Topology and its Applications 268 (2019) 106898

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On Rančin's problem

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ARTICLE INFO

Article history:
Received 4 January 2017
Received in revised form 19 June 2019
Accepted 16 September 2019
Available online 25 September 2019

MSC:
54A25
54A35
54D55

Keywords:
Compact spaces
Countable tightness
c-Sequential
Sequential
Hušek number
Cardinality

ABSTRACT

Few observations on a paper of Arhangel'skiĭ and Buzyakova led us to consider Rančin's problem. The main result here is the construction under $\diamondsuit$ of a compact c-sequential space that is not sequential.

1. Hušek number and c-sequentiality

All spaces are assumed T_2. For undefined notions we refer to [6]. Given a space X and a point $x \in X$, the Hušek number $Hus(x, X)$ (as defined in [1]) is the smallest cardinal κ such that for any set $A \subseteq X \setminus \{x\}$ of regular cardinality $|A| \geq \kappa$ there exists an open neighborhood U of x such that $|A| = |A \setminus U|$. Clearly, we always have $Hus(x, X) \leq \psi(x, X)^+$. As is standard, $Hus(X) = \sup\{Hus(x, X) : x \in X\}$.

A space is linearly Lindelöf if every open cover which is totally ordered by inclusion has a countable subcover. Equivalently, X is linearly Lindelöf if every subset of uncountable regular cardinality has a complete accumulation point.

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[1] The research that led to the present paper was partially supported by a grant of the group GNSAGA of INdAM.
[2] The research was supported by the NSF grant No. NSF-DMS 1501506.

https://doi.org/10.1016/j.topol.2019.106898
0166-8641/

Proposition 1. *[1] (Proposition 4) Let X be a compact space and $x \in X$. Then $Hus(x, X) \leq \omega_1$ if and only if $X \setminus \{x\}$ is linearly Lindelöf.*

Since a compact space of uncountable tightness contains an uncountable convergent free sequence [11], we immediately get:

Proposition 2. *A compact space X such that $Hus(X) \leq \omega_1$ has countable tightness.*

Since there are locally compact linearly Lindelöf spaces which are not Lindelöf [12] and [13], a compact space X satisfying $Hus(x, X) \leq \omega_1$ may fail to be first countable at x. However, the following remains open:

Question 3. *[1] Is a compact space X satisfying $Hus(X) \leq \omega_1$ always first countable?*

Arhangel'skiĭ and Buzyakova pointed out in [1] (Theorem 6) that there is a positive answer to Question 3 under CH. This result can be improved as follows:

Proposition 4 *($2^{\aleph_0} < \aleph_\omega$). A compact space X satisfying $Hus(X) \leq \omega_1$ is first countable.*

Proof. If X is not first countable, then there is a set $A \subseteq X$ such that $|A| \leq \omega_1$ and $\chi(p, \overline{A}) \geq \omega_1$ for some $p \in \overline{A}$ (see 6.14b in [9]). Since X is countably tight, the weight of the subspace $\overline{A}$ does not exceed $2^{\aleph_0} < \aleph_\omega$. Thus, $\chi(p, \overline{A})$ is an uncountable regular cardinal κ. Now, the compactness of $\overline{A}$ implies the existence of a sequence of length κ in $\overline{A} \setminus \{p\}$ converging to p. As $Hus(\overline{A}) \leq Hus(X)$, we reach a contradiction. □

A weaker question is:

Question 5. *[1] (Question 4) Let X be a compact space such that $Hus(X) \leq \omega_1$. Is it true that $|X| \leq 2^{\aleph_0}$?*

Recall that a space X is tame if $|\overline{A}| \leq 2^{|A|}$ holds for every $A \subseteq X$ [10]. Here we call a space X countably tame if every separable subspace has cardinality at most the continuum. Of course every sequential space is tame.

Proposition 6. *Let X be a compact space satisfying $Hus(X) \leq \omega_1$. If X is countably tame, then $|X| \leq 2^{\aleph_0}$.*

Proof. Assume by contradiction that $|X| > 2^{\aleph_0}$. Since X has countable tightness and is countably tame, there exists a closed subspace Y satisfying $|Y| = (2^{\aleph_0})^+$. Since a space is linearly Lindelöf if and only if every open cover has a subcover of countable cofinality, we see that a linearly Lindelöf space of cardinality $(2^{\aleph_0})^+$ has Lindelöf degree not exceeding $2^{\aleph_0}$. Therefore for every $x \in Y$ we must have $\chi(x, Y) \leq 2^{\aleph_0}$. For each $x \in Y$ let $\mathcal{U}_x$ be a base of open neighborhoods at x satisfying $|\mathcal{U}_x| \leq 2^{\aleph_0}$. Since Y is countably tight and countably tame, we can construct a non-decreasing collection $\{F_\alpha : \alpha < \omega_1\}$ of closed subsets of Y in such a way that:

1) $|F_\alpha| \leq 2^{\aleph_0}$ for each α;

2) if $Y \setminus \bigcup \mathcal{V} \neq \emptyset$ for a finite $\mathcal{V} \subseteq \bigcup\{\mathcal{U}_x : x \in F_\alpha\}$, then $F_{\alpha+1} \setminus \bigcup \mathcal{V} \neq \emptyset$.

As Y has countable tightness, the set $F = \bigcup\{F_\alpha : \alpha < \omega_1\}$ is closed. Since $|F| \leq 2^{\aleph_0}$, we must have $F \neq Y$. Now, the usual closing-off argument leads to a contradiction with condition 2. □

A space X is c-sequential [15] if for any closed set $F \subseteq X$ and any non-isolated point $x \in F$ there is a sequence in $F \setminus \{x\}$ converging to x.

Proposition 7. *[1] (Theorem 13) A countably compact space X satisfying $Hus(X) \leq \omega_1$ is c-sequential.*

In [1], page 163, the authors claimed that Martin's Axiom implies that a compact c-sequential space is sequential. They then conclude (Corollary 14) that under Martin's Axiom every compact space X satisfying $Hus(X) \leq \omega_1$ is sequential. While the latter assertion may well be true (even in ZFC), the former is false. As we will see in the next section, even CH is not enough.

2. Rančin's problem

Rančin in [15] formulated the following:

Question 8. *Is a compact c-sequential space sequential?*

The fact that a compact space of uncountable tightness has a convergent uncountable free sequence [11] implies that a compact c-sequential space is countably tight. Hence, Rančin's question has a positive answer under PFA [2] and in some models of CH [7]. Malykhin announced in 1990 [14] the existence of a counterexample in a model satisfying $(t) + 2^{\omega} < 2^{\omega_1}$, but he never published this result. During the preparation of this note, he replied to a request for more information about it by saying "I left topology in 1999 and I do not remember if I have ever proved that fact". However, a much stronger counterexample (also in a model in which Martin's Axiom fails) of a compact C-closed non-sequential space is described in [5]. A space X is C-closed [8] if every countably compact subset is closed. A C-closed space is necessarily c-sequential. Here we will present a negative answer to Rančin's problem under $\diamondsuit$. Notice that, in this model every compact C-closed space is sequential.

Theorem 9. $\diamondsuit$ *implies there exists a compact c-sequential space that is not sequential.*

The remainder of this section is dedicated to the proof of this theorem. We will construct a closed subset X of the uncountable product 2^{ω_1} as the inverse limit of the system $\langle X_\alpha : \alpha \in \omega_1 \rangle$ with the usual projection maps being the bonding maps. One could think of the construction of Fedorchuk's space as a good prototype.

We will ensure that X has cardinality $\aleph_1$ and is the union of two disjoint subsets. There will be a dense countably compact subset of points of countable character. These will be identified and labeled as the points $\{x_\alpha : \alpha \in \omega_1\}$. This set of points will be dense but proper, and since it is countably compact this ensures that X is not sequential.

The complement, call it Y, in X of that dense first countable subset will be indexed as $\{y_\alpha : \alpha \in \omega_1\}$. We will ensure that any subset of the dense first countable subset that is not compact, will have infinitely many of the y_α in its closure. Also, we ensure that if A is a non-discrete subset of $\{y_\alpha : \alpha \in \omega_1\}$ then each non-isolated point of A will be the limit of a converging sequence from A.

These properties ensure that X is c-sequential. Indeed, suppose that $F \subset X$ is closed and let z be a non-isolated point of F. We have to show there is a sequence from F converging to z. If z has countable character, then this is obvious. This means that z is equal to y_α for some $\alpha \in \omega_1$. Also let A denote the set of y_β that are in F. By our assumption, A will have a sequence converging to y_α if we prove that y_α is a limit point of A. To see this, let W be any clopen neighborhood of y_α. Since y_α is a limit point of $W \cap F$, we have that $W \cap F \cap \{x_\beta : \beta \in \omega_1\}$ is not compact, and by assumption, has infinitely many limit points in A.

Let E denote the stationary set consisting of limit of limits. Let $\{L_\xi : \xi \in \omega_1 \setminus E\}$ enumerate the infinite countable subsets of ω_1 in such a way that $L_\xi \subset \xi$. For technical convenience we arrange that for each $\beta \in E$ and $\ell \in \omega$, $L_{\beta+\ell} = \omega$.

Suppose that there is a partition $\{E_0, E_1, E_2\}$ of E into disjoint stationary sets, and that there is a sequence $\{a_\alpha : \alpha \in \omega_1\}$ such that, for each α, a_α is a subset of α, and for all sets $A \subset \omega_1$, the set $E_i(A) = \{\delta \in E_i : a_\delta = A \cap \delta\}$ is stationary for each $i = 0, 1, 2$. We omit the straightforward verification that this assumption is equivalent to $\diamondsuit$.

As a technical device, for each $\beta \in \omega_1$, let e_β be any bijection from β to ω.

We define, (as we said), $X_\alpha \subset 2^\alpha$, as well as, $x^\alpha_\beta, y^\alpha_\beta \in X_\alpha$ (for $\beta \le \alpha$). We also define countable sets $\tau_\alpha \subset \alpha$ and ordinals γ_α satisfying these inductive assumptions (the role of the τ_α are to ensure that there are converging sequences in Y). For each $\omega \le \delta \le \beta \le \alpha$,

(1) X_α is a compact subset of 2^α that projects onto X_β,
(2) $X_n = 2^n$ for all $n \in \omega$ and $X_\omega = 2^\omega$,
(3) $\{x^\omega_n : n \in \omega\}$ and $\{y^\omega_n : n \in \omega\}$ are arbitrary disjoint dense subsets of X_ω,
(4) $x^\alpha_\beta, y^\alpha_\beta$ are points in X_α such that $x^\alpha_\beta \restriction \beta = x^\beta_\beta$ and $y^\alpha_\beta \restriction \beta = y^\beta_\beta$,
(5) x^α_β is the only point in X_α that projects onto x^β_β,
(6) $\{x^\alpha_\xi : \xi \le \alpha\}$ and $\{y^\alpha_\xi : \xi \le \alpha\}$ are disjoint and dense in X_α,
(7) if $\beta < \alpha$, then the set $\{x^\alpha_\xi : \xi \in L_\beta\}$ has a limit point in $\{x^\alpha_\gamma : \gamma \le \alpha\}$,
(8) τ_β is an infinite subset of β, and $\{y^\alpha_\xi : \xi \in \tau_\beta\}$ converges to $y^\alpha_{\gamma_\beta}$,
(9) if $\alpha \in E_0$, and if the point χ_{a_α} (the characteristic function of a_α) is a point of X_α that is not an element of $\{y^\alpha_\beta : \beta < \alpha\}$, then x^α_α is chosen to be χ_{a_α},
(10) if $\delta \in E_1$ and if there is a unique $\zeta_\delta < \delta$ such that $y^\delta_{\zeta_\delta}$ is in the closure of $\{x^\delta_\xi : \xi \in a_\delta\}$, then, for all $\ell \in \omega$ such that $\delta + \ell \le \alpha$, each of the points $y^\alpha_{\zeta_\delta}$ and $y^\alpha_{\delta+\ell}$ are limits of $\{x^\alpha_\xi : \xi \in a_\delta\}$,
(11) if $\delta \in E_2$ and if $\gamma < \delta$ is such that y^δ_γ is a limit point of $\{y^\delta_\xi : \xi \in a_\delta\}$, then
    (a) if $\alpha < \delta + e_\delta(\gamma)$, then y^α_γ is still a limit point of $\{y^\alpha_\xi : \xi \in a_\delta\}$, and
    (b) if $\beta = \delta + e_\delta(\gamma) \le \alpha$, then $\gamma_\beta = \gamma$ and $\tau_\beta \subset a_\delta$ and (by clause 8) $\{y^\alpha_\xi : \xi \in \tau_\beta\}$ converges to y^α_γ.

Before actually carrying out the induction, let us verify that the resulting space $X_{\omega_1} = X$ has the desired properties. Naturally, for each $\beta \in \omega_1$, we let the points x_β and y_β respectively, denote the limit of the cohering sequences $\{x^\alpha_\beta : \beta \le \alpha \in \omega_1\}$ and $\{y^\alpha_\beta : \beta \le \alpha \in \omega_1\}$.

Clause (5) ensures that each x_β is a G_δ and so, a point of countable character in X. Clause (7) ensures that the subset $\{x_\beta : \beta \in \omega_1\}$ is countably compact. As above, let Y denote the set $\{y_\beta : \beta \in \omega_1\}$. Now we show that $X \setminus Y$ is the set $\{x_\beta : \beta \in \omega_1\}$. Let x be any point of $X \setminus Y$ and let A denote the set of values $\xi \in \omega_1$ such that $x(\xi) = 1$. In other words, x is the characteristic function of A. Recall that $E_0(A)$ is a stationary subset of E_0 and this is the set of $\delta \in E_0$ such that $a_\delta = A \cap \delta$. Since $x \in X$ we have that $x \restriction \delta$ is a point of X_δ for all δ. Assume there is a $\delta \in E_0(A)$ such that $x \restriction \delta$ is not an element of $\{y^\delta_\beta : \beta < \delta\}$. By property (9), we then have that $x^\delta_\delta = x \restriction \delta$, and then by property (5), $x = x_\delta$. So suppose there is no such δ. Then, for each $\delta \in E_0(A)$, there is $\beta_\delta < \delta$ such that $x \restriction \delta = y^\delta_{\beta_\delta}$. By the pressing down lemma, there is (essentially) a single such β. But then it follows that $x = y_\beta$.

Now we just have to prove those two properties of Y described in the third paragraph. First, suppose that $A \subset \omega_1$ satisfies that $\{x_\alpha : \alpha \in A\}$ is a closed but not compact subset of $X \setminus Y$. Of course this means that there is a $\beta \in \omega_1$ such that y_β is a limit point of $\{x_\alpha : \alpha \in A\}$. We have to prove that this set has infinitely many limit points in Y. In fact, by intersecting with a clopen neighborhood of y_β, it is easy to see that it suffices to prove that it has more than one limit. If y_β were the unique limit of $\{x_\alpha : \alpha \in A\}$, then there is a cub C satisfying that for all $\delta \in C$, the point y^δ_β is the unique limit point in $\{y^\delta_\xi : \xi \in \delta\}$ of the set $\{x^\delta_\alpha : \alpha \in A \cap \delta\}$. This follows from the fact that we just need witnessing basic clopen neighborhoods with support below δ for each $\beta \neq \xi \in \delta$. Choose any $\delta \in E_1(A) \cap C$. However, now property (10) ensures that, for some $\delta \in C$, $\{x_\xi : \xi \in a_\delta \subset A\}$ has infinitely many limits in Y.

Finally we consider a subset A of ω_1 such that $\{y_\alpha : \alpha \in A\}$ is not discrete. Fix any $\beta \in \omega_1$ such that y_β is a limit. Again, it is a basic exercise to show that there is a cub C such that for all $\delta \in C$, $y_\beta \restriction \delta$ is a limit of the set $\{y_\xi \restriction \delta : \xi \in A \cap \delta\}$. Here is the proof of that (not using elementary submodels): define an increasing function f from ω_1 to ω_1 so that for each $\gamma \in \omega_1$ and each finite $H \subset \gamma$, $f(\gamma)$ is large enough so that there is a $\xi \in A \cap f(\gamma)$ such that y_ξ is in the basic clopen neighborhood y_β obtained by restricting to the coordinates in H. If δ satisfies that $f(\gamma) < \delta$ for all $\gamma < \delta$, then $\delta \in C$. Now choose any $\delta \in E_2(A) \cap C$

and check that clause (11) guarantees that with $\ell = e_\delta(\beta)$, $L_{\delta+\ell} \subset A \cap \delta$ and the sequence $\{y_\xi : \xi \in L_{\delta+\ell}\}$ converges to y_β.

Now it remains to carry out the induction. We can use this next lemma in each step.

Lemma 10. *Assume that X is a compact 0-dimensional metric space. Let z be any non-isolated point of X. Assume that $\{\sigma_n : n \in \omega\}$ are sequences that converge to z, and that $\{\tau_n : n \in \omega\}$ are sets that have z as a limit. Further assume that for each $n, m \in \omega$, z is a limit of $\tau_n \setminus \bigcup\{\sigma_k : k < m\}$. Then there is a partition U, W of $X \setminus \{z\}$ into non-compact open sets satisfying that for each $n \in \omega$ σ_n is almost contained in U, while, for each n, m, z is a limit point of each of $U \cap \tau_n \setminus \bigcup\{\sigma_k : k < m\}$ and $W \cap \tau_n \setminus \bigcup\{\sigma_k : k < m\}$.*

Proof. Let $\{B_\ell : \ell \in \omega\}$ enumerate the family of all compact open subsets of $X \setminus \{z\}$. For convenience, let A_ℓ denote the union of the family $\{B_k : k \leq \ell\}$. Another assumption that we make for convenience is that we assume that the family of sequences $\{\sigma_k : k \in \omega\}$ is increasing. We will recursively choose a sequence $\{\ell_k : k \in \omega\}$ so that the sequence $\{B_{\ell_k} : k \in \omega\}$ is pairwise disjoint and converges to z. That is, if W is the union of any infinite subsequence of this sequence then we will have that $U = X \setminus (\{z\} \cup W)$ will be open. Choose ℓ_0 to be minimal such that B_{ℓ_0} meets τ_0 and is disjoint from σ_0. At stage k, we choose ℓ_k to be minimal so that

(1) B_{ℓ_k} is disjoint from $A_{\ell_{k-1}}$,
(2) B_{ℓ_k} is disjoint from σ_k,
(3) B_{ℓ_k} meets τ_j for each $j \leq k$.

To see there is such a value ℓ, we just note that z is a limit of each of the sets $\tau_j \setminus \big(\sigma_k \cup A_{\ell_{k-1}}\big)$. For each such $j \leq k$, choose a point z^k_j from each of these sets, and there is an ℓ such that $\{z^k_j : j \leq k\} \subset B_\ell$ while B_ℓ is disjoint from $\sigma_k \cup A_{\ell_{k-1}}$.

Finally, set W equal to $\bigcup\{B_{\ell_{2k}} : k \in \omega\}$. By construction W is almost disjoint from each σ_n. Additionally, W meets $\tau_n \setminus (\sigma_k \cup A_k)$ for each pair n, k and so $W \cap \tau_n \setminus \sigma_k$ has z in its closure. It follows similarly that $U \cap \tau_n \setminus \sigma_k$ has z in its closure for each n, k. $\square$

Now we show how to select $\{x^\alpha_\beta : \beta \leq \alpha\}$, $\{y^\alpha_\beta : \beta \leq \alpha\}$ and $\tau_\alpha, \gamma_\alpha$ depending on the value of $\omega \leq \alpha \in \omega_1$. Let δ denote the largest limit such that $\delta \leq \alpha$ and let $\bar{\ell} \in \omega$ be fixed so that $\alpha = \delta + \bar{\ell}$. If $\delta = \alpha$, then let X_α denote the intersection of the family $\{X_\beta \times 2^{\alpha\setminus\beta} : \beta < \alpha\}$ (i.e. the inverse limit). Also, for each $\beta < \alpha$, let $x^\alpha_\beta, y^\alpha_\beta$ denote the unique points in X_α satisfying that $x^\alpha_\beta \restriction \gamma = x^\gamma_\beta$ and $y^\alpha_\beta \restriction \gamma = y^\gamma_\beta$ for each $\beta < \gamma < \alpha$.

We proceed in cases:

Case 1.1: $\alpha = \delta \notin E$. Clearly items (9)-(11) do not apply in this case. Since the closure of $\{y^\alpha_n : n \in \omega\}$ maps onto X_ω, we can choose a point $y^\alpha_\alpha \notin \{x^\alpha_\beta : \beta < \alpha\}$ in this closure. Also choose $\tau_\alpha \subset \omega$ so that $\{y^\alpha_n : n \in \tau_\alpha\}$ converges to y^α_α, and set $\gamma_\alpha = \alpha$. Similarly we can choose $x^\alpha_\alpha \in X_\alpha$ simply so that it is not in $\{y^\alpha_\beta : \beta \leq \alpha\}$. Now we verify the inductive conditions (1)-(8). Items (1)-(4) and item (6) are immediate. Item (5) holds by the induction hypothesis and because we are at a limit step. Item (7) is vacuous, and τ_α was chosen so that (8) holds when we set $\gamma_\alpha = \alpha$.

Case 1.2: If $0 < \bar{\ell}$, then let β be the predecessor of α. Clearly we have already chosen points x^β_ξ, y^β_ξ in X_β for all $\xi \leq \beta$. Since $\delta \notin E$, the main task is to ensure item (7). If $\{x^\beta_\xi : \xi \in L_\beta\}$ has a limit point z that is not in $\{y^\beta_\xi : \xi \leq \beta\}$, then the construction is trivial. We let $X_\alpha = X_\beta \times \{0\}$ and, for all $\xi \leq \beta$, both x^α_ξ and y^α_ξ are the unique points of X_α that project onto x^β_ξ, y^β_ξ respectively. We let x^α_α denote the unique point of X_α that projects onto z. The choice of y^α_α is again taken to be any limit point (not among $\{x^\alpha_\xi : \xi \leq \alpha\}$) of $\{y^\alpha_n : n \in \omega\}$ and we let $\tau_\alpha \subset \omega$ be chosen so that $\{y^\alpha_n : n \in \tau_\alpha\}$ converges to y^α_α. Set $\gamma_\alpha = \alpha$. The verification of the inductive conditions proceeds as in Case 1.1.

So now assume that z is in the set $\{y^\beta_\xi : \xi \leq \beta\}$. It is possible that z is the unique limit of the set $\{x^\beta_\xi : \xi \in L_\beta\}$ and so we will "double" the point z before assigning a value to x^α_α. Let $\{\sigma_n : n \in \omega\}$ enumerate the (possibly finitely) many sequences of the form $\{y^\beta_\xi : \xi \in \tau_\gamma\}$ $(\gamma \leq \beta)$ that converge to z. Notice that $\{x^\beta_\xi : \xi \in L_\beta\}$ is disjoint from each σ_n. Apply Lemma 10 to choose the open subsets U and W of $X_\beta \setminus \{z\}$ as indicated in the conclusion of the Lemma, namely that $W \cap \{x^\beta_\xi : \xi \in L_\beta\}$ has z as a limit point, and that U mod finite contains σ_n for each n as well as having that z is a limit of $U \cap \{x^\beta_\xi : \xi \in L_\beta\}$. We define X_α to be $((U \cup \{z\}) \times \{0\}) \cup ((W \cup \{z\}) \times \{1\})$ as a subspace of 2^α. We define y^α_α to equal $z^\frown 0$ and we let x^α_α be $z^\frown 1$. Similarly, y^α_ξ is equal to y^α_α for any $\xi < \alpha$ such that y^β_ξ is equal to z. Evidently, every point of $X_\beta \setminus \{z\}$ has a unique extension in X_α, hence the definition of x^α_ξ for all $\xi < \alpha$ and similarly for all $y^\alpha_\xi \neq z$ is clear. By the induction assumption (6), we can choose a sequence $\tau_\alpha \subset \alpha$ so that $\{y^\alpha_\xi : \xi \in \tau_\alpha\}$ converges to y^α_α as required in item (8), and set $\gamma_\alpha = \alpha$.

Case 2.1: $\delta = \alpha \in E_0$. In this case we have already defined X_α and all the points in $\{x^\alpha_\xi, y^\alpha_\xi : \xi < \alpha\}$. If χ_{a_α} is as described in item (9), then x^α_α is equal to χ_{a_α}. Otherwise, we let x^α_α be any point of $X_\alpha \setminus \{y^\alpha_\xi : \xi < \alpha\}$. Next, let y^α_α be any point of $X_\alpha \setminus \{x^\alpha_\xi : \xi \leq \alpha\}$ and choose $\tau_\alpha \subset \alpha$ so that $\{y^\alpha_\xi : \xi \in \tau_\alpha\}$ converges to y^α_α.

Case 2.2: $\delta \in E_0$ and $\delta < \alpha$. There are few requirements for this case. Let $\alpha = \beta + 1$ and set X_α equal $X_\beta \times \{0\}$. For each $\xi < \alpha$ the definitions of x^α_ξ and y^α_ξ are immediate. Choose $x^\alpha_\alpha, y^\alpha_\alpha$ to be distinct points of $X_\alpha \setminus \{x^\alpha_\xi, y^\alpha_\xi : \xi < \alpha\}$. Finally, choose $\tau_\alpha \subset \alpha$ so that $\{y^\alpha_n : n \in \tau_\alpha\}$ converges to y^α_α.

Case 3.1: δ is in E_1 and there is a unique $\zeta_\delta < \delta$ such that $y^\delta_{\zeta_\delta}$ is in the closure of $\{x^\delta_\xi : \xi \in a_\delta\}$. If $\alpha = \delta$, then let y^α_α be equal to $y^\delta_{\zeta_\delta}$ and also let $\tau_\alpha = \tau_{\zeta_\delta}$. Choose any x^α_α in $X_\alpha \setminus \{y^\alpha_\xi : \xi \leq \alpha\}$.

If $\alpha = \beta+1$, then let z denote $y^\beta_{\zeta_\delta}$ and apply Lemma 10 to choose disjoint open U, W so that $\{y^\beta_\xi : \xi \in \tau_\gamma\}$ is almost contained in U for all $\gamma < \alpha$ for which $y^\alpha_\gamma = y^\alpha_{\zeta_\delta}$. Also, by Lemma 10, ensure that z is a limit of each of $U \cap \{x^\delta_\xi : \xi \in a_\delta\}$ and $W \cap \{x^\delta_\xi : \xi \in a_\delta\}$. Define X_α to equal $((U \cup \{z\}) \times \{0\}) \cup ((W \cup \{z\}) \times \{1\})$ and set $y^\alpha_{\zeta_\delta} = z^\frown 0$ and $y^\alpha_\alpha = z^\frown 1$. Choose $\tau_\alpha \subset \alpha$ as usual, as well as x^α_α in $X_\alpha \setminus \{y^\alpha_\xi : \xi \leq \beta\}$. By our assumption that L_β is equal to ω, item (7) is immediate.

Case 3.2: $\delta \in E_1$ and it is not the case that there is a unique $\zeta_\delta < \delta$ such that $y^\delta_{\zeta_\delta}$ is in the closure of $\{x^\delta_\xi : \xi \in a_\delta\}$. Then we proceed as in Case 2.2. If $\delta < \beta+1 = \alpha$, then set X_α equal $X_\beta \times \{0\}$. For each $\xi < \alpha$ the definitions of x^α_ξ and y^α_ξ are immediate. Choose $x^\alpha_\alpha, y^\alpha_\alpha$ to be distinct points of $X_\alpha \setminus \{x^\alpha_\xi, y^\alpha_\xi : \xi < \alpha\}$. Finally, choose $\tau_\alpha \subset \alpha$ so that $\{y^\alpha_\xi : \xi \in \tau_\alpha\}$ converges to y^α_α.

Case 4. $\delta \in E_2$. For easier reference we restate the key requirements for this case:

(a) if $\alpha < \delta + e_\delta(\gamma)$, then y^α_γ is still a limit point of $\{y^\alpha_\xi : \xi \in a_\delta\}$, and
(b) if $\beta = \delta + e_\delta(\gamma) \leq \alpha$, then $\gamma_\beta = \gamma$ and $\tau_\beta \subset a_\delta$ and (by clause 8) $\{y^\alpha_\xi : \xi \in \tau_\beta\}$ converges to y^α_γ.

If $\alpha = \delta$, we have already defined X_α. Otherwise, choose β so that $\alpha = \beta + 1$, and define X_α to be $X_\beta \times \{0\}$. For all $\gamma < \alpha$, define x^α_γ and y^α_γ in the obvious way. We have clearly preserved the inductive requirement that $\{y^\alpha_\xi : \xi \in \tau_\zeta\}$ converges to $y^\alpha_{\gamma_\zeta}$ for all $\zeta < \alpha$. It is also immediate that we have preserved that y^α_ζ is a limit of $\{y^\alpha_\xi : \xi \in a_\delta\}$ for any $\zeta < \delta$ such that y^δ_ζ was a limit of $\{y^\delta_\xi : \xi \in a_\delta\}$ for any $\zeta < \delta$. Choose $\gamma < \delta$ so that $e_\delta(\gamma) = \bar{\ell}$. We have, by induction assumption, that y^α_γ is a limit point of $\{y^\alpha_\xi : \xi \in a_\delta\}$, so choose $\tau_\alpha \subset a_\delta$ so that $\{y^\alpha_\xi : \xi \in \tau_\alpha\}$ converges to y^α_γ and set $\gamma_\alpha = \gamma$. Choose $y^\alpha_\alpha \in X_\alpha \setminus \{x^\alpha_\xi : \xi < \alpha\}$ arbitrarily. Similarly choose $x^\alpha_\alpha \in X_\alpha \setminus \{y^\alpha_\xi : \xi \leq \alpha\}$.

This completes the proof of Theorem 9.

3. One more remark

Recall that a space X is said to be weakly Whyburn provided that for any non-closed set A there is a set $B \subseteq A$ such that $|\overline{B} \setminus A| = 1$. Clearly, a space is sequential if and only if it is weakly Whyburn and c-sequential.

A space X is pseudoradial if for any non-closed set A there is a well-ordered net $S \subseteq A$ converging to a point outside A. In [3] it was observed that any compact weakly Whyburn space is pseudoradial. Much harder it is to show that the previous implication is not reversible [4] (Theorem 2.3). The space we constructed in Theorem 9 is sequentially compact, being a compact space of cardinality $\aleph_1$. Since the continuum hypothesis implies that a compact sequentially compact space is pseudoradial [16], we obtain another example of a compact pseudoradial non weakly Whyburn space. This new example is in addition c-sequential and of size $\aleph_1$.

Notice that, the one-point compactification of Ostaszewski's space provides a compact weakly Whyburn (hence pseudoradial) space of countable tightness which is not c-sequential.

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