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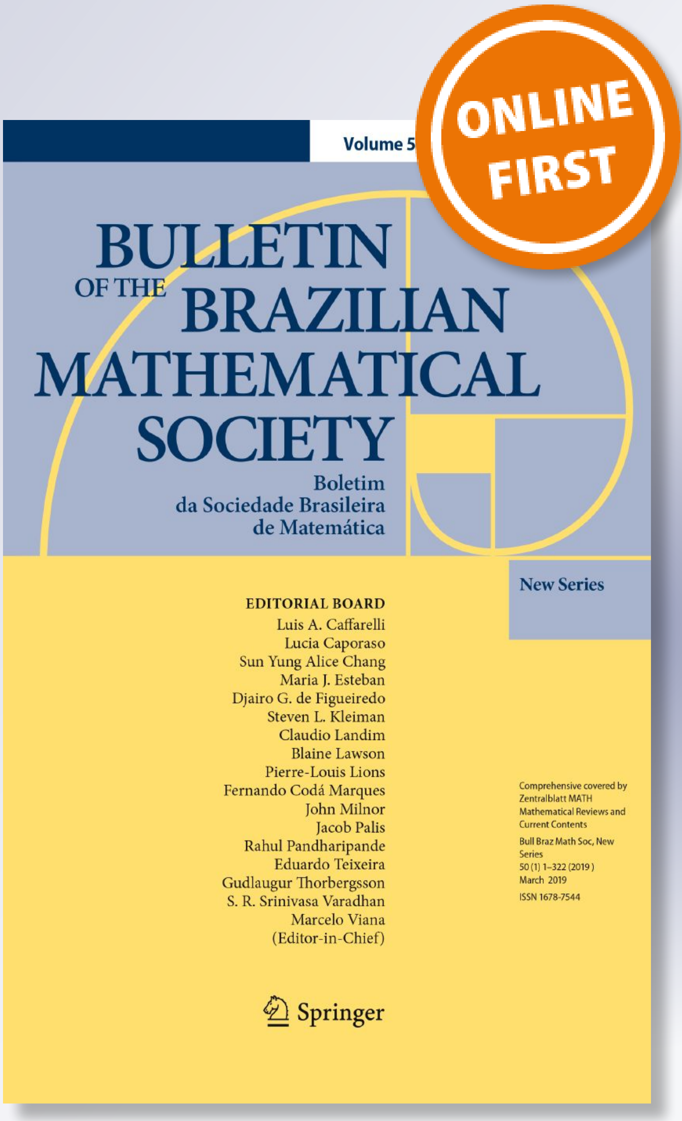

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A Regularity Criterion of Weak Solutions to the 3D Boussinesq Equations

Sadek Gala[1,2] · **Maria Alessandra Ragusa**[2,3]

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Abstract

The paper deals with the regularity criteria for the weak solutions to the 3D Boussinesq equations in terms of the partial derivatives in Besov spaces. It is proved that the weak solution (u, θ) becomes regular provided that

$$(\nabla_h \widetilde{u}, \nabla_h \theta) \in L^1(0, T; \dot{B}^{0}_{\infty,\infty}(\mathbb{R}^3))$$

Our results improve and extend the well-known result of Dong and Zhang (Nonlinear Anal 11:2415–2421, 2010) for the Navier–Stokes equations.

Keywords Boussinesq equations · Regularity criterion · Weak solutions · Besov space

Mathematics Subject Classification 35Q35 · 76D03

1 Introduction and Main Result

This paper is devoted to the study of the Cauchy problem for the Boussinesq equations in $\mathbb{R}^3 \times (0, T)$:

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$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = \theta e_3, \\ \partial_t \theta - \Delta \theta + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \\ u\,(x, 0) = u_0\,(x)\,, \quad \theta\,(x, 0) = \theta_0\,(x)\,, \end{cases} \tag{1.1}$$

where $u = u(x, t)$ is the velocity of the fluid, $\theta = \theta(x, t)$ is the scalar temperature variation in a gravity field, in which case the forcing term θe_3 in the momentum equation (1.1) describes the action of the buoyancy force on fluid motion, $\pi = \pi(x, t)$ is the scalar pressure, while u_0 and θ_0 are given initial velocity and initial temperature with $\nabla \cdot u_0 = 0$ in the sense of distributions. $e_3 = (0, 0, 1)^T$ denotes the vertical unit vector.

The Cauchy problem (1.1) for the Boussinesq equation has been studied extensively by many authors (see, for example, Alghamdi et al. 2017; Abidi and Hmidi 2007; Brandolese and Schonbek 2012; Cannon and Dibenedetto 1980; Chae and Nam 1997; Chae et al. 1999; Dong et al. 2012; Fan and Ozawa 2009; Fan and Zhou 2009; Gala 2011; Gala et al. 2017; Gala and Ragusa 2016b; Gala et al. 2014; Guo and Gala 2012; Hou and Li 2005; Ishimura and Morimoto 1999 and references therein).

When $\theta = 0$, (1.1) is the well-known Navier–Stokes equations, which the global regularity is an outstanding open problem, as well as the famous millennium prize problem. Since the global existence of weak solutions is well-known and strong solutions are unique and smooth in $(0, T)$, it is an interesting problem on the regularity criterion of the weak solutions if some partial derivatives of the velocity satisfy certain growth conditions (see, e.g. Berselli 2002; Dong and Zhang 2010; Gala 2009; Chen and Gala 2011; Gala and Ragusa 2016a; Kukavica and Zinae 2007; Skalák 2015; Zhou 2005; Zhou and Pokorny 2010). One of the most significant achievements in this direction is the celebrated Dong–Zhang criterion (Dong and Zhang 2010). More precisely, they showed that a weak solution with H^1-data is a strong solution provided that

$$\nabla_h \widetilde{u} \in L^1(0, T; \dot{B}^{0}_{\infty,\infty}(\mathbb{R}^3)), \tag{1.2}$$

where $\nabla_h = (\partial_1, \partial_2)$ denotes the horizontal gradient operator, $\widetilde{u} = (u_1, u_2, 0)$ and $\dot{B}^{0}_{\infty,\infty}$ denotes the homogeneous Besov space.

Motivated by the reference mentioned above, our aim of the present paper is to extend the above regularity criterion (1.2) to the Boussinesq equations (1.1).

Our main result reads as follows.

Theorem 1.1 *Suppose $T > 0$, $(u_0, \theta_0) \in H^1(\mathbb{R}^3)$ with* div $u_0 = 0$ *in $\mathbb{R}^3$, in the sense of distributions. Let (u, θ) be a weak solution of (1.1) in $(0, T)$. Assume that*

$$(\nabla_h \widetilde{u}, \nabla_h \theta) \in L^1(0, T; \dot{B}^{0}_{\infty,\infty}(\mathbb{R}^3)), \tag{1.3}$$

then the solution (u, θ) is regular on $\mathbb{R}^3 \times (0, T]$.

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Remark 1.1 If we set $\theta = 0$ in the Boussinesq system, the above theorem reduces to the well-known Dong and Zhang result (Dong and Zhang 2010) for the Navier–Stokes equations.

We start by recalling the basic existence result of weak solutions to the system (1.1), see (Brandolese and Schonbek 2012).

Proposition 1.2 *Let* $(u_0, \theta_0) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *with* $\mathrm{div}\, u_0 = 0$ *in* $\mathbb{R}^3$, *in the sense of distributions. There exists a weak solution* (u, θ) *of the Boussinesq system* (1.1) *with data* (u_0, θ_0) *continuous from* $\mathbb{R}^+$ *to* L^2 *with the weak topology, such that for any* $T > 0$

$$(u, \theta) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)).$$

Such a solution satisfies, for all $t \in [0, T]$, *the energy inequalities*

$$\|\theta(\cdot, t)\|_{L^2}^2 + 2\int_0^t \|\nabla\theta(\cdot, \tau)\|_{L^2}^2\, d\tau \le \|\theta_0\|_{L^2}^2,$$

and

$$\|u(\cdot, t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\cdot, \tau)\|_{L^2}^2\, d\tau \le C(\|u_0\|_{L^2}^2 + t^2\, \|\theta_0\|_{L^2}^2),$$

for all $t \ge 0$ *and some constant* $C > 0$.

Next, we recall some tools from the theories of the Besov spaces, for details see (Triebel 1983). By $\mathcal{S}(\mathbb{R}^3)$ we denote the class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\widehat{f} = \mathcal{F}(f)$ is defined by

$$\widehat{f}(\omega) = \int\limits_{\mathbb{R}^3} f(x) e^{-2\pi i x\cdot\omega} dx$$

and for any given $g \in \mathcal{S}(\mathbb{R}^3)$, its inverse Fourier transform $\widetilde{f} = \mathcal{F}^{-1}(f)$ is defined by

$$\widetilde{f}(x) = \int\limits_{\mathbb{R}^3} f(\omega) e^{2\pi i x\cdot\omega} d\omega.$$

Let us choose a nonnegative radial function $\varphi \in \mathcal{S}(\mathbb{R}^3)$ such that

$$0 \le \widehat{\varphi}(\omega) \le 1 \quad \text{and} \quad \widehat{\varphi}(\omega) = \begin{cases} 1, & \text{if} \quad |\omega| \le 1, \\ 0, & \text{if} \quad |\omega| \ge 2, \end{cases}$$

and let

$$\psi(x) = \varphi(x) - 2^{-3}\varphi\left(\frac{x}{2}\right), \quad \varphi_j(x) = 2^{3j}\varphi(2^j x), \quad \psi_j(x) = 2^{3j}\psi(2^j x), \quad j \in \mathbb{Z}$$

For $j \in \mathbb{Z}$, the Littlewood–Paley projection operators S_j and Δ_j are respectively defined by

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$$S_j f = \varphi_j * f,$$
$$\Delta_j f = \psi_j * f.$$

Informally, Δ_j is a frequency projection to the annulus $\{|\omega| \sim 2^j\}$, while S_j is a frequency projection to the ball $\{|\omega| \lesssim 2^j\}$. Observe that $\Delta_j = S_j - S_{j-1}$. Also, if f is an L^2 function then $S_j f \to 0$ in L^2 as $j \to -\infty$ and $S_j f \to f$ in L^2 as $j \to +\infty$ (this is an easy consequence of Parseval's theorem). By telescoping the series, we thus have the Littlewood–Paley decomposition

$$f = \sum_{j=-\infty}^{+\infty} \Delta_j f$$

for all $f \in L^2$, where the summation is in the L^2 sense. Notice that

$$\Delta_j f = \sum_{l=j-2}^{l=j+2} \Delta_l (\Delta_j f) = \sum_{l=j-2}^{l=j+2} \psi_l * \psi_j * f,$$

then from the Young inequality, it follows that

$$\left\| \Delta_j f \right\|_{L^q} \leq C 2^{3j(\frac{1}{p}-\frac{1}{q})} \left\| \Delta_j f \right\|_{L^p}, \tag{1.4}$$

where $1 \leq p \leq q \leq \infty$, C is a constant independent of f, j.

With the introduction of Δ_j, we recall the definition of the homogeneous Besov space.

Definition 1.3 The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined by

$$\dot{B}^s_{p,q}(\mathbb{R}^3) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3) \backslash \mathcal{P}(\mathbb{R}^3) : \|f\|_{\dot{B}^s_{p,q}} < \infty \right\},$$

for $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. where

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \sum_{j \in \mathbb{Z}} \left(2^{jsq} \left\| \Delta_j f \right\|^q_{L^p} \right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \left\| \Delta_j f \right\|_{L^p}, & \text{if } q = \infty, \end{cases}$$

and $\mathcal{S}'(\mathbb{R}^3)$, $\mathcal{P}(\mathbb{R}^3)$ are the spaces of all tempered distributions on $\mathbb{R}^3$ and the set of all scalar polynomials defined on $\mathbb{R}^3$, respectively.

It is of interest to note that the homogeneous Besov space $\dot{B}^s_{2,2}(\mathbb{R}^3)$ is equivalent to the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$, which is equipped with the norm:

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$$\|f\|^2_{\dot{H}^s} = \sum_{j\in\mathbb{Z}} 2^{2js} \|\Delta_j f\|^2_{L^2}.$$

1.1 Proof of Theorem 1.1

In this section, we shall give the proof of Theorem 1.1, we first need to prove the following lemma.

Lemma 1.4 *Let (u,θ) be a smooth solution to (1.1). Then, there exists a positive universal constant C such that the following a priori estimates hold:*

$$\begin{aligned}
&\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u dx + \int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta\theta dx \\
&\le C\int_{\mathbb{R}^3}|\nabla_h\widetilde{u}|\,|\nabla u|^2\,dx + C\int_{\mathbb{R}^3}|\nabla_h\widetilde{u}|\,|\nabla\theta|^2\,dx \\
&\quad + C\int_{\mathbb{R}^3}|\nabla_h\theta|\,|\nabla u|\,|\nabla\theta|\,dx,
\end{aligned}\tag{1.5}$$

Proof Due to the divergence-free condition $\nabla\cdot u = 0$, one shows that

$$\begin{aligned}
\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i\partial_k u_j\partial_k u_j dx &= \frac{1}{2}\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i(\partial_k u_j)^2 dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^3}\left(\sum_{i=1}^{3}\partial_i u_i\right)\left(\sum_{j,k=1}^{3}(\partial_k u_j)^2\right)dx = 0,
\end{aligned}$$

and

$$\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i\partial_k\theta\partial_k\theta dx = 0.$$

As a consequence, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u dx + \int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta\theta dx \\
&= -\int_{\mathbb{R}^3}\nabla(u\cdot\nabla)u\cdot\nabla u dx - \int_{\mathbb{R}^3}\nabla(u\cdot\nabla)\theta\cdot\nabla\theta dx \\
&= -\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u\cdot\nabla u\cdot\partial_k u dx - \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u\cdot\nabla\theta\cdot\partial_k\theta dx \\
&= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j dx - \sum_{i,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_i\theta\partial_k\theta dx \\
&= L_1 + L_2.
\end{aligned}$$

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To estimate L_1, we write the integrand explicitly

$$\begin{aligned} L_1 = & - \sum_{i,j,k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx - \sum_{i,j=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_i \partial_i u_j \partial_3 u_j dx \\ & - \sum_{i,k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_3 \partial_k u_3 dx - \sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_i \partial_i u_3 \partial_3 u_3 dx \\ & - \sum_{j,k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j dx - \sum_{j=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_j \partial_3 u_j dx \\ & - \sum_{k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_3 \partial_k u_3 dx \\ = & \sum_{m=1}^{7} L_{1m} \end{aligned}$$

Taking advantage of the definition of $\nabla_h \widetilde{u}$, we have

$$\left| \sum_{m=1}^{4} L_{1m} \right| \leq C \int_{\mathbb{R}^3} |\nabla_h \widetilde{u}| \, |\nabla u|^2 \, dx. \tag{1.6}$$

Since $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$, it readily follows that

$$\begin{aligned} \left| \sum_{m=5}^{7} L_{1m} \right| & \leq C \int_{\mathbb{R}^3} |-\partial_1 u_1 - \partial_2 u_2| \, |\nabla u|^2 \, dx \\ & \leq C \int_{\mathbb{R}^3} |\nabla_h \widetilde{u}| \, |\nabla u|^2 \, dx. \end{aligned}$$

Thus, we get

$$\int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u dx \leq C \int_{\mathbb{R}^3} |\nabla_h \widetilde{u}| \, |\nabla u|^2 \, dx.$$

Following the same line as L_1, we see that

$$\begin{aligned} L_2 & = - \sum_{i,k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_i \partial_i \theta \partial_k \theta dx \\ & = - \sum_{i,k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_i \partial_i \theta \partial_k \theta dx - \sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_i \partial_i \theta \partial_3 \theta dx \end{aligned}$$

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$$
\begin{aligned}
& -\sum_{k=1}^{2}\int_{\mathbb{R}^3} \partial_k u_3 \partial_3 \theta \partial_k \theta dx \\
& \leq C\int_{\mathbb{R}^3} |\nabla_h \widetilde{u}|\, |\nabla\theta|^2\, dx + C\int_{\mathbb{R}^3} |\nabla u|\, |\nabla_h\theta|\, |\nabla\theta|\, dx. \qquad (1.7)
\end{aligned}
$$

Then, it follows from (1.6) and (1.7) that estimate (1.5) is established.
This completes the proof of Lemma 1.4. □

We are ready to present the proof of Theorem 1.1.

Proof Since the initial data $(u_0, \theta_0) \in H^1(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in $\mathbb{R}^3$, there exists a unique local strong solution (u, θ) of the 3D Boussinesq equations on $(0, T)$ (see Abidi and Hmidi 2007; Chae and Nam 1997; Chae et al. 1999; Hou and Li 2005). By using a standard method, we only need to show the following a priori estimate

$$
\sup_{0\leq t\leq T}(\|u(\cdot,t)\|_{H^1}^2 + \|\theta(\cdot,t)\|_{H^1}^2) \leq (e + \|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2 + CT)e^{C\mathcal{R}(t)}, \qquad (1.8)
$$

where we set

$$
\mathcal{R}(t) = \int_0^T (\|\nabla_h \widetilde{u}(\tau)\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla_h\theta(\tau)\|_{\dot{B}^0_{\infty,\infty}})d\tau.
$$

Taking the L^2-inner product of the first equation and the second equation in (1.1) with $(-\Delta u)$ and $(-\Delta\theta)$, respectively, and integrating by parts, we obtain by Lemma 1.4

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) + \|\Delta u(t)\|_{L^2}^2 + \|\Delta\theta(t)\|_{L^2}^2 \\
& = \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u dx + \int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta\theta dx - \int_{\mathbb{R}^3}\theta e_3\cdot\Delta u dx \\
& \leq C\int_{\mathbb{R}^3}|\nabla_h\widetilde{u}|\,|\nabla u|^2\,dx + C\int_{\mathbb{R}^3}|\nabla_h\widetilde{u}|\,|\nabla\theta|^2\,dx \\
& \quad + C\int_{\mathbb{R}^3}|\nabla_h\theta|\,|\nabla u|\,|\nabla\theta|\,dx + \|\Delta u\|_{L^2}\,\|\theta\|_{L^2} \\
& \leq C\mathcal{I}_1 + C\mathcal{I}_2 + C\mathcal{I}_3 + \frac{1}{4}\|\Delta u\|_{L^2}^2 + C, \qquad (1.9)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_1 = \int_{\mathbb{R}^3}|\nabla_h\widetilde{u}|\,|\nabla u|^2\,dx,\ \mathcal{I}_2 &= \int_{\mathbb{R}^3}|\nabla_h\widetilde{u}|\,|\nabla\theta|^2\,dx \text{ and } \mathcal{I}_3 \\
&= \int_{\mathbb{R}^3}|\nabla_h\theta|\,|\nabla u|\,|\nabla\theta|\,dx.
\end{aligned}
$$

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In the following, we will estimate each term on right-hand side of (1.9) separately. Invoking the homogeneous Littlewood-Paley decomposition, we decompose $\nabla_h \widetilde{u}$ and $\nabla_h \theta$ into the three parts in the phase variables as follows

$$\begin{cases} \nabla_h \widetilde{u} = \sum_{j=-\infty}^{+\infty} \Delta_j(\nabla_h \widetilde{u}) = \sum_{j<-N} \Delta_j(\nabla_h \widetilde{u}) + \sum_{j=-N}^{N} \Delta_j(\nabla_h \widetilde{u}) + \sum_{j>N} \Delta_j(\nabla_h \widetilde{u}), \\ \nabla_h \theta = \sum_{j=-\infty}^{+\infty} \Delta_j(\nabla_h \theta) = \sum_{j<-M} \Delta_j(\nabla_h \theta) + \sum_{j=-M}^{M} \Delta_j(\nabla_h \theta) + \sum_{j>M} \Delta_j(\nabla_h \theta), \end{cases} \tag{1.10}$$

where N and M are positive integers that will be chosen later. Substituting this into $\mathcal{I}_1$, one has

$$\begin{aligned} \mathcal{I}_1 \le & \sum_{j<-N} \int_{\mathbb{R}^3} \left|\Delta_j(\nabla_h \widetilde{u})\right| |\nabla u|^2 \, dx + \sum_{j=-N}^{N} \int_{\mathbb{R}^3} \left|\Delta_j(\nabla_h \widetilde{u})\right| |\nabla u|^2 \, dx \\ & + \sum_{j>N} \int_{\mathbb{R}^3} \left|\Delta_j(\nabla_h \widetilde{u})\right| |\nabla u|^2 \, dx \\ = & \, \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}. \end{aligned}$$

For $\mathcal{I}_{11}$, from the Hölder inequality, (1.4) and Cauchy inequalities, we obtain that

$$\begin{aligned} \mathcal{I}_{11} & \le \|\nabla u\|_{L^2}^2 \sum_{j<-N} \left\|\Delta_j(\nabla_h \widetilde{u})\right\|_{L^\infty} \\ & \le C \|\nabla u\|_{L^2}^2 \sum_{j<-N} 2^{\frac{3}{2}j} \left\|\Delta_j(\nabla_h \widetilde{u})\right\|_{L^2} \\ & \le C \|\nabla u\|_{L^2}^2 \left(\sum_{j<-N} 2^{3j}\right)^{\frac{1}{2}} \left(\sum_{j<-N} \left\|\Delta_j(\nabla_h \widetilde{u})\right\|_{L^2}^2\right)^{\frac{1}{2}} \\ & \le C 2^{-\frac{3}{2}N} \|\nabla u\|_{L^2}^3 . \end{aligned} \tag{1.11}$$

For $\mathcal{I}_{12}$, by the Hölder inequality, (1.4) and the definition of Besov space, we have

$$\mathcal{I}_{12} \le C \|\nabla u\|_{L^2}^2 \sum_{j=-N}^{N} \left\|\Delta_j(\nabla_h \widetilde{u})\right\|_{L^\infty} \le C N \|\nabla u\|_{L^2}^2 \|\nabla_h \widetilde{u}\|_{\dot{B}^0_{\infty,\infty}} .$$

For $\mathcal{I}_{13}$, from the Hölder inequality, (1.4) and the Gagliardo–Nirenberg inequality, it follows that

$$\begin{aligned} \mathcal{I}_{13} & \le \|\nabla u\|_{L^3}^2 \sum_{j>N} \left\|\Delta_j(\nabla_h \widetilde{u})\right\|_{L^3} \\ & \le C \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \sum_{j>N} 2^{\frac{j}{2}} \left\|\Delta_j(\nabla_h \widetilde{u})\right\|_{L^2} \end{aligned}$$

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$$
\begin{aligned}
&\le C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\left(\sum_{j>N}2^{-j}\right)^{\frac12}\left(\sum_{j>N}2^{2j}\left\|\Delta_j(\nabla_h\widetilde{u})\right\|_{L^2}^2\right)^{\frac12}\\
&\le C2^{-\frac{N}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^2.
\end{aligned}
$$

Combining the above inequalities $\mathcal{I}_{11}, \mathcal{I}_{12}$ and $\mathcal{I}_{13}$ and inserting into $\mathcal{I}_1$, we obtain

$$
\mathcal{I}_1 \le C2^{-\frac32 N}\|\nabla u\|_{L^2}^3 + CN\|\nabla u\|_{L^2}^2\|\nabla_h\widetilde{u}\|_{\dot{B}^0_{\infty,\infty}} + C2^{-\frac{N}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^2.
$$

Following similar computations as in $\mathcal{I}_1$, the term $\mathcal{I}_2$ can be bounded as

$$
\begin{aligned}
\mathcal{I}_2 &\le C2^{-\frac32 N}\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^2 + CN\|\nabla\theta\|_{L^2}^2\|\nabla_h\widetilde{u}\|_{\dot{B}^0_{\infty,\infty}}\\
&\quad + C2^{-\frac{N}{2}}\|\nabla\theta\|_{L^2}\|\Delta u\|_{L^2}\|\Delta\theta\|_{L^2}\\
&\le C2^{-\frac32 N}\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^2 + CN\|\nabla\theta\|_{L^2}^2\|\nabla_h\widetilde{u}\|_{\dot{B}^0_{\infty,\infty}}\\
&\quad + C2^{-\frac{N}{2}}\|\nabla\theta\|_{L^2}(\|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2).
\end{aligned}
$$

Now we turn to estimate $\mathcal{I}_3$. Using the decomposition $(1.10)_2$, $\mathcal{I}_3$ can be written as

$$
\begin{aligned}
\mathcal{I}_3 &\le \sum_{j<M}\int_{\mathbb{R}^3}\left|\Delta_j(\nabla_h\theta)\right||\nabla u||\nabla\theta|\,dx + \sum_{j=-M}^{M}\int_{\mathbb{R}^3}\left|\Delta_j(\nabla_h\theta)\right||\nabla u||\nabla\theta|\,dx\\
&\quad + \sum_{j>M}\int_{\mathbb{R}^3}\left|\Delta_j(\nabla_h\theta)\right||\nabla u||\nabla\theta|\,dx\\
&= \mathcal{I}_{31} + \mathcal{I}_{32} + \mathcal{I}_{33}.
\end{aligned}
$$

Then by the same procedure leading to $\mathcal{I}_1$, we get

$$
\begin{aligned}
\mathcal{I}_{31} &\le \|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}\sum_{j<M}\left\|\Delta_j(\nabla_h\theta)\right\|_{L^\infty}\\
&\le C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}\sum_{j<M}2^{\frac32 j}\left\|\Delta_j(\nabla_h\theta)\right\|_{L^2}\\
&\le C2^{-\frac32 M}\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^2.\\
\mathcal{I}_{32} &\le C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}\sum_{j=-M}^{M}\left\|\Delta_j(\nabla_h\theta)\right\|_{L^\infty}\\
&\le CM\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}\|\nabla_h\theta\|_{\dot{B}^0_{\infty,\infty}}.
\end{aligned}
$$

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$$
\begin{aligned}
\mathcal{I}_{33} &\le \|\nabla u\|_{L^6} \|\nabla\theta\|_{L^2} \sum_{j>M} \left\|\Delta_j(\nabla_h\theta)\right\|_{L^3} \\
&\le C\|\nabla\theta\|_{L^2} \left\|\nabla^2 u\right\|_{L^2} \sum_{j>M} 2^{\frac{j}{2}} \left\|\Delta_j(\nabla_h\theta)\right\|_{L^2} \\
&\le C\|\nabla\theta\|_{L^2} \|\Delta u\|_{L^2} \left(\sum_{j>M} 2^{-j}\right)^{\frac{1}{2}} \left(\sum_{j>M} 2^{2j} \left\|\Delta_j(\nabla_h\theta)\right\|_{L^2}^2\right)^{\frac{1}{2}} \\
&\le C2^{-\frac{M}{2}} \|\nabla\theta\|_{L^2} \|\Delta u\|_{L^2} \|\Delta\theta\|_{L^2} \\
&\le C2^{-\frac{M}{2}} \|\nabla\theta\|_{L^2} (\|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2).
\end{aligned}
$$

Substituting $\mathcal{I}_{31}, \mathcal{I}_{32}$ and $\mathcal{I}_{33}$ into $\mathcal{I}_5$, we obtain

$$
\begin{aligned}
\mathcal{I}_3 \le{}& C2^{-\frac{3}{2}M} \|\nabla u\|_{L^2} \|\nabla\theta\|_{L^2}^2 + CM \|\nabla u\|_{L^2} \|\nabla\theta\|_{L^2} \|\nabla_h\theta\|_{\dot{B}^0_{\infty,\infty}} \\
&+ C2^{-\frac{M}{2}} \|\nabla\theta\|_{L^2} (\|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2).
\end{aligned}
$$

Inserting the above estimates into (1.9), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla\theta(\cdot,t)\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 \\
&\le C(2^{-\frac{3}{2}N} + 2^{-\frac{3}{2}M})(\|\nabla u\|_{L^2}^3 + \|\nabla\theta\|_{L^2}^3) \\
&\quad + C(N+M)(\|\nabla_h\widetilde{u}\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla_h\theta\|_{\dot{B}^0_{\infty,\infty}})(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) \\
&\quad + C(2^{-\frac{N}{2}} + 2^{-\frac{M}{2}})(\|\nabla u\|_{L^2} + \|\nabla\theta\|_{L^2})(\|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2).
\end{aligned}
\tag{1.12}
$$

Now we choose N and M in (1.12) sufficiently large so that

$$
C2^{-\frac{N}{2}}(\|\nabla u\|_{L^2} + \|\nabla\theta\|_{L^2}) \le \frac{1}{4} \quad \text{and} \quad C2^{-\frac{M}{2}}(\|\nabla u\|_{L^2} + \|\nabla\theta\|_{L^2}) \le \frac{1}{4},
$$

that is,

$$
N \ge \frac{2\log^+(C(\|\nabla u\|_{L^2} + \|\nabla\theta\|_{L^2}))}{\log 2} + 4
$$

and

$$
M \ge \frac{2\log^+(C(\|\nabla u\|_{L^2} + \|\nabla\theta\|_{L^2}))}{\log 2} + 4,
$$

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where $\log^+ t = \log t$ for $t > e$ and $\log^+ t = 1$ for $0 < t \le e$. Then (1.12) implies that

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) + \|\Delta u(t)\|_{L^2}^2 + \|\Delta\theta(t)\|_{L^2}^2 \\
&\le C(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)\left(\|\nabla_h \widetilde{u}\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla_h\theta\|_{\dot{B}^0_{\infty,\infty}}\right) \\
&\quad \times \log(e + \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + C.
\end{aligned} \tag{1.13}$$

for all $0 < t < T$. Integrating in time and applying the Gronwall inequality, we infer that

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2 \le C(\|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2 + CT) \\
&\quad \times \exp\left(C\int_0^t (\|\nabla_h\widetilde{u}(\tau)\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla_h\theta(\tau)\|_{\dot{B}^0_{\infty,\infty}})\log(e + \|\nabla u(\tau)\|_{L^2}^2 + \|\nabla\theta(\tau)\|_{L^2}^2)d\tau\right).
\end{aligned} \tag{1.14}$$

Defining

$$\mathcal{Z}(t) = \log(e + \|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla\theta(\cdot,t)\|_{L^2}^2),$$

Substitute the function $\mathcal{Z}(t)$ into (1.14), it follows that

$$\begin{aligned}
\mathcal{Z}(t) &\le \log(e + \|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2 + CT) \\
&\quad \times \exp\left(C\int_0^t (\|\nabla_h\widetilde{u}(\tau)\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla_h\theta(\tau)\|_{\dot{B}^0_{\infty,\infty}})d\tau\right),
\end{aligned} \tag{1.15}$$

Thus, (1.14) yields

$$\begin{aligned}
&\sup_{0\le t\le T}(\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla\theta(\cdot,t)\|_{L^2}^2) \\
&\le (e + \|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2 + CT)^{\exp\left(C\int_0^T (\|\nabla_h\widetilde{u}(\tau)\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla_h\theta(\tau)\|_{\dot{B}^0_{\infty,\infty}})d\tau\right)}.
\end{aligned} \tag{1.16}$$

On the other hand, (u,θ) satisfies the energy inequality, i.e.

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\int_0^t (\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla\theta(\tau)\|_{L^2}^2)d\tau \le \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2, \\
&\forall 0 \le t \le T.
\end{aligned} \tag{1.17}$$

From (1.16) and (1.17), we obtain the desired estimate (1.8). Therefore, by the standard arguments of continuation of local solutions, it is easy to conclude that the estimate

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(1.8) implies that the solution $(u(x,t), \theta(x,t))$ can be smoothly extended beyond T. This completes the proof of Theorem 1.1. □

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