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Optimal road maintenance investment in traffic networks with random demands

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Abstract
The paper deals with a traffic network with random demands in which some of the roads need maintenance jobs. Due to budget constraints, a central authority has to choose which of them are to be maintained in order to decrease as much as possible the average total travel time spent by all the users, assuming that the network flows are distributed according to the Wardrop equilibrium principle. This optimal road maintenance problem is modeled as an integer nonlinear program, where the objective function evaluation is based on the solution of a stochastic variational inequality. We propose a regularization and approximation procedure for its computation and prove its convergence. Finally, the results of preliminary numerical experiments on some test networks are reported.

Keywords Traffic network equilibrium · Random demand · Stochastic variational inequality · Investment optimization

1 Introduction

In this paper we deal with the problem of optimizing road maintenance investments. Indeed, in a traffic network some of the roads usually need improvement jobs but, due to the limited amount of money available, some decision makers have to find out the optimal allocation of resources, i.e., they have to choose the roads to be maintained in such a way that the resulting impact on the traffic in the network is the best according so some criteria, or performance indices. To provide a useful performance index, we assume that the flows in the network are distributed according to Wardrop equilibrium, which implies that travelers choose their road so as to minimize their journey time,

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and all the roads *actually used* to connect a given origin to a given destination share the same travel time. Moreover, to be closer to concrete applications, we also assume that the traffic demand can be randomly perturbed according to a given probability distribution. Accordingly, the quantities of interest in our analysis are the mean values with respect to the given probability distribution, related to Wardrop equilibrium. The performance index that we choose is the mean value of the total travel time spent by all the users in the network, which can be directly connected to the pollution released by all the vehicles and can be also thought of as a *social cost*, because it actually represents the total time subtracted to work or to personal leisure. In the literature, the terms travel time and travel cost are considered on an equal footing, although a general cost may include aspects different from the pure travel time. A performance index only based on a weighted sum of topological parameters of a rail network has been recently put forward in [20]. To model congestion in the network, we make use of the link-cost functions in the form given by the Bureau of Public Roads (BPR) [3], which explicitly contain the capacity u_i of each road represented by a link a_i in the network. In this model, the maintenance of link a_i improves its capacity from u_i to $\gamma_i u_i$, where $\gamma_i > 1$ is called the *enhancement ratio* of link a_i. The case where $\gamma_i = \gamma$ for all the roads was considered in [14] in a deterministic framework. For each set of links that can be maintained, under the budget constraint, we update the mean total travel time mentioned above and compute its relative variation. The decision makers can then assess the impact of each intervention, with respect to the investment required, and select a small number of eligible alternatives for the final choice.

The paper is organized as follows. In Sect. 2, we brief the reader on the concept of Wardrop equilibrium and on the variational inequality formulation of the traffic equilibrium problem. We also analyze the relationship between the link and the path formulations, with respect to the monotonicity properties of the cost operator. Section 3 provides the reader with the essential background on variational inequalities in probability spaces and gives a stochastic variational inequality formulation of the traffic equilibrium problem with random perturbations of the data. In Sect. 4, we define our performance index, i.e., the mean value of the total cost, and give an approximation procedure for its computation together with its convergence proof. Later, the optimal road maintenance investment problem is modeled as an integer nonlinear optimization program. Finally, Sect. 5 is devoted to some numerical experiments showing the approximated solutions of the optimization model together with the impact of different probability distributions of the traffic demand on the approximated average total cost. The paper ends with an appendix, where we describe the approximation procedure for our stochastic variational inequality, in order to enable the interested reader to implement the model.

2 Traffic network equilibrium and efficiency measure

For a comprehensive treatment of the mathematical aspects of the traffic network equilibrium problem, we refer the reader to the book [17]. In this section, we focus on the basic definitions and on the variational inequality formulation of a user equilibrium flow (see, e.g. [4,21]). Throughout the paper, $a^\top b$ denotes the scalar product between vectors a and b, while $M^\top$ denotes the transpose of a matrix M. A traffic network consists of a triple $G = (N, A, W)$, where $N = \{N_1, \dots, N_p\}$ is the set of nodes, $A = \{a_1, \dots, a_n\} \subseteq N \times N$ is the set of arcs (or links) connecting pairs of nodes and $W = \{W_1, \dots, W_m\} \subseteq N \times N$ is the set of the origin-destination (O-D) pairs. The flow on the link a_i is denoted by f_i and the link flows are grouped into a vector $f = (f_1, \dots, f_n)$. A path (or route) is defined as a set of consecutive links and we suppose that each O-D pair W_j is connected by r_j paths, whose set is denoted by P_j. All the paths of the network are grouped into a vector $(R_1, \dots, R_k)$. The structure of the paths can be described by using the link-path incidence matrix $\Delta = (\delta_{ir})$, with $i = 1, \dots, n$ and $r = 1, \dots, k$, where $\delta_{ir} = 1$ if $a_i \in R_r$ and 0 otherwise. The flow on path R_r is denoted by F_r. All the path flows are grouped into the network path flow vector $(F_1, \dots, F_k)$. The flow f_i on the link a_i is equal to the sum of the flows on the paths containing a_i, so that $f = \Delta F$. The unit cost of traveling through link a_i is a non-negative real function $c_i(f)$ of the flows on the network, so that $c(f) = (c_1(f), \dots, c_n(f))$ denotes the link cost vector of the network. The meaning of the cost is usually that of travel time and, in the simplest case, the generic component c_i only depends on f_i. In our model, we use the BPR form of the link cost function which explicitly takes into account the link capacities. More precisely, the travel cost for link a_i is given by

$$c_i(f_i) = t_i^0 \left[1 + k \left(\frac{f_i}{u_i} \right)^\beta \right], \tag{1}$$

where t_i^0 is the free flow travel time on link a_i, u_i describes the capacity of link a_i, while k and β are positive parameters. Analogously, $C(F) = (C_1(F), \dots, C_k(F))$ is the path cost vector, where the cost $C_r(F)$ of path R_r is the sum of the costs on the links which build that path, i.e.

$$C_r(F) = \sum_{i=1}^{n} \delta_{ir} c_i(f),$$

or in compact form $C(F) = \Delta^\top c(\Delta F)$. For each pair W_j, there is a given traffic demand $D_j > 0$, so that $D = (D_1, \dots, D_m)$ is the demand vector of the network. Feasible path flows are non-negative flows such that the traffic demands are satisfied, i.e., they belong to the set

$$K = \{F \in \mathbb{R}^k : \ F \geq 0 \ \text{ and } \ \Phi F = D\}, \tag{2}$$

where Φ is the pair-path incidence matrix, whose entries are

$$\varphi_{jr} = \begin{cases} 1, & \text{if the path } R_r \text{ connects the pair} W_j, \\ 0, & \text{elsewhere,} \end{cases}$$

with $j = 1, \dots, m$ and $r = 1, \dots, k$. The notion of a user traffic equilibrium is given by the following definition.

Definition 1 A network flow $H \in K$ is a *Wardrop equilibrium* if, for each O-D pair W_j and for each pair of paths R_r, R_s which connect W_j, the following implication holds:

$$C_r(H) > C_s(H) \Longrightarrow H_r = 0;$$

that is, if traveling along the path R_r takes more time than traveling along R_s, then the flow along R_r vanishes.

It follows from the previous definition that the travel cost of the paths which connect a given O-D pair is the same (and minimum) for all paths with positive flow. Hence, H is a Wardrop equilibrium if for each O-D pair W_j there exists a scalar λ_j (representing the equilibrium cost shared by all the used paths connecting W_j) such that

$$C_r(H) \begin{cases} = \lambda_j, & \text{if } H_r > 0, \\ \geq \lambda_j, & \text{if } H_r = 0. \end{cases}$$

The variational inequality formulation of the Wardrop equilibrium is given by the following result (see, e.g., [17]).

Theorem 1 *A network flow $H \in K$ is a Wardrop equilibrium iff it solves the variational inequality*

$$C(H)^\top (F - H) \geq 0, \qquad \forall\, F \in K. \tag{3}$$

Notice that the variational inequality (3) can be rewritten by decomposing the scalar product according to the various O-D pairs as follows:

$$\sum_{j=1}^{m} \sum_{r \in P_j} C_r(H)\,(F_r - H_r) \geq 0, \qquad \forall\, F \in K.$$

For the subsequent development, the monotonicity properties of the cost operators will be exploited.

Definition 2 A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is said *monotone* if

$$(\,(T(x) - T(y)\,)^\top (x - y) \geq 0, \qquad \forall\, x, y \in \mathbb{R}^n,$$

and *strictly monotone* if the equality holds only for $x = y$; T is said *strongly monotone* if there exists $\alpha > 0$ such that

$$(\,(T(x) - T(y)\,)^\top (x - y) \geq \alpha \|x - y\|^2, \qquad \forall\, x, y \in \mathbb{R}^n.$$

The strict monotonicity assumption of the link-cost functions is commonly used because it models the congestion effect. However, this does not necessarily imply the strict monotonicity of the path-cost functions, this needs an extra condition as the following lemma shows.

Lemma 1 (a) *If c is monotone, then C is monotone.*
(b) *If c is strictly monotone and Δ has full column rank, then C is strictly monotone.*
(c) *If c is strongly monotone and Δ has full column rank, then C is strongly monotone.*

Proof (a) If $F^1, F^2 \in K$, then

$$\begin{aligned}[F^1 - F^2]^\top[C(F^1) - C(F^2)] &= [F^1 - F^2]^\top \Delta^\top[c(\Delta F^1) - c(\Delta F^2)] \\ &= [\Delta F^1 - \Delta F^2]^\top[c(\Delta F^1) - c(\Delta F^2)] \\ &\geq 0.\end{aligned}$$

(b) If $F^1 \neq F^2$, then $\Delta F^1 \neq \Delta F^2$ since Δ has full column rank, hence

$$\begin{aligned}[F^1 - F^2]^\top[C(F^1) - C(F^2)] &= [F^1 - F^2]^\top \Delta^\top[c(\Delta F^1) - c(\Delta F^2)] \\ &= [\Delta F^1 - \Delta F^2]^\top[c(\Delta F^1) - c(\Delta F^2)] \\ &> 0.\end{aligned}$$

(c) If $F^1, F^2 \in K$, then there exists $\alpha > 0$ such that

$$\begin{aligned}[F^1 - F^2]^\top[C(F^1) - C(F^2)] &= [F^1 - F^2]^\top \Delta^\top[c(\Delta F^1) - (\Delta F^2)] \\ &= [\Delta F^1 - \Delta F^2]^\top[c(\Delta F^1) - c(\Delta F^2)] \\ &\geq \alpha\|\Delta F^1 - \Delta F^2\|^2 \\ &= \alpha(F^1 - F^2)^\top \Delta^\top \Delta(F^1 - F^2) \\ &\geq \alpha\lambda_{min}(\Delta^T \Delta)\|F^1 - F^2\|^2,\end{aligned}$$

where $\lambda_{min}(\Delta^\top \Delta)$, which denotes the minimum eigenvalue of $\Delta^\top \Delta$, is positive since Δ has full column rank. □

A useful network efficiency index is the total travel time (or total cost), when a Wardrop equilibrium H is reached:

$$TC = \sum_{j=1}^{m} \sum_{r \in P_j} C_r(H)\, H_r = \sum_{j=1}^{m} \lambda_j D_j. \tag{4}$$

It has to be noted that an enhancement of capacity of a link can result in an increase of the total cost as a consequence of the well known Braess paradox [2].

In our model, we wish to include the possibility that the traffic demand can be affected by a random perturbation. As a result, we model the traffic equilibrium problem as a stochastic variational inequality. Thus, the total cost at equilibrium becomes a random variable as well, whose expectation is then defined as the efficiency index of our network. In the next section, we brief the reader on the basic facts on the variational inequality theory in probability spaces (for more details on this topic see, e.g., [7–10]).

3 An outline of stochastic variational inequalities and their application to the traffic equilibrium problem

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $A, B : \mathbb{R}^k \to \mathbb{R}^k$ two given mappings, and $b, c \in \mathbb{R}^k$ two given vectors in $\mathbb{R}^k$. Moreover, let R and S be two real-valued random variables defined on Ω, D a random vector in $\mathbb{R}^m$, and $G \in \mathbb{R}^{m\times k}$ a given matrix. For $\omega \in \Omega$, we define a random set $M(\omega) := \{x \in \mathbb{R}^k : \ Gx \leq D(\omega)\}$. Consider the following stochastic variational inequality: for almost every $\omega \in \Omega$, find $\hat{x} := \hat{x}(\omega) \in M(\omega)$ such that

$$(S(\omega)\, A(\hat{x}) + B(\hat{x})\,)^\top (z - \hat{x}) \geq (R(\omega)\, c + b\,)^\top (z - \hat{x}), \qquad \forall\, z \in M(\omega). \tag{5}$$

To facilitate the foregoing discussion, we set $T(\omega, x) := S(\omega)\, A(x) + B(x)$. We assume that A, B and S are such that the map $T : \Omega \times \mathbb{R}^k \mapsto \mathbb{R}^k$ is a Carathéodory function.

We also assume that $T(\omega, \cdot)$ is monotone for every $\omega \in \Omega$.

Since we are only interested in solutions with finite first- and second-order moments, our approach is to consider an integral variational inequality instead of the parametric variational inequality (5).

Thus, for a fixed $p \geq 2$, consider the Banach space $L^p(\Omega, P, \mathbb{R}^k)$ of random vectors V from Ω to $\mathbb{R}^k$ such that the expectation (p-moment) is given by $E^P(\|V\|^p) = \int_\Omega \|V(\omega)\|^p dP(\omega) < \infty$. For subsequent developments, we need the following growth condition

$$\|T(\omega, z)\| \leq \alpha(\omega) + \beta(\omega)\|z\|^{p-1}, \qquad \forall\, z \in \mathbb{R}^k, \tag{6}$$

where $\alpha \in L^q(\Omega, P)$ and $\beta \in L^\infty(\Omega, P)$. Due to the above growth condition, the Nemytskii operator $\hat{T}$ associated to T, acts from $L^p(\Omega, P, \mathbb{R}^k)$ to $L^q(\Omega, P, \mathbb{R}^k)$, where $p^{-1} + q^{-1} = 1$, and is defined by $\hat{T}(V)(\omega) := T(\omega, V(\omega))$, for any $\omega \in \Omega$. Assuming $D \in L^p_m(\Omega) := L^p(\Omega, P, \mathbb{R}^m)$, we introduce the following nonempty, closed and convex subset of $L^p_k(\Omega)$

$$M^P := \{V \in L^p_k(\Omega) : G\, V(\omega) \leq D(\omega),\ \ P - a.s.\}.$$

Let $S(\omega) \in L^\infty$, $0 < \underline{s} < S(\omega) < \overline{s}$, and $R(\omega) \in L^q$. Equipped with these notations, we consider the following L^p formulation of (5). Find $\hat{U} \in M^P$ such that for every $V \in M^P$, we have

$$\int_\Omega (S(\omega)\, A[\hat{U}(\omega)] + B[\hat{U}(\omega))]^\top (V(\omega) - \hat{U}(\omega))\, dP(\omega) \geq \int_\Omega (b + R(\omega)\, c)^\top (V(\omega) - \hat{U}(\omega)) dP(\omega). \tag{7}$$

A general theorem for the solvability of (7) is given at the end of the Appendix.

To get rid of the abstract sample space Ω, we consider the joint distribution $\mathbb{P}$ of the random vector (R, S, D) and work with the special probability space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P})$, where $d := 2 + m$ and $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^d$. For simplicity, we assume that R, S, and D are independent random vectors. We set

$$r = R(\omega), \quad s = S(\omega), \quad t = D(\omega), \quad y = (r, s, t).$$

For each $y \in \mathbb{R}^d$, we define the set $M(y) := \{x \in \mathbb{R}^k : \; Gx \leq t\}$.

Consider the space $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ and introduce the closed and convex set $M_{\mathbb{P}} := \{v \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) : \; Gv(r, s, t) \leq t, \; \mathbb{P} - a.s.\}$. Without any loss of generality, we assume that $R \in L^q(\Omega, P)$ and $D \in L^p(\Omega, P, \mathbb{R}^m)$ are non-negative.

Moreover, we assume that the support (i.e., the set of possible outcomes) of $S \in L^\infty(\Omega, P)$ is the interval $[\underline{s}, \overline{s}[\subset (0, \infty)$. With these ingredients, we consider the variational inequality problem of finding $\hat{u} \in M_{\mathbb{P}}$ such that for every $v \in M_{\mathbb{P}}$ we have

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} (s\, A[\hat{u}(y)] + B[\hat{u}(y)])^\top (v(y) - \hat{u}(y))\, d\mathbb{P}(y) \geq \int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} (b + r\, c)^\top (v(y) - \hat{u}(y))\, d\mathbb{P}(y). \tag{8}$$

For the reader's convenience, we provide some details on the numerical approximation of the solution $\hat{u}$ in the Appendix. Here, we only mention that the set $M_{\mathbb{P}}$ can be approximated by a sequence $\{M^n_{\mathbb{P}}\}$ of finite dimensional sets, and the functions r and s can be approximated by the sequences $\{\rho_n\}$ and $\{\sigma_n\}$ of step functions, with $\rho_n \to \rho$ in L^p and $\sigma_n \to \sigma$ in L^∞, respectively, where $\rho(r, s, t) = r$ and $\sigma(r, s, t) = s$. When the solution of (8) is unique, we can compute a sequence of step functions $\hat{u}_n$ which converges strongly to $\hat{u}$, under suitable hypotheses, by solving, for $n \in \mathbb{N}$, the following discretized variational inequality: find $\hat{u}_n := \hat{u}_n(y) \in M^n_{\mathbb{P}}$ such that, for every $v_n \in M^n_{\mathbb{P}}$, we have

$$\begin{aligned} &\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} (\sigma_n(y)\, A[\hat{u}_n(y)] + B[\hat{u}_n(y)])^\top (v_n(y) - \hat{u}_n(y))\, d\mathbb{P}(y) \\ &\quad \geq \int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} (b + \rho_n(y)\, c)^\top (v_n(y) - \hat{u}_n(y))\, d\mathbb{P}(y). \end{aligned} \tag{9}$$

In absence of strict monotonicity, the solution of (7) and (8) can be not unique and the previous approximation procedure must be coupled with a regularization scheme as follows. We choose a sequence $\{\varepsilon_n\}$ of regularization parameters and choose the regularization map to be the duality map $J : L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) \to L^q(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$. We assume that $\varepsilon_n > 0$ for every $n \in \mathbb{N}$ and that $\varepsilon_n \downarrow 0$ as $n \to \infty$.

We can then consider the following regularized stochastic variational inequality: for $n \in \mathbb{N}$, find $w_n = w_n^{\varepsilon_n}(y) \in M^n_{\mathbb{P}}$ such that, for every $v_n \in M^n_{\mathbb{P}}$, we have

$$\begin{aligned} &\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} \big(\sigma_n(y)\, A[w_n(y)] + B[w_n(y)] + \varepsilon_n J(w_n(y))\big)^\top (v_n(y) - w_n(y))\, d\mathbb{P}(y) \\ &\qquad \geq \int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} (b + \rho_n(y)\, c)^\top (v_n(y) - w_n(y))\, d\mathbb{P}(y). \end{aligned} \tag{10}$$

As usual, the solution w_n will be referred to as the regularized solution. Weak and strong convergence of w_n to the minimal-norm solution of (8) can be proved under suitable hypotheses (see the Appendix).

In traffic network equilibrium problems, the demand and the cost are often modeled as random variables (see, e.g., [1,5,7]).

Thus, let Ω be a sample space and P be a probability measure on Ω, and consider the following feasible set which takes into consideration random fluctuations of the demand:

$$K(\omega) = \{F \in \mathbb{R}^k : \ F \geq 0, \ \ \Phi F = D(\omega)\}, \qquad \omega \in \Omega.$$

Moreover, let $C : \Omega \times \mathbb{R}^k \mapsto \mathbb{R}^k$ be the random cost function. We can thus introduce ω as a random parameter in (3) and consider the problem of finding a vector $H(\omega) \in K(\omega)$ such that, $P-$ a.s:

$$C(\omega, H(\omega))^\top (F - H(\omega)) \geq 0, \qquad \forall\, F \in K(\omega). \tag{11}$$

Definition 3 A random vector $H \in K(\omega)$ is a *random Wardrop equilibrium* if for P-almost every $\omega \in \Omega$, for each O-D pair W_j and for each pair of paths R_r, R_s which connect W_j, we get

$$C_r(\omega, (H(\omega)) > C_s(\omega, (H(\omega))) \Longrightarrow H_r(\omega) = 0.$$

Consider then the set

$$K_P = \{F \in L^p(\Omega, P, \mathbb{R}^k) : \ F_r(\omega) \geq 0, P. - \text{a.s.}, \forall\, r = 1, \ldots, k, \ \Phi F(\omega) = D(\omega), P. - \text{a.s.}\},$$

which is is convex, closed and bounded, hence weakly compact. Furthermore, assume that the cost function C satisfies the growth condition:

$\|C(\omega, z)\| \leq \alpha(\omega) + \beta(\omega)\|z\|^{p-1}$, for any $z \in \mathbb{R}^k$, $P. -$ a.s.,

for some $\alpha \in L^q(\Omega, P)$, $\beta \in L^\infty(\Omega, P)$, $p^{-1} + q^{-1} = 1$.

The Carathéodory function C gives rise to a Nemytskii map $\hat{C} : L^p(\Omega, P, \mathbb{R}^k) \to L^q(\Omega, P, \mathbb{R}^k)$ defined through the usual position $\hat{C}(F)(\omega) = C(\omega, F((\omega))$, and, with abuse of a notation, instead of $\hat{C}$, the same symbol C is often used for both the Carathéodory function and the Nemytskii map that it induces. We thus consider the following integral variational inequality: find $H \in K_P$ such that

$$\int_\Omega C(\omega, H(\omega))^\top (F - H(\omega))\,dP(\omega) \geq 0, \qquad \forall\, F \in K_P. \tag{12}$$

A solution of (12) satisfies the random Wardrop conditions in the sense shown by the following lemma (see [11] for the proof).

Lemma 2 *If $H \in K_P$ is a solution of* (12)*, then H is a random Wardrop equilibrium.*

As a consequence of the previous lemma, we get that there exists a vector function $\lambda \in L^p(\Omega, P, \mathbb{R}^m)$ such that

$$C_r(\omega, H(\omega)) = \lambda_j(\omega) \tag{13}$$

for all paths R_r which connect w_j, with $H_r(\omega) > 0$, P-almost surely.

We assume that the operator is the sum of a purely deterministic term and of a random term where randomness act as a modulation:

$$C(\omega, H(\omega)) = S(\omega)A[H(\omega)] + B[H(\omega)] - b - R(\omega)c,$$

where $S \in L^\infty(\Omega, P)$, $R \in L^q(\Omega)$, $A, B : L^p(\Omega, P, \mathbb{R}^k) \to L^q(\Omega, P, \mathbb{R}^k)$, $b, c \in \mathbb{R}^k$.

The integral variational inequality now reads: find $H \in K_P$ such that, for all $F \in K_P$, we have

$$\int_\Omega (S(\omega)(A[H(\omega)])^\top + (B[H(\omega)])^\top)(F - H(\omega))dP(\omega) \geq \int_\Omega (b^\top + R(\omega)c^\top)(F - H(\omega))dP(\omega). \tag{14}$$

4 Average total cost at equilibrium and optimal road maintenance investment

We are now ready to define the mean values of two important quantities: the (unit) cost at equilibrium and the total cost at equilibrium. Then, the latter will be used to formulate the optimal road maintenance investment problem as an integer nonlinear optimization program.

4.1 Average unit and total costs at equilibrium

Let the traffic demand between the origin and destination be a random function $D : \Omega \to \mathbb{R}^m$ and $\hat{C} : L^p(\Omega, P, \mathbb{R}^k) \to L^q(\Omega, P, \mathbb{R}^k)$ be the cost operator. As usual, we denote by P the probability measure on Ω, while E_P is the expectation (or mean value) with respect to the probability P. We assume that $D \in L^p(\Omega, P, \mathbb{R}^m)$. We consider the following definitions:

1. The *average cost at equilibrium* is defined as

$$E_P[\lambda] = \int_\Omega \lambda(\omega)dP(\omega), \tag{15}$$

where $\lambda = \lambda(\omega) = (\lambda_1(\omega), \ldots, \lambda_m(\omega))$ is defined as in (13).

Remark 1 Let us note that the integral in (15) is different from zero under the natural assumption that in each path R_r there is a link where the cost is bounded from below by a positive number (uniformly in $\omega \in \Omega$). This hypothesis is fulfilled in real networks because the cost is positive for positive flows, but also the cost at zero flow (called the free flow time) is positive, because it represents the travel time without congestion.

2. The *average total cost at equilibrium* is defined as

$$E_P[TC] = \int_\Omega \sum_{j=1}^{m} \sum_{r\in P_j} C_r[\omega, H(\omega)]\, H_r(\omega) dP(\omega) = \int_\Omega \sum_{j=1}^{m} \lambda_j(\omega)\, D_j(\omega) dP(\omega), \tag{16}$$

it is finite (because of Hölder inequality) and different from zero for the same reason as above and because the demands are assumed strictly positive.

As explained in Sect. 3, the random vector $t = D(\omega)$ and the two random variables $r = R(\omega)$ and $s = S(\omega)$ generate a probability $\mathbb{P}$ in the image space $\mathbb{R}^{2+m}$ of (r, s, t) from the probability P on the abstract sample space Ω. Hence, we can express the earlier defined quantities in terms of the image space variables, thus obtaining functions which can be approximated through a discretization procedure. The integration now runs over the image space variables, but to keep notation simple we just write $\int$ instead of $\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+}$. The transformed expressions read as follows:

$$E_{\mathbb{P}}[\lambda] = \int \lambda(r, s, t) d\mathbb{P}(r, s, t), \tag{17}$$

$$E_{\mathbb{P}}[TC] = \int \sum_{j=1}^{m} \sum_{l\in P_j} C_l[r, s, H(r, s, t)]\, H_l(r, s, t) d\mathbb{P}(r, s, t) = \int \sum_{j=1}^{m} \lambda_j(r, s, t)\, t_j\, d\mathbb{P}(r, s, t). \tag{18}$$

Let us recall that the solution $H = H(r, s, t)$ of the stochastic variational inequality which describes the network equilibrium can be approximated using the procedure explained in the Appendix by a sequence $\{H^n\}_n$ of step functions such that $H^n \to H$ in L^p, as $n \to \infty$. In the theorem that follows, we give converging approximations for the mean values defined previously.

Theorem 2 *Let, for all* $n \in \mathbb{N}$, $C^n[\rho_n, \sigma_n, H^n(r, s, t)] = \sigma_n A[H^n(r, s, t)] + B[H(r, s, t)] - b - \rho_n c$ *and let* $\lambda^n(r, s, t) = (\lambda^n_1(r, s, t), \ldots, \lambda^n_m(r, s, t))$, *where* $\lambda^n_j(r, s, t) = C^n_l[\rho_n, \sigma_n, H^n(r, s, t)]$ *for all paths* R_l *which connect* w_j, *for which* $H^n_l(r, s, t) > 0$, $\mathbb{P}$*-a.s.. Let* $\rho(r, s, t) = r$, $\sigma(r, s, t) = s$ *and* $\rho_n \to \rho$ *strongly in* L^q *and* $\sigma_n \to \sigma$ *strongly in* L^∞, *as* $n \to \infty$. *Let* $H^n \to H$ *strongly in* L^p. *We then have:*

1. *The sequence*

$$\{E_{\mathbb{P}}[\lambda^n]\}_n = \left\{ \int \lambda^n(r, s, t) d\mathbb{P}(r, s, t) \right\}_n$$

converges to $E_{\mathbb{P}}[\lambda]$, *as* $n \to \infty$.

2. *The sequence*

$$\{E_{\mathbb{P}}[TC^n]\}_n = \left\{ \int \sum_{j=1}^{m} \sum_{l\in P_j} C_l(\rho_n, \sigma_n, H^n(r, s, t))\, H^n_l(r, s, t) d\mathbb{P}(r, s, t) \right\}_n$$

converges to $E_{\mathbb{P}}[TC]$, *as* $n \to \infty$.

Proof 1. Since $H^n \to H$ strongly in L^p, it follows that $A[H^n] \to A[H]$ and $B[H^n] \to B[H]$, strongly in $L^q = L^{\frac{p}{p-1}}$ because of the continuity of the Nemytskii operators A and B. Moreover, $\rho_n \to \rho$ strongly in L^q and $\sigma_n \to \sigma$ strongly in L^∞. As a consequence,

$$\sigma_n A[H^n] + B[H^n] - b - \rho_n c \;\to\; \sigma A[H] + B[H] - b - \rho c$$

strongly in L^q, and also strongly in L^1 because $\mathbb{P}$ is a probability measure. Hence, for each $i = 1, \ldots, k$, we get $C_i^n[\rho_n, \sigma_n, H^n] \to C_i[r, s, H]$ strongly in L^1 and, by the definitions of λ and λ^n, the thesis is proved.

2. From the previous proof we got that, for each $i = 1, \ldots, k$, $C_i^n[\rho_n, \sigma_n, H^n] \to C_i[r, s, H]$ strongly in L^q, as $n \to \infty$. This, together with $H^n \to H$ in L^p, yields to

$$C_i(\rho_n, \sigma_n, H^n(r, s, t))\, H_i^n(r, s, t) \to C_i(r, s, H(r, s, t))\, H_i^n(r, s, t)$$

strongly in L^1 and the second claimed is proved. □

The following corollary is a straightforward consequence of the previous theorem, together with Remark 1 and the fact that the traffic demand is assumed strictly positive ($\mathbb{P}$ a.s.).

Corollary 1 *Let C' be another cost operator in the random traffic problem (but with the same functional form as C). We then have that*

$$\frac{\{E_{\mathbb{P}}[TC^n]\}_n - \{E_{\mathbb{P}}[TC'^n]\}_n}{\{E_{\mathbb{P}}[TC^n]\}_n} \to \frac{\{E_{\mathbb{P}}[TC] - \{E_{\mathbb{P}}[TC']\}\}}{\{E_{\mathbb{P}}[TC]\}}, \qquad as\ n \to \infty. \tag{19}$$

4.2 The optimal road maintenance investment model

We can now formalize the optimal road maintenance investment problem. Let us suppose that a public authority has allocated an amount of money I for road maintenance. The improvement process involves a subset of links $\{a_i : i \in \mathcal{L}\}$, where $\mathcal{L} \subset \{1, \ldots, n\}$, and I_i is the investment required to enhance the capacity of link a_i by a given ratio $\gamma_i > 1$. Since the available budget does not allow to maintain all roads, the central authority aims to find the optimal subset of links to be maintained in order to improve as much as possible the average total cost at equilibrium (18) with respect to the current situation of the network, while satisfying the budget constraint.

This optimal investment problem can be formulated as an integer nonlinear optimization program as follows. We introduce for any $i \in \mathcal{L}$ a binary variable x_i, which is equal to 1 if the investment is actually carried out on link a_i, and 0 otherwise. Thus, a vector $x = (x_i)_{i\in\mathcal{L}}$ is feasible if the budget constraint $\sum_{i\in\mathcal{L}} I_i x_i \le I$ is satisfied. Given a feasible vector x, the capacity of each link a_i becomes equal to

$$u_i(x) := \gamma_i\, u_i\, x_i + (1 - x_i)\, u_i,$$

i.e., $u_i(x) = \gamma_i u_i > u_i$ when $x_i = 1$ and $u_i(x) = u_i$ when $x_i = 0$. The objective function to be maximized is the relative variation of the average total cost with respect to the current situation of the network, defined as

$$f(x) = 100 \cdot \frac{E_{\mathbb{P}}[TC] - E_{\mathbb{P}}[TC](x)}{E_{\mathbb{P}}[TC]}, \tag{20}$$

where $E_{\mathbb{P}}[TC]$ is the average total cost at equilibrium (18) before the maintenance job, while $E_{\mathbb{P}}[TC](x)$ is the average total cost at equilibrium corresponding to the improved network. Therefore, the resulting optimization model is as follows:

$$\begin{cases} \max & f(x) \\ \text{s.t.} & \sum_{i \in \mathcal{L}} I_i x_i \le I \\ & x_i \in \{0, 1\} \quad i \in \mathcal{L}. \end{cases} \tag{21}$$

We remark that the computation of the nonlinear function f at a given x requires to find a random Wardrop equilibrium for both the original and the improved network. Thus, model (21) can be considered as a stochastic nonlinear knapsack problem. Several authors considered in the literature different stochastic versions of the knapsack problem (see, e.g., [6,13,19]).

5 Numerical experiments

In this section, we consider the random Wardrop equilibrium problem and the related optimal road maintenance investment problem on two medium-size networks, assuming that the traffic demands are affected by random perturbations, while the arc cost functions are supposed to be exactly known. Hence, the average total cost at equilibrium (18) depends only on the random vector $t = D(\omega)$. The numerical computation of random Wardrop equilibria has been performed by implementing in Matlab 2018a the discretization procedure described in Sect. 4, and possibly the regularization procedure shown in Sect. 3, combined with the algorithm designed in [15] for deterministic Wardrop equilibria. The nonlinear knapsack problem (21) has been solved by a complete enumeration algorithm, i.e., evaluating the objective function at all the feasible solutions.

Example 1 We consider the grid network shown in Fig. 1 consisting of 36 nodes and 60 links. The link cost functions are of the BPR form (1) with $k = 0.15$ and $\beta = 4$ for all the links, while $t_i^0 = 1$ and $u_i = 100$ for any $i = 1, \ldots, 30$, and $t_i^0 = 5$ and $u_i = 200$ for any $i = 31, \ldots, 60$. We consider five O-D pairs: (1,12), (7,18), (13,24), (19,30), (25,36). We assume that the traffic demand for the first two O-D pairs is $D_j = d'_j + \delta'$, with $j = 1, 2$, where $d' = (150, 200)$ and δ' is a random variable which varies in the interval $[-100, 100]$ with either uniform distribution or truncated normal with mean 0 and standard deviation 50. Moreover, the traffic demand for the last three O-D pairs is $D_j = d''_j + \delta''$, for $j = 3, 4, 5$, where $d'' = (100, 200, 100)$ and δ'' is a random variable which varies in the interval $[-50, 50]$ with either uniform

Fig. 1 Grid network of Example 1

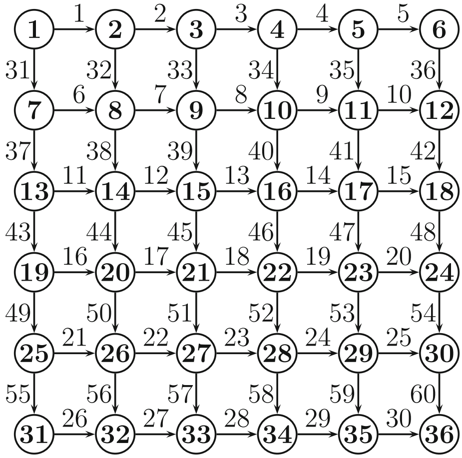

Table 1 The impact of different probability densities on the approximated average total cost of Example 1

N^d	Approximated avg total costs			
	U-U	U-N	N-U	N-N
10	9777.273	9673.016	9524.207	9428.736
20	9784.510	9680.161	9530.686	9435.027
50	9786.537	9682.170	9532.516	9436.810
100	9786.827	9682.457	9532.778	9437.065

distribution or truncated normal with mean 0 and standard deviation 25. The four different combinations of probability densities of δ' and δ'' are denoted by U-U, U-N, N-U and N-N; for instance, U-N means that δ' has a uniform distribution, while δ'' has a truncated normal distribution, and so on.

Notice that each O-D pair is connected by 6 paths and any arc a_i, with $i = 31, \ldots, 60$, belongs to a unique path, thus the link-path incidence matrix Δ has full column rank. Lemma 1 guarantees that the path cost operator is strongly monotone, hence there exists a unique random Wardrop equilibrium and the regularization procedure is not needed for this instance.

Both intervals $[-100, 100]$ and $[-50, 50]$ have been partitioned into N^d subintervals in the approximation procedure. Table 1 shows the convergence of the approximated average total costs for different values of N^d by using for four different combinations of probability densities.

We now consider the optimal investment problem in road maintenance. We assume that the available budget $I = 30$ k€, while the subset $\mathcal{L}$ of links to be maintained together with the values of γ_i and I_i are shown in Table 2.

Table 3 reports the ten best feasible solutions with the approximated value of the objective function f and the corresponding investment $I(x) = \sum_{i \in \mathcal{L}} I_i x_i$. The approximated values of $f(x)$ have been computed by partitioning the intervals $[-100, 100]$ and $[-50, 50]$ into 50 subintervals and assuming that random variables δ' and δ'' are uniformly distributed.

Table 2 Link capacities and investments for Example 1

$\mathcal{L}$	12	16	20	23	25	29	31	53
γ_i	1.4	1.8	1.3	1.5	1.7	1.4	1.1	1.5
I_i	6	12	4	6	10	5	2	6

Table 3 The ten best feasible solutions of the optimal road maintenance investment problem for Example 1

x	$f(x)$	$I(x)$
(1,0,0,0,1,1,1,1)	5.306	29
(0,0,0,1,1,1,1,1)	5.300	29
(1,0,1,1,0,1,1,1)	5.279	29
(1,0,0,1,1,1,1,0)	5.128	29
(1,0,0,1,1,0,1,1)	5.073	30
(1,0,1,1,0,1,0,1)	4.945	27
(1,0,1,0,1,1,1,0)	4.904	27
(1,0,0,0,1,1,0,1)	4.900	27
(0,0,1,1,1,1,1,0)	4.898	27
(0,0,0,1,1,1,0,1)	4.894	27

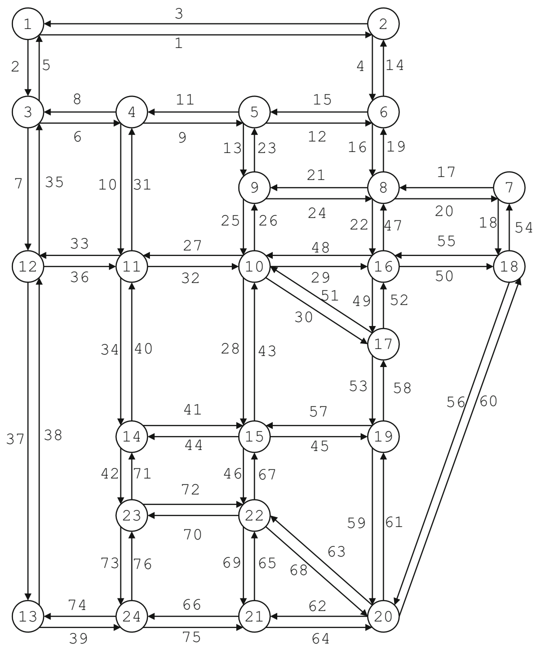

Fig. 2 Sioux-Falls network of Example 2

Example 2 We consider the Sioux-Falls network shown in Fig. 2 consisting of 24 nodes and 76 links. The link cost functions are of the BPR form (1) with $k = 0.15$ and $\beta = 1$ for all the links, while the parameters t_i^0 and u_i are given in [16]. We assume that the traffic demand for the 528 O-D pairs is $D_j = d_j + \delta$ if $d_j \geq 7$ and $D_j = d_j$ otherwise,

Table 4 The impact of different probability densities on the approximated average total cost of Example 2

N^d	ε	Approximated avg total costs U	N
10	1.0e-02	1560.71	1536.89
20	2.5e-03	1206.28	1189.09
50	4.0e-04	1103.39	1088.72
100	1.0e-04	1088.46	1074.24
200	2.5e-05	1084.72	1070.59
500	4.0e-06	1083.67	1069.57
1000	1.0e-06	1083.52	1069.43

Table 5 Link capacities and investments for Example 2

Links	Scenario 1		Scenario 2	
	γ_i	I_i	γ_i	I_i
25	1.2	5	1.4	6
26	1.5	6	1.8	7
28	1.1	10	1.3	12
43	1.3	5	1.5	6
45	1.4	4	1.7	5
46	1.2	8	1.4	10
56	1.1	6	1.3	7
57	1.5	2	1.8	2.5
60	1.4	3	1.7	3.5
67	1.3	2	1.5	2.5

where the deterministic demand d is given in [16] and δ is a random variable which varies in the interval $[-5, 5]$ with either uniform distribution (U) or truncated normal with mean 0 and standard deviation 0.5 (N).

Notice that in this case the link-path incidence matrix Δ has not full column rank and the path cost operator is monotone but not strongly monotone. Hence, the discretization procedure must be coupled with the regularization scheme described in Sect. 3.

The interval $[-5, 5]$ has been partitioned into N^d subintervals in the approximation procedure and the regularization parameter ε has been chosen equal to $1/(N^d)^2$. Table 4 shows the convergence of the approximated average total costs for different values of N^d and ε for any of the two probability densities.

We now consider the optimal investment problem in road maintenance. We assume that the available budget $I = 40\ k€$ and $\mathcal{L} = \{25, 26, 28, 43, 45, 46, 56, 57, 60, 67\}$ is the subset of links to be maintained. We consider two different scenarios: a low quality maintenance scenario, with an average enhancement ratio close to 1.3, and a high quality maintenance one with a ratio close to 1.55. The values of γ_i and I_i of the two scenarios are shown in Table 5.

Table 6 reports the ten best feasible solutions for the two scenarios. The approximated values of $f(x)$ have been computed by partitioning the interval $[-5, 5]$ into

Table 6 The ten best feasible solutions of the optimal road maintenance investment problem in the two scenarios of Example 2

Scenario 1			Scenario 2		
x	$f(x)$	$I(x)$	x	$f(x)$	$I(x)$
(0,1,1,1,1,1,0,1,1,1)	2.75	40	(0,1,1,1,1,0,0,1,1,1)	3.82	38
(1,1,0,1,1,1,0,1,1,1)	2.74	35	(1,1,0,1,1,0,1,1,1,1)	3.81	40
(1,1,1,1,1,0,0,1,1,1)	2.65	37	(0,1,0,1,1,1,0,1,1,1)	3.78	36
(0,1,0,1,1,1,1,1,1,1)	2.62	36	(1,1,0,1,1,1,0,1,0,1)	3.78	39
(1,1,0,1,1,1,1,1,0,1)	2.57	38	(1,1,1,1,0,0,0,1,1,1)	3.77	40
(1,1,0,1,0,1,1,1,1,1)	2.55	37	(1,1,0,1,0,1,0,1,1,1)	3.74	38
(0,1,0,1,1,1,0,1,1,1)	2.54	30	(1,1,0,1,1,1,0,0,1,1)	3.67	40
(0,1,1,1,1,0,1,1,1,1)	2.53	38	(0,1,0,1,1,1,1,1,0,1)	3.64	40
(1,1,0,1,1,0,1,1,1,1)	2.52	33	(0,1,0,1,0,1,1,1,1,1)	3.61	38
(1,0,1,1,1,1,0,1,1,1)	2.52	39	(0,1,1,1,0,1,0,1,0,1)	3.60	40

50 subintervals and assuming that the regularization parameter is $\varepsilon = 1/2500$ and the random variable δ is uniformly distributed.

Let us note that the value of the ten best solutions in the first scenario varies between around 2.5% and 2.7%, while that in second scenario between around 3.6% and 3.8%. Thus, an improvement in the quality of maintenance leads to a greater reduction in the average total cost.

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Appendix

In this section, we provide some details for the numerical approximation of the solution $\hat{u}$ of (8). First, we need a discretization of the space $X := L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$. We introduce a sequence $\{\pi_n\}_n$ of partitions of the support

$$\Upsilon := [0, \infty[\times[\underline{s}, \overline{s}[\times\mathbb{R}^m_+$$

of the probability measure $\mathbb{P}$ induced by the random elements R, S, and D. For this, we set

$$\pi_n = (\pi_n^R, \pi_n^S, \pi_n^D),$$

where

$$\pi_n^R := (r_n^0, \ldots, r_n^{N_n^R}), \quad \pi_n^S := (s_n^0, \ldots, s_n^{N_n^S}), \quad \pi_n^{D_i} := (t_{n,i}^0, \ldots, t_{n,i}^{N_n^{D_i}})$$

$$
\begin{aligned}
&0 = r_n^0 < r_n^1 < \dots r_n^{N_n^R} = n, \;\; \underline{s} = s_n^0 < s_n^1 < \dots s_n^{N_n^S} = \overline{s}, \\
&0 = t_{n,i}^0 < t_{n,i}^1 < \dots t_{n,i}^{N_n^{D_i}} = n \;\; (i = 1, \dots, m) \\
&|\pi_n^R| := \max\{r_n^j - r_n^{j-1} : j = 1, \dots, N_n^R\} \to 0 \;\; (n \to \infty) \\
&|\pi_n^S| := \max\{s_n^k - s_n^{k-1} : k = 1, \dots, N_n^S\} \to 0 \;\; (n \to \infty) \\
&|\pi_n^{D_i}| := \max\{t_{n,i}^{h_i} - t_{n,i}^{h_i-1} : h_i = 1, \dots, N_n^{D_i}\} \to 0 \;\; (i = 1, \dots, m; \; n \to \infty) .
\end{aligned}
$$

These partitions give rise to an exhausting sequence $\{\Upsilon_n\}$ of subsets of Υ, where each Υ_n is given by the finite disjoint union of the intervals:

$$
I_{jkh}^n := [r_n^{j-1}, r_n^j[\times[s_n^{k-1}, s_n^k[\times I_h^n,
$$

where we use the multi-index $h = (h_1, \cdots, h_m)$ and

$$
I_h^n := \prod_{i=1}^m [t_{n,i}^{h_i-1}, t_{n,i}^{h_i}[.
$$

For each $n \in \mathbb{N}$, we consider the space of the $\mathbb{R}^l$-valued step functions ($l \in \mathbb{N}$) on Υ_n, extended by 0 outside of Υ_n:

$$
X_n^l := \left\{ v_n : \; v_n(r, s, t) = \sum_j \sum_k \sum_h v_{jkh}^n 1_{I_{jkh}^n}(r, s, t), \;\; v_{jkh}^n \in \mathbb{R}^l \right\},
$$

where 1_I denotes the $\{0, 1\}$-valued characteristic function of a subset I.

To approximate an arbitrary function $w \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R})$, we employ the mean value truncation operator μ_0^n associated to the partition π_n given by

$$
\mu_0^n w := \sum_{j=1}^{N_n^R} \sum_{k=1}^{N_n^S} \sum_h (\mu_{jkh}^n w) \, 1_{I_{jkh}^n} , \tag{22}
$$

where

$$
\mu_{jkh}^n w := \begin{cases} \dfrac{1}{\mathbb{P}(I_{jkh})} \displaystyle\int_{I_{jkh}^n} w(y) \, d\mathbb{P}(y), & \text{if } \mathbb{P}(I_{jkh}^n) > 0, \\ 0, & \text{otherwise.} \end{cases}
$$

Analogously, for a L^p vector function $v = (v_1, \dots, v_l)$, we define

$$
\mu_0^n v := (\mu_0^n v_1, \dots, \mu_0^n v_l),
$$

for which one can prove that $\mu_0^n v$ converges to v, in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$.

To construct approximations for

$$M_{\mathbb{P}} = \left\{ v \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) : Gv(r,s,t) \le t \,,\ \mathbb{P} - \text{a.s.} \right\},$$

we introduce the orthogonal projector $q : (r,s,t) \in \mathbb{R}^d \mapsto t \in \mathbb{R}^m$ and define for each elementary cell I^n_{jkh},

$$\overline{q}^n_{jkh} = (\mu^n_{jkh} q) \in \mathbb{R}^m, \quad (\mu^n_0 q) = \sum_{jkh} \overline{q}^n_{jkh}\, 1_{I^n_{jkh}} \in X^m_n.$$

This leads to the following sequence of convex and closed sets of the polyhedral type:

$$M^n_{\mathbb{P}} := \{ v \in X^k_n : \ \ Gv^n_{jkh} \le \overline{q}^n_{jkh}\,,\ \forall j,k,h \}.$$

Since our objective is to approximate the random variables R and S, we introduce

$$\rho_n = \sum_{j=1}^{N^R_n} r^{j-1}_n \, 1_{[r^{j-1}_n, r^j_n[} \in X_n \qquad \text{and} \qquad \sigma_n = \sum_{k=1}^{N^S_n} s^{k-1}_n \, 1_{[s^{k-1}_n, s^k_n[} \in X_n.$$

Notice that

$$\sigma_n(r,s,t) \to \sigma(r,s,t) = s \ \text{ in } L^\infty(\mathbb{R}^d, \mathbb{P}) \text{ and } \rho_n(r,s,t) \to \rho(r,s,t) = r \text{ in } L^p(\mathbb{R}^d, \mathbb{P}).$$

Combining the above ingredients, for $n \in \mathbb{N}$, we consider the following discretized variational inequality: Find $\hat{u}_n := \hat{u}_n(y) \in M^n_{\mathbb{P}}$ such that for every $v_n \in M^n_{\mathbb{P}}$, we have

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^d} [\sigma_n(y)\, A(\hat{u}_n) + B(\hat{u}_n)]^\top [v_n - \hat{u}_n]\, d\mathbb{P}(y) \ge \int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^d} [b + \rho_n(y)\, c]^\top [v_n - \hat{u}_n]\, d\mathbb{P}(y)\,. \tag{23}$$

We also assume that the probability measures P_R, P_S, and P_{D_i} have the probability densities φ_R, φ_S, and φ_{D_i}, $i = 1, \ldots, m$, respectively. Therefore, for $i = 1, \ldots, m$, we have

$$dP_R(r) = \varphi_R(r)\, dr, \qquad dP_S(s) = \varphi_S(s)\, ds, \qquad dP_{D_i}(t_i) = \varphi_{D_i}(t_i)\, dt_i.$$

It turns out that (23) can be split in a finite number of finite dimensional variational inequalities: For every $n \in \mathbb{N}$, and for every j,k,h, find $\hat{u}^n_{jkh} \in M^n_{jkh}$ such that

$$[\tilde{T}^n_k(\hat{u}^n_{jkh})]^\top [v^n_{jkh} - \hat{u}^n_{jkh}] \ge [\tilde{c}^n_j]^\top [v^n_{jkh} - \hat{u}^n_{jkh}], \qquad \forall\, v^n_{jkh} \in M^n_{jkh}, \tag{24}$$

where

$$M^n_{jkh} := \{ v^n_{jkh} \in \mathbb{R}^k : Gv^n_{jkh} \le \overline{q}^n_{jkh} \}, \quad \tilde{T}^n_k := s^{k-1}_n\, A + B, \quad \tilde{c}^n_j := b + r^{j-1}_n\, c.$$

Clearly, we have

$$\hat{u}_n = \sum_j \sum_k \sum_h \hat{u}^n_{jkh} \, 1_{I^n_{jkh}} \in X^k_n.$$

We recall the following convergence result (see [9]).

Theorem 3 *Assume that* $T(\omega, \cdot)$ *is strongly monotone, uniformly with respect to* $\omega \in \Omega$*, that is*

$$(T(\omega, x) - T(\omega, y))^\top (x - y) \geq \alpha \|x - y\|^2 \quad \forall\, x,\ y,\ a.e.\ \omega \in \Omega,$$

where $\alpha > 0$ *and that the growth condition* (6) *holds. Then the sequence* $\{\hat{u}_n\}$*, where* $\hat{u}_n$ *is the unique solution of* (23)*, converges strongly in* $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ *to the unique solution* $\hat{u}$ *of* (8)*.*

In absence of strict monotonicity, the solution of (8) is not unique and we resort to a regularization technique as follows (see [10] for the details and proofs).

We will regularize the above discrete variational inequality and show that its continuous analogue is recovered by the limiting process. For this, we choose a sequence $\{\varepsilon_n\}$ of regularization parameters and choose the regularization map to be the duality map $J : L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) \to L^q(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$. We assume that $\varepsilon_n > 0$ for every $n \in \mathbb{N}$ and that $\varepsilon_n \downarrow 0$ as $n \to \infty$. We consider the following regularized stochastic variational inequality: For $n \in \mathbb{N}$, find $w_n = w_n^{\varepsilon_n}(y) \in M^n_{\mathbb{P}}$ such that for every $v_n \in M^n_{\mathbb{P}}$, we have

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} [\sigma_n(y)\, A(w_n(y)) + B(w_n(y)) + \varepsilon_n J(w_n(y))]^\top (v_n(y) - w_n(y))\, d\mathbb{P}(y)$$
$$\geq \int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} [b + \rho_n(y)\, c]^\top (v_n(y) - w_n(y))\, d\mathbb{P}(y). \tag{25}$$

As usual, the solution w_n will be referred to as the regularized solution.

The following theorems highlight some of the features of the regularized solutions.

Theorem 4 *The following statements hold.*

1. *For every* $n \in \mathbb{N}$*, the regularized stochastic variational inequality* (10) *has the unique solution* w_n*.*
2. *Any weak limit of the sequence* $\{w_n\}$ *of the regularized solutions is a solution of* (8)*.*

3. *The sequence of the regularized solutions* $\{w_n\}$ *is bounded provided that the following coercivity condition holds: There exists a bounded sequence* $\{\delta_n\}$ *with* $\delta_n \in M^n_{\mathbb{P}}$ *such that*

$$\frac{\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} [\sigma_n(y)\, A(u_n(y)) + B(u_n(y))]^\top (u_n(y) - \delta_n(y))\, d\mathbb{P}(y)}{\|u_n\|} \to \infty \quad as\ \|u_n\| \to \infty. \tag{26}$$

To obtain strong convergence we need to use the concept of fast Mosco convergence, as given by the following definition.

Definition 4 Let X be a normed space, let $\{K_n\}$ be a sequence of closed and convex subsets of X and let $K \subset X$ be closed and convex. Let ε_n be a a sequence of positive real numbers such that $\varepsilon_n \to 0$. The sequence $\{K_n\}$ is said to converge to K in the fast Mosco sense, related to ε_n, if

1. For each $x \in K$, $\exists \{x_n\} \in K_n$ such that $\varepsilon_n^{-1}\|x_n - x\| \to 0$;
2. For $\{x_m\} \subset X$, if $\{x_m\}$ weakly converges to $x \in K$, then $\exists \{z_m\} \in K$ such that $\varepsilon_m^{-1}(z_m - x_m)$ weakly converges to 0.

Theorem 5 *Assume that* $M^n_{\mathbb{P}}$ *converges to* $M_{\mathbb{P}}$ *in the fast Mosco sense related to* ε_n. *Moreover assume that* $\varepsilon_n^{-1}\|\sigma_n - \sigma\| \to 0$, *and* $\varepsilon_n^{-1}\|\rho_n - \rho\| \to 0$ *as* $n \to \infty$. *Then the sequence of regularized solutions* $\{w_n\}$ *converges strongly to the minimal-norm solution of stochastic variational inequality* (8) *provided that* w_n *is bounded.*

We conclude this section by recalling the following general result that ensures the solvability of an infinite dimensional variational inequality like (7) or (8) (see [12] for a recent survey on existence results for variational inequalities).

Theorem 6 *Let* E *be a reflexive Banach space and let* E^* *denote its topological dual space. We denote the duality pairing between* E *and* E^* *by* $\langle \cdot, \cdot \rangle_{E,E^*}$. *Let* K *be a nonempty, closed, and convex subset of* E, *and* $A : K \longrightarrow E^*$ *be monotone and continuous on finite dimensional subspaces of* K. *Consider the variational inequality problem of finding* $u \in K$ *such that*

$$\langle Au, v - u \rangle_{E,E^*} \geq 0, \qquad \forall\, v \in K.$$

Then, a necessary and sufficient condition for the above problem to be solvable is the existence of $\delta > 0$ *such that at least a solution of the variational inequality:*

$$\textit{find } u_\delta \in K_\delta \textit{ such that } \langle Au_\delta, v - u_\delta \rangle_{E,E^*} \geq 0, \quad \forall\, v \in K_\delta$$

satisfies $\|u_\delta\| < \delta$, *where* $K_\delta = \{v \in K : \|v\| \leq \delta\}$.

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