently proved in [58, Lemma 5] , see also [59, Lemma 5] for the manifold constrained case.

Proposition 2.5. *Let $a(x)$, q and p satisfy the same conditions in Theorem 2.1 and let for $A \subset B_T^+$*

$$\mathcal{H}(w,A) := \int_A H(x,w)dx,\ \ H(x,z) := |z|^p + a(x)|z|^q.$$

$u \in W^{1,p}(B_T^+)$ be a given function with

$$\int_{B_T^+}\left(|Du|^p + a(x)|Du|^q\right)^{1+\delta_0}dx < \infty,$$

for some δ_0 >. Assume that $v \in W^{1,p}(B^+(T))$ be a local minimizer of $\mathcal{H}$ in the class

$$\{w \in W^{1,p}(B_T^+)\,;\; u = w \;\text{ on }\; \Gamma_T\}$$

Then, for any $S \in (0,T)$, there exists a constants $\delta \in (0,\delta_0)$ and $C > 0$ such that for any $y \in B_S^+$ and $R \in (0, T-S)$ we have

$$\left(\fint_{\Omega_{R/2}} \big(H(x,Dv)\big)^{1+\delta}\,dx\right)^{\frac{1}{1+\delta}} \le C\fint_{\Omega_R} H(x,Dv)dx + C\left(\fint_{\Omega_R} \big(H(x,Du)\big)^{1+\delta}\,dx\right)^{\frac{1}{1+\delta}}.$$

Proof. For convenience, we extend u, v, Du, Dv to be zero in $B_T \setminus B_T^+$. Of course, because extended u, v may have discontinuity on Γ_T, they are not always in $W^{1,p}_{\mathrm{loc}}(B_T)$, and therefore Du, Dv do not necessarily coincide with distributional derivatives of u, v on $B(T)$. On the other hand, since $u = v$ on $\Gamma(T)$, $u - v$ is in the class $W^{1,p}(B(S))$ and $Du - Dv$ can be regarded as the weak derivatives of $u - v$ on $B(S)$ for any $S < T$.

Let R be a positive constant satisfying $R \le (T-S)/2$. For $x_0 \in B_S^+$, we treat the two cases $x_0^n \le \frac{3}{4}R$ and $x_0^n > \frac{3}{4}R$ separately.

Case 1. Suppose that $x_0^n \le \frac{3}{4}R$. Take radii s, t so that $0 < R/2 \le t < s \le R$ and choose a $\eta \in C_0^\infty(B_T)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_t, supp $\eta \subset B_s$ and $|D\eta| \le 2/(s-t)$. Defining

$$\varphi := \eta(v-u),$$

we see that $\varphi \in W_0^{1,1}(B_T^+)$ with supp $\varphi \subset B_s$, and that

$$D(v-\varphi) = (1-\eta)Dv - (v-u)D\eta + \eta Du.$$

Then, by virtue of the minimality of v, for a positive constant c_4 depending only on q, we have

$$\begin{aligned}
\int_{\Omega_t} H(x,Dv)dx &\le \int_{\Omega_s} H(x,Dv)dx \le \int_{\Omega_s} H(x, D(v-\varphi))dx \\
&= \int_{\Omega_s} \left(|D(v-\varphi)|^p + a(x)|D(v-\varphi)|^q\right)dx \\
&\le c_4 \int_{\Omega_s\setminus\Omega_t} \left(|Dv|^p + a(x)|Dv|^q\right)dx + c_4\int_{\Omega_s}\left(|Du|^p + a(x)|Du|^q\right)dx \\
&\quad + c_4\int_{\Omega_s}\left(\left(\frac{2}{s-t}\right)^p |v-u|^p + a(x)\left(\frac{2}{s-t}\right)^q |v-u|^q\right)dx \\
&\le c_4 \int_{\Omega_s\setminus\Omega_t} \left(|Dv|^p + a(x)|Dv|^q\right)dx + c_4\int_{\Omega_s}\left(|Du|^p + a(x)|Du|^q\right)dx \\
&\quad + c_4\left(\frac{2}{s-t}\right)^p \int_{\Omega_s}|v-u|^p dx + c_4\left(\frac{2}{s-t}\right)^q\int_{\Omega_s} a(x)|v-u|^q dx.
\end{aligned}$$

Now, we use *the hole filling method* as in the proof of Proposition 2.4. Namely, adding

$$c_4\int_{\Omega_t}\left(|Dv|^p + a(x)|Dv|^q\right)dx$$

and dividing both side by $c_4 + 1$, we obtain

$$\int_{\Omega_t} H(x,Dv)dx \le \frac{c_4}{c_4+1}\left(\int_{\Omega_s} H(x,Dv)dx + \int_{\Omega_s} H(x,Du)dx + \frac{1}{(s-t)^p}\int_{\Omega_s}|v-u|^p dx + \frac{1}{(s-t)^q}\int_{\Omega_s} a(x)|v-u|^q dx\right),$$

Using the iteration lemma [55, Lemma 6.1], we get for some constant $C = C(c_4, p, q)$

$$\int_{\Omega_t} H(x, Dv)dx \le C \int_{\Omega_s} H(x, Du)dx + \frac{C}{(s-t)^p} \int_{\Omega_s} |v-u|^p dx + \frac{C}{(s-t)^q} \int_{\Omega_s} a(x)|v-u|^q dx.$$

Putting $t = R/2$ and $s = R$, we have

$$\int_{\Omega_{R/2}} H(x, Dv)dx \le C \int_{\Omega_R} H\left(x, \frac{v-u}{R}\right) dx + C \int_{\Omega_R} H(x, Du)dx.$$

Let us now consider the mean integral in all the terms, we obtain

$$\fint_{\Omega_{R/2}} H(x, Dv)dx \le C \fint_{\Omega_R} H(x, Du)dx + C \fint_{\Omega_R} H\left(x, \frac{v-u}{R}\right) dx.$$

Since we are assuming that $x_0^n \le \frac{3}{4}R$ we can apply Corollary 2.2 with a constant independent on R for the last term in the right hand side and get

$$\fint_{\Omega_{R/2}} H(x, Dv)dx \le C \fint_{\Omega_R} H(x, Du)dx + C \left(\fint_{\Omega_R} (H(x, D(v-u)))^{d_2} dx \right)^{\frac{1}{d_2}}.$$

Taking into consideration that $d_2 < 1$ we share in the last term Dv and Du, apply Hölder inequality for the integral of $H(x, Du)^{d_2}$, and obtain

$$\fint_{\Omega_{R/2}} H(x, Dv)dx \le C \fint_{\Omega_R} H(x, Du)dx + C \left(\fint_{\Omega_R} \left(H(x, Dv)\right)^{d_2} dx \right)^{\frac{1}{d_2}}. \tag{2.26}$$

Case 2. Let us deal with the case that $x_0^n > \frac{3}{4}R$. In this case, since $B_{3R/4}(x_0) \Subset B_T^+$, we can proceed as in [1, 9. Proof of Theorem 1.1:(1.8)], slightly modifying the radii, to get

$$\begin{aligned}
\fint_{\Omega_{R/2}} H(x, Dv)dx &= \fint_{B_{R/2}} H(x, Dv)dx \\
&\le C \left(\fint_{B_{3R/4}} \left(H(x, Dv)\right)^{d_2} dx \right)^{\frac{1}{d_2}} \le C' \left(\fint_{\Omega_R} \left(H(x, Dv)\right)^{d_2} dx \right)^{\frac{1}{d_2}}.
\end{aligned} \tag{2.27}$$

Thus, we see that (2.26) holds for every $0 < R < (S-T)/2$. Now, the reverse Hölder inequality allows us to obtain

$$\left(\fint_{\Omega_R} \left(H(x, Dv)\right)^{1+\delta} dx \right)^{\frac{1}{1+\delta}} \le C \fint_{\Omega_{\frac{R}{2}}} H(x, Dv)dx + C \left(\fint_{\Omega_R} \left(H(x, Du)\right)^{1+\delta} dx \right)^{\frac{1}{1+\delta}}.$$

□

By virtue of [1, Theorem 1.1] and Proposition 2.5, we have the following global higher integrability for functions which minimize $\mathcal{H}$ with Dirichlet boundary condition.

Corollary 2.6. *Let $a(x)$, q and p satisfy the same conditions in Theorem 2.1 and $\delta_2 \in (0, 1)$ be a some constant. Assume that $u \in W^{1,(1+\delta_1)p}(B_R(y))$ be a given function with*

$$\int_{B_R(y)} H(x, Du)^{1+\delta_1} dx := \int_{B_R(y)} \left(|Du|^p + a(x)|Dv|^q\right)^{1+\delta_1} dx \le C$$

for some constant $C > 0$. Let $v \in W^{1,p}(B_R(y))$ be a minimizer of

$$\mathcal{H}(w, B_R(y)) := \int_{B_R(y)} H(x, Dw)dx$$

in the class

$$u + W_0^{1,p}(B_R(y)) = \{w \in W^{1,p}(B_R(y))\,;\ u - w \in W_0^{1,p}(B_R(x_0))\}.$$

Then, for some $\delta_2 \in (0, \delta_1)$ and for any $\delta_3 \in (0, \delta_2)$, we have $H(x, Dv) \in L^{1+\delta}(B_R(y))$ and

$$\int_{B_R} \big(H(x, Dv)\big)^{1+\delta_3}\, dx \le C \int_{B_R} \big(H(x, Du)\big)^{1+\delta_3}\, dx. \tag{2.28}$$

Proof. From [1, Theorem 1.1], Proposition 2.5 and covering argument, we have

$$\left(\fint_{B_R} \big(H(x, Dv)\big)^{1+\delta} dx\right)^{\frac{1}{1+\delta}} \le C \fint_{B_R} H(x, Dv)dx + C\left(\fint_{B_R} \big(H(x, Du)\big)^{1+\delta} dx\right)^{\frac{1}{1+\delta}}$$

and then, by the minimality of v,

$$\left(\fint_{B_R} \big(H(x, Dv)\big)^{1+\delta} dx\right)^{\frac{1}{1+\delta}} \le C \fint_{B_R} H(x, Du)dx + C\left(\fint_{B_R} \big(H(x, Du)\big)^{1+\delta} dx\right)^{\frac{1}{1+\delta}}$$

Once again we use the Hölder inequality for the first term of the right-hand side that gives us the assertion. □

3 Proof of the main theorem

In this section we prove Theorem 1.2. We employ the so-called *direct approach*, namely we consider a *frozen functional* for which the regularity theory has been established in [1] and compare a local minimizer of the frozen functional with u under consideration.

For a constant $p > 1$, let us define the auxiliary vector field $V_p : \mathbb{R}^n \to \mathbb{R}^n$ as

$$V_p(z) := |z|^{p-2} z. \tag{3.1}$$

Let mention that V_p satisfies

$$|V_p(z)|^2 = |z|^p \ \text{ and } \ |V_p(z_1) - V_p(z_2)| \approx (|z_1| + |z_2|)^{\frac{p-2}{2}} |z_1 - z_2|. \tag{3.2}$$

Proof of Theorem 1.2. We divide the proof into two parts. We prove the Hölder continuity of u in **Part 1**, and of the gradient Du in **Part 2**.

Part 1. Let K and $B_{R_0}(x_0)$, are as in the Proposition 2.4. For $B_R(y) \subset B_{2R}(y) \subset B_{R_0}(x_0)$, let us define p_i and q_i as in the Proposition 2.4. We define a *frozen functional* $\mathcal{F}_0$ as

$$F_0(x, z) := |z|^{p_2} + a(x)^{\frac{q_2}{q(x)}} |z|^{q_2} \tag{3.3}$$

$$\mathcal{F}_0(w, D) = \int_{B_R(y)} F_0(x, Dw)dx. \tag{3.4}$$

In what follows, let us abbreviate $\tilde{a}(x) = \big(a(x)\big)^{\frac{q_2}{q(x)}}$ as in the proof of Proposition 2.4.

Let $v \in W^{p_2}(B_R(y))$ be a minimizer of $\mathcal{F}_0$ in the class

$$u + W_0^{p_2}(B_R(y)) := \{w \in W^{p_2}(B_R(y))\,;\ w - u \in W_0^{p_2}(B_R(y))\}.$$

Then, by [1, Theorem1.3], for any $\gamma \in (0, 1)$ there exists a constant $C > 0$ dependent on $n, p_2, q_2, \lambda, \Lambda, [\tilde{a}]_{0,\beta}, \|\tilde{a}\|_\infty, \|Dv\|_{L^{p_2}(B_R(y))}$ and γ such that

$$\int_{B_\rho(y)} F_0(x, Dv)dx \le C\left(\frac{\rho}{R}\right)^{n-\gamma} \int_{B_R(y)} F_0(x, Dv)dx \le C\left(\frac{\rho}{R}\right)^{n-\gamma} \int_{B_R(y)} F_0(x, Du)dx, \tag{3.5}$$

where we used the minimality of v. Here, we mention that by the coercivity of the functional and the minimality of v we have the following:

$$\|Dv\|^{p_2}_{L^{p_2}(B_R(y))} \leq \mathcal{F}_0(v, B_R(y)) \leq \mathcal{F}_0(u, B_R(y)). \tag{3.6}$$

On the other hand, since we are taking $R > 0$ sufficiently small so that (2.25) holds, there exists a constant $C(p_2, q_2) > 0$ such that

$$F_0(x, \xi) \leq C(p_2, q_2)(1 + F(x, \xi))^{1+\delta_0} \tag{3.7}$$

holds for any $(x, \xi) \in B_R(y) \times \mathbb{R}^{nN}$. Now, by virtue of above 2 estimates and Proposition 2.4, we can see, for a constant $C > 0$ depending only on the given data on the functional, that

$$\|Dv\|^{p_2}_{L^{p_2}(B_R(y))} \leq \mathcal{F}_0(v, B_R(y)) \leq C\left(1 + \mathcal{F}(u, K)\right)^{1+\delta}. \tag{3.8}$$

Because of the local minimality of u, the last quantity is finite. Consequently, we can regard the constant in (3.5) is a constant depending only on given data and $\mathcal{F}(u, K)$.

For further convenience, let us mention that from (3.5), is nothing to see that

$$\begin{aligned}\int_{B_\rho(y)} (1 + F_0(x, Dv))dx &\leq C\left(\frac{\rho}{R}\right)^{n-y} \int_{B_R(y)} (1 + F_0(x, Dv))dx \\ &\leq C\left(\frac{\rho}{R}\right)^{n-y} \int_{B_R(y)} (1 + F_0(x, Du))dx.\end{aligned} \tag{3.9}$$

Let us compare Du and Dv. Mentioning the elementary equality for a twice differentiable function

$$f(1) - f(0) = f'(0) + \int_0^1 (1-t)f''(t)dt,$$

as [21, (9)], and using the fact that v satisfies the Euler-Lagrange equation of $\mathcal{F}_0$, we can see that

$$\begin{aligned}\mathcal{F}_0(u) - \mathcal{F}_0(v) &= \int_{B_R(y)} \frac{d}{dt}F_0(x, tDu - (1-t)Dv)\Big|_{t=0}dx \\ &\quad + \int_{B_R(y)} dx \int_0^1 (1-t)\frac{d^2}{dt^2}F_0(x, tDu + (1-t)Dv)dt \\ &= \int_{B_R(y)} D_zF_0(x, Dv)(Du - Dv) \\ &\quad + \int_{B_R(y)} dx \int_0^1 (1-t)D_zD_zF_0(x, tDu + (1-t)Dv)(Du - Dv)(Du - Dv)dt \\ &\geq C \int_{B_R(y)} dx \int_0^1 (1-t)\Big[|tDu + (1-t)Dv|^{p_2-2} \\ &\qquad + \tilde{a}(x)|tDu + (1-t)Dv|^{q_2-2}\Big]|Du - Dv|^2 dt \\ &\geq C \int_{B_R(y)} \left(|Du|^{p_2-2} + |Dv|^{p_2-2}\right)|Du - Dv|^2 dx \\ &\quad + \int_{B_R(y)} \tilde{a}(x)\left(|Du|^{q_2-2} + |Dv|^{q_2-2}\right)|Du - Dv|^2 dx.\end{aligned} \tag{3.10}$$

On the other hand, by the minimality of v, we have

$$\mathcal{F}_0(u) - \mathcal{F}_0(v) \le \mathcal{F}_0(u) - \mathcal{F}(u, B_R(y)) + \mathcal{F}(v, B_R(y)) - \mathcal{F}_0(v). \tag{3.11}$$

Since we are assuming $p(x), q(x) \in C^{0,\sigma}$, using the inequality [21, (7)], we can see that, for any $\varepsilon \in (0, 1)$, there exists a positive constant C such that

$$\begin{aligned}
\mathcal{F}_0(u) - \mathcal{F}(u, B_R(y)) &\le \int_{B_R(y)} \left[\left(|Du|^{p_2} - |Du|^{p(x)}\right) + \left(\left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{q_2} - \left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{q(x)}\right)\right] dx \\
&\le C(\varepsilon)R^\sigma \int_{B_R(y)} \left(1 + |Du|^{(1+\varepsilon)p_2}\right) dx \\
&\quad + C(\varepsilon)R^\sigma \int_{B_R(y)} \left(1 + \left(a(x)^{\frac{1}{q(x)}}|Du|\right)^{(1+\varepsilon)q_2}\right) dx \\
&\le CR^{n+\sigma} + C(\varepsilon)R^\sigma \int_{B_R(y)} \left(1 + |Du|^{p_2(1+\varepsilon)} + \left(1 + \tilde{a}(x)|Du|^{q_2}\right)^{1+\varepsilon}\right) dx \\
&\le CR^{n+\sigma} + C(\varepsilon)R^\sigma \int_{B_R(y)} F_0(x, Du)^{1+\varepsilon} dx
\end{aligned} \tag{3.12}$$

Similarly we have

$$\begin{aligned}
\mathcal{F}(v, B_R(y)) - \mathcal{F}_0(v) &\le \int_{B_R(y)} \left[\left(|Dv|^{p_2} - |Dv|^{p(x)}\right) + \left(\left(a(x)^{\frac{1}{q(x)}}|Dv|\right)^{q_2} - \left(a(x)^{\frac{1}{q(x)}}|Dv|\right)^{q(x)}\right)\right] dx \\
&\le C(\varepsilon)R^\sigma \int_{B_R(y)} \left(1 + |Dv|^{(1+\varepsilon)p_2}\right) dx \\
&\quad + C(\varepsilon)R^\sigma \int_{B_R(y)} \left(1 + \left(a(x)^{\frac{1}{q(x)}}|Dv|\right)^{(1+\varepsilon)q_2}\right) dx \\
&\le CR^{n+\sigma} + C(\varepsilon)R^\sigma \int_{B_R(y)} \left(1 + |Dv|^{p_2(1+\varepsilon)} + \left(1 + \tilde{a}(x)|Dv|^{q_2}\right)^{1+\varepsilon}\right) dx \\
&\le CR^{n+\sigma} + C(\varepsilon)R^\sigma \int_{B_R(y)} F_0(x, Dv)^{1+\varepsilon} dx.
\end{aligned} \tag{3.13}$$

Now, for δ_0 of Proposition 2.4, choose $\delta_3 > 0$ so that (2.28) of Corollary 2.6 holds, and let us take ε so that $\varepsilon \in (0, \min\{\delta_0/2, \delta_3\}/2)$. Since we are choosing R so that (2.25) holds, we have

$$F_0(x, \cdot)^{1+\varepsilon} \le (1 + F_0(x, \cdot))^{1+\min\{\delta_0/2, \delta_3\}} \le C(1 + F(x, \cdot))^{1+\delta_0}. \tag{3.14}$$

By Proposition 2.4 and (3.14), we deduce from (3.12) that

$$\begin{aligned}
\mathcal{F}_0(u) - \mathcal{F}(u, B_R(y)) &\le CR^{n+\sigma} + C(\varepsilon)R^\sigma \int_{B_R(y)} \left(1 + F(x, Du)\right)^{1+\delta_0} dx \\
&\le CR^{n+\sigma} + CR^\sigma \int_{B_R(y)} F(x, Du)^{1+\delta_0} dx \\
&\le CR^{n+\sigma} + CR^{\sigma-n\varepsilon} \left(\int_{B_{2R}(y)} F(x, Du) dx\right)^{1+\delta_0} \\
&\le CR^{n+\sigma} + CR^{\sigma-n\varepsilon} \int_{B_{2R}(y)} F(x, Du) dx,
\end{aligned} \tag{3.15}$$

where we used the fact that

$$\int_{B_{2R}(y)} F(x, Du)dx \le \int_K F(x, Du)dx \le M_0$$

for some constant M_0. The existence of M_0 guaranteed by the local minimality of u.

For (3.13) we use Proposition 2.6, Proposition 2.4 and (3.14), to get

$$\begin{aligned}\mathcal{F}(v, B_R(y)) - \mathcal{F}_0(v) &\le CR^{n+\sigma} + C(\varepsilon)R^{\sigma} \int_{B_R(y)} F_0(x, Du)^{1+\varepsilon} dx \\ &\le CR^{n+\sigma} + CR^{\sigma-n\varepsilon} \int_{B_{2R}(y)} F(x, Du)dx. \end{aligned} \tag{3.16}$$

On the other hand, by the definition of F_0, we have

$$F(x, Du) \le C\left(1 + F_0(x, Du)\right).$$

So we have, combining (3.10), (3.11), (3.15) and (3.16), that

$$\begin{aligned}&\int_{B_R(y)} \left(|Du|^{p_2-2} + |Dv|^{p_2-2}\right)|Du - Dv|^2 dx + \int_{B_R(y)} \tilde{a}(x)\left(|Du|^{q_2-2} + |Dv|^{q_2-2}\right)|Du - Dv|^2 dx \\ &\le \mathcal{F}_0(u) - \mathcal{F}_0(v) \\ &\le CR^{n+\sigma} + CR^{\sigma-n\varepsilon} \int_{B_{2R}(y)} (1 + F_0(x, Du))dx. \end{aligned} \tag{3.17}$$

By virtue of (3.2) and (3.9), we can see that

$$\begin{aligned}\int_{B_\rho(y)} (1 + F_0(x, Du))dx &= \int_{B_\rho(y)} (1 + F_0(x, Dv))dx + \int_{B_\rho(y)} \left(F_0(x, Du) - F_0(x, Dv)\right) dx \\ &\le C\left(\frac{\rho}{R}\right)^{n-y} \int_{B(y)} (1 + F_0(x, Dv))dx \\ &\quad + \int_{B_\rho(y)} \left[|V_{p_2}(Du)|^2 + \tilde{a}(x)|V_{q_2}(Du)|^2 - \left(|V_{p_2}(Dv)|^2 + \tilde{a}(x)|V_{q_2}(Dv)|^2\right)\right] dx \\ &\le C\left(\frac{\rho}{R}\right)^{n-y} \int_{B(y)} (1 + F_0(x, Dv))dx \\ &\quad + \int_{B_R(y)} \left[\left(|V_{p_2}(Du)|^2 - |V_{p_2}(Dv)|^2\right) + \tilde{a}(x)\left(|V_{q_2}(Du)|^2 - |V_{q_2}(Dv)|^2\right)\right] dx \\ &\le C\left(\frac{\rho}{R}\right)^{n-y} \int_{B(y)} (1 + F_0(x, Dv))dx \\ &\quad + \int_{B_R(y)} |V_{p_2}(Du) - V_{p_2}(Dv)|^2 dx + \int_{B_R(y)} \tilde{a}(x)|V_{q_2}(Du) - V_{q_2}(Dv)|^2 dx \\ &\le C\left(\frac{\rho}{R}\right)^{n-y} \int_{B(y)} (1 + F_0(x, Dv))dx \\ &\quad + \int_{B_R(y)} \left(|Du|^{p_2-2} + |Dv|^{p_2-2}\right)|Du - Dv|^2 dx \\ &\quad\quad + \int_{B_R(y)} \tilde{a}(x)\left(|Du|^{q_2-2} + |Dv|^{q_2-2}\right)|Du - Dv|^2 dx \end{aligned}$$

$$\leq C\left(\frac{\rho}{R}\right)^{n-y}\int_{B_R(y)}(1+F_0(x,Dv))dx$$
$$+CR^{n+\sigma}+CR^{\sigma-n\varepsilon}\int_{B_{2R}(y)}(1+F_0(x,Du))dx$$
$$\leq C\left[\left(\frac{\rho}{R}\right)^{n-y}+R^{\sigma-n\varepsilon}\right]\int_{B_{2R}(y)}(1+F_0(x,Du))dx+CR^{n+\sigma}. \tag{3.18}$$

Using well-known lemma (see for example [60, Lemma 5.13]), for sufficiently small $R>0$, we can see that for any $y'\in(y,1)$ there exists a constant C depending given data and ζ such that

$$\int_{B_\rho(y)}F_0(x,Du)dx\leq C\left(\frac{\rho}{R}\right)^{n-y'}\int_{B_{2R}(y)}F_0(x,Du)dx+C\rho^{n-y'} \tag{3.19}$$

hold for any $\rho\in(0,R)$. Now, since (3.9) holds for any $y\in(0,1)$, we can choose $y'\in(0,1)$ arbitrarily in (3.19). On the other hand, since we are supposing that $p(x)\geq p_0>1$, for any $\zeta\in(0,1)$, choosing $y'\in(0,1)$ so that $y'\leq p_0(1-\zeta)$, we see that there exists a positive constant C dependent on the given data, $K\Subset\Omega$ and $\mathcal{F}(u,K)$ such that

$$\int_{B_\rho(y)}|Du|^{p_0}dx\leq C\rho^{n-p_0(1-\zeta)}$$

holds for any $B_\rho(y)$ with $4\rho\leq\operatorname{dist}(K,\partial\Omega)$. So, we conclude that $u\in C^{0,\zeta}_{\mathrm{loc}}(\Omega)$ for any $\zeta\in(0,1)$ by virtue of Morrey's theorem.

Part 2. Now, we are going to show the Hölder continuity of the gradient Du. For $y\in\mathring{K}$ let $R_1\in(0,R_0)$ be a constant such that $B_{R_1}(y)\subset K$, and for $0<R<R_1/4$ let v be as in *Part 1*. Then, by the estimate given by Colombo-Mingione at [1, p.484, l.-6], we see that there exist constants $C>0$, dependent on $n,p_2,q_2,\lambda,\Lambda,\|\tilde{\alpha}\|_\infty$, $\operatorname{dist}(K,\partial\Omega)$, $\mathcal{F}_0(v,B_R(y))$ and $\tilde{\alpha}\in(0,1)$

$$⨍_{B_\rho(y)}|Dv-(Dv)_\rho|^{p_2}dx\leq C\rho^{\frac{\tilde{\alpha}\beta}{64n}}, \tag{3.20}$$

holds for any $\rho\leq R/2$. Here, as in **Part 1**, let us mention that $\mathcal{F}_0(v,B_R(y))$ can be controlled by $\mathcal{F}(u,K)$ as (3.8). So, we can choose the above constant in (3.20) to be dependent only on the given data of the functional, the local minimizer u under consideration and K.

In what follows, let us abbreviate

$$\bar{\alpha}:=\frac{\tilde{\alpha}\beta}{64n}.$$

By virtue of (3.20), for ρ and R as above, we get

$$\int_{B_\rho(y)}|Du-(Du)_\rho|^{p_2}dx\leq C\int_{B_\rho(y)}|Du-(Dv)_\rho|^{p_2}dx\leq C\left(\int_{B_\rho(y)}\left|Dv-(Dv)_\rho\right|^{p_2}dx+C\int_{B_\rho(y)}|Du-Dv|^{p_2}dx\right)$$
$$\leq C\rho^{n+\bar{\alpha}}+C\int_{B_R(y)}|Du-Dv|^{p_2}dx. \tag{3.21}$$

For the case that $p_2\geq 2$, since there exists a constant such that

$$|z_1-z_2|^{p_2}\leq C\left(|z_1|^{p_2-2}+|z_2|^{p_2-2}\right)|z_1-z_2|^2$$

for any $z_1,z_2\in\mathbb{R}^n$, using (3.17), we can estimate the last term of the right hand side of (3.21) as

$$\int_{B_R(y)}|Du-Dv|^{p_2}dx\leq CR^{n+\sigma}+CR^{\sigma-n\varepsilon}\int_{B_{2R}(y)}F_0(x,Du)dx. \tag{3.22}$$

We use (3.19) replacing ρ by $2R$ and R by R_0 to see that

$$\int_{B_{2R}(y)} F_0(x, Du)dx \leq CR^{n-\zeta} R_0^{\zeta} ⨍_{B_{R_0}} F_0(x, Du)dx + CR^{n-\zeta}.$$

Since R_0 is determined in the beginning of the proof, we can regard $R_0^{\zeta} ⨍_{B_{R_0}} F_0(x, Du)dx$ as a constant. So, we get

$$\int_{B_{2R}(y)} F_0(x, Du)dx \leq CR^{n-\zeta}. \tag{3.23}$$

By (3.22) and (3.23), we obtain

$$\int_{B_R(y)} |Du - Dv|^{p_2} dx \leq CR^{n+\sigma} + CR^{n-\zeta+\sigma-n\varepsilon} \leq CR^{n-\zeta+\sigma-n\varepsilon}. \tag{3.24}$$

When $1 < p_2 < 2$, using Hölder's inequality, (3.2) and (3.17), we can see that

$$\begin{aligned}
\int_{B_R(y)} |Du - Dv|^{p_2} dx &\leq C \int_{B_R(y)} \left|V_{p_2}(Du) - V_{p_2}(Dv)\right|^{p_2} (|Du| + |Dv|)^{\frac{p_2(2-p_2)}{2}} dx \\
&\leq C \left(\int_{B_R(y)} \left|V_{p_2}(Du) - V_{p_2}(Dv)\right|^2 dx \right)^{\frac{p_2}{2}} \left(\int_{B_R(y)} (|Du| + |Dv|)^{\frac{p_2}{2}} dx \right)^{\frac{2-p_2}{2}} \\
&\leq \left(\int_{B_R(y)} (|Du| + |Dv|)^{p_2-2} |Du - Dv|^2 dx \right)^{p_2} \left(\int_{B_R(y)} F_0(x, Du)dx \right)^{\frac{2-p_2}{2}} \\
&\leq \left(CR^{n+\sigma} + CR^{\sigma-n\varepsilon} \int_{B_{2R}(y)} F_0(x, Du)dx \right)^{\frac{p_2}{2}} \left(\int_{B_{2R}(y)} F_0(x, Du)dx \right)^{\frac{2-p_2}{2}} \\
&\leq CR^{\frac{(n+\sigma)p_2}{2}} \left(\int_{B_{2R}(y)} F_0(x, Du)dx \right)^{\frac{2-p_2}{2}} + CR^{\frac{(\sigma-n\varepsilon)p_2}{2}} \int_{B_{2R}(y)} F_0(x, Du)dx.
\end{aligned} \tag{3.25}$$

By (3.25) and (3.23), we obtain

$$\begin{aligned}
\int_{B_R(y)} |Du - Dv|^{p_2} dx &\leq CR^{\frac{p_2(n+\sigma)}{2}} R^{\frac{(2-p_2)(n-\zeta)}{2}} + CR^{\frac{(\sigma-n\varepsilon)p_2}{2}} R^{n-\zeta} \\
&= CR^{n-\zeta+\frac{p_2(\sigma+\zeta)}{2}} + CR^{n-\zeta+\frac{p_2(\sigma-n\varepsilon)}{2}} \\
&\leq 2CR^{n-\zeta+\frac{p_2(\sigma-n\varepsilon)}{2}} \leq 2CR^{n-\zeta+\frac{(\sigma-n\varepsilon)}{2}}.
\end{aligned} \tag{3.26}$$

For the last inequality we used the following facts:

$$0 < R \leq 1, \quad 0 < \sigma - n\varepsilon, \quad p_2 > 1.$$

Mentioning the above facts again and comparing (3.24) and (3.26), we see that, for $p_2 > 2$, the estimate (3.26) holds. Now, combining (3.21) and (3.26), we obtain

$$\int_{B_\rho(y)} |Du - (Du)_\rho|^{p_2} dx \leq C \left(\rho^{n+\bar{\alpha}} + R^{n-\zeta+\frac{\sigma-n\varepsilon}{2}} \right).$$

This holds for any $0 < \rho < R/2 \leq R_0/8$. For $k > 1$, let us put $\rho = R^k/2$ (bearing in mind that $R^k/2 \leq R/2$ holds for $k > 1$), then

$$\rho^{n+\bar{\alpha}} + R^{n-\zeta+\frac{\sigma-n\varepsilon}{2}} = \rho^{n+\bar{\alpha}} + (2\rho)^{\frac{2n-2\zeta+\sigma-n\varepsilon}{2k}}.$$

So, we have

$$\int_{B_\rho(y)} |Du - (Du)_\rho|^{p_2}\, dx \leq \rho^{n+\bar{\alpha}} + (2\rho)^{\frac{2n-2\zeta+\sigma-n\varepsilon}{2k}}. \tag{3.27}$$

Since

$$\bar{\alpha} = \frac{\tilde{\alpha}}{64n}\beta = \frac{\tilde{\alpha}}{64n}\min\{\alpha, \sigma\} \leq \frac{\sigma}{64},$$

we can take ε sufficiently small so that $\bar{\alpha} < (\sigma - n\varepsilon)/2$ then, for sufficiently small ζ,

$$n - \zeta + \frac{\sigma - n\varepsilon}{2} > n + \bar{\alpha}$$

holds. Now, for such a choice of ε and ζ, putting

$$k = \frac{2n - 2\zeta + \sigma - n\varepsilon}{2(n + \bar{\alpha})} \ (> 1)$$

in (3.27), we get

$$\int_{B_\rho(y)} |Du - (Du)_\rho|^{p_2}\, dx \leq C\rho^{n+\bar{\alpha}},$$

and therefore we obtain the Hölder continuity of Du by virtue of the Campanato's theorem. □

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