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Hypersurfaces with vanishing hessian via Dual Cayley Trick

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ABSTRACT

We present a general construction of hypersurfaces with vanishing hessian, starting from any irreducible non-degenerate variety whose dual variety is a hypersurface and based on the so called Dual Cayley Trick. The geometrical properties of these hypersurfaces are different from the series known until now. In particular, their dual varieties can have arbitrary codimension in the image of the associated polar map.

0. Introduction

If $X = V(f) \subset \mathbb{P}^N$ is a hypersurface, the hessian determinant of f (from now on simply called the hessian of f or, by abusing language, the hessian of X) is the determinant of the hessian matrix of f.

Hypersurfaces with vanishing hessian were studied systematically for the first time in the fundamental paper [15], where P. Gordan and M. Noether analysed O. Hesse's claims in [17,18] according to which these hypersurfaces should be necessarily cones. Clearly the claim is true if $\deg(f) = 2$ so that the first relevant case for the problem is that of cubic hypersurfaces. The cubic hypersurface $V(x_0x_3^2 + x_1x_3x_4 + x_2x_4^2) \subset \mathbb{P}^4$ has vanishing hessian but it is not a cone, see [22]. By adding the term $\sum_{i=5}^{N} x_i^3$ we get examples of irreducible cubic hypersurfaces in $\mathbb{P}^N$ with $N \geq 4$ with vanishing hessian which are not cones.

Notwithstanding, the question is quite subtle because, as it was firstly pointed out in [15], Hesse's claim is true for $N \leq 3$, see also [20,11] and [25, Section 7]. Moreover, hypersurfaces with vanishing hessian and the Gordan-Noether Theory developed in [15] have a wide range of applications in different areas of mathematics

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such as Algebraic and Differential Geometry (see [1,2,10]), Commutative Algebra and the theory of EDP (see [7,25,12]), Approximation Theory and Theoretical Physics (see [10,1]) and Combinatorics (see [14]).

Hypersurfaces with vanishing hessian have been forgotten by algebraic geometers for a long time and recently they were rediscovered in other contexts. For example the cubic hypersurface in $\mathbb{P}^4$ recalled above is celebrated nowadays in the modern differential geometry literature as the *Bourgain-Sacksteder Hypersurface* (see [2,1,10]).

Many classes of hypersurfaces with vanishing hessian, which are not cones, are ruled by a family of linear spaces along which the hypersurface is not developable. In particular, this ruling is different from the one given by the fibers of the Gauss map. These examples and their generalizations are known in differential geometry as *twisted planes*, see for example [10]. Despite the huge number of papers dedicated to this subject by differential geometers very few classification or structure results have been obtained. Moreover, the global point of view provided by polarity, used systematically in this paper, has been completely overlooked in other areas.

The known series of examples of hypersurfaces $X \subset \mathbb{P}^N$ with vanishing hessian, which are not cones, have been constructed by Gordan and Noether, Perazzo, Franchetta, Permutti, see [15,22,9,23,24], and later have been revisited and generalised in [3, Section 2], see also [5,6].

All these examples share several geometrical behaviours, see in particular Subsection 2.3. For instance there exists a linear subspace $L \subset \operatorname{Sing} X$, dubbed the *core of* X in [3], such that, letting $L_\alpha = \mathbb{P}^{k+1} \supset L$ and letting $L_\alpha \cap X = \mu_\alpha L \cup X_\alpha$ for some $\mu_\alpha \in \mathbb{N}$, the variety X_α is a cone with vertex $V_\alpha = \operatorname{Vert}(X_\alpha) \subset L$ tangent to $Z_X^* \subset L$, where $Z_X \subsetneq (\mathbb{P}^N)^*$ is the closure of the image of the polar map of X. When the cones X_α split into a union of linear spaces, the hypersurface is a twisted plane. Furthermore, the dual variety $X^* \subset (\mathbb{P}^N)^*$ of most of the examples in these series tends to be a divisor in Z_X. From the perspective of Segre's Formula, recalled in Section 1, this means that the rank of the hessian matrix of a homogeneous polynomial with vanishing hessian determinant should be equal to the rank of the hessian matrix modulo the ideal generated by f, see Section 1 for precise definitions. This seemed to be the most natural and general behaviour, at least at a first glance.

On the other hand, if $Y \subset (\mathbb{P}^N)^*$ is an arbitrary non-degenerate irreducible variety of dimension $n \geq 1$, then, after identifying $\mathbb{P}^N$ with $(\mathbb{P}^N)^{**}$, the dual variety $X = Y^* \subset \mathbb{P}^N$ is not a cone and, in general, one expects that X is a hypersurface with non vanishing hessian. If this is the case, $Z_X = (\mathbb{P}^N)^*$ and $\operatorname{codim}(X^*, Z_X) = \operatorname{codim}(Y, (\mathbb{P}^N)^*) = N - n$ is arbitrary large.

These remarks motivate the search of hypersurfaces with vanishing hessian $X \subset \mathbb{P}^N$ such that X^* has arbitrary codimension in $Z_X \subsetneq (\mathbb{P}^N)^*$. Here we shall present a general construction of such hypersurfaces, starting from any irreducible non-degenerate variety whose dual variety is a hypersurface and based on the Dual Cayley Trick. The geometrical properties of these hypersurfaces are different from those described above (for example V_α is not contained in L) and their dual varieties can have arbitrary codimension in the image of the polar map. The ubiquity of the examples suggests that the classification of hypersurfaces with vanishing hessian for $N \geq 5$ might be very intricate, perhaps requiring a completely different approach not based (only) on Gordan–Noether Theory, which worked for $N \leq 4$, see [15,9,11,25]. Other interesting series of examples of hypersurfaces with vanishing hessian such that $\operatorname{codim}(X^*, Z_X)$ is large has been recently constructed in [5,6] (see also Remark 4.3 for a possible geometrical description of this series of examples).

The paper is organised as follows. In Section 1 we fix the notation and introduce the main definitions. Section 2 is devoted to the construction of the first series of examples, leading to Theorem 2.1 and ending with the description of the geometrical properties of the examples. In Section 3 we briefly recall the definitions of resultant and discriminant and we apply them to calculate explicitly the dual of rational normal scroll surfaces in Theorem 3.1. Then we introduce the Cayley Trick and the Dual Cayley Trick and apply the Dual Cayley Trick to a generalisation of the series of examples constructed in Section 2 (see Theorem 3.5), also providing a new conceptual proof of the part (ii) of Theorem 2.1. In Section 4 we prove that the duals of internal projections from a point of Scorza Varieties have vanishing hessian and we describe the geometrical properties of these hypersuperfaces and of their polar maps.

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1. Preliminaries and definitions

Let $f(x_0, \ldots, x_N) \in \mathbb{K}[x_0, \ldots, x_N]_d$ be a homogeneous polynomial of degree $d \geq 1$ without multiple irreducible factors and let $X = V(f) \subset \mathbb{P}^N$ be the associated projective hypersurface. We shall always assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero. Let

$$\mathrm{H}(f) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{0 \leq i,j \leq N}$$

be the *hessian matrix of* f (or of X).

Clearly $\mathrm{H}(f) = \mathbf{0}_{(N+1)\times(N+1)}$ if and only if $d = 1$. Thus, from now on, we shall suppose $d \geq 2$. Let

$$\mathrm{hess}_f = \det(\mathrm{H}(f))$$

be the *hessian (determinant) of* f (or of X, in which case it will be denoted by hess_X, which is defined modulo multiplication by a non zero element in $\mathbb{K}$).

There are two possibilities:

(1) either $\mathrm{hess}_f = 0$ or
(2) $\mathrm{hess}_f \in \mathbb{K}[x_0, \ldots, x_N]_{(N+1)(d-2)}$.

We shall be interested in case (1), that is in *hypersurfaces with vanishing hessian* (*determinant*).

1.1. The polar map

Let

$$\nabla_f = \nabla_X : \mathbb{P}^N \dashrightarrow (\mathbb{P}^N)^*$$

be *the polar (or gradient) map of* $X = V(f) \subset \mathbb{P}^N$ which associates to $p \in \mathbb{P}^N$ the *polar hyperplane to* X *with respect to* p. In coordinates it is defined by

$$\nabla_f(p) = \Big(\frac{\partial f}{\partial x_0}(p) : \cdots : \frac{\partial f}{\partial x_N}(p)\Big).$$

The base locus scheme of ∇_f is the scheme

$$\mathrm{Sing}(X) := V\Big(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_N}\Big) \subset \mathbb{P}^N.$$

Let

$$Z_X := \overline{\nabla_f(\mathbb{P}^N)} \subseteq (\mathbb{P}^N)^*$$

be *the polar image of* $\mathbb{P}^N$.

We can consider the rational map ∇_f as the quotient by the natural $\mathbb{K}^*$-action of the affine morphism

$$\nabla_f : \mathbb{K}^{N+1} \to \mathbb{K}^{N+1}$$

defined in the same way. Thus we have the following key formula:

$$\mathrm{H}(f) = \mathrm{Jac}(\nabla_f : \mathbb{K}^{N+1} \to \mathbb{K}^{N+1}), \tag{1}$$

that is, the hessian matrix of f is the Jacobian matrix of the affine morphism ∇_f. Hence, $\mathrm{hess}_f = 0$ if and only if $Z_X \subsetneq \mathbb{P}^N$ so that $\mathrm{hess}_f = 0$ if and only if $\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}$ are algebraically dependent.

The restriction of ∇_f to X is the Gauss map of X:

$$\begin{array}{rccc} \mathcal{G}_X = \nabla_{f|X} : & X & \dashrightarrow & (\mathbb{P}^N)^* \\ & X_{\mathrm{reg}} \ni p & \longmapsto & \mathcal{G}_X(p) = [T_pX] \end{array}$$

which to a non-singular point $p \in X_{\mathrm{reg}}$ associates the point $[T_pX] \in (\mathbb{P}^N)^*$ representing the projective tangent hyperplane T_pX to X at the smooth point p. Then, by definition,

$$X^* := \overline{\mathcal{G}_X(X)} \subseteq Z_X$$

is the *dual variety of* X.

If A is a matrix with entries in $\mathbb{K}[x_0, \ldots, x_N]$ and if $f \in \mathbb{K}[x_0, \ldots, x_N]$, then $\mathrm{rk}_{(f)} A$ denotes the *rank of* A *modulo* (f), that is the maximal order of a minor not belonging to the ideal generated by f. With this notation, obviously $\mathrm{rk}\, A = \mathrm{rk}_{(0)} A$.

Lemma 1.1. ([26]) *Let $X = V(f) = X_1 \cup \cdots \cup X_r \subset \mathbb{P}^N$ be a reduced hypersurface with $X_i = V(f_i)$, $f = f_1 \cdots f_r$ and f_i irreducible. Then:*

(i) *If $p_i \in X_i$ is general, then*

$$\mathrm{rk}(\mathrm{d}\,\mathcal{G}_X)_{p_i} = \mathrm{rk}_{(f_i)} H(f) - 2. \tag{2}$$

In particular,

$$\dim(X_i^*) = \mathrm{rk}(\mathrm{d}\,\mathcal{G}_X)_{p_i} = \mathrm{rk}_{(f_i)} H(f) - 2 \leq \mathrm{rk}\, H(f) - 2 = \dim(Z_X) - 1. \tag{3}$$

(ii) *If X is irreducible, then $f^{N-\dim(X^*)-1}$ divides* hess_f.

We point out an immediate consequence for further reference.

Corollary 1.2. *Let $X = V(f) \subset \mathbb{P}^N$ be a reduced hypersurface with vanishing hessian. Then*

$$X^* \subsetneq Z_X \subsetneq (\mathbb{P}^N)^*.$$

We shall also need the following remark.

Lemma 1.3. ([3, Lemma 3.10]) *Let $X = V(f) \subset \mathbb{P}^N$ be a hypersurface. Let $H = \mathbb{P}^{N-1}$ be a hyperplane not contained in X, let $h = H^*$ be the corresponding point in $(\mathbb{P}^N)^*$ and let π_h denote the projection from the point h. Then we have a commutative diagram:*

$$\begin{array}{ccc} H & \overset{\nabla_{X\cap H}}{\dashrightarrow} & H^* \\ {\scriptstyle (\nabla_X)|_H} \downarrow & \nearrow {\scriptstyle \pi_h} & \\ (\mathbb{P}^N)^* & & \end{array}$$

In particular, $\overline{\nabla_{X\cap H}(H)} \subseteq \pi_h(Z_X)$.

2. Hypersurfaces with vanishing hessian constructed from any non-degenerate variety

2.1. Hypersurfaces with vanishing hessian constructed from duals of arbitrary non-degenerate subvarieties

Let us consider $\mathbb{P}^{2n+1}$ with homogeneous coordinates

$$(u : v : x_1 : \cdots : x_n : y_1 : \cdots : y_n),$$

$\mathbb{P}^1$ with homogeneous coordinates $(s : t)$ and $\mathbb{P}^{n-1}_{\mathbf{z}}$ with homogeneous coordinates $(z_1 : \cdots : z_n)$, where $\mathbf{z} = (z_1, \ldots, z_n)$. Let $\mathbf{x} = (x_1, \ldots, x_n)$, let $\mathbf{y} = (y_1, \ldots, y_n)$, let

$$\phi_1 : \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^1$$

be the rational map defined by

$$\phi_1(u : v : \mathbf{x} : \mathbf{y}) = (u : v)$$

and let $\phi_2 : \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^{n-1}_{\mathbf{z}}$ be the rational map defined by

$$\phi_2(u : v : \mathbf{x} : \mathbf{y}) = (ux_1 - vy_1 : \cdots : ux_n - vy_n). \tag{4}$$

Let $g(z_1, \ldots, z_n) \in \mathbb{K}[z_1, \ldots, z_n]_d$ be a reduced irreducible polynomial such that the associated irreducible hypersurface of degree d

$$Y^* = V(g) \subset \mathbb{P}^{n-1}_{\mathbf{z}}$$

is not a cone. This is equivalent to

$$Y = Y^{**} = \overline{\nabla_g(Y^*)} \subset (\mathbb{P}^{n-1}_{\mathbf{z}})^*$$

being non-degenerate.

Let

$$f(u, v, \mathbf{x}, \mathbf{y}) = g(ux_1 - vy_1, \ldots, ux_n - vy_n) \in \mathbb{K}[u, v, \mathbf{x}, \mathbf{y}]_{2d}$$

and let $X = V(f) \subset \mathbb{P}^{2n+1}$. Clearly,

$$V(f) = \overline{\phi_2^{-1}(V(g))}. \tag{5}$$

The partial derivatives of f are linearly independent over $\mathbb{K}$, due to the hypothesis on g, so that $X = V(f) \subset \mathbb{P}^{2n+1}$ is not a cone. One verifies that

$$\mathrm{Sing}(X) = V(u, v) \cup (\mathbb{P}^1 \times \mathbb{P}^n) \cup \phi_2^{-1}(\mathrm{Sing}(V(g))),$$

where $\mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ is the Segre variety defined by the equations

$$\operatorname{rk}\begin{pmatrix} v & x_1 & \dots & x_n \\ u & y_1 & \dots & y_n \end{pmatrix} = 1.$$

From

$$\frac{\partial f}{\partial x_i} = u\frac{\partial g}{\partial z_i}(u\mathbf{x} - v\mathbf{y}) \tag{6}$$

and

$$\frac{\partial f}{\partial y_j} = -v\frac{\partial g}{\partial z_j}(u\mathbf{x} - v\mathbf{y}), \tag{7}$$

we deduce that, for every $i \neq j$,

$$\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial y_j} - \frac{\partial f}{\partial x_j}\frac{\partial f}{\partial y_i} = 0. \tag{8}$$

Thus $X \subset \mathbb{P}^{2n+1}$ has vanishing hessian since the partial derivates of f are algebraically dependent.

2.2. Polar image and dual variety of $X = V(g(u\mathbf{x} - v\mathbf{y})) \subset \mathbb{P}^{2n+1}$

We need to introduce some more notation. Let

$$(u' : v' : x'_1 : \dots : x'_n : y'_1 : \dots : y'_n)$$

be homogenous coordinates on $(\mathbb{P}^{2n+1})^*$, dual to the coordinates chosen on $\mathbb{P}^{2n+1}$. Let

$$L' = V(x'_1, \dots, x'_n, y'_1, \dots, y'_n) = \mathbb{P}^1_{u',v'} \subset (\mathbb{P}^{2n+1})^*$$

and let

$$W = \mathbb{P}^1 \times \mathbb{P}^{n-1} \subset V(u', v') = \mathbb{P}^{2n-1}_{\mathbf{x}',\mathbf{y}'} \subset (\mathbb{P}^{2n+1})^*$$

be the Segre variety defined by the equations:

$$\operatorname{rk}\begin{pmatrix} x'_1 & \dots & x'_n \\ y'_1 & \dots & y'_n \end{pmatrix} = 1. \tag{9}$$

Let $S(L', W) \subset (\mathbb{P}^{2n+1})^*$ be the cone with vertex L' over the Segre variety $W = \mathbb{P}^1 \times \mathbb{P}^{n-1}$. Thus $\dim(S(L', W)) = n + 2$ and $S(L', W) \subset (\mathbb{P}^{2n+1})^*$ is defined by the equations (9) of W.

Let $\mathbb{P}^{n-1}_{\mathbf{z}'}$ with homogeneous coordinates $(z'_1 : \dots : z'_n)$ be the dual of $\mathbb{P}^{n-1}_{\mathbf{z}}$ and consider $\mathbb{P}^n_{\mathbf{z}'}$ with homogeneous coordinates $(z'_0 : z'_1 : \dots : z'_n)$. With this notation $\mathbb{P}^{n-1}_{\mathbf{z}'} \subset \mathbb{P}^n_{\mathbf{z}'}$ is the hyperplane of equation $z'_0 = 0$. Given $Y \subset \mathbb{P}^{n-1}_{\mathbf{z}'}$, let $\widetilde{Y} \subset \mathbb{P}^n_{\mathbf{z}'}$ be the cone over Y with vertex $(1 : 0 : \dots : 0)$.

Theorem 2.1. *Let the hypothesis and the notation be as above, let*

$$Z_{Y^*} = \overline{\nabla_g(\mathbb{P}^{n-1}_{\mathbf{z}})} \subseteq \mathbb{P}^{n-1}_{\mathbf{z}'},$$

let

$$\mathbb{P}^1 \times Z_{Y^*} \subset (\mathbb{P}^{2n-1})^*$$

be the Segre embedding and let $X = V(g(u\mathbf{x} - v\mathbf{y})) \subset \mathbb{P}^{2n+1}$. *Then:*

(i) $Z_X = \overline{\nabla_f(\mathbb{P}^{2n+1})} = S(L', \mathbb{P}^1 \times Z_{Y^*}) \subset (\mathbb{P}^{2n+1})^*$;
(ii) $X^* = \mathbb{P}^1 \times \widetilde{Y} \subset (\mathbb{P}^{2n+1})^*$ *Segre embedded;*
(iii) $\operatorname{codim}(X^*, Z_X) = \operatorname{codim}(Y, Z_{Y^*}) + 1$.

In particular, if $g(\mathbf{z})$ *has non-vanishing hessian determinant, then* $Z_{Y^*} = \mathbb{P}^{n-1}_{\mathbf{z}'}$ *and* $\operatorname{codim}(X^*, Z_X) = \operatorname{codim}(Y, \mathbb{P}^{n-1}) + 1$.

Proof. By definition $\nabla_f : \mathbb{P}^{2n+1} \dashrightarrow (\mathbb{P}^{2n+1})^*$ is given by

$$\left(\frac{\partial f}{\partial u} : \frac{\partial f}{\partial v} : \frac{\partial f}{\partial x_1} : \cdots : \frac{\partial f}{\partial x_n} : \frac{\partial f}{\partial y_1} : \cdots : \frac{\partial f}{\partial y_n}\right).$$

From (8) and from (9) we deduce that

$$\overline{\nabla_f(\mathbb{P}^{2n+1})} \subseteq S(L', \mathbb{P}^1 \times Z_{Y^*}). \tag{10}$$

We also have

$$\frac{\partial f}{\partial u} = \sum_{i=1}^{n} x_i \frac{\partial g}{\partial z_i}(u\mathbf{x} - v\mathbf{y}), \tag{11}$$

$$\frac{\partial f}{\partial v} = -\sum_{j=1}^{n} y_j \frac{\partial g}{\partial z_j}(u\mathbf{x} - v\mathbf{y}). \tag{12}$$

Let

$$p = (\tilde{u}' : \tilde{v}' : \tilde{\mathbf{z}}' : \lambda\tilde{\mathbf{z}}') \in S(L', \mathbb{P}^1 \times Z_{Y^*})$$

be a general point. In particular, we can suppose $\tilde{u}' \neq \lambda\tilde{v}'$ and that $\tilde{\mathbf{z}}' \neq \mathbf{0}$. Then $[\tilde{\mathbf{z}}'] \in Z_{Y^*}$ is general and by definition there exists $\tilde{\mathbf{z}} \in \mathbb{P}^{n-1}_{\mathbf{z}}$ such that $\nabla_g(\tilde{\mathbf{z}}) = \tilde{\mathbf{z}}'$. Looking at (6) and (7), we impose $v = -\lambda u$. If $u(\mathbf{x} + \lambda\mathbf{y}) = \tilde{\mathbf{z}}$ and if $v = -\lambda u$ hold, then $\frac{\partial f}{\partial x_i} = u\tilde{\mathbf{z}}'_i$ and $\frac{\partial f}{\partial y_i} = u\lambda\tilde{\mathbf{z}}'_i$. Hence, to find q such that $\nabla_f(q) = p$, it is sufficient to determine a solution of the system of equations

$$\begin{cases} u(\mathbf{x} + \lambda\mathbf{y}) = & \tilde{\mathbf{z}} \\ \sum_{i=1}^{n} x_i \frac{\partial g}{\partial z_i}(\tilde{\mathbf{z}}) = & u\tilde{u}' \\ -\sum_{i=1}^{n} y_i \frac{\partial g}{\partial z_i}(\tilde{\mathbf{z}}) = & u\tilde{v}' \end{cases} \tag{13}$$

From

$$u\tilde{u}' = \sum_{i=1}^{n} \left(\frac{\tilde{z}_i}{u} - \lambda y_i\right) \frac{\partial g}{\partial z_i}(\tilde{\mathbf{z}}) = \frac{1}{u} d \cdot g(\tilde{\mathbf{z}}) + \lambda u\tilde{v}',$$

we deduce

$$u^2 = \frac{d \cdot g(\tilde{\mathbf{z}})}{\tilde{u}' - \lambda\tilde{v}'}.$$

If $\tilde{u}$ is a solution of this last equation and if $\tilde{\mathbf{a}} = (\tilde{a}_1, \ldots, \tilde{a}_n)$ is a solution of the last linear equation in (13), then

$$p = \nabla_f(\tilde{u} : -\lambda\tilde{u} : \frac{\tilde{\mathbf{z}}}{\tilde{u}} - \lambda\tilde{\mathbf{a}} : \tilde{\mathbf{a}}),$$

yielding equality in (10).

By restricting ∇_f to X, we deduce that

$$X^* = \overline{\nabla_f(X)} \subseteq S(L', \mathbb{P}^1 \times Z_{Y^*}).$$

Let $T = \mathbb{P}^1 \times \mathbb{P}^n \subset (\mathbb{P}^{2n+1})^*$ be the Segre variety defined by the equations

$$\operatorname{rk}\begin{pmatrix} v' & x_1' & \ldots & x_{n'}' \\ u' & y_1' & \ldots & y_n' \end{pmatrix} = 1.$$

Since $\deg(f) = 2d$, Euler's Formula gives

$$(2d)f = u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} + \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n} y_i \frac{\partial f}{\partial y_i} = 2(u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}). \tag{14}$$

Thus, for every $p \in X$, we have

$$u(p)\frac{\partial f}{\partial u}(p) + v(p)\frac{\partial f}{\partial v}(p) = 0, \tag{15}$$

yielding

$$\overline{\nabla_f(X)} = X^* \subseteq T \subset (\mathbb{P}^{2n+1})^*.$$

Indeed, for $i \neq j$ the equations $x_i'y_j' - x_jy_i' = 0$ are satisfied by any point $\nabla_f(p)$ due to (8). Due to (15), the equations $v'y_i' - u'x_i' = 0$ are satisfied by $\nabla_f(p)$ for every $p \in X$. Furthermore, for every $(\mu : \nu) \in \mathbb{P}^1_{\mathbb{K}}$, the hypersurface $X \cap V(\mu u + \nu v)$ is singular so that $(\mu : \nu : \mathbf{0} : \mathbf{0}) \in X^* \cap ((\mu : \nu) \times \mathbb{P}^n)$. By fixing u, v, by restricting to $X \cap V(u\mathbf{x} - v\mathbf{y})$ and by taking into account (6), (7) and (15) one deduces that $X^* = \mathbb{P}^1 \times \widetilde{Y} \subset T$ (see also the next sections for more details). The other conclusions are now clear. □

Remark 2.2. To prove equality in (10) one could have argued also in this way. Letting $\rho = \operatorname{rk}(\mathrm{H}(g)) = \dim(Z_{Y^*})+1$, it is sufficient to prove that $\operatorname{rk}(\mathrm{H}(f)) = \dim(Z_X)+1$ is equal to $\rho+3 = \dim(S(L', \mathbb{P}^1 \times Z_{Y^*}))+1$.

Clearly $\mathrm{H}(f)$ is a $(2n+2) \times (2n+2)$ matrix, whose rank can be computed in this way. The $(2n) \times (2n)$ submatrix corresponding to the second partial derivatives with respect to the variables x_i and y_j has rank ρ by (6) and (7). The $2 \times 2n$ submatrix of $\mathrm{H}(f)$ corresponding to the second partial derivatives with respect to the variables $(u, v) \times (x_i, y_j)$ increases the rank by 1 by (11) and by (12). The 2×2 submatrix corresponding to the second partial derivatives with respect to the variables u, v increases the rank by 2. In conclusion $\operatorname{rk}(\mathrm{H}(f)) = \rho + 1 + 2 = \rho + 3$ so that equality holds in (10) (see also Remark 3.6).

Remark 2.3. Obviously also other similar changes of variables, for example like $\mathbf{z} \to (u\mathbf{x} - v\mathbf{y})^k$ with $k \geq 2$, will produce other interesting hypersurfaces with vanishing hessian. Instead of pursuing further these generalizations, we prefer to focus on the geometrical properties of the previous examples and on the connections with the so called *Dual Cayley Trick*.

2.3. Geometrical properties of $X = V(g(u\mathbf{x} - v\mathbf{y})) \subset \mathbb{P}^{2n+1}$, of Z_X^* and of the associated polar map

Let notation be as above and suppose that $V(g) \subset \mathbb{P}^{n-1}$ has non-vanishing hessian. Then

$$Z_X = S(L', W) = S(L', \mathbb{P}^1 \times \mathbb{P}^{n-1}) \subset (\mathbb{P}^{2n+1})^*$$

and

$$Z_X^* \simeq \mathbb{P}^1 \times \mathbb{P}^{n-1} \subset V(u, v) = \langle Z_X^* \rangle \subset \mathbb{P}^{2n+1}$$

is the Segre variety defined in $V(u, v)$ by the equations:

$$\operatorname{rk} \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} = 1.$$

In the terminology of [3, Section 2.2], the linear space

$$\Pi = V(u, v) = \mathbb{P}^{2n-1} \subset \operatorname{Sing}(X)$$

is the *core of* X. If $p = (0 : 0 : \mathbf{x} : \mathbf{y}) \in V(u, v)$, then we can take $(\mathbf{x} : \mathbf{y})$ as coordinates on $V(u, v)$. Let L be the line of equations $\mathbf{x} = \mathbf{0} = \mathbf{y}$, let

$$\xi = (u : v : \mathbf{0} : \mathbf{0}) \in L,$$

and let

$$\Pi_\xi = \langle \Pi, \xi \rangle \subset \mathbb{P}^{2n+1}.$$

For a fixed ξ, the points of the hyperplane Π_ξ can be parametrized by $(tu : tv : \mathbf{x} : \mathbf{y})$ so that $(t : \mathbf{x} : \mathbf{y})$ can be taken as coordinates on Π_ξ. Then $\Pi_\xi \cap X$ has the following equation in the hyperplane Π_ξ:

$$t^d g(u\mathbf{x} - v\mathbf{y}) = 0$$

with u, v fixed. Since $t = 0$ is the equation of $\Pi \subset \Pi_\xi$, $\Pi_\xi \cap X$ contains Π with multiplicity d, while

$$V(g(u\mathbf{x} - v\mathbf{y})) \subset \Pi_\xi$$

is a cone with vertex a $\mathbb{P}^n$, which is not contained in Π. The change of variable $u\mathbf{x} - v\mathbf{y} \mapsto \mathbf{x}$, $\mathbf{y} \mapsto \mathbf{y}$, $t \mapsto t$ shows that the resulting equation does not depend on the variables $\mathbf{y}$ and t, yielding that the vertex of the cone is the linear subspace of Π_ξ given by the n linear equations $u\mathbf{x} - v\mathbf{y} = \mathbf{0}$.

Varying the hyperplane in the pencil of hyperplanes through $V(u, v)$, the vertices of the corresponding cones describe a Segre variety $\mathbb{P}^1 \times \mathbb{P}^n$, which is the dual of $T \subset (\mathbb{P}^{2n+1})^*$ and which cuts $V(u, v)$ along Z_X^*.

In particular, the series of examples constructed in this section is completely different from those known up to now, which we shall simply call of *Gordan-Noether-Perazzo-Permutti-CRS type*. Indeed, in all these examples the intersection of the linear spaces $\Pi_\xi = \mathbb{P}^{c+1}$ through the core $\Pi =< Z_X^* >= \mathbb{P}^c$ with the hypersurface X consists of the core with a suitable multiplicity and of a cone, whose vertex is a linear space contained in Π.

From the algebraic point of view this new phenomenon means that there does not exist a suitable linear change of coordinates such that we can *separate the variables in the equation via the core*. We recall that Gordan-Noether-Perazzo-Permutti-CRS type hypersurfaces in $\mathbb{P}^4$ exhaust the list of hypersurfaces with vanishing hessian that are not cones.

Example 2.4. In $\mathbb{P}^7$ with homogeneous coordinates $(u : v : x_1 : x_2 : x_3 : y_1 : y_2 : y_3)$, let

$$X = V((x_1u - y_1v)^2 + (x_2u - y_2v)^2 + (x_3u - y_3v)^2) \subset \mathbb{P}^7$$

and let $Y = V(z_1^2 + z_2^2 + z_3^2) \subset \mathbb{P}^2$ be the self dual Fermat conic. Letting the notation be as above, we have $X^* = \mathbb{P}^1 \times \widetilde{Y} \subset (\mathbb{P}^7)^*$. Let us remark that the construction of irreducible hypersurfaces of this kind starts from $\mathbb{P}^7$. Specialisations of above examples have interesting applications, see [8, Example 2.3] and [7].

Example 2.5. In $\mathbb{P}^5$ with homogeneous coordinates $(u : v : x_1 : x_2 : y_1 : y_2)$, let

$$X = V((x_1u - y_1v)^2 + (x_2u - y_2v)^2 + u^4) \subset \mathbb{P}^5.$$

It is not difficult to see that $Z_X = V(\tilde{x}_1\tilde{y}_2 - \tilde{x}_2\tilde{y}_1) \subset \mathbb{P}^5$ and that, taking into account the previous remarks, $X \subset \mathbb{P}^5$ is the first example of a hypersurface with vanishing hessian that is not a cone and that it is not of Gordan-Noether-Perazzo-Permutti-CRS type.

3. Hypersurfaces with vanishing hessian constructed from cones with vertex a $\mathbb{P}^{k-1}$ via Dual Cayley Trick

3.1. Resultants and discriminants

We recall some well known facts on resultants and discriminants. A reference for most of the properties listed below is [16], see also [4].

Let $f_i(x_0, \dots, x_N)$, $i \in \{0, \dots, N\}$, be $N + 1$ universal homogeneous polynomials of degree $d_i \geq 1$. Then the *resultant of* $f_0, \dots, f_N$, indicated by

$$\mathrm{Res}(f_0, \dots, f_N),$$

is a polynomial in the coefficients of the f_i's, which is homogeneous of degree $d_0 \cdots d_{j-1}d_{j+1} \cdots d_N$ in the variables corresponding to f_j and which has degree

$$d_0 \cdots d_N \sum_{i=0}^{N} \frac{1}{d_i}. \tag{16}$$

The polynomial $\mathrm{Res}(f_0, \dots, f_N)$ has the following property: given homogeneous polynomials $g_0, \dots, g_N \in \mathbb{K}[x_0, \dots, x_N]$ with $\deg(g_i) = d_i$, the value of $\mathrm{Res}(f_0, \dots, f_N)$ on the coefficients of $g_0, \dots, g_N$ is zero if and only if $g_0 = \ldots = g_N = 0$ has a non-zero solution in $\mathbb{K}^{N+1}$ (or equivalently $V(g_0) \cap \ldots \cap V(g_N) \neq \emptyset$ where $V(g_i) \subset \mathbb{P}^N_{\mathbb{K}}$ is the projective hypersurface defined by g_i), see [16].

For a universal $f \in \mathbb{K}[x_0, \dots, x_N]_d$, let

$$\Delta_{N,d} = \mathrm{Res}(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_N}),$$

which is a homogeneous polynomial of degree $(N + 1)(d - 1)^N$ in the $\binom{N+d}{d}$ coefficients of the universal $f \in \mathbb{K}[x_0, \dots, x_N]_d$. By the previous geometrical property of the resultant we deduce that the (geometrically irreducible) hypersurface $V(\Delta_{N,d}) \subset \mathbb{P}(\mathbb{K}[x_0, \dots, x_N]_d)$, called the *discriminant hypersurface*, is well defined and it describes the locus of singular projective hypersurfaces of degree d.

3.2. Dual varieties of rational normal scrolls surfaces and some explicit examples of hypersurfaces with vanishing hessian

Let $1 \leq a \leq b$ be integers and let

$$S(a,b) \subset \mathbb{P}^{a+b+1}$$

be a rational normal scroll of degree $d = a + b$. The surface $S(a,b) \subset \mathbb{P}^{a+b+1}$ is smooth and projectively generated by a rational normal curve $C_a = \nu_a(\mathbb{P}^1) \subset \mathbb{P}^a$ and a rational normal curve $C_b = \nu_b(\mathbb{P}^1) \subset \mathbb{P}^b$ with $\langle C_a \rangle \cap \langle C_b \rangle = \emptyset$, by taking the union of the lines $\langle \nu_a(p), \nu_b(p) \rangle$, $p \in \mathbb{P}^1$.

We shall choose coordinates $(x_0 : \cdots : x_a : y_0 : \cdots : y_b)$ on $\mathbb{P}^{a+b+1}$ such that $V(y_0, \ldots, y_b) = \langle C_a \rangle$ and such that $V(x_0, \ldots, x_a) = \langle C_b \rangle$. Accordingly, $C_a \subset \mathbb{P}^{a+b+1}$ has parametrization $(s^a : s^{a-1}t : \cdots : st^{a-1} : t^a : 0 : 0 : \cdots : 0)$ and $C_b \subset \mathbb{P}^{a+b+1}$ has parametrization $(0 : 0 : \cdots : 0 : s^b : s^{b-1}t : \cdots : st^{b-1} : t^b)$.

The following result is well known, see for example [16, Example 3.6], and it shows the existence of a lot of significative examples of hypersurfaces with vanishing hessian, not cones. Special projections of such examples in $\mathbb{P}^4$ produce examples of Gordan-Noether-Perazzo-Permutti-CRS hypersurfaces (see [3]).

Theorem 3.1. *Let notation be as above and let $(w_0 : \cdots : w_a : z_0 : \cdots : z_b)$ be dual coordinates to $(x_0 : \cdots : x_a : y_0 : \cdots : y_b)$. Then*

$$S(a,b)^* = V(\mathrm{Res}(f,g)) \subset (\mathbb{P}^{a+b+1})^*$$

is a hypersurface of degree $a+b$, where $f = w_0 s^a + w_1 s^{a-1}t + \cdots + w_a t^a \in \mathbb{K}[s,t]_a$ is a general binary form of degree a and where $g = z_0 s^b + z_1 s^{b-1}t + \cdots + z_b t^b \in \mathbb{K}[s,t]_b$ is a general binary form of degree b.

In particular,

$$S(1,b)^* = V((-w_1)^b z_0 + (-w_1)^{b-1} w_0 z_1 + \cdots - w_1 w_0^{b-1} + w_0^b z_b) \subset (\mathbb{P}^{b+2})^*$$

is a hypersurface of degree $b+1$, which is not a cone and which for $b \geq 2$ has vanishing hessian.

Proof. Let notation be as above. To calculate the parametric equations of the tangent plane to $S(a,b)$ at a general point $q = \lambda\nu_a(s,t) + \mu\nu_b(s,t)$ with $(s:t) \in \mathbb{P}^1$ we shall suppose $(s:t) = (1:t)$ and $(\lambda : \mu) = (1 : \mu)$. In particular we can suppose $x_0 = 1$ and $y_0 = \mu$. Thus the projective tangent space $T_q S(a,b)$ is spanned by the rows of the following matrix:

$$\begin{pmatrix} 1 & t & \ldots & t^a & \mu & \mu t & \ldots & \mu t^b \\ 0 & 1 & \ldots & at^{a-1} & 0 & \mu & \ldots & \mu b t^{b-1} \\ 0 & 0 & \ldots & 0 & 1 & t & \ldots & t^b \end{pmatrix},$$

and hence it is also spanned by the rows of the matrix

$$\begin{pmatrix} 1 & t & \ldots & t^a & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & at^{a-1} & 0 & \mu & \ldots & \mu b t^{b-1} \\ 0 & 0 & \ldots & 0 & 1 & t & \ldots & t^b \end{pmatrix}. \tag{17}$$

If $(w_0 : \cdots : w_a : z_0 : \cdots : z_b)$ are dual coordinates on $(\mathbb{P}^{a+b+1})^*$, we get that a point of $(\mathbb{P}^{a+b+1})^*$ belongs to $S(a,b)^*$ if and only if

$$\begin{cases} w_0 + w_1 t + \cdots + w_a t^a & = & 0 \\ w_1 + 2w_2 t + \cdots + a w_a t^{a-1} + \cdots + \mu z_1 + 2\mu t z_2 + \cdots + b\mu t^{b-1} z_b & = & 0 \\ z_0 + t z_1 + \cdots + t^b z_b & = & 0 \end{cases}$$

has a solution (t, μ). Since the second equation is linear in μ, this happens if and only if

$$\begin{cases} w_0 s^a + w_1 s^{a-1} t + \cdots + w_a t^a & = & 0 \\ z_0 s^b + z_1 s^{b-1} t + \cdots + z_b t^b & = & 0 \end{cases}$$

has a solution $(s, t) \neq (0, 0)$. In conclusion, the equation of $S(a, b)^*$ is the resultant of two general homogeneous forms of degree a and degree b in the variables (s, t). Therefore, $S(a, b)^*$ is a hypersurface of degree $d = a + b$ by (16), whose equation can be explicitly written (for example by using Sylvester Formula).

For $a = 1$, the first equation is $sw_0 + tw_1 = 0$, whose roots are $(-w_1, w_0)$. Thus the equation of $S(1, b)^*$ is obtained by imposing that $(-w_1, w_0)$ is a solution of the second equation, that is

$$S(1, b)^* = V((-w_1)^b z_0 + (-w_1)^{b-1} w_0 z_1 + \cdots + w_0^b z_b) \subset (\mathbb{P}^{b+2})^*.$$

If $b > 1$, then the partial derivatives of the equation of $S(1, b)^*$ with respect to z_i are algebraically dependent so that the hypersurface $S(1, b)^* \subset (\mathbb{P}^{b+2})^*$ has vanishing hessian and is not a cone. □

The previous analysis admits obvious generalizations we shall only mention without proofs. The surface $S(a, b) \subset \mathbb{P}^{a+b+1}$ can be seen as the embedding of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))$ into $\mathbb{P}^{a+b+1} = \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(a)) \oplus \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(b)))$ by the tautological line bundle $\mathcal{O}(1)$. Thus, letting $r \geq 1$ and letting

$$\mathbb{P}^{N(a_0,\ldots,a_r)} = \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(a_0)) \oplus \ldots \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(a_r))),$$

we shall suppose $1 \leq a_0 \leq \ldots \leq a_r$ and consider

$$X(a_0, \ldots, a_r) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^r}(a_0) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^r}(a_r)) \subset \mathbb{P}^{N(a_0,\ldots,a_r)}$$

embedded by the tautological line bundle $\mathcal{O}(1)$. This smooth manifold is a $\mathbb{P}^r$-bundle over $\mathbb{P}^r$, which is projectively generated by the $r + 1$ varieties $\nu_{a_i}(\mathbb{P}^r)$ lying in disjoint linear subspaces of $\mathbb{P}^{N(a_0,\ldots,a_r)}$.

The same calculations used in the proof of Theorem 3.1 above prove that

$$X(a_0, \ldots, a_r)^* = V(\mathrm{Res}(f_0, \ldots, f_r)),$$

where f_i is a generic polynomial of degree a_i for $i = 0, \ldots, r$. Moreover,

$$\deg(X(a_0, \ldots, a_r)^*) = a_0 \cdots a_r \sum_{i=0}^{r} \frac{1}{a_i}$$

by (16).

In particular, if $a_0 = \ldots = a_{r-1} = 1$ and if $a_r = a \geq 2$, then

$$X(1, \ldots, 1, a)^* \subset (\mathbb{P}^{N(1,\ldots,1,a)})^*$$

is a hypersurface of degree $r \cdot a + 1$ with vanishing hessian which is not a cone.

Remark 3.2. These are the first instances of a general method, which has been dubbed *the Cayley Trick for mixed resultants* in [16, Ch. 3, sections 2, 3, 4], for calculating the explicit equations of dual varieties of $\mathbb{P}^r$-bundles of the form $\mathbb{P}(\mathcal{E})$ embedded by the tautological line bundle $\mathcal{O}(1)$ with $\mathcal{E}$ a very ample rank $r+1$ locally free sheaf over an irreducible projective variety X of dimension r.

We now introduce and apply the classical *Cayley Trick* and its dual variant, the so called *Dual Cayley Trick* to calculate explicitly the equations of some dual varieties.

3.3. Cayley Trick

Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be an irreducible non-degenerate variety of dimension $n \geq 1$. Let $\mathbb{G}(r, \mathbb{P}(V))$ denote the Grassmann variety of r-dimensional projective subspaces of $\mathbb{P}(V)$. If $L = \mathbb{P}(U) \subset \mathbb{P}(V)$ has dimension $r \geq 0$, then $L^\perp = \mathbb{P}(\mathrm{Ann}(U)) \subset \mathbb{P}(V^*)$ has dimension $N - r - 1$ and we have a natural isomorphism $\mathbb{G}(r, \mathbb{P}(V)) \simeq \mathbb{G}(N - r - 1, \mathbb{P}(V^*))$, defined by sending $[L]$ to $[L^\perp]$. We have two natural rational maps:

$$q : \mathbb{P}(\mathbb{K}^{r+1} \otimes V) \dashrightarrow \mathbb{G}(r, \mathbb{P}(V))$$

and

$$p : \mathbb{P}(\mathbb{K}^{N-r} \otimes V^*) \dashrightarrow \mathbb{G}(r, \mathbb{P}(V)),$$

corresponding, respectively, to the parametric equations and to the cartesian equations of a subspace $L = \mathbb{P}(U) \subset \mathbb{P}(V)$. The rational maps are defined on the open sets of elements of maximal rank and on these open sets they are the quotient maps of the natural action via left multiplication of the group of invertible matrices.

Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be as above and let $e = \deg(X) \geq 2$. Following [16], let

$$Z(X) = \{[L] \in \mathbb{G}(N - n - 1, \mathbb{P}(V)) : \ L \cap X \neq \emptyset\} \subset \mathbb{G}(N - n - 1, \mathbb{P}(V))$$

be the *associated hypersurface of* X. Indeed,

$$\mathrm{codim}(Z(X), \mathbb{G}(N - n - 1, \mathbb{P}(V)) = 1,$$

see [16, Proposition 2.2, Chap. 3], and $Z(X)$ is given by a homogeneous element of degree $e = \deg(X)$ in the homogeneous coordinate ring of

$$\mathbb{G}(N - n - 1, \mathbb{P}(V)) \subset \mathbb{P}(\Lambda^{N-n} V),$$

defined modulo Plücker relations and dubbed the *Chow form of* X.

Example 3.3. Let $f_0, \ldots, f_N \in \mathbb{K}[x_0, \ldots, x_N]_d$ be homogeneous forms of degree $d \geq 1$. Then $\mathrm{Res}(f_0, \ldots, f_N)$ is a homogeneous polynomial of degree $(N+1)d^N$ in the $(N+1) \times \binom{N+d}{d}$ variables, which are the coefficients of the universal f_i's or equivalently the homogeneous coordinates on $\mathbb{P}(\mathbb{K}[x_0, \ldots, x_N]_d)$.

Under this assumption, if $[a_{i,j}] \in GL_{N+1}(\mathbb{K})$ and if

$$h_i = \sum_{j=0}^{N} a_{i,j} g_j,$$

then one proves that

$$\mathrm{Res}(h_0,\dots,h_N)=\det([a_{i,j}])^{d^N}\,\mathrm{Res}(g_0,\dots,g_N). \tag{18}$$

In particular, $\mathrm{Res}(f_0,\dots,f_N)$ is an invariant for the action by left multiplication of $SL_{N+1}(\mathbb{K})$ on the set of $(N+1)\times\binom{N+d}{d}$ matrices with entries in the coefficients of $f_0,\dots,f_N$.

By the *First Theorem of Invariant Theory* the polynomial $\mathrm{Res}(f_0,\dots,f_N)$ can be written as a polynomial of degree d^N in the $(N+1)\times(N+1)$ minors of the $(N+1)\times\binom{N+d}{d}$ matrix associated to $\{f_0,\dots,f_N\}$, that is in the Plücker coordinates of the matrix. This polynomial is the (dual) Chow form of $\nu_d(\mathbb{P}(V))\subset\mathbb{P}(S^d(V))=\mathbb{P}^{N(d)}$, defined modulo Plücker coordinates, and its restriction to $\mathbb{G}(N(d)-N-1,\mathbb{P}(S^d(V)))$ defines $Z(\nu_d(\mathbb{P}(V)))$ by the geometrical interpretation of the resultant.

Letting

$$p:\mathbb{P}(\mathbb{K}^{N+1}\otimes(S^dV)^*)\dashrightarrow\mathbb{G}(N(d)-N-1,\mathbb{P}(S^d(V)))$$

be the natural map defined above, we deduce

$$V(\mathrm{Res}(f_0,\dots,f_N))=\overline{p^{-1}(Z(\nu_d(\mathbb{P}(V)))}.$$

Let $y_0,\dots,y_N$ be other variables and let

$$y_0f_0+\cdots+y_Nf_N\in\mathbb{K}[x_0,\dots,x_N,y_0,\dots,y_N]_{d+1},$$

which is also a bihomogeneous polynomial of bidegree $(d,1)$. Then the classical *Cayley Trick* is the formula:

$$\mathrm{Res}(f_0,\dots,f_N)=\Delta(y_0f_0+\cdots+y_Nf_N), \tag{19}$$

a useful remark which dates back to Cayley.

The geometrical translation of the Cayley Trick is the following: if

$$\mathbb{P}(\mathbb{K}^{N+1})\times\nu_d(\mathbb{P}(V))\subset\mathbb{P}^{(N+1)(N(d)+1)-1}=\mathbb{P}(\mathbb{K}^{N+1}\otimes S^d(V))$$

is the Segre embedding of $\mathbb{P}^N\times\nu_d(\mathbb{P}^N)$, then

$$(\mathbb{P}^N\times\nu_d(\mathbb{P}(V)))^*=\overline{p^{-1}(Z(\nu_d(\mathbb{P}(V))))}.$$

Indeed, formula (19) says that if a hyperplane $H\subset\mathbb{P}(\mathbb{K}^{N+1}\otimes S^d(V))$ contains $\mathbb{P}^N\times p$, $p\in\nu_d(\mathbb{P}^N)$ (the condition on the left), then there exists $q\in\mathbb{P}^N$ such that $q\times T_p\nu_d(\mathbb{P}^N)\subset H$ so that H is tangent to $\mathbb{P}^N\times\nu_d(\mathbb{P}^N)$ at (q,p) (the condition on the right), yielding $[H]\in(\mathbb{P}^n\times\nu_d(\mathbb{P}^N))^*$.

Nowadays the geometrical version of the Cayley Trick has been generalized by Gelfand, Weyman and Zelevinsky to arbitrary irreducible varieties.

Theorem 3.4. (Cayley Trick, [16, Theorem 2.7, Chap. 3]) *Let $X\subset\mathbb{P}^N=\mathbb{P}(V)$ be an irreducible non-degenerate variety of dimension $n\geq 1$ and let $\mathbb{P}^n\times X\subset\mathbb{P}(\mathbb{K}^{n+1}\otimes V)$ be the Segre embedding. Then*

$$(\mathbb{P}^n\times X)^*=\overline{p^{-1}(Z(X))}, \tag{20}$$

where $p:\mathbb{P}(\mathbb{K}^{n+1}\otimes V^*)\dashrightarrow\mathbb{G}(N-n-1,\mathbb{P}(V))$ is the quotient map corresponding to cartesian equations of linear subspaces of dimension $N-n-1$ of $\mathbb{P}(V)$.

3.4. Dual Cayley Trick

We now present the so called *Dual Cayley trick*, introduced by Weyman and Zelevinsky in [28], see also [19].

Let $\widetilde{Y} \subset \mathbb{P}^N = \mathbb{P}(V)$ be an irreducible variety of dimension $n \geq 1$ such that $\widetilde{Y}^* \subset \mathbb{P}(V^*)$ has dimension $N - 1 - r$. Let $\mathbb{P}^r = \mathbb{P}(T)$ and let

$$\mathbb{P}^r \times \widetilde{Y} \subset \mathbb{P}^{(r+1)(N+1)-1} = \mathbb{P}(T \otimes V)$$

be the Segre embedding of $\mathbb{P}(T) \times \widetilde{Y}$. Then by [28, Corollary 3.3] the dual variety $(\mathbb{P}^r \times \widetilde{Y})^* \subset (\mathbb{P}^{(r+1)(N+1)-1})^*$ is a hypersurface

$$V(f) \subset (\mathbb{P}^{(r+1)(N+1)-1})^* = \mathbb{P}((T \otimes V)^*) = \mathbb{P}(T^* \otimes V^*),$$

which can be computed in this way.

Let

$$q : (\mathbb{P}^{(r+1)(N+1)-1})^* = \mathbb{P}(T^* \otimes V^*) \dashrightarrow \mathbb{G}(r, \mathbb{P}(V^*))$$

be the natural rational map defined above and corresponding to parametric equations of linear subspaces of dimension r of $\mathbb{P}(V^*)$. This map, in the natural coordinates, sends a rank $r+1$ matrix $\mathbf{X} \in \mathbb{K}^{r+1,N+1}$ to its Plücker coordinates. It is thus given by forms of degree $r+1$ in the natural coordinates.

Let

$$Z(\widetilde{Y}^*) \subset \mathbb{G}(r, \mathbb{P}(V^*))$$

be the associated hypersurface of $\widetilde{Y}^*$. Then [28, Proposition 4.2.b] yields the following formula:

$$(\mathbb{P}(T) \times \widetilde{Y})^* = \overline{q^{-1}(Z(\widetilde{Y}^*))}. \tag{21}$$

3.5. Hypersurfaces with vanishing hessian constructed from cones with vertex a $\mathbb{P}^{r-1}$

We now apply the Dual Cayley Trick to generalize part (ii) of Theorem 2.1, giving also a different and more theoretical proof of this result.

Let $r \geq 1$ be an integer and let $\widetilde{Y} \subset \mathbb{P}^N = \mathbb{P}(V)$ be a cone with vertex a $\mathbb{P}^{r-1} = \mathbb{P}(U)$ over a non-degenerate variety $Y \subset \mathbb{P}^{N-r} = \mathbb{P}(W)$ without dual defect, that is $Y^* = V(g) \subset (\mathbb{P}^{N-r})^* = \mathbb{P}(W^*)$ is a hypersurface with $g = g(z_0, \ldots, z_{N-r}) \in \mathbb{K}[z_0, \ldots, z_{N-r}]_d$ for some $d \geq 2$. By definition $V = U \oplus W$.

Let $\mathbb{P}^r = \mathbb{P}(T)$ and let

$$\mathbb{P}(T) \times \widetilde{Y} \subset \mathbb{P}(T) \times \mathbb{P}(V) \subset \mathbb{P}(T \otimes V) = \mathbb{P}^{(r+1)(N+1)-1},$$

be the Segre embedding of $\mathbb{P}^r \times \widetilde{Y}$.

We can identify points of $(\mathbb{P}^{(r+1)(N+1)-1})^* = \mathbb{P}((T \otimes V)^*)$ with matrices

$$\mathbf{X} = \left[\begin{array}{c|c} \text{A'} & \text{B'} \end{array} \right], \tag{22}$$

where $A' \in \mathbb{K}^{r+1,N-r+1}$ and $B' \in \mathbb{K}^{r+1,r}$. This decomposition corresponds to the natural decomposition

$$\mathrm{Hom}(T, V^*) \simeq \mathrm{Hom}(T, W^*) \oplus \mathrm{Hom}(T, U^*) \simeq (T^* \otimes W^*) \oplus (T^* \otimes U^*).$$

The linear system of equations $B' = 0_{(r+1)\times r}$ defines the linear span $\mathbb{P}(\mathrm{Hom}(T, W^*)) = \mathbb{P}(T^* \otimes W^*)$ of the Segre variety

$$R = \mathbb{P}(T^*) \times \mathbb{P}(W^*) \subset (\mathbb{P}^{(r+1)(N+1)-1})^*.$$

There are exactly $N - r + 1$ minors B'_j of order $r + 1$, obtained from $\mathbf{X}$ as in (22) by adding the jth column of A' to B', $j = 0, \dots, N - r$. Define

$$\phi_1 : (\mathbb{P}^{(r+1)(N+1)-1})^* \dashrightarrow \mathbb{P}^{(r+1)r-1} = \mathbb{P}(\mathrm{Hom}(T, U^*)),$$

by

$$\phi_1(\mathbf{X}) = B',$$

and

$$\phi_2 : (\mathbb{P}^{(r+1)(N+1)-1})^* \dashrightarrow (\mathbb{P}^{N-r})^* = \mathbb{P}(W^*),$$

by

$$\phi_2(\mathbf{X}) = (B'_0 : \cdots : B'_{N-r}) \in (\mathbb{P}^{N-r})^*.$$

The point $\phi_2(\mathbf{X})$ is the intersection of $\mathbb{P}(W^*)$ with the r-dimensional linear subspace of $\mathbb{P}(V^*)$ corresponding to $\mathbf{X}$.

Theorem 3.5. *Let the hypothesis and the notation be as above, let*

$$Z_{Y^*} = \overline{\nabla_g(\mathbb{P}(W^*))} \subseteq \mathbb{P}(W) = \mathbb{P}^{N-r},$$

let

$$\mathbb{P}^r \times Z_{Y^*} \subset \mathbb{P}(T \otimes W)$$

be the Segre embedding, let $X = V(g(B'_0, \dots, B'_{N-r})) \subset \mathbb{P}(T^* \otimes V^*)$ and let $f = g(B'_0, \dots, B'_{N-r})$. Then:

(i) $(\mathbb{P}^r \times \widetilde{Y})^* = V(g(B'_0, \dots, B'_{N-r})) \subset \mathbb{P}(T^* \otimes V^*)$;
(ii) $Z_X = \overline{\nabla_f(\mathbb{P}(T^* \otimes V^*))} \subseteq S(\mathbb{P}(T \otimes U), \mathbb{P}(T) \times Z_{Y^*}) \subset \mathbb{P}(T \otimes V)$ *and X has vanishing hessian.*

Proof. One can argue as in the proof of Theorem 2.1 but we prefer to deduce this more general result from (21), specializing the Dual Cayley Trick to our situation.

Recall that in this case $\widetilde{Y}^* = Y^* \subset (\mathbb{P}^{N-r})^* = \mathbb{P}(W^*) \subset \mathbb{P}(V^*)$. Moreover, we can define a rational map

$$\psi : \mathbb{G}(r, \mathbb{P}(V^*)) \dashrightarrow (\mathbb{P}^{N-r})^* = \mathbb{P}(W^*),$$

by

$$\psi([L]) = [L \cap \mathbb{P}(W^*)].$$

The map ψ is not defined along $\mathbb{G}(r, \mathbb{P}(W^*)) \subset \mathbb{G}(r, \mathbb{P}(V^*))$ and along the Schubert cycles given by the $[L]$'s such that $\dim(L \cap \mathbb{P}(W^*)) > 0$.

Since, by hypothesis, $\widetilde{Y}^* = Y^* = V(g)$ is a hypersurface in $\mathbb{P}(W^*)$, it follows that

$$\overline{\psi^{-1}(Y^*)} = Z(\widetilde{Y}^*).$$

Using the coordinates introduced above, we deduce that

$$\phi_2 = \psi \circ q, \tag{23}$$

where $q : \mathbb{P}(T^* \otimes V^*) \dashrightarrow \mathbb{G}(r, \mathbb{P}(V^*))$ is the natural rational map considered above.

Then (21) gives

$$(\mathbb{P}(T) \times \widetilde{Y})^* = \overline{q^{-1}(Z(\widetilde{Y}^*))}.$$

Using (23) we get

$$\overline{q^{-1}(Z(\widetilde{Y}^*))} = \overline{q^{-1}(\psi^{-1}(\widetilde{Y}^*))} = \overline{\phi_2^{-1}(Y^*)} = V(g(B'_0, \dots, B'_{N-r})) = X.$$

Let notation be as in (22), let $a'_{i,j}$, with $i = 0, \dots, r$ and with $j = 0, \dots, N - r$, be the homogeneous coordinates corresponding to A' and let $b'_{i,k}$ with $k = 1, \dots, r$ be the coordinates corresponding to B'. By definition of B'_j, $j = 0, \dots, N - r$, we deduce from Laplace Formula applied to the first column of B'_j:

$$B'_j = \sum_{i=0}^{r} (-1)^i a_{i,j} C_i$$

yielding

$$\frac{\partial B'_j}{\partial a_{i,j}} = (-1)^i C_i.$$

Moreover, for $m \neq j$, we have

$$\frac{\partial B'_j}{\partial a_{i,m}} = 0.$$

From

$$\frac{\partial f}{\partial a_{i,j}} = (-1)^i C_i \frac{\partial g}{\partial z_j}(B'_0, \dots, B'_{N-r}) \tag{24}$$

we deduce that, for every $i \neq k$ and for every $l \neq m$,

$$\frac{\partial f}{\partial a_{i,l}} \frac{\partial f}{\partial a_{k,m}} - \frac{\partial f}{\partial a_{i,m}} \frac{\partial f}{\partial a_{k,l}} = 0. \tag{25}$$

Thus $X \subset \mathbb{P}(T^* \otimes V^*)$ has vanishing hessian since the partial derivates of f are algebraically dependent and more precisely

$$Z_X = \overline{\nabla_f(\mathbb{P}(T^* \otimes V^*))} \subseteq S(\mathbb{P}(T \otimes U), \mathbb{P}(T) \times Z_{Y^*}) \subset \mathbb{P}(T \otimes V). \quad \square \tag{26}$$

Let us remark that for $r = 1$ the base locus of ψ is exactly $\mathbb{G}(1, (\mathbb{P}(W^*))$ since a line cuts $\mathbb{P}(W^*)$ in one point if and only if it is not contained in $\mathbb{P}(W^*)$. Thus in this case the expression of ϕ_2 is, modulo the obvious identifications, that given in (4).

Remark 3.6. One could ask if equality holds in (26). Letting $\rho = \mathrm{rk}(\mathrm{H}(g)) = \dim(Z_{Y^*}) + 1$, it would be sufficient (indeed equivalent) to prove that

$$\mathrm{rk}(\mathrm{H}(f)) - 1 = \dim(Z_X) = \dim(S(\mathbb{P}(T \otimes U), \mathbb{P}(T) \times Z_{Y^*})) = r(r+1) + r + \rho - 1,$$

i.e. that $\mathrm{rk}(\mathrm{H}(f)) = r(r+1) + r + \rho$.

Since this analysis is quite delicate (and also intricate) we preferred to skip the details and to concentrate on the interesting connections with the Dual Cayley Trick in order to produce the new examples, which generalize to arbitrary $r \geq 2$ the case $r = 1$ considered in part (ii) of Theorem 2.1. Last but not least, we point out that Theorem 2.1 is sufficient to construct examples of hypersurfaces with vanishing hessian with $\mathrm{codim}(X^*, Z_X)$ arbitrary large. Assuming equality in (26), one would deduce $\mathrm{codim}(X^*, Z_X) = \mathrm{codim}(Y, Z_{Y^*}) + r^2$ and there is no advantage in solving the previous equation instead of the simpler $\mathrm{codim}(X^*, Z_X) = \mathrm{codim}(Y, Z_{Y^*}) + 1$.

3.6. The dual variety of $\mathbb{P}^1 \times Y \subset \mathbb{P}^{2n+3}$ with $Y = V(f) \subset \mathbb{P}^{n+1}$ an irreducible hypersurface

Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be an irreducible projective variety of dimension $n = \dim(X)$ and degree $e \geq 2$. Let $\mathbb{P}^{n-i} = \mathbb{P}(T)$, $i \in \{0, \ldots, n\}$ and let

$$\mathbb{P}^{n-i} \times X \subset \mathbb{P}^{(n-i+1)(N+1)+1} = \mathbb{P}(T \otimes V)$$

be the Segre embedding of $\mathbb{P}(T) \times X$. Let $Z_i(X) \subset \mathbb{G}(N - n + i - 1, \mathbb{P}(V))$ be the ith *higher associated variety of* X in the sense of [16], i.e. it is the closure of the set

$$\{[L] \in \mathbb{G}(N - n + i - 1, \mathbb{P}(V)) : \exists\, x \in X_{\mathrm{reg}} : x \in L, \dim(L \cap T_x X) \geq i\}.$$

Clearly $Z_0(X) = Z(X)$ is the Chow hypersurface of X and

$$Z_n(X) = X^* \subset \mathbb{P}(V^*) = \mathbb{G}(N - 1, \mathbb{P}(V)).$$

Let us recall that $Z_i(X)$ is a hypersurface in $\mathbb{G}(N - n + i - 1, \mathbb{P}(V))$ if and only if $i \leq n - \mathrm{codim}(X^*) + 1$, see [16,28] and [19]. In particular $Z_{n-1}(X)$ is a hypersurface if and only if $\mathrm{codim}(X^*) \in \{1, 2\}$.

Let

$$p : (\mathbb{P}^{(n-i+1)(N+1)-1})^* = \mathbb{P}(T^* \otimes V^*) \dashrightarrow \mathbb{G}(N - n + i - 1, \mathbb{P}(V))$$

be the rational map defined in Subsection 3.3 by considering the cartesian equations of a linear subspace of $\mathbb{P}(V)$. We defined also the rational map

$$q : (\mathbb{P}^{(n-i+1)(N+1)-1})^* = \mathbb{P}(T^* \otimes V^*) \dashrightarrow \mathbb{G}(n - i, \mathbb{P}(V^*))$$

associated to the representation of a linear subspace of $\mathbb{P}(V)^*$ via parametric equations. Then [28, Proposition 4.2.a], see also [19], yields the following formulas:

$$(\mathbb{P}^{n-i} \times X)^* = \overline{p^{-1}(Z_i(X))}, \tag{27}$$

$$(\mathbb{P}^{n-i} \times X)^* = \overline{q^{-1}(Z_{n-\mathrm{codim}(X^*)-i+1}(X^*))}. \tag{28}$$

Suppose now that $Y \subset \mathbb{P}^{n+1} = \mathbb{P}(V)$ is a hypersurface such that $Y^* \subset \mathbb{P}(V^*)$ is a hypersurface. Then the previous formulas give

$$(\mathbb{P}^1 \times Y)^* = \overline{p^{-1}(Z_{n-1}(Y))} = \overline{q^{-1}(Z_1(Y^*))}. \tag{29}$$

Let $Y \subset \mathbb{P}(V) = \mathbb{P}^{n+1}$ be an irreducible hypersurface of degree at least two, then $Z_1(Y) \subset \mathbb{G}(1, \mathbb{P}(V))$ is a hypersurface parametrizing the tangent lines to Y. It is given by a polynomial in the Plücker coordinates of $\mathbb{G}(1, \mathbb{P}(V))$. To determine the degree e of this polynomial let us remark that a general line $L \subset \mathbb{G}(1, \mathbb{P}(V))$ consists of the lines $l \subset \mathbb{P}(V)$ passing through a general point $p \in \mathbb{P}(V)$ and contained in a general plane $\Pi \subset \mathbb{P}(V)$ with $p \in \Pi$. Then $e = \#(L \cap Z_1(Y))$ equals the number of tangent lines to $Y \cap \Pi$ passing through p, that is $e = \deg((Y \cap \Pi)^*)$.

Example 3.7. Let $Y = V(x_0^2 + \cdots + x_{n+1}^2) \subset \mathbb{P}^{n+1}$ be the Fermat quadric hypersurface. Then $Z_1(Y) \subset \mathbb{G}(1, \mathbb{P}^{n+1})$ is a hypersurface of degree 2 whose equation is quadratic and, modulo Plücker relations, is the quadratic Fermat in the Plücker coordinates, that is

$$Z_1(V(x_0^2 + \cdots + x_{n+1}^2)) = V(\sum_{0 \le i < j \le n+1} p_{i,j}^2) \subset \mathbb{G}(1, \mathbb{P}^{n+1}).$$

Let $f(x_0, \dots, x_{n+1}) = x_0^2 + x_1^2 + \cdots + x_{n+1}^2$, then $Y^* = V(f(y_0, \dots, y_{n+1})) \subset (\mathbb{P}^{n+1})^*$. Let $\mathbf{a} = (a_0 : \cdots : a_{n+1})$ and $\mathbf{b} = (b_0 : \cdots : b_{n+1})$ and let $(\mathbf{a} : \mathbf{b})$ the natural coordinates on $\mathbb{P}^{2n+3}$ in such a way that the Plücker coordinates $q_{i,j}$ of a matrix whose first row is $\mathbf{a}$ and whose second row is $\mathbf{b}$ are $q_{i,j} = a_i b_j - a_j b_i$ for $0 \le i < j \le n+1$. Then

$$Z_1(V(y_0^2 + \cdots + y_{n+1}^2)) = V(\sum_{0 \le i < j \le n+1} q_{i,j}^2) \subset \mathbb{G}(1, (\mathbb{P}^{n+1})^*)$$

and

$$(\mathbb{P}^1 \times Y)^* = \overline{q^{-1}(Z_1(Y^*))} = V(\sum_{0 \le i < j \le n+1} (a_i b_j - a_j b_i)^2) \subset (\mathbb{P}^{2n+3})^*.$$

By Lagrange's Identity

$$\sum_{0 \le i < j \le n+1} (a_i b_j - a_j b_i)^2 = ||\mathbf{a}||^2 \cdot ||\mathbf{b}||^2 - (\mathbf{a} \bullet \mathbf{b})^2.$$

Thus the dual of $\mathbb{P}^1 \times V(x_0^2 + \cdots + x_{n+1}^2) \subset \mathbb{P}^{2n+3}$ has a *Cauchy-Schwartz* equation:

$$(\mathbb{P}^1 \times V(x_0^2 + \cdots + x_{n+1}^2))^* = V(||\mathbf{a}||^2 \cdot ||\mathbf{b}||^2 - (\mathbf{a} \bullet \mathbf{b})^2).$$

This is a *homaloidal polynomial*, that is the associated polar map is a Cremona transformation of $\mathbb{P}^{2n+3}$. In particular, the hessian of this quartic polynomial is different from zero and one can verify that it has a unique irreducible factor equal to the polynomial itself.

4. Duals of internal projections of Scorza varieties from a point have vanishing hessian

The series of varieties

$$\mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{n^2+2n} \text{ (Segre embedded, } n \ge 2),$$

$$\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n^2+3n}{2}} \text{ (quadratic Veronese embedding, } n \ge 2),$$

$$\mathbb{G}(1, \mathbb{P}^{2m+1}) \subset \mathbb{P}^{2m^2+3m} \text{ (Plücker embedding, } m \ge 2)$$

together with the Severi variety $E^{16} \subset \mathbb{P}^{26}$ are the so called *Scorza varieties.* These varieties and their duals have a uniform description via linear algebra and via the theory of determinantal varieties we now briefly recall.

4.1. Generic determinantal Scorza varieties

Let

$$\mathbb{P}^{n^2+2n} = \mathbb{P}(M_{(n+1)\times(n+1)}(\mathbb{K})).$$

We shall indicate the generic matrix in $M_{(n+1)\times(n+1)}(\mathbb{K})$ by

$$X = [x_{i,j}],$$

$i, j = 0, \dots, n$ and, by abusing notation, we shall also consider

$$(x_{0,0} : \cdots : x_{n,n})$$

as homogeneous coordinates on $\mathbb{P}^{n^2+2n}$. Analogously, we shall indicate by $Y = [y_{i,j}]$ the matrices in the dual space $\mathbb{P}((M_{(n+1)\times(n+1)}(\mathbb{K}))^*)$ in such a way that $(y_{0,0} : \cdots : y_{n,n})$ are homogeneous coordinates dual to the previous ones.

For every $r = 1, \dots, n+1$ we can define the variety

$$X_r = \{[X] \ : \ \mathrm{rk}(X) \leq r\} \subset \mathbb{P}(M_{(n+1)\times(n+1)}(\mathbb{K}));$$

the variety $Y_r \subset \mathbb{P}((M_{(n+1)\times(n+1)}(\mathbb{K}))^*)$ is defined in the same way. With this notation we have

$$X_1 = \mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{n^2+2n}$$

Segre embedded and

$$X_n = V(\det(X)) \subset \mathbb{P}^{n^2+2n}$$

is a hypersurface of degree $n+1$. For simplicity, let

$$f = \det(X) \in \mathbb{K}[x_{i,j}]_{n+1}$$

and consider

$$\nabla_f : \mathbb{P}(M_{(n+1)\times(n+1)}(\mathbb{K})) \dashrightarrow \mathbb{P}((M_{(n+1)\times(n+1)}(\mathbb{K}))^*).$$

Letting

$$X^\# \in M_{(n+1)\times(n+1)}(\mathbb{K})$$

be the matrix defined by the Laplace formula:

$$X \cdot X^\# = \det(X) \cdot I_{(n+1)\times(n+1)} = X^\# \cdot X, \tag{30}$$

the identity

$$(X^{\#})^{\#} = \det(X)^{n-1} \cdot X \tag{31}$$

shows that $\nabla_f([X]) = [X^{\#}]^t$ is birational outside X_n. The map is not defined along $X_{n-1} = \mathrm{Sing}(X_n)$ and its ramification divisor is given by the determinant of the Jacobian matrix of ∇_f, which is $\mathrm{hess}(f)$. Since $f = \det(X)$ is an irreducible polynomial, we deduce from (31) that the ramification divisor of ∇_f is supported on X_n, yielding

$$\mathrm{hess}(f) = \alpha \cdot f^{(n+1)(n-1)}$$

with $\alpha \in \mathbb{K}^*$. This property was proved in a similar (but not identical) way by B. Segre in [27, Teorema 1]. By evaluating the previous identity on particular matrices B. Segre also deduced $\alpha = (-1)^{\frac{n(n-1)}{2}} n$, see [27, Teorema 1].

If $[X] \notin X_n$, then $[X^{\#}] \notin Y_n$ while for $[X] \in X_n \setminus X_{n-1}$, $[X^{\#}] \in Y_1$ from which it easily follows that

$$X_n^* = Y_1 = \mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}((M_{(n+1)\times(n+1)}(\mathbb{K}))^*).$$

By the definition of the Gauss map, for every $[X] \in X_n \setminus X_{n-1}$, we have that $\nabla_f([X]) = [T_{[X]}X_n]$ and, recalling that the closure of the fibers of the Gauss map are linear spaces, that $\overline{\nabla_f^{-1}([T_{[X]}X_n])}$ is linear space of dimension $n^2 - 1$.

We are now ready to prove the next result.

Proposition 4.1. *Let notation be as above, let $n \geq 2$ and let $[X] \in X_n \setminus X_{n-1}$. Then*

$$\overline{\nabla_f(T_{[X]}X_n)}$$

is a hypersurface of degree n which is a cone with vertex

$$\overline{\nabla_f^{-1}([T_{[X]}X_n])}^{\perp} = \mathbb{P}^{2n}$$

over the dual of a Segre variety $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

Proof. Since $X_n \setminus X_{n-1}$ is homogeneous, it is sufficient to verify the assertion for X with $x_{i,j} = \delta_{i,j}$ for $i, j = 0, \dots, n-1$ and $x_{n,j} = x_{i,n} = 0$ for every i, j. Then

$$X^{\#} = (0 : 0 : \cdots : 0 : 1),$$

so $T_{[X]}X_n$ has equation $x_{n,n} = 0$. Letting $h = \det(Y)$, (31) implies $\nabla_f^{-1} = \nabla_h$ as rational maps. Then

$$\overline{\nabla_f(V(x_{n,n}))} = V(\frac{\partial h}{\partial y_{n,n}})$$

is the determinant of the $n \times n$ matrix with entries $y_{i,j}$, $i, j = 0, \dots, n-1$ which does not depend on the $2n + 1$ variables $y_{n,i}$ and $y_{j,n}$. Hence it is a cone with vertex $\overline{\nabla_f^{-1}([T_{[X]}X_n])}^{\perp}$ over the dual of the Segre variety $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \subset \mathbb{P}^{n^2-1}$ corresponding to the n^2 variables $y_{i,j}$, $i, j = 0, \dots, n-1$. □

Corollary 4.2. *Let notation be as above, let $n \geq 2$, let*

$$p \in Y_1 = \mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{n^2+2n} = \mathbb{P}((M_{(n+1)\times(n+1)}(\mathbb{K}))^*)$$

and let $T_n \subset \mathbb{P}^{n^2+2n-1}$ *be the projection of* $Y_1 = \mathbb{P}^n \times \mathbb{P}^n$ *from* p*. Let* $q \in X_n \setminus X_{n-1}$ *be such that* $T_q X_n = p^\perp$ *and let*

$$R_n = X_n \cap p^\perp \subset p^\perp = \mathbb{P}^{n^2+2n-1}.$$

Then:

(i) $R_n \subset \mathbb{P}^{n^2+2n-1}$ *is an irreducible hypersurface of degree* $n+1$ *with vanishing hessian and such that* $R_n^* = T_n$*.*
(ii) $Z_{R_n} \subset \mathbb{P}^{n^2+2n-1}$ *is a hypersurface of degree* n *which is a cone with vertex a* $\mathbb{P}^{2n-1}$ *over the dual of a Segre variety* $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$*.*
(iii) $\operatorname{codim}(R_n^*, Z_{R_n}) = n^2 - 2$*.*

Proof. The hypersurface $R_n \subset \mathbb{P}^{n^2+2n-1}$ is connected, it is also normal by Serre's Criterion being non-singular in codimension 1 ($\operatorname{Sing}(R)$ is the closure of the contact locus $\mathbb{P}_q^{n^2-1}$ of $T_q X_n$ defined above) and hence it is irreducible. The variety T_n has no dual defect so that T_n^* is an irreducible hypersurface contained in R_n (see for example [25, Exercise 1.5.22]), yielding $R_n = T_n^*$. Since $\overline{\nabla_f(T_q X_n)}$ is a cone such that $p \in \operatorname{Vert}(\overline{\nabla_f(T_q X_n)})$, we deduce that Z_{R_n} is the projection of $\overline{\nabla_f(T_q X_n)}$ from p. Thus (ii) follows from Proposition 4.1 and Lemma 1.3. □

Remark 4.3. The previous result has been discovered for $n = 2$ in [13], see also [25, Example 7.6.11]. Part (ii) has been proved algebraically also in [21, Proposition 4.9]. By passing to a suitable linear section of X_n obtained by putting some more variable equal to zero in the matrix X such that $f = \det(X)$, Cunha, Ramos and Simis produced explicit irreducible polynomials with vanishing hessian and such that $\operatorname{codim}(X^*, Z_X)$ is a function of n. These examples can be also described geometrically as the duals of some explicit projections of Y_1. Clearly the examples in [5] are of Gordan-Noether-Perazzo-Permutti-CRS type since one can *separate the variables* via Laplace formula for the expansion of the determinant.

4.2. Symmetric determinantal Scorza varieties

Let

$$W_n = \{S \in M_{(n+1)\times(n+1)}(\mathbb{K}) \ : \ S = S^t\} \subset M_{(n+1)\times(n+1)}(\mathbb{K}).$$

Although for $n \geq 1$ the subspace W_n is not a subalgebra of $M_{(n+1)\times(n+1)}(\mathbb{K})$, we have $S^\# \in W_n$ for every $S \in W_n$. Let $S = [s_{i,j}]$ be the generic matrix in W_n and let $(s_{0,0} : \cdots : s_{n,n})$ be the corresponding homogeneous coordinates on

$$\mathbb{P}(W_n) = \mathbb{P}^{\frac{n^2+3n}{2}}.$$

Let

$$g = \det(S) \in \mathbb{K}[s_{i,j}]_{n+1}.$$

The operation $\#$ on $M_{(n+1)\times(n+1)}(\mathbb{K})$ induces by restriction to W_n a birational involution

$$\nabla_g : \mathbb{P}(W_n) \dashrightarrow \mathbb{P}(W_n^*),$$

defined by $\nabla_g([S]) = [S^\#]^t$.

For every $r = 1, \dots, n+1$ we can define the variety

$$S_r = \{[S] \in \mathbb{P}(W_n) \ : \ \mathrm{rk}(S) \leq r\} \subset \mathbb{P}(W_n);$$

the variety $U_r \subset \mathbb{P}(W_n^*)$ is defined in the same way. With this notation we have

$$S_1 = \nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n^2+3n}{2}} = \mathbb{P}(W_n)$$

Veronese embedded and that

$$S_n = V(\det(S)) \subset \mathbb{P}^{\frac{n^2+3n}{2}}$$

is a hypersurface of degree $n+1$.

The rational map ∇_g is not defined along $S_{n-1} = \mathrm{Sing}(S_n)$ and its ramification divisor is given by the determinant of the Jacobian matrix of ∇_g, which is $\mathrm{hess}(g)$. Since $g = \det(S)$ is an irreducible polynomial, we deduce from (31) that the ramification divisor of ∇_g is supported on S_n, yielding

$$\mathrm{hess}(g) = \beta \cdot g^{\frac{(n+2)(n-1)}{2}}$$

with $\beta \in \mathbb{K}^*$. This property was proved in a similar (but not identical) way by B. Segre in [27, Theorem 2]. By evaluating the previous identity on particular matrices B. Segre deduced $\beta = (-1)^{\frac{n(n-1)}{2}} \cdot 2^{\frac{(n+1)n}{2}} \cdot n$, see [27, Teorema 2].

If $[S] \notin S_n$, then $[S^\#] \notin U_n$ while for $[S] \in S_n \setminus S_{n-1}$, $[S^\#] \in U_1$. From this it follows that

$$S_n^* = U_1 = \nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n^2+3n}{2}} = \mathbb{P}(W_n^*).$$

By the definition of the Gauss map, for every $[S] \in S_n \setminus S_{n-1}$, we have $\nabla_g([S]) = [T_{[S]}S_n]$ and $\overline{\nabla_g^{-1}([T_{[S]}S_n])}$ is an $((n^2+n-2)/2)$-dimensional projective space.

We are now ready to prove the next result and its Corollary, whose proofs will be omitted being analogous to those presented in Proposition 4.1 and in Corollary 4.2.

Proposition 4.4. *Let notation be as above, let $n \geq 2$ and let $[S] \in S_n \setminus S_{n-1}$. Then*

$$\overline{\nabla_g(T_{[S]}S_n)}$$

is a hypersurface of degree n which is a cone with vertex

$$\overline{\nabla_f^{-1}([T_{[X]}X_n])}^{\perp} = \mathbb{P}^n$$

over the dual of a Veronese variety $\nu_2(\mathbb{P}^{n-1})$.

Corollary 4.5. *Let notation be as above, let $n \geq 2$, let*

$$p \in S_1 = \nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n^2+3n}{2}} = \mathbb{P}(W_n^*)$$

and let $V_n \subset \mathbb{P}^{\frac{n^2+3n-2}{2}}$ be the projection of $S_1 = \nu_2(\mathbb{P}^n)$ from p. Let $q \in S_n \setminus S_{n-1}$ be such that $T_qS_n = p^\perp$ and let

$$Q_n = S_n \cap p^\perp \subset p^\perp = \mathbb{P}^{\frac{n^2+3n-2}{2}}.$$

Then:

(i) $Q_n \subset \mathbb{P}^{\frac{n^2+3n-2}{2}}$ *is an irreducible hypersurface of degree* $n+1$ *with vanishing hessian and such that* $Q_n^* = V_n$.
(ii) $Z_{Q_n} \subset \mathbb{P}^{\frac{n^2+3n-2}{2}}$ *is a hypersurface of degree* n *which is a cone with vertex a* $\mathbb{P}^{n-1}$ *over the dual of a Veronese variety* $\nu_2(\mathbb{P}^{n-1})$.
(iii) $\operatorname{codim}(Q_n^*, Z_{Q_n}) = \frac{n^2+n-4}{2}$.

Remark 4.6. The varieties $V_n \subset \mathbb{P}^{\frac{n^2+3n-2}{2}}$ are smooth being isomorphic to $\mathrm{Bl}_p \mathbb{P}^n$ and their duals are hypersurfaces with vanishing hessian. The classification of hypersurfaces with vanishing hessian whose dual is smooth seems to be an intriguing question, also due to the lack of known examples.

The first element of the series, $V_2 \subset \mathbb{P}^4$ is nothing but $S(1,2)$. The cubic hypersurface $S(1,2)^* \subset \mathbb{P}^4$, whose equation we computed explicitly in Theorem 3.1, is surely the easiest and simplest counterexample to Hesse's Claim. As far as we know, the fact that this example was the first member of an infinite series of hypersurfaces with vanishing hessian has been noticed by us for the first time several years ago. The very recent paper [6] deals with similar phenomena treated from a purely algebraic point of view.

4.3. Skew-symmetric determinantal Scorza varieties

Let

$$M_n = \{A \in M_{(n+1)\times(n+1)}(\mathbb{K}) \ : \ A = -A^t\} \subset M_{(n+1)\times(n+1)}(\mathbb{K}).$$

For $n \geq 1$ the subspace M_n is not a subalgebra of $M_{(n+1)\times(n+1)}(\mathbb{K})$ but $A^{\#} \in M$ for every $A \in M$. Let $A = [a_{i,j}]$ be the generic matrix in M_n and let $(a_{0,1} : \cdots : a_{n-1,n})$ be the corresponding homogeneous coordinates on

$$\mathbb{P}(M_n) = \mathbb{P}^{\frac{n^2+n-2}{2}}.$$

From now on suppose that $n+1 = 2m+2$ with $m \geq 2$ so that $\frac{n^2+n-2}{2} = 2m^2+3m$ and $n = 2m+1$. Then

$$\det(A) = \mathrm{Pf}^2 \in \mathbb{K}[a_{i,j}]_{2m+2},$$

with $\mathrm{Pf} \in \mathbb{K}[s_{i,j}]_{m+1}$. The operation $\#$ on $M_{(n+1)\times(n+1)}(\mathbb{K})$ induces by restriction to M_{2m+1} a birational involution

$$\nabla_{\mathrm{Pf}} : \mathbb{P}(M_{2m+1}) \dashrightarrow \mathbb{P}(M_{2m+1}^*),$$

defined by $\nabla_{\mathrm{Pf}}([A]) = [A^{\#}]^t$.

For every $r = 1, \ldots, m+1$ we can define the variety

$$A_{2r} = \{[A] \in \mathbb{P}(M_{2m+1}) \ : \ \mathrm{rk}(A) \leq 2r\} \subset \mathbb{P}(M_{2m+1});$$

the variety $C_{2r} \subset \mathbb{P}(M_{2m+1}^*)$ is defined in the same way. With this notation we have

$$A_2 = \mathbb{G}(1, \mathbb{P}^{2m+1}) \subset \mathbb{P}^{2m^2+3m} = \mathbb{P}(M_{2m+1})$$

Plücker embedded and

$$A_{2m} = V(\mathrm{Pf}) \subset \mathbb{P}^{2m^2+3m}$$

is a hypersurface of degree $m+1$.

The rational map ∇_{Pf} is not defined along $A_{2m-2} = \mathrm{Sing}(A_{2m})$ and its ramification divisor is given by the determinant of the Jacobian matrix of ∇_{Pf}, which is hess(Pf). Since Pf is an irreducible polynomial, we deduce from (31) that the ramification divisor of ∇_{Pf} is supported on A_{2m}, yielding

$$\mathrm{hess}(\mathrm{Pf}) = \gamma \cdot \mathrm{Pf}^{(2m+1)(m-1)}$$

with $\gamma \in \mathbb{K}^*$. This property was proved in a similar (but not identical) way by B. Segre in [27, Theorem 3]. By evaluating the previous identity on particular matrices B. Segre deduced $\gamma = (-1)^m \cdot m$, see [27, Teorema 3].

If $[A] \notin A_{2m}$, then $[A^\#] \notin C_{2m}$ while for $[A] \in A_{2m} \setminus A_{2m-2}$, $[A^\#] \in C_2$ so that

$$A^*_{2m} = C_2 = \mathbb{G}(1, \mathbb{P}^{2m+1}) \subset \mathbb{P}^{2m^2+3m} = \mathbb{P}(M^*_{2m+2}).$$

By the definition of the Gauss map, for every $[A] \in A_{2m} \setminus A_{2m-2}$, we have $\nabla_{\mathrm{Pf}}([A]) = [T_{[A]}A_{2m}]$ and $\overline{\nabla^{-1}_{\mathrm{Pf}}([T_{[A]}A_{2m}])}$ is a $(2m^2 - m - 1)$-dimensional projective space.

We are now ready to prove the next result and its Corollary, whose proofs will be omitted being analogous to those presented above.

Proposition 4.7. *Let notation be as above, let $m \geq 2$ and let $[A] \in A_{2m} \setminus A_{2m-2}$. Then*

$$\overline{\nabla_{\mathrm{Pf}}(T_{[A]}A_{2m})}$$

is a hypersuface of degree m which is a cone with vertex

$$\overline{\nabla^{-1}_{f}([T_{[A]}A_{2m}])}^{\perp} = \mathbb{P}^{4m}$$

over the dual of a Grassmann variety $\mathbb{G}(1, \mathbb{P}^{2m-1})$.

Corollary 4.8. *Let notation be as above, let $m \geq 2$, let*

$$p \in C_2 = \mathbb{G}(1, \mathbb{P}^{2m+1}) \subset \mathbb{P}^{2m^2+3m} = \mathbb{P}(M^*_{2m+2})$$

and let $G_m \subset \mathbb{P}^{2m^2+3m-1}$ be the projection of C_2 from p. Let $q \in A_{2m} \setminus A_{2m-2}$ be such that $T_qA_{2m} = p^\perp$ and let

$$F_m = A_{2m} \cap p^\perp \subset p^\perp = \mathbb{P}^{2m^2+3m-1}.$$

Then:

(i) *$F_m \subset \mathbb{P}^{2m^2+3m-1}$ is an irreducible hypersurface of degree $m+1$ with vanishing hessian and such that $F^*_m = G_m$.*
(ii) *$Z_{F_m} \subset \mathbb{P}^{2m^2+3m-1}$ is a hypersurface of degree m which is a cone with vertex a $\mathbb{P}^{4m-1}$ over the dual of a Grassmann variety $\mathbb{G}(1, \mathbb{P}^{2m-1})$.*
(iii) $\mathrm{codim}(F^*_m, Z_{F_m}) = 2m^2 - m - 2$.

Remark 4.9. The three series of hypersurfaces R_n, Q_n and F_m are of Gordan-Noether-Perazzo-Permutti-CRS type and such that the duals of their polar images are of the same type of their duals. Indeed, the first property easily follows from Laplace formula and by the determinantal description of their equation in a suitable coordinate system while Z^* is of the same type by part (ii) of the previous Corollaries 4.2, 4.5, and 4.8.

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