# Minimal free resolutions for homogeneous ideals with Betti numbers 1, n, n, 1 

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#### Abstract

We investigate the standard generalized Gorenstein algebras of homological dimension three, giving a structure theorem for their resolutions. Moreover in many cases we are able to give a complete description of their graded Betti numbers.


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## 1. Introduction

One of the most important tool for studying projective schemes is the minimal free resolution of their defining ideals. When the scheme has homological dimension 2, there is the Hilbert-Burch structure theorem which allows a deep knowledge of these schemes (see [4]). When we study schemes having homological dimension 3, the matter becomes a little intriguing even in the simplest case of the arithmetically Cohen Macaulay schemes. In this last case, when the rank of the last syzygy module is 1 , the situation becomes analogous to the homological dimension 2 case according to the structure theorem by Buchsbaum and Eisenbud (see [1]). These are the arithmetically Gorenstein schemes of codimension 3.

So it seems completely natural to study all the schemes of homological dimension 3 such that the rank of the last syzygy module is 1 , i.e. schemes whose resolution is of the type $1, n, n 1$, even for codimension less than 3 . The goal of this paper is just to study such schemes, which we will call generalized Gorenstein schemes (of course we omit the trivial case when the codimension is 1 ).

In literature there are paper which deal with a similar theme. In particular, Y. Kamoi in [6], gives a description of ideals with a resolution of type $1, n, n, 1$ in terms of Koszul complexes associated to the entries of the resolution. J. Weyman in [10], in a very general context, gives a structure for the rings with a resolution of type $1, n, n, 1$, in terms of some maps which come from a comparison of the resolution with a Koszul complex on the generators. P. Pragacz and J. Weyman in [7], give a description of general rings with resolutions of type $1, n, n, 1$ using a suitable decomposition on Schur functors.

[^0]In this paper we work on standard graded algebras and mainly we focus on the graded aspects of their resolutions. These schemes arise frequently in different contexts. For instance the disjoint union of two complete intersection curves in $\mathbb{P}^{3}$ and some subschemes of star configurations (see [9]) are generalized Gorenstein schemes. As in the Gorenstein case, the central map of the resolution is represented by a square matrix $M$ of submaximal rank, which is named presentation matrix. This matrix, once more, completely determines the resolution of the defining ideals.

Therefore, in Section 2, after studying properties of presentation matrices $n \times m, n \leq m$, we give a first characterization of such matrices in Proposition 2.6.

Section 3 is dedicated to producing a structure theorem for the resolutions of the generalized Gorenstein algebras of homological dimension 3 and the main result on this direction is Theorem 3.8. Moreover, we study the special case in which the ideal associated to the last map in the resolution is minimally generated by 3 elements. In Propositions 3.14 and 3.15 we give a complete characterization of such schemes and in Theorem 3.17 we give a nice geometrical description of these schemes.

All these results permit us to study the graded Betti numbers for generalized Gorenstein schemes of homological dimension 3. The main result in Section 4 is in Theorem 4.5, in which all the graded Betti numbers are characterized in the case in which the defining ideal is minimally generated by $n=3$ elements. The main results of Section 5 are in Theorem 5.14, where we give a complete description of the graded Betti sequences for those schemes which have $n$ generators and syzygies with concentrated degrees, with $n$ odd, and in Proposition 5.15 where some necessary conditions and some sufficient conditions are given in the even case.

## 2. Matrices of submaximal rank

Throughout the article $k$ will be a field and $R:=k\left[x_{1}, \ldots, x_{r}\right], r \geq 3$, will be the standard graded polynomial $k$-algebra.

The aim of this paper is to investigate graded minimal free resolutions of type

$$
0 \longrightarrow R(-s) \longrightarrow \bigoplus_{j=1}^{n} R\left(-b_{j}\right) \longrightarrow \bigoplus_{i=1}^{n} R\left(-a_{i}\right) \longrightarrow R .
$$

We give the following definition.
Definition 2.1. Let $M \in R^{n, m}$ be a matrix with $m \geq n$. We say that $M$ is a presentation matrix if it is associated to a map $\varphi$ in a presentation of type

$$
R^{m} \xrightarrow{\varphi} R^{n} \longrightarrow R .
$$

Note that if $M \in R^{n, m}$ is a presentation matrix, then rank $M=n-1$.
At first in this section we would like to study properties of presentation matrices.
Let $S$ be a UFD and let $H \in S^{n, n-1}$. We set $g_{i}(H)=(-1)^{i+1} H_{i}$ where $H_{i}$ is the minor of $H$ obtained by deleting the $i$-th row of $H$. The following lemmas will play a key role along the article.

Lemma 2.2. Let $S$ be a UFD, $M \in S^{n, m}, n \leq m$, with rank $M=n-1$. Let $N$ and $N^{\prime}$ be two submatrices of $M$ of size $n \times(n-1)$ of rank $n-1$. Let $d_{N}=\operatorname{GCD}\left(g_{1}(N), \ldots, g_{n}(N)\right)$ and $d_{N^{\prime}}=$ $\operatorname{GCD}\left(g_{1}\left(N^{\prime}\right), \ldots, g_{n}\left(N^{\prime}\right)\right)$. Then there exists a unit $a \in S$ such that

$$
\left(\frac{g_{1}(N)}{d_{N}}, \ldots, \frac{g_{n}(N)}{d_{N}}\right)=a\left(\frac{g_{1}\left(N^{\prime}\right)}{d_{N^{\prime}}}, \ldots, \frac{g_{n}\left(N^{\prime}\right)}{d_{N^{\prime}}}\right)
$$

Proof. Let $Q(S)$ be the field of fractions of $S$. Let us consider

$$
v=\left(\frac{g_{1}(N)}{d_{N}}, \ldots, \frac{g_{n}(N)}{d_{N}}\right) \in Q(S)^{n}, v^{\prime}=\left(\frac{g_{1}\left(N^{\prime}\right)}{d_{N^{\prime}}}, \ldots, \frac{g_{n}\left(N^{\prime}\right)}{d_{N^{\prime}}}\right) \in Q(S)^{n},
$$

then we have that $\langle v\rangle=\left\langle v^{\prime}\right\rangle$, as vector subspaces of $Q(S)^{n}$, since rank $M=n-1$. Therefore $b v=$ $b^{\prime} v^{\prime}$, with $b, b^{\prime} \in S, \operatorname{GCD}\left(b, b^{\prime}\right)=1$, so $b$ is a divisor of each component of the vector $v^{\prime}$, hence $b$ is unit. Analogously $b^{\prime}$ is a unit too, so $v=a v^{\prime}$, with $a$ a unit.

By Lemma 2.2, using the same notation, we can set $g_{i}(M)=g_{i}(N) / d_{N}$ and we will set

$$
\gamma(M)=\left(g_{1}(M), \ldots, g_{n}(M)\right)
$$

Note that $\gamma(M)$ is determined up to a unit and GCD $(\gamma(M))=1$. So $\gamma(M)$ generates an ideal $I_{M}$, with depth $I_{M} \geq 2$. In the sequel $I_{M}$ will be called the ideal associated to $M$. Moreover $\gamma(M)$ defines a map

$$
\gamma(M): R^{n} \longrightarrow R .
$$

When $F$ and $G$ are graded free modules, rank $F \geq \operatorname{rank} G$ and $\varphi: F \longrightarrow G$, is a map associated to a matrix $M$ (with respect to some bases), with rank $\varphi=\operatorname{rank} G-1$, we set $\gamma(\varphi)=\gamma(M)$. So $\gamma(\varphi): G \longrightarrow R$. By construction we have

$$
\gamma(\varphi) \varphi=0
$$

Corollary 2.3. With the same notation of Lemma 2.2, if $\left(h_{1}, \ldots, h_{n}\right) M=0$ then $\left(h_{1}, \ldots, h_{n}\right)=$ $\lambda \gamma(M)$ for some $\lambda \in S$.

Proof. As in the proof of Lemma $2.2 b\left(h_{1}, \ldots, h_{n}\right)=a \gamma(M)$ with $a, b \in S$. Now since $\operatorname{GCD}(\gamma(M))=1$ we see that $b$ divides $a$, so the conclusion follows.

The following proposition deals with the special case when $I_{M}$ is perfect of height 2 .
Proposition 2.4. Let $S$ be a UFD and $M \in S^{n, m}, n \leq m$, with rank $M=n-1$. Let us suppose that a submatrix $M^{\prime}$ obtained taking $n-1$ columns of $M$ has rank $n-1$ and its maximal minors are coprime. Then $I_{M}$ is a perfect ideal of height two and each column of $M$ is in the module generated by the columns of $M^{\prime}$.

Proof. By the hypothesis, $M^{\prime}$ is a Hilbert-Burch matrix of $I_{M}=\left(g_{1}, \ldots, g_{n}\right)$, where the $g_{i}$ 's are the maximal minors of $M^{\prime}$. So $I_{M}$ is perfect ideal of height two and the columns of $M^{\prime}$ generate the syzygy module on $\left(g_{1}, \ldots, g_{n}\right)$. Since any column of $M$ is a syzygy on $\left(g_{1}, \ldots, g_{n}\right)$, the conclusion follows.

Remark 2.5. In particular, when $M \in R^{n, m}, n \leq m$, is a presentation matrix with a submatrix $M^{\prime}$ as in Proposition 2.4, then the presentation defined by $M$ is not minimal.

Proposition 2.6. Let $M \in R^{n, m}, m \geq n$ with rank $M=n-1$. Let $C(M)$ be the module generated by the columns of $M$.
$M$ is a presentation matrix iff every syzygy on $\gamma(M)$ belongs to the module $C(M)$.
Proof. If $M$ is a presentation matrix then there exists an exact complex

$$
\begin{equation*}
R^{m} \xrightarrow{\varphi} R^{n} \xrightarrow{\psi} R, \tag{1}
\end{equation*}
$$

where $\varphi$ is the map associated to $M$. Let $\left(f_{1}, \ldots, f_{n}\right)=\operatorname{Im} \psi$. Then

$$
\left(f_{1} \ldots f_{n}\right) M=0
$$

and by Corollary $2.3\left(f_{1}, \ldots, f_{n}\right)=\lambda \gamma(M)$, for some $\lambda \in R$. Thus, every syzygy on $\gamma(M)$ is a syzygy on $\left(f_{1}, \ldots, f_{n}\right)$ too. Since the complex (1) is exact, we are done.

Vice versa, let us consider the following complex

$$
\begin{equation*}
R^{m} \xrightarrow{\varphi} R^{n} \xrightarrow{\gamma(\varphi)} R, \tag{2}
\end{equation*}
$$

where $\varphi$ is the map associated to $M$. Since, by the hypothesis, $\operatorname{Ker} \gamma(\varphi) \subseteq \operatorname{Im} \varphi, M$ is a presentation matrix.

Example 2.7. Let $R=k[x, y, z, t]$ and let

$$
M=\left(\begin{array}{cccc}
y & -x & 0 & 0 \\
0 & z & -y & 0 \\
0 & 0 & t & -z \\
-t & 0 & 0 & x
\end{array}\right)
$$

We have that $\gamma(M)=(z t, x t, x y, y z)=(x, z) \cap(y, t)$, so it is easy to verify that $M$ is a presentation matrix, whereas $M^{T}$ is not. In fact $\gamma\left(M^{T}\right)=(x, y, z, t)$, so it does not satisfy the hypothesis of Proposition 2.6.

## 3. The structure of the resolution

In this section we will deal with the case in which $M$ is a presentation square matrix of size $n \geq$ 3. Now we define the class of algebras which we will investigate along this paper.

Definition 3.1. A graded standard $R$-algebra $R / I$ of homological dimension 3 is called generalized Gorenstein algebra if the rank of the last syzygy module is 1 in a its minimal graded free resolution.
Theorem 3.2. Let $M \in R^{n, n}$ be a presentation square matrix. Then $R / I_{M}$ is a generalized Gorenstein algebra whose a free resolution is

$$
\begin{equation*}
0 \longrightarrow R^{*} \xrightarrow{\gamma\left(\varphi^{*}\right)^{*}} R^{n} \xrightarrow{\varphi} R^{n} \xrightarrow{\gamma(\varphi)} R \longrightarrow R / I_{M} \longrightarrow 0, \tag{3}
\end{equation*}
$$

where $\varphi$ is the map associated to the matrix $M$.
Proof. Since $M$ is a presentation matrix, using Proposition 2.6, we get $\operatorname{Im} \varphi=\operatorname{Ker} \gamma(\varphi)$. Furthermore $\gamma\left(\varphi^{*}\right) \varphi^{*}=0$ implies that $\varphi \gamma\left(\varphi^{*}\right)^{*}=0$. Now let $u \in \operatorname{Ker} \varphi$. It defines a map $f_{u}$ : $R \longrightarrow R^{n}$. Therefore we have that $\varphi f_{u}=0$, hence $f_{u}^{*} \varphi^{*}=0$. Using Corollary 2.3 we deduce that $f_{u}^{*}=a \gamma\left(\varphi^{*}\right)$, with $a \in R$, consequently $f_{u}=a \gamma\left(\varphi^{*}\right)^{*}$, i.e. $u \in \operatorname{Im} \gamma\left(\varphi^{*}\right)^{*}$.

Finally, by Lemma 2.6 in [8], since $\operatorname{rank} \varphi=n-1, \operatorname{Ker} \varphi$ is a free module of rank 1.
Definition 3.3. Whenever the resolution of Theorem 3.2 is minimal, we say that $M$ is a minimal presentation matrix.

Remark 3.4. If $M \in R^{n, n}$ is a presentation matrix then ht $I_{M}$ is either 2 or 3 . When ht $I_{M}=3, I_{M}$ is a Gorenstein ideal. Consequently if $n$ is even, then ht $I_{M}=2$.

Remark 3.5. If $M \in R^{n, n}$ is a presentation matrix, by the exactness of the complex (3), then depth $\left(\gamma\left(M^{T}\right)\right) \geq 3$ and depth $\left(I_{n-1}(M)\right) \geq 2$, by the exactness criterion in [2].

Using Theorem 3.2 we are able to characterize all the standard generalized Gorenstein algebras of homological dimension 3.

Corollary 3.6. Let I be an ideal of $R$. Then $R / I$ is a generalized Gorenstein algebra of homological dimension 3 iff there exists a presentation matrix $M$ such that $I=I_{M}$.

Proof. Theorem 3.2 says that every $I_{M}$ associated to a presentation matrix is a generalized Gorenstein ideal. Conversely, if $I$ is a generalized Gorenstein ideal in $R$ of homological dimension 3 then there is a resolution of $R / I$ of type

$$
0 \longrightarrow R \longrightarrow R^{n} \xrightarrow{M} R^{n} \longrightarrow R \longrightarrow R / I \longrightarrow 0
$$

So $M$ is a presentation matrix hence, again by Theorem 3.2, Coker $(M)=I_{M}$. From which we get $I=I_{M}$.

Thus it is important to recognize when $M$ is a presentation matrix. The next results will give another characterization of such matrices.
Lemma 3.7. Let $M \in R^{n, n}$ be a matrix of rank $n-1$. We set $\gamma(M)=\left(g_{1}, \ldots, g_{n}\right)$ and $\gamma\left(M^{T}\right)=$ $\left(h_{1}, \ldots, h_{n}\right)$. Let $M^{C}$ be the cofactor matrix of $M$. Then $M^{C}=u\left(g_{i} h_{j}\right)$, where $u \in R$.

Proof. Let us consider the following complex

$$
0 \longrightarrow R^{\left(h_{1} \cdots h_{n}\right)^{T}} R^{n} \xrightarrow{M} R^{n} \xrightarrow{\left(g_{1} \cdots g_{n}\right)} R .
$$

We denote by $c_{i j}$ the cofactor of $M$ in the position $(i, j)$. Since $\left(g_{1} \ldots g_{n}\right) M=0$ and $\left(c_{1 j} \ldots c_{n j}\right) M=0$, by Corollary 2.3 we get

$$
\left(c_{1 j}, \ldots, c_{n j}\right)=\lambda_{j}\left(g_{1}, \ldots, g_{n}\right)
$$

On the other hand since $\left(h_{1} \ldots h_{n}\right) M^{T}=0$ and $\left(c_{i 1} \ldots c_{i n}\right) M=0$, again by Corollary 2.3 we get

$$
\left(c_{i 1}, \ldots, c_{i n}\right)=\mu_{i}\left(h_{1}, \ldots, h_{n}\right)
$$

By these equalities follows that

$$
\lambda_{j}\left(g_{1}, \ldots g_{n}\right)=h_{j}\left(\mu_{1}, \ldots, \mu_{n}\right) ; \mu_{i}\left(h_{1}, \ldots, h_{n}\right)=g_{i}\left(\lambda_{1}, \ldots, \lambda_{n}\right) .
$$

Since GCD $\left(g_{1}, \ldots g_{n}\right)=\operatorname{GCD}\left(h_{1}, \ldots h_{n}\right)=1$, we obtain that

$$
c_{i j}=\alpha_{j} h_{j} g_{i}=\beta_{i} g_{i} h_{j},
$$

for some $\alpha_{j}, \beta_{i} \in R$, so $\alpha_{j}=\beta_{i}$ for every $i$ and $j$. Let

$$
u=\alpha_{1}=\ldots=\alpha_{n}=\beta_{1}=\ldots=\beta_{n},
$$

thus $c_{i j}=u g_{i} h_{j}$.
Theorem 3.8. Let $M \in R^{n, n}$ be a matrix of rank $n-1$. Let $\gamma(M)=\left(g_{1}, \ldots, g_{n}\right), \gamma\left(M^{T}\right)=\left(h_{1}, \ldots, h_{n}\right)$ and let $J$ be the ideal generated by $\gamma\left(M^{T}\right)$. Let $M^{C}$ be the cofactor matrix of $M$.

The matrix $M$ is a presentation matrix iff depth $J \geq 3$ and $M^{C}=u\left(g_{i} h_{j}\right)$ where $u$ is a unit.
Proof. We set $I=\left(g_{1}, \ldots, g_{n}\right)$. If $M$ is a presentation matrix then by Theorem 3.2 the complex

$$
\begin{equation*}
0 \longrightarrow R^{\left(h_{1} \ldots h_{n}\right)^{T}} R^{n} \xrightarrow{M} R^{n^{\left(g_{1} \ldots g_{n}\right)}} R \tag{4}
\end{equation*}
$$

is exact, so by the Buchsbaum-Eisenbud criterion, we get depth $J \geq 3$ and depth $I_{n-1}(M) \geq 2$.

Note that $I_{n-1}(M)$ is generated by the entries of $M^{C}$, so by Lemma 3.7, $I_{n-1}(M)=u I J$, consequently $u$ is a unit.

Conversely let us suppose that depth $J \geq 3$ and $M^{C}=u\left(g_{i} h_{j}\right)$ where $u$ is a unit. To show that $M$ is a presentation matrix it is enough to show that the complex (4) is exact. To do this, we will use the Buchsbaum-Eisenbud criterion. The conditions about the ranks of the modules are trivially satisfied. We have that depth $I \geq 2$, by definition of $\gamma(M)$, and depth $J \geq 3$ by the hypothesis. It remains only to prove that depth $I_{n-1}(M) \geq 2$. Since $M^{C}=u\left(g_{i} h_{j}\right)$ where $u$ is a unit, we have that $I_{n-1}(M)=I J$. Let $f \in I J, f \neq 0$. Since depth $I \geq 2$ and depth $J \geq 3$, there exist $g \in I$ and $h \in J$ such that $(f, g)$ and $(f, h)$ are regular sequences. Hence $(f, g h)$ is a regular sequence in $I J$ and we are done.

Remark 3.9. Let $M$ be an alternating matrix of odd size $n$ and of rank $n-1$. Then $\gamma(M)=\gamma\left(M^{T}\right)$ and $M^{C}=\left(p_{i} p_{j}\right)$, where $p_{h}$ is the $h$-th submaximal pfaffian of $M$. Let $P_{M}$ be the ideal generated by the submaximal pfaffians of $M$. Suppose that depth $P_{M}=3$. Then $M$ is a presentation matrix and $P_{M}=I_{M}$. So Theorem 3.8 allows us to recover the well known characterization of the Gorenstein ideals of heighth 3 of Buchsbaum and Eisenbud [1].

Remark 3.10. By Theorem 3.8, let $M$ be a presentation matrix and let $\left(h_{1}, \ldots, h_{n}\right)=\gamma\left(M^{T}\right)$. Then $h_{j}=0$ iff the submatrix obtained from $M$ by removing the $j$-th column has not maximal rank.

The following propositions put into relation a presentation square matrix $M$ with the kernel of the associated map.

Proposition 3.11. Let $M \in R^{n, n}$ be a presentation square matrix and let $\varphi: R^{n} \longrightarrow R^{n}$ be the associated map. Let $J=\operatorname{Im} \gamma\left(\varphi^{*}\right)$. Let us suppose that

$$
R^{m} \xrightarrow{\psi} R^{n *} \xrightarrow{\nu\left(\varphi^{*}\right)} R
$$

is a presentation of J. Then there exists a map $\beta: R^{m *} \longrightarrow R^{n}$, such that $\varphi=\beta \psi^{*}$.
Proof. At first we dualize the resolution of Theorem 3.2. We get the complex

$$
0 \longrightarrow R^{*} \xrightarrow{\gamma(\varphi)^{*}} R^{n *} \xrightarrow{\varphi^{*}} R^{n *} \xrightarrow{\gamma\left(\varphi^{*}\right)} R .
$$

By the exactness of the presentation we get the factorization $\varphi^{*}=\psi \alpha$ for a suitable $\alpha$ : $R^{n *} \longrightarrow R^{m}$. Consequently we have $\varphi=\alpha^{*} \psi^{*}$, so $\alpha^{*}$ is the required map $\beta$.

Proposition 3.12. Let $M \in R^{n, n}$ be a presentation square matrix and let $\varphi: R^{n} \longrightarrow R^{n}$ be the associated map. Let us suppose that $\gamma\left(M^{T}\right)=\left(h_{1}, \ldots, h_{t}, 0, \ldots, 0\right)$. Let $J=\left(h_{1}, \ldots, h_{t}\right)$. Take a presentation of $J$

$$
\begin{equation*}
R^{m} \xrightarrow{\psi} R^{t^{*}} \xrightarrow{\tau} R \tag{5}
\end{equation*}
$$

where $\tau\left(e_{i}^{*}\right)=h_{i}$, for $1 \leq i \leq t$. Then there exist two maps $\beta: R^{m *} \longrightarrow R^{n}$ and $\delta: R^{n-t} \longrightarrow R^{n}$ such that $\varphi=\left(\beta \psi^{*}\right) \oplus \delta$.

Proof. By the hypotheses $R / I_{M}$ has a resolution of the type

$$
0 \longrightarrow R^{*} \xrightarrow{\left(\tau^{*}, 0\right)} R^{t} \oplus R^{n-t} \xrightarrow{\varphi=\varphi_{1} \oplus \delta} R^{n} \longrightarrow R \longrightarrow R / I_{M} \longrightarrow 0 .
$$

Consequently we have $\varphi_{1} \tau^{*}=0$, so we get the complex

$$
R^{n *} \xrightarrow{\varphi_{1}^{*}} R^{t^{*}} \xrightarrow{\tau} R .
$$

By the exactness of (5), we get the factorization $\varphi_{1}^{*}=\psi \alpha$ for a suitable $\alpha: R^{n *} \longrightarrow R^{m}$, so $\varphi_{1}=$ $\alpha^{*} \psi^{*}$ and $\varphi=\left(\alpha^{*} \psi^{*}\right) \oplus \delta$.

The next results will be useful for studying generalized Gorenstein algebras.
Lemma 3.13. Let $M$ be a minimal presentation matrix and let $J$ be the ideal generated by $\gamma\left(M^{T}\right)$. Then there exists a resolution of $I_{M}$ of the type

$$
0 \longrightarrow R \xrightarrow{\rho} R^{n} \longrightarrow R^{n} \longrightarrow R \longrightarrow R / I_{M} \longrightarrow 0
$$

such that $\rho(1)=\left(h_{1}, \ldots, h_{s}, 0, \ldots, 0\right)$ where $h_{1}, \ldots, h_{s}$ minimally generate $J$.
Proof. Let $\left(h_{1}, \ldots, h_{n}\right)=\gamma\left(M^{T}\right)$. Let us suppose that $h_{n}=\sum_{i=1}^{n-1} a_{i} h_{i}$. Let $\varphi: R^{n} \longrightarrow R^{n}$ be the map associated to the matrix $M$. We change the basis in the domain of $\varphi$ from $\left(e_{1}, \ldots, e_{n}\right)$ to $\left(v_{1}, \ldots, v_{n}\right)$, where $v_{i}=e_{i}+a_{i} e_{n}$, for $1 \leq i \leq n-1$ and $v_{n}=e_{n}$. Then $\rho(1)=\sum_{i=1}^{n-1} h_{i} v_{i}$. By iterating this procedure we get the stated result.

According to Lemma 3.13 we will use the following notation. Let $I$ be a generalized Gorenstein ideal $I$ of homological dimension 3 and let

$$
0 \longrightarrow R \xrightarrow{\rho} R^{n} \longrightarrow R^{n} \longrightarrow R \longrightarrow R / I \longrightarrow 0
$$

be a minimal free resolution. We define $\zeta(I)=\nu(I)-\nu(I(\rho))$.
Note that $0 \leq \zeta(I) \leq \nu(I)-3$.
Now we would like to study minimal free resolutions for generalized Gorenstein ideals $I$ of homological dimension 3 with maximal $\zeta(I)$. Observe that in this case, using the same notation as before, $\rho(1)=\left(h_{1}, h_{2}, h_{3}, 0, \ldots, 0\right)$, where $\left(h_{1}, h_{2}, h_{3}\right)$ is a regular sequence.

Using Proposition 3.12, R/I has a minimal free resolution of the type

$$
\begin{equation*}
0 \longrightarrow R \xrightarrow{(\tau, 0)} R^{3} \oplus R^{n-3} \xrightarrow{(\alpha \kappa) \oplus \delta} R^{n} \longrightarrow R, \tag{6}
\end{equation*}
$$

where $\tau(1)=\left(h_{1}, h_{2}, h_{3}\right), \kappa: R^{3} \longrightarrow R^{3}$ is the Koszul map on $h_{1}, h_{2}, h_{3}$ and $\alpha: R^{3} \longrightarrow R^{n}, \delta:$ $R^{n-3} \longrightarrow R^{n}$ are suitable maps. Consequently $I=I_{M}$, where $M$ is a minimal presentation matrix having the structure $M=(A K \mid C)$, with

$$
K=\left(\begin{array}{ccc}
0 & h_{3} & -h_{2} \\
-h_{3} & 0 & h_{1} \\
h_{2} & -h_{1} & 0
\end{array}\right),
$$

for some $A \in R^{n, 3}$ and $C \in R^{n, n-3}$. In the next result we will give the structure of the generators of such ideals.

Proposition 3.14. Let $I=I_{M}$ be a generalized Gorenstein ideal of homological dimension 3 with maximal $\zeta(I)$, where $M=(A K \mid C)$. Then I is generated by the maximal minors obtained by deleting one by one the first $n$ rows of the $(n+1) \times n$-matrix

$$
B=\left(\begin{array}{lllllll} 
& A & & & & C & \\
& & & & & \\
h_{1} & h_{2} & h_{3} & 0 & \ldots & 0
\end{array}\right) \text {. }
$$

Proof. To compute a minimal set of generators for $I$, for instance $\gamma(M)$, it is enough to compute the maximal minors of a submatrix of $M$ obtained by choosing a submatrix of $M$ of size $n \times$ ( $n-1$ ) of rank $n-1$ (see Lemma 2.2). Note that, since rank $(A K)=2$, to obtain such a submatrix, we are forced to remove one of the first three columns.

Let $M_{(i, j)}$ be the minor of $M$ obtained by deleting the row $i$ and the column $j$. The following computation will show that, for some $s$,

$$
M_{(i, j)}=(-1)^{s} h_{j} B_{i}
$$

where $B_{i}$ is the minor of $B$ obtained by deleting the row $i$. Hence $I=\left(B_{1}, \ldots, B_{n}\right)$.
In fact we write the matrix $A$ by columns $A=\left(A_{1} A_{2} A_{3}\right)$ and $M$ in this way

$$
M=\left(-h_{3} A_{2}+h_{2} A_{3}\left|h_{3} A_{1}-h_{1} A_{3}\right|-h_{2} A_{1}+h_{1} A_{2} \mid C\right) .
$$

Moreover we will write $A_{j}^{(i)}$ the submatrix obtained by $A_{j}$ by removing the $i$-th row. So

$$
\begin{aligned}
M_{i, 1}= & \left|h_{3} A_{1}^{(i)}-h_{2} A_{1}^{(i)} C\right|+\left|h_{3} A_{1}^{(i)} h_{1} A_{2}^{(i)} C\right|+\left|-h_{1} A_{3}^{(i)}-h_{2} A_{1}^{(i)} C\right|+ \\
& +\left|-h_{1} A_{3}^{(i)} h_{1} A_{2}^{(i)} C\right|=h_{1}\left(h_{1}\left|A_{2}^{(i)} A_{3}^{(i)} C\right|-h_{2}\left|A_{1}^{(i)} A_{3}^{(i)} C\right|+h_{3}\left|A_{1}^{(i)} A_{2}^{(i)} C\right|\right) \\
= & (-1)^{s} h_{1} B_{i} .
\end{aligned}
$$

similarly we get $M_{i, 2}$ and $M_{i, 3}$.
In order to reverse Proposition 3.14, we need to fix some notation. Let $B \in R^{n+1, n}$ be a minimal Hilbert-Burch matrix, such that a row, say the last row, is $H=\left(h_{1}, h_{2}, h_{3}, 0, \ldots, 0\right)$. Then $B$ has the following shape

$$
B=\left(\begin{array}{ccccccc} 
& & & & &  \tag{7}\\
& A & & & C & \\
h_{1} & h_{2} & h_{3} & 0 & \ldots & 0
\end{array}\right) \text {. }
$$

Moreover we write $B_{i}$ for the minor obtained from $B$ by removing the $i$-th row, multiplied by $(-1)^{i}$.
Proposition 3.15. With the above notation let $B \in R^{n+1, n}$ be a minimal Hilbert-Burch matrix, such that the last row is $H=\left(h_{1}, h_{2}, h_{3}, 0, \ldots, 0\right)$ with $\left(h_{1}, h_{2}, h_{3}\right)$ a regular sequence. Let I be the ideal generated by $B_{1}, \ldots, B_{n}$.

Then $I$ is a generalized Gorenstein ideal of homological dimension 3 with maximal $\zeta(I)$.
Proof. Let us consider the complex

$$
0 \longrightarrow R \xrightarrow{\rho} R^{n} \xrightarrow{\varphi} R^{n} \xrightarrow{\gamma} R
$$

where $\rho(1)=\left(h_{1}, h_{2}, h_{3}, 0, \ldots, 0\right), \varphi$ is represented by the matrix $M=(A K \mid C)$, where $K$ is the matrix of the central map of the Koszul complex on $\left(h_{1}, h_{2}, h_{3}\right)$ and $\gamma$ is the map defined by the row $\left(B_{1}, \ldots, B_{n}\right)$. We have to check that this complex is exact. According to our hypotheses it is useful to rewrite it as follows.

$$
\begin{equation*}
0 \longrightarrow R^{\rho=(\tau, 0)} R^{3} \oplus R^{n-3} \xrightarrow{\varphi=\alpha \kappa \oplus \delta} R^{n} \xrightarrow{\gamma} R, \tag{8}
\end{equation*}
$$

where $\tau(1)=\left(h_{1}, h_{2}, h_{3}\right)$ and $\alpha, \kappa$ and $\delta$ are the maps represented respectively by $A, K$ and $C$.
Of course $\rho$ is injective and $\operatorname{Im} \rho \subseteq \operatorname{Ker} \varphi$.
Now we show that $\operatorname{Ker} \varphi \subseteq \operatorname{Im} \rho$. Let $u=\left(u_{1}, u_{2}\right) \in \operatorname{Ker} \varphi, u_{1} \in R^{3}, u_{2} \in R^{n-3}$ i.e. $\alpha \kappa\left(u_{1}\right)=$ 0 and $\delta\left(u_{2}\right)=0$. Since $B$ is minimal, $\operatorname{det}(A \mid C) \neq 0$, so $\alpha$ is injective, therefore $\kappa\left(u_{1}\right)=0$. Consequently $u_{1} \in \operatorname{Ker} \kappa=\operatorname{Im} \tau$. Since $B$ is an Hilbert-Burch matrix, $C$ has maximal rank, hence $\delta$ is injective i.e. $u_{2}=0$, therefore $u=\left(u_{1}, u_{2}\right) \in \operatorname{Im} \rho$.

By Proposition 3.14 we have that $\operatorname{Im} \varphi \subseteq \operatorname{Ker} \gamma$, so we need to show that $\operatorname{Ker} \gamma \subseteq \operatorname{Im} \varphi$. Let $\left(v_{1}, \ldots, v_{n}\right) \in \operatorname{Ker} \gamma$. Then $\left(v_{1}, \ldots, v_{n}, 0\right)$ is a syzygy on $\left(B_{1}, \ldots, B_{n}, \operatorname{det}(A \mid C)\right)$. So ( $\left.v_{1}, \ldots, v_{n}, 0\right)$ belongs to the module generated by the columns of $B$ i.e.

$$
\left(\begin{array}{c}
v_{1} \\
\ldots \\
v_{n} \\
0
\end{array}\right)=\sum_{i=1}^{3} \lambda_{i}\left(\begin{array}{c}
b_{1 i} \\
\ldots \\
b_{n i} \\
h_{i}
\end{array}\right)+\sum_{i=4}^{n} \lambda_{i}\left(\begin{array}{c}
b_{1 i} \\
\ldots \\
b_{n i} \\
0
\end{array}\right) \Rightarrow
$$

$\left(v_{1}, \ldots, v_{n}\right)=\alpha\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)+\delta(\lambda)$, where $\lambda=\left(\lambda_{4}, \ldots, \lambda_{n}\right)$. Moreover, since $\lambda_{1} h_{1}+\lambda_{2} h_{2}+\lambda_{3} h_{3}=$ 0 , we deduce that $\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)=\kappa(z)$, for some $z \in R^{3}$. Therefore $\left(v_{1}, \ldots, v_{n}\right)=\alpha \kappa(z)+\delta(\lambda)$, consequently $\varphi(z, \lambda)=\left(v_{1}, \ldots, v_{n}\right)$.

By Proposition $3.15 I_{M}$ is generated by $n$ among $n+1$ maximal minors of the matrix $B$ in (7). The next result will allow us to give a structure for $I_{M}$ in terms of intersection of two simpler ideals.
Lemma 3.16. Let $B \in R^{n+1, n}$, with rank $B=n$. Let $B_{i}, 1 \leq i \leq n+1$ be the maximal minors of $B$. Let $I(B)=\left(B_{1}, \ldots, B_{n+1}\right)$, such that ht $I(B)=2$. Let $\left(h_{1}, \ldots, h_{n}\right)$ be the last row in $B$. Then

$$
I(B) \cap\left(h_{1}, \ldots, h_{n}\right)=\left(B_{1}, \ldots, B_{n}\right) \Longleftrightarrow B_{n+1} \text { is regular in } R /\left(h_{1}, \ldots, h_{n}\right) .
$$

Proof. Take $a_{n+1} \in R$, such that $a_{n+1} B_{n+1} \in\left(h_{1}, \ldots, h_{n}\right)$. By assumption there exist $a_{1}, \ldots, a_{n} \in R$ such that $\sum_{i=1}^{n} a_{i} B_{i}=a_{n+1} B_{n+1}$. Since $B$ is an Hilbert-Burch matrix, $\left(a_{1}, \ldots, a_{n+1}\right)$ belongs to the $R$-module generated by the columns of $B$. In particular $a_{n+1} \in\left(h_{1}, \ldots, h_{n}\right)$.

Conversely we have only to prove that $I(B) \cap\left(h_{1}, \ldots, h_{n}\right) \subseteq\left(B_{1}, \ldots, B_{n}\right)$. Let $f \in I(B) \cap$ $\left(h_{1}, \ldots, h_{n}\right)$, so $f=\sum_{i=1}^{n+1} a_{i} B_{i}$. Since $B_{i} \in\left(h_{1}, \ldots, h_{n}\right)$ for $1 \leq i \leq n$, we get $a_{n+1} B_{n+1} \in\left(h_{1}, \ldots, h_{n}\right)$. Then, by the assumption, $a_{n+1} \in\left(h_{1}, \ldots, h_{n}\right)$, i.e. $a_{n+1}=\sum_{i=1}^{n} u_{i} h_{i}$. For $1 \leq j \leq n, \sum_{i=1}^{n} b_{i j} B_{i}=$ $-h_{j} B_{n+1}$, hence $\sum_{i=1}^{n} u_{j} b_{i j} B_{i}=-u_{j} h_{j} B_{n+1}$. Summing up we get

$$
\sum_{j=1}^{n} \sum_{i=1}^{n} u_{j} b_{i j} B_{i}=-a_{n+1} B_{n+1},
$$

so $a_{n+1} B_{n+1} \in\left(B_{1}, \ldots, B_{n}\right)$, i.e. $f \in\left(B_{1}, \ldots, B_{n}\right)$.
Using Lemma 3.16, we can give a geometric description of projective schemes having a minimal free resolution of type (8).

Theorem 3.17. Let $X \subset \mathbb{P}^{r}, r \geq 3$ be a closed projective scheme, whose defining ideal $I_{X}$ has a graded minimal free resolution of type (8). Let $Z$ be the complete intersection defined by $I(\rho)=$ $I(\tau)$, let $S=V(\operatorname{det}(\alpha \oplus \delta))$ and let $Y$ be the scheme defined by $I\left(\alpha \oplus \delta, \rho^{*}\right)$. If $\operatorname{codim}(S \cap Z)=4$ then $X=Y \cup Z$.

Proof. It is enough to observe that since codim $(S \cap Z)=4, \operatorname{det}(\alpha \oplus \delta)$ is regular in $R / I_{Z}$. So we can apply Lemma 3.16 to have our assertion.

Remark 3.18. Note that when $\operatorname{det}(\alpha \bigoplus \delta)$ is a unit, $Y=\emptyset$ and $X=Z$. When $\operatorname{det}(\alpha \oplus \delta)$ is not a unit, then $X$ is a union of an aCM scheme of codimension 2 and a complete intersection scheme of codimension 3 .

## 4. The case $\boldsymbol{n}=\mathbf{3}$

Now we will apply the results of previous sections and we will provide an explicit characterization of the graded Betti numbers for generalized Gorenstein ideals having a graded minimal free resolution of the type

$$
\begin{equation*}
0 \longrightarrow F_{3} \xrightarrow{\rho} F_{2} \xrightarrow{\varphi} F_{1} \xrightarrow{\psi} R \longrightarrow R / I \longrightarrow 0 \tag{9}
\end{equation*}
$$

with $\operatorname{rank} F_{1}=\operatorname{rank} F_{2}=3$ (and consequently rank $F_{3}=1$ ).
The three-generated ideals were studied initially in the article [3] by Buchsbaum and Eisenbud. Here we focus on graded Betti numbers for such ideals.

We start by observing that, by the exactness criterion, $\operatorname{Im} \rho$ is generated by a regular sequence ( $h_{1}, h_{2}, h_{3}$ ).

Let $M$ be the matrix associated to $\varphi$ with respect suitable bases. Since $\left(h_{1}, h_{2}, h_{3}\right)$ is a regular sequence, its first syzygy module is generated by the rows of the following matrix

$$
K=\left(\begin{array}{ccc}
0 & h_{3} & -h_{2} \\
-h_{3} & 0 & h_{1} \\
h_{2} & -h_{1} & 0
\end{array}\right) .
$$

As $\varphi \rho=0$, we get $M=A K$, where $A \in R^{3,3}$. Consequently the resolution (9) can be written in the following way

$$
\begin{equation*}
0 \longrightarrow F_{3} \xrightarrow{\rho} F_{2} \xrightarrow{\varphi=\alpha \kappa} F_{1} \xrightarrow{\psi} R \longrightarrow R / I \longrightarrow 0 \tag{10}
\end{equation*}
$$

where $\alpha$ and $\kappa$ are the maps associated to the mentioned matrices $A$ and $K$.
Proposition 4.1. If

$$
0 \longrightarrow F_{3} \xrightarrow{\rho} F_{2} \xrightarrow{\varphi} F_{1} \xrightarrow{\psi} R
$$

is a graded minimal free resolution and $M$ is the matrix associated to $\varphi$ with respect suitable bases, then $I_{M}$ is generated by the maximal minors obtained by deleting one by one the first 3 rows of the $4 \times 3$-matrix

$$
B=\left(\begin{array}{lll} 
& & \\
& A & \\
h_{1} & h_{2} & h_{3}
\end{array}\right),
$$

where $A$ is the matrix defined above and $\left(h_{1}, h_{2}, h_{3}\right)$ generates $\operatorname{Im} \rho$.
Proof. This is a particular case of Proposition 3.14, when $\zeta(I)=0$.
In order to reverse Proposition 4.1, we need to fix some notation. Let $B \in R^{4,3}$ be a HilbertBurch matrix. Let us consider a row of $B$, say $H=\left(h_{1}, h_{2}, h_{3}\right)$ and let $\hat{B}$ be the matrix obtained from $B$ by deleting the row $H$. Let $B_{1}, B_{2}, B_{3}$ be the maximal minors of $B$ including the row $H$.
Proposition 4.2. With the above notation let $B \in R^{4,3}$ be a Hilbert-Burch matrix, providing a minimal set of generators, such that one of its rows $H=\left(h_{1}, h_{2}, h_{3}\right)$ is a regular sequence. Let $J$ be the ideal generated by $B_{1}, B_{2}, B_{3}$. Then a graded minimal free resolution of $R / J$ is

$$
0 \longrightarrow F_{3} \xrightarrow{\rho} F_{2} \xrightarrow{\varphi} F_{1} \xrightarrow{\psi} R
$$

where $\operatorname{Im} \rho$ is generated by $\left(h_{1}, h_{2}, h_{3}\right), \varphi=\alpha \kappa$ where $\alpha$ is the map associated to the matrix $\hat{B}, \kappa$ is the central map of the Koszul complex on $\left(h_{1}, h_{2}, h_{3}\right), \psi$ is the map defined by the row $\left(B_{1}, B_{2}, B_{3}\right)$ and $F_{1}, F_{2}$ are graded free modules of rank three.

Proof. This is a particular case of Proposition 3.15, when $\zeta(I)=0$.
The next proposition describes the graded Betti numbers for an ideal $I \subset R$ whose resolution is of type (10).

Proposition 4.3. Let $I \subset R$ be a generalized Gorenstein ideal, ht $I \geq 2$. Then there exist six integers $d_{1}, d_{2}, d_{3}, a_{1}, a_{2}, a_{3}$, with $d_{i}>0$ and $a_{i} \geq 0$, such that the graded minimal free resolution of $R / I$ is

$$
0 \longrightarrow R(-d-a) \longrightarrow \oplus_{i=1}^{3} R\left(d_{i}-d-a\right) \longrightarrow \bigoplus_{i=1}^{3} R\left(a_{i}-d_{i}-a\right) \longrightarrow R
$$

where $d=d_{1}+d_{2}+d_{3}$ and $a=a_{1}+a_{2}+a_{3}$.
Conversely if we choose six integers $d_{1}, d_{2}, d_{3}, a_{1}, a_{2}, a_{3}$, with $d_{i}>0$ and $a_{i} \geq 0$, then there exists a generalized Gorenstein ideal, ht $I \geq 2$, such that $R / I$ has the following minimal graded free resolution

$$
0 \longrightarrow R(-d-a) \longrightarrow \oplus_{i=1}^{3} R\left(d_{i}-d-a\right) \longrightarrow \oplus_{i=1}^{3} R\left(a_{i}-d_{i}-a\right) \longrightarrow R
$$

where $d=d_{1}+d_{2}+d_{3}$ and $a=a_{1}+a_{2}+a_{3}$.
Proof. Since $R / I$ has a minimal free resolution of type (10), we set $d_{1}, d_{2}, d_{3}$, the degrees of the complete intersection $I(\rho)$ and $\alpha: \oplus_{i=1}^{3} R\left(-e_{i}\right) \longrightarrow \oplus_{j=1}^{3} R\left(-e_{j}^{\prime}\right)$. We set $a_{i}=e_{i}-e_{i}^{\prime}$, for $1 \leq i \leq 3$. By Proposition 4.1 we see that the degrees of the minimal generators of $I$ are $d_{1}+a_{2}+a_{3}, d_{2}+$ $a_{1}+a_{3}, d_{3}+a_{1}+a_{2}$ i.e. $a+d_{i}-a_{i}$ for $1 \leq i \leq 3$. Furthermore since $\varphi=\alpha \kappa$, a simple computation shows that the shifts of the second module are $e_{1}-e_{1}^{\prime}+d_{2}+\left(a+d_{1}-a_{1}\right)=a+d_{1}+d_{2}$, $e_{2}-e_{2}^{\prime}+d_{3}+\left(a+d_{2}-a_{2}\right)=a+d_{2}+d_{3}, e_{3}-e_{3}^{\prime}+d_{1}+\left(a+d_{3}-a_{3}\right)=a+d_{1}+d_{3}$, consequently they are $a+d-d_{i}$, for $1 \leq i \leq 3$. Since the map $\rho$ is the map of the complete intersection of type $\left(d_{1}, d_{2}, d_{3}\right)$ the last graded Betti number is $a+d$.

Conversely let $J=\left(h_{1}, h_{2}, h_{3}\right)$ be a complete intersection with $\operatorname{deg} h_{i}=d_{i}$ for $1 \leq i \leq 3$ and we choose three forms $g_{i}$, $\operatorname{deg} g_{i}=a_{i}$ for $1 \leq i \leq 3$. $I=\left(h_{1} g_{2} g_{3}, h_{2} g_{1} g_{3}, h_{3} g_{1} g_{2}\right)$ is a required ideal. Namely if we consider the matrix

$$
B=\left(\begin{array}{ccc}
g_{1} & 0 & 0 \\
0 & g_{2} & 0 \\
0 & 0 & g_{3} \\
h_{1} & h_{2} & h_{3}
\end{array}\right)
$$

it satisfies the hypotheses of Proposition 4.2.
In order to avoid trivial cases in the sequel we will use the following definition.
Definition 4.4. A Betti sequence is said to be essential if it occurs for R/I where I is a homogeneous ideal with ht $I \geq 2$.

The next theorem will characterize the Betti sequences for generalized Gorenstein ideals of homological dimension 3 and ht $I \geq 2$.

Theorem 4.5. A sequence ( $\left.a_{1}, a_{2}, a_{3} ; b_{1}, b_{2}, b_{3} ; s\right)$ with $a_{1} \leq a_{2} \leq a_{3}$ and $b_{1} \geq b_{2} \geq b_{3}$ is a essential Betti sequence iff

1. $s=\sum_{j=1}^{3} b_{j}-\sum_{i=1}^{3} a_{i}$;
2. $\sum_{i=1}^{3} a_{i}<b_{2}+b_{3}$;
3. $a_{j}+b_{j} \leq \sum_{i=1}^{3} a_{i}$, for $1 \leq j \leq 3$.

Proof. Let $\left(a_{1}, a_{2}, a_{3} ; b_{1}, b_{2}, b_{3} ; s\right)$ be a essential Betti sequence then there is an ideal $I$ of height at least two, such that $R / I$ has the following graded minimal free resolution

$$
0 \longrightarrow R(-s) \longrightarrow \bigoplus_{j=1}^{3} R\left(-b_{j}\right) \longrightarrow \bigoplus_{i=1}^{3} R\left(-a_{i}\right) \longrightarrow R
$$

Trivially $s=\sum_{j=1}^{3} b_{j}-\sum_{i=1}^{3} a_{i}$.
Moreover the last map of the resolution is defined by a regular sequence $h_{1}, h_{2}, h_{3}$ with $\operatorname{deg} h_{j}=s-b_{j}$, for $1 \leq j \leq 3$. Consequently $s-b_{1}>0$, i.e. $\sum_{i=1}^{3} a_{i}<b_{2}+b_{3}$.

Let $M=\left(m_{i j}\right)$ be a matrix associated to the central map of the above resolution, with $\operatorname{deg} m_{i j}=b_{j}-a_{i}$. We have that $\sum_{j=1}^{3} m_{i j} h_{j}=0$, for $1 \leq i \leq 3$. Consequently

$$
\left(m_{i 1}, m_{i 2}, m_{i 3}\right) \in\left(\left(0, h_{3},-h_{2}\right),\left(-h_{3}, 0,-h_{2}\right),\left(h_{2},-h_{1}, 0\right)\right), 1 \leq i \leq 3 .
$$

Therefore $\operatorname{deg} m_{i j} \geq \operatorname{deg} h_{k}$, with $i \neq j \neq k \neq i$, i.e.

$$
b_{j}-a_{i} \geq s-b_{k} \Rightarrow b_{i} \leq a_{j}+a_{k} \Rightarrow a_{j}+b_{j} \leq \sum_{i=1}^{3} a_{i}, \text { for } 1 \leq j \leq 3
$$

Conversely let us suppose that the sequence $\left(a_{1}, a_{2}, a_{3} ; b_{1}, b_{2}, b_{3} ; s\right)$ satisfies the conditions 1,2 , 3 above. We set $c_{j}=s-b_{j}$, for $1 \leq j \leq 3$. Then $c_{3} \geq c_{2} \geq c_{1}$ and $c_{1}>0$ by Assumption 2. Hence $c_{j}>0$ for $1 \leq j \leq 3$. Now we set $t_{j}=\sum_{i=1}^{3} a_{i}-a_{j}-b_{j}$. By Assumption $3, t_{j} \geq 0$, for $1 \leq j \leq 3$. Note that $\sum_{j=1}^{3} t_{j}+\sum_{j=1}^{3} c_{j}=s, \quad \sum_{i=1}^{3} t_{i}+\sum_{i=1}^{3} c_{i}-c_{j}=b_{j}$ and $\sum_{i=1}^{3} t_{i}+c_{j}-t_{j}=a_{j}$. Now, applying Proposition 4.3 to the integers $c_{1}, c_{2}, c_{3}, t_{1}, t_{2}, t_{3}$ we get that $\left(a_{1}, a_{2}, a_{3} ; b_{1}, b_{2}, b_{3} ; s\right)$ is an essential Betti sequence.

## 5. Graded Betti numbers for ideals $I_{M}$

In this section we study the graded Betti numbers for generalized Gorenstein ideals $I_{M}$, arising from a minimal presentation matrix $M=\left(m_{i j}\right)$. A graded minimal resolution for such ideals can be written in the following way

$$
\begin{equation*}
0 \longrightarrow R(-s) \longrightarrow \oplus_{j=1}^{n} R\left(-b_{j}\right) \longrightarrow \oplus_{i=1}^{n} R\left(-a_{i}\right) \longrightarrow R \longrightarrow R / I_{M} \longrightarrow 0 \tag{11}
\end{equation*}
$$

where $a_{1} \leq \ldots \leq a_{n}, b_{1} \geq \ldots \geq b_{n}$ and $s=\sum_{j=1}^{n} b_{j}-\sum_{i=1}^{n} a_{i}$. We will set also $c_{j}=s-b_{j}$, for $1 \leq$ $j \leq n$. Note that $C=\left(c_{1} \ldots c_{n}\right)^{T}$ is the degree vector of the leftmost map of the resolution. It is easy to check that $a_{i}<b_{n+1-i}$ for $1 \leq i \leq n$, and $a_{2}<b_{n}, a_{3}<b_{n-1}$. Moreover $b_{n-2}<s \leq a_{1}+a_{2}+$ $a_{3}$. Now we set $d_{i j}=\operatorname{deg} m_{i j}$. Note that $d_{i j}=b_{j}-a_{i}$, so $d_{i j} \geq d_{i+1 j}$ and $d_{i j} \geq d_{i j+1}$ for $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$. So if $d_{h k}=0$ for some ( $h, k$ ) then $m_{i j}=0$ for every $i \geq h$ and $j \geq k$. The matrix $D=\left(d_{i j}\right) \in \mathbb{Z}^{n, n}$ is called the degree matrix of $M$. The degree matrix of $M$ does not determine, in general, the graded Betti numbers of $I_{M}$.
Example 5.1. Let us consider the ideals

$$
I=(x y z, y z t, z t u, t u v, u v x, v x y), J=(x y z t, y z t u, z t u v, t u v x, u v x y, v x y z) .
$$

Their graded minimal free resolutions are

$$
0 \longrightarrow R(-6) \longrightarrow R(-4)^{6} \longrightarrow R(-3)^{6} \longrightarrow R \longrightarrow R / I \longrightarrow 0
$$

and

$$
0 \longrightarrow R(-6) \longrightarrow R(-5)^{6} \longrightarrow R(-4)^{6} \longrightarrow R \longrightarrow R / J \longrightarrow 0 .
$$

However if we know one of the $c_{j}$ 's in addition to the degree matrix, then the graded Betti numbers are determined.

Proposition 5.2. Let $M$ be a minimal presentation matrix and let $D=\left(d_{i j}\right)$ be the degree matrix of M. Let $c_{r}$ be the degree of the $r$-th component of $C$. Then the graded Betti numbers of $R / I_{M}$ are $s=$ $\sum_{i=1}^{n} d_{i i} ; b_{j}=s+d_{r j}-d_{r r}-c_{r}$, for $1 \leq j \leq n ; a_{i}=s-d_{i r}-c_{r}$, for $1 \leq i \leq n$.

Proof. By the exactness of (11) $s=\sum_{i=1}^{n}\left(b_{i}-a_{i}\right)=\sum_{i=1}^{n} d_{i i}$.
Furthermore

$$
\begin{aligned}
& a_{i}=a_{i}+s-b_{r}-\left(s-b_{r}\right)=s-\left(b_{r}-a_{i}\right)-c_{r}=s-d_{r i}-c_{r} . \\
& b_{j}=a_{r}+d_{r j}=s-d_{r r}-c_{r}+d_{r j} .
\end{aligned}
$$

Proposition 5.3. Let

$$
\left(a_{1}, \ldots, a_{n} ; b_{1}, \ldots, b_{n} ; s\right)
$$

be an essential Betti sequence. Let $u_{i} \geq 0,1 \leq i \leq n$ be any integers. We set $u=\sum_{i=1}^{n} u_{i}$. Then the sequence

$$
\left(a_{1}+u-u_{1}, \ldots, a_{n}+u-u_{n} ; b_{1}+u, \ldots, b_{n}+u ; s+u\right)
$$

is an essential Betti sequence.
Proof. By the assumptions there exists an ideal $I \subset R=k\left[x_{1}, \ldots, x_{r}\right]$, ht $I \geq 2$, having a resolution of the type

$$
0 \longrightarrow R(-s) \longrightarrow \bigoplus_{j=1}^{n} R\left(-b_{j}\right) \longrightarrow \bigoplus_{i=1}^{\varphi} R\left(-a_{i}\right) \longrightarrow R \longrightarrow R / I \longrightarrow 0,
$$

where $\varphi\left(e_{i}\right)=\left(m_{1 i}, \ldots, m_{n i}\right)$ and $\left(e_{1}, \ldots, e_{n}\right)$ is a basis of $\oplus_{j=1}^{n} R\left(-b_{j}\right)$. Let $S=R\left[y_{1}, \ldots, y_{n}\right]$, where the $y_{i}$ 's are new variables. Let

$$
\varphi^{\prime}: \bigoplus_{j=1}^{n} S\left(-b_{j}-u\right) \longrightarrow \bigoplus_{i=1}^{n} S\left(-a_{i}-u+u_{i}\right)
$$

be the map defined by

$$
\varphi^{\prime}\left(e_{i}^{\prime}\right)=\left(m_{1 i} y_{1}^{u_{1}}, \ldots, m_{n i} y_{n}^{u_{n}}\right) .
$$

By Theorem 3.8 one sees that the matrix $M^{\prime}=\left(m_{i j} y_{j}^{u_{j}}\right)$ (matrix associated to $\varphi^{\prime}$ ) is a minimal presentation matrix, so it defines an ideal $I_{M^{\prime}}$, whose minimal free resolution looks like

$$
0 \longrightarrow S(-s-u) \longrightarrow \bigoplus_{j=1}^{n} S\left(-b_{j}-u\right) \xrightarrow{\varphi^{\prime}} \oplus_{i=1}^{n} S\left(-a_{i}-u+u_{i}\right) \longrightarrow S \longrightarrow S / I_{M^{\prime}} \longrightarrow 0 .
$$

Moreover if $I_{M}=\left(g_{1}, \ldots, g_{n}\right)$, then $I_{M^{\prime}}=\left(g_{1}^{\prime}, \ldots, g_{n}^{\prime}\right)$, where

$$
g_{i}^{\prime}=g_{i} \prod_{j \neq i=1}^{n} y_{j}^{u_{j}}
$$

Definition 5.4. We will say that a Betti sequence $\left(a_{1}, \ldots, a_{n} ; b_{1}, \ldots, b_{n} ; s\right)$ is minimal if the $n$ sequences $\left(a_{1}, a_{2}-1 \ldots, a_{n}-1 ; b_{1}-1, \ldots, b_{n}-1 ; s-1\right), \ldots,\left(a_{1}-1, a_{2}-1 \ldots, a_{n} ; b_{1}-1, \ldots, b_{n}-1 ; s-1\right)$ are not Betti sequences.

Of course, by Proposition 5.3, it is enough to find the minimal Betti sequences for determine all Betti sequences for ideals $I_{M}$.

In order to give more information about Betti sequences for ideals $I_{M}$, we give the following definition, which arises from perfect ideals of height 2 (see [5]).

Definition 5.5. We will say that a sequence $\left(a_{1} \leq \ldots \leq a_{n} ; b_{1} \geq \ldots \geq b_{n} ; s\right)$ is a Gaeta sequence if $s=\sum_{j=1}^{n} b_{j}-\sum_{i=1}^{n} a_{i}$ and $b_{n+2-i}>a_{i}$ for $2 \leq i \leq n$.

Remark 5.6. Of course not every Gaeta sequence is an essential Betti sequence. For instance the sequence

$$
(3,3,3,3 ; 5,5,5,5 ; 8)
$$

is a Gaeta sequence. If it were an essential Betti sequence there should be an ideal $I \subset R=$ $k[x, y, z]$, of height at least 2 , having this Betti sequence. But $H_{R / I}(6)=0$, so $R / I$ is an Artinian algebra, therefore $R / I$ is a Gorenstein algebra of codimension 3, which is a contradiction because $I$ has an even number of minimal generators.

Our aim is to understand when a Gaeta sequence is an essential Betti sequence.
The next result will permit us to reduce the study of the essential Betti sequences to the Gaeta sequences.

Theorem 5.7. Let ( $\left.a_{1} \leq \ldots \leq a_{n} ; b_{1} \geq \ldots \geq b_{n} ; s\right)$ be a sequence such that $b_{n+2-t} \leq a_{t}$ for some $t \geq$ 4. We set $d=\sum_{i=t}^{n}\left(b_{n+1-i}-a_{i}\right)$. It is an essential Betti sequence iff

1. $s=\sum_{j=1}^{n} b_{j}-\sum_{i=1}^{n} a_{i}$;
2. ( $\left.a_{1}-d, \ldots, a_{t-1}-d ; b_{n+2-t}-d, \ldots b_{n}-d ; s-d\right)$ is an essential Betti sequence;
3. $b_{n+1-i}>a_{i}$ for $i \geq t$.

Proof. Let us suppose that ( $a_{1} \leq \ldots \leq a_{n} ; b_{1} \geq \ldots \geq b_{n} ; s$ ) is an essential Betti sequence, so there is an ideal $I$, ht $I \geq 2$, such that the graded minimal free resolution of $R / I$ looks like

$$
0 \longrightarrow R(-s) \longrightarrow \oplus_{j=1}^{n} R\left(-b_{j}\right) \longrightarrow \oplus_{i=1}^{n} R\left(-a_{i}\right) \longrightarrow R .
$$

The conditions 1 and 3 are well known for general facts. Let $M=\left(m_{i j}\right)$ be a matrix associated to the central map of the resolution, such that $\operatorname{deg} m_{i j}=b_{j}-a_{i}$. Let $\left(g_{1}, \ldots, g_{n}\right)=\gamma(M)$. Since $b_{n+2-t} \leq a_{t}, m_{i j}=0$ for $i \geq t$ and $j \geq n+2-t$. Let $M^{\prime}=\left(m_{i j}\right)$ be the square submatrix of $M$ where $1 \leq i \leq t-1$ and $n+2-t \leq j \leq n$ and $D=\left(m_{i j}\right)$ where $t \leq i \leq n$ and $1 \leq j \leq n+1-t$. Note that rank $M^{\prime} \leq t-2$ since $\left(g_{1} \ldots g_{t-1}\right) M^{\prime}=0$. Furthermore rank $M^{\prime} \geq t-2$ since rank $M=$ $n-1$. So rank $M^{\prime}=t-2$. Moreover $\operatorname{det} D \neq 0$. Indeed, because of the vanishing of the maximal minors of the submatrix of $M$ obtained by removing a column $C_{j}$ with $1 \leq j \leq n+1-t$, there is a column $C_{k}$ with $n+2-t \leq k \leq n$ such that the maximal minors of the submatrix of $M$ obtained by removing $C_{k}$ are not vanishing multiple of $g_{1}, \ldots, g_{n}$. Since such minors are multiple of $\operatorname{det} D$ we get that $\operatorname{det} D \neq 0$. Consequently $\gamma\left(M^{\prime}\right)=(\operatorname{det} D)^{-1}\left(g_{1}, \ldots, g_{t-1}\right)$. Now since every syzygy on $\gamma\left(M^{\prime}\right)$ is also a syzygy on $g_{1}, \ldots, g_{n}$ and since $\operatorname{det} D \neq 0$ this syzygy must be in the span of $C_{j}$ for $n+2-t \leq j \leq n$. So by Proposition 2.6, $M^{\prime}$ is a presentation matrix and the Betti sequence of $I_{M^{\prime}}$ is $\left(a_{1}-d, \ldots, a_{t-1}-d ; b_{n+2-t}-d, \ldots b_{n}-d ; s-d\right)$.

Conversely we suppose that the conditions 1,2 and 3 are satisfied. In particular, by condition 2 there exists a presentation matrix $M^{\prime}=\left(m_{i j}^{\prime}\right)$, with $\operatorname{deg} m_{i j}^{\prime}=\left(b_{n+1-t-j}-d\right)-\left(a_{i}-d\right)=b_{n+1-t-j}-a_{i}$ of size $t-1$ such that $I_{M^{\prime}}$ has Betti sequence $\left(a_{1}-d, \ldots, a_{t-1}-d ; b_{n+2-t}-d, \ldots b_{n}-d ; s-d\right)$. Now we define a square matrix $M=\left(m_{i j}\right)$, of size $n$, in the following way

$$
m_{i j}= \begin{cases}m_{i, j-(n+1-t)}^{\prime} & \text { for } 1 \leq i \leq t-1, n+2-t \leq j \leq n \\ y_{i}^{b_{j}-a_{i}} & \text { for } i+j=n, t-1 \leq i \leq n-1 \\ z_{i}^{b_{j}-a_{i}} & \text { for } i+j=n+1, t \leq i \leq n \\ 0 & \text { elsewhere }\end{cases}
$$

where $y_{j}$ and $z_{j}$ are new variables for every $j$. The condition 3 guarantees that the exponents of $y_{j}$ and $z_{j}$ are positive integers. Since rank $M^{\prime}=t-2$, we have rank $M=n-1$. We set $\left(g_{1}^{\prime}, \ldots, g_{t-1}^{\prime}\right)=\gamma\left(M^{\prime}\right)$. Now if we set $\left(g_{1}, \ldots, g_{n}\right)=\gamma(M)$, we see that $g_{i}=g_{i}^{\prime} \prod_{i=t}^{n} z_{i}^{b_{n-i}-a_{i}}$ for $1 \leq$ $i \leq t-1$ and $g_{i}=g_{t-1}^{\prime} \prod_{h=t-1}^{i-1} y_{h}^{b_{n-h}-a_{h}} \prod_{h=i+1}^{n} z_{h}^{b_{n+1-h}-a_{h}}$ for $t \leq i \leq n$. Note that $\operatorname{deg} g_{i}=a_{i}$ for $1 \leq$ $i \leq n$. In order to show that $M$ is a minimal presentation matrix, we will use Theorem 3.8. We set $\left(h_{1}^{\prime}, \ldots, h_{t-1}^{\prime}\right)=\gamma\left(M \prime^{T}\right)$. By Theorem 3.8, the ideal $J^{\prime}$ generated by the components of $\gamma\left(M \prime^{T}\right)$ has depth $J^{\prime} \geq 3$. Since $\gamma\left(M^{T}\right)=\left(0, \ldots, 0, h_{1}^{\prime}, \ldots, h_{t-1}^{\prime}\right)$, the ideal $J$ generated by the components of $\gamma\left(M^{T}\right)$ coincides with $J^{\prime}$, so it has depth greater than or equal to 3 too. By Lemma 3.7, $M^{C}=$ $u \gamma(M)^{T} \gamma\left(M^{T}\right)$. So we need only to show that $u$ is a unit. To do this we compute the cofactor $M_{1 n}$ of the entry in position $(1, n)$.

$$
M_{1 n}=(-1)^{n+1} g_{1}^{\prime} h_{t-1}^{\prime} \prod_{i=t}^{n} z_{i}^{b_{n-i}-a_{i}}=(-1)^{n+1} g_{1} h_{t-1}^{\prime}
$$

since $h_{t-1}^{\prime}$ is the $n$-th component of $\gamma\left(M^{T}\right)$, we are done.
Corollary 5.8. Let $\left(a_{1} \leq \ldots \leq a_{n} ; b_{1} \geq \ldots \geq b_{n} ; s\right)$ be a sequence such that $b_{n-2} \leq a_{4}$. We set $d=$ $\sum_{i=4}^{n}\left(b_{n+1-i}-a_{i}\right)$. It is an essential Betti sequence iff

1. $s=\sum^{n} b_{j}-\sum^{n} a_{i}$;
2. $a_{j}+\dot{b}_{n}^{=1}-3+j+\dot{\bar{a}}^{1} \leq a_{1}+a_{2}+a_{3}<b_{n-1}+b_{n}+d$, for $j=1,2,3$.
3. $b_{n+1-i}>a_{i}$ for $i \geq 4$.

Proof. According to Theorem 5.7, we need to show that $\left(a_{1}-d, a_{2}-d, a_{3}-d\right.$; $\left.b_{n-2}-d, b_{n-1}-d, b_{n}-d ; s-d\right)$ is a essential Betti sequence. Now it is enough to use Theorem 4.5 to verify this fact.

Remark 5.9. Note that by iterating the procedure of Theorem 5.7 any sequence $\beta=\left(a_{1} \leq \ldots \leq\right.$ $\left.a_{n} ; b_{1} \geq \ldots \geq b_{n} ; s\right)$ can be transformed in a Gaeta sequence $\beta^{\prime}=\left(a_{1}^{\prime} \leq \ldots \leq a_{m}^{\prime} ; b_{1}^{\prime} \geq \ldots \geq b_{m}^{\prime} ; s^{\prime}\right)$.

Corollary 5.10. Let $\beta=\left(a_{1} \leq \ldots \leq a_{n} ; b_{1} \geq \ldots \geq b_{n} ; s\right)$ be a sequence. Using the same notation of Remark 5.9, $\beta$ is an essential Betti sequence iff the Gaeta sequence $\beta^{\prime}$ is an essential Betti sequence.

Proof. Taking into account Remark 5.9, it is an easy application of Theorem 5.7.
Now we study the essential Betti sequences of the type

$$
(a, \ldots, a ; b, \ldots, b ; s)
$$

Definition 5.11. Let $M$ be a square matrix of size $n$. The matrix $M=\left(m_{i j}\right)$ is said to be bidiagonal iff $m_{i j}=n_{i j}=0$ for $j \neq i, i+1,1 \leq i \leq n$ (here $m_{n, n+1}$ means $m_{n 1}$ ).

Lemma 5.12. Let $S=k\left[x_{1}, \ldots, x_{r}, y_{1}, \ldots, y_{s}\right]$. Let $M=\left(m_{i j}\right)$ and $N=\left(n_{i j}\right)$ be two minimal presentation bidiagonal square matrices of size $n$ such that $m_{i j}$ are forms of degree d in $k\left[x_{1}, \ldots, x_{r}\right], n_{i j}$ are forms of degree $e$ in $k\left[y_{1}, \ldots, y_{s}\right]$. Let $M * N=\left(t_{i j}\right)$ be the matrix such that $t_{i i}=m_{i i} n_{i i}$ and $t_{i, i+1}=$ $-m_{i, i+1} n_{i i+1}$ and $t_{i j}=0$ otherwise.

Then $M * N$ is a presentation bidiagonal matrix. Moreover if $\gamma\left(M^{T}\right)=\left(h_{1}, \ldots, h_{n}\right)$ and $\gamma\left(N^{T}\right)=\left(k_{1}, \ldots, k_{n}\right)$ then $\gamma\left((M * N)^{T}\right)=\left(h_{1} k_{1}, \ldots, h_{n} k_{n}\right)$.

Proof. Note that

$$
\begin{aligned}
\operatorname{det}(M * N) & =\prod_{i=1}^{n} t_{i i}+(-1)^{n+1} \prod_{i=1}^{n} t_{i, i+1}= \\
& =\prod_{i=1}^{n} m_{i i} n_{i i}+(-1)^{n+1}(-1)^{n} \prod_{i=1}^{n} m_{i, i+1} n_{i, i+1}= \\
& =\prod_{i=1}^{n} m_{i i} n_{i i}-\prod_{i=1}^{n} m_{i, i+1} n_{i, i+1}=0 .
\end{aligned}
$$

To show that $M * N$ is a minimal presentation matrix we use Theorem 3.8. At first we need to compute the cofactors of $M * N$. Such a computation can be found, for instance, in the article [9] on page 281. From this computation follows immediately that $\operatorname{det}(M * N)_{i j}=\operatorname{det} M_{i j} \operatorname{det} N_{i j}$ (where with the index $i j$ we mean the submatrix obtained by removing $i$-th row and $j$-th column). Consequently we get that $\gamma\left((M * N)^{T}\right)=\left(h_{1} k_{1}, \ldots, h_{n} k_{n}\right)$. Since $\gamma\left((M)^{T}\right)$ and $\gamma\left((N)^{T}\right)$ consisting of forms living in polynomial rings in different variables we deduce that the ideal generated by $\gamma\left((M * N)^{T}\right)$ has depth at least 3. From the same computation follows also that $(M * N)^{C}=\gamma((M * N))^{T} \gamma\left((M * N)^{T}\right)$.

Proposition 5.13. If a sequence $\left(a_{1}, \ldots, a_{n} ; b_{1}, \ldots, b_{n} ; s\right)$ of positive integers, with $a_{1}=\ldots=a_{n}=a$, $b_{1}=\ldots=b_{n}=b$, is a essential Betti sequence then

1. $s=n(b-a)$;
2. $n a<(n-1) b \leq(n+1) a$; moreover, when $n$ is even, $(n-1) b<(n+1) a$.

Proof. Let us suppose that $\left(a_{1}, \ldots, a_{n} ; b_{1}, \ldots, b_{n} ; s\right)$ with $a_{1}=\ldots=a_{n}=a, b_{1}=\ldots=b_{n}=b$, is a essential Betti sequence. Then there exists an ideal $I$, ht $I \geq 2$, whose resolution is

$$
0 \longrightarrow R(-s) \longrightarrow R(-b)^{n} \longrightarrow R(-a)^{n} \longrightarrow R \longrightarrow R / I \longrightarrow 0
$$

The condition 1 is trivial. Moreover, since $s>b, n(b-a)>b$, so $n a<(n-1) b$.
Since depth $(R / I)=$ depth $(R)-3$ we can reduce to a ring in only 3 variables. So we can suppose that $R=k\left[x_{1}, x_{2}, x_{3}\right]$. Of course we have that $H_{R / I}(s-2) \geq 0$. Therefore

$$
\begin{aligned}
0 \leq H_{R / I}(s-2) & =\binom{s}{2}-n\binom{s-a}{2}+n\binom{s-b}{2}-\binom{0}{2} \\
& =\frac{1}{2} n(b-a)[a(n+1)-b(n-1)]
\end{aligned}
$$

that implies $(n-1) b \leq(n+1) a$. Moreover, when $n$ is even, since $R / I$ cannot be Gorenstein, hence it cannot be Artinian, so $H_{R / I}(s-2)>0$, so for $n$ even we have $(n-1) b<(n+1) a$.

Theorem 5.14. A sequence $\left(a_{1}, \ldots, a_{n} ; b_{1}, \ldots, b_{n} ; s\right)$ of positive integers, with $a_{1}=\ldots=a_{n}=a, b_{1}=$ $\ldots=b_{n}=b, n$ odd is an essential Betti sequence iff

1. $s=n(b-a)$;
2. $n a<(n-1) b \leq(n+1) a$.

Proof. The condition is necessary by Proposition 5.13.
Conversely let $\left(a_{1}, \ldots, a_{n} ; b_{1}, \ldots, b_{n} ; s\right)$ with $a_{1}=\ldots=a_{n}=a, b_{1}=\ldots=b_{n}=b, n$ odd a sequence satisfying the conditions 1 ) and 2 ).

By subtracting ( $n-1$ )a, the condition 2) becomes

$$
\frac{n-1}{2}(b-a) \leq a \leq(n-1)(b-a)-1 .
$$

Now we work by induction on $b-a$. For $b-a=1$ our condition becomes $s=n$ and $\frac{n-1}{2} \leq a \leq$ $n-2$. Using Corollary 3.11 in [9] we can produce an ideal $I$, ht $I \geq 2$, in $R$ such that $R / I$ has the requested Betti sequence. Let us suppose that we have realized an algebra $R / I$ having the requested Betti sequence when $b-a=h$. We need to construct algebras $R / I$ with Betti sequence satisfying $\frac{(n-1)}{2}(h+1) \leq a \leq(n-1)(h+1)-1$ and $s=n(h+1)$. Using Proposition 5.3 for $u_{i}=1$ for every $i$, we realize the Betti sequences satisfying $s=n(h+1)$ and $\frac{(n-1)(h+2)}{2} \leq a \leq(n-1)(h+$ 1) -1 . So, it remains to build the Betti sequences such that

$$
s=n(h+1) \text { and } \frac{(n-1)}{2}(h+1) \leq a \leq \frac{(n-1)}{2}(h+2)-1
$$

i.e. $\frac{n-1}{2} h+1 \leq s-b \leq \frac{n-1}{2}(h+1)$. We are interested on the integer $s-b$ since it is the degree of the components of vector $\gamma\left(M^{T}\right)$, where $M$ is the presentation matrix which we will use to realize these Betti sequences. Note that $h$ and $s-b$ determine all the Betti sequence. For $h=1$ we have realized every Betti sequence such that $1 \leq s-b \leq \frac{n-1}{2}$, using bidiagonal matrices. Moreover, by the inductive hypothesis, we have also realized every Betti sequence such that $b-a=h$ and $1 \leq$ $s-b \leq h \frac{n-1}{2}$, using again bidiagonal matrices. Now let $1 \leq t \leq \frac{n-1}{2}$ and let $M$ be a minimal presentation bidiagonal matrix realizing the Betti sequence such that $s=n, b-a=1$ and $s-b=t$. Let $N$ be a minimal presentation bidiagonal matrix realizing the Betti sequence such that $s=n h$, $b-a=h$ and $s-b=h \frac{n-1}{2}$. Applying Lemma 5.12 we get a matrix $M * N$ realizing the Betti sequence such that $s=n(h+1), b-a=h+1$ and $s-b=h \frac{n-1}{2}+t$.

Proposition 5.15. A sequence $\left(a_{1}, \ldots, a_{n} ; b_{1}, \ldots, b_{n} ; s\right)$ of positive integers, with $a_{1}=\ldots=a_{n}=a$, $b_{1}=\ldots=b_{n}=b$, n even is an essential Betti sequence provided that

1. $s=n(b-a)$;
2. $n a<(n-1) b \leq n a+\frac{n-2}{2}(b-a)$.

Proof. At first we observe that the condition 2 is equivalent to

$$
\frac{n}{2}(b-a) \leq a \leq(n-1)(b-a)-1 \Longleftrightarrow 1 \leq s-b \leq \frac{n-2}{2}(b-a) .
$$

We proceed analogously to the proof of Theorem 5.14. Now we work by induction on $b-a$. For $b-a=1$ our conditions become $s=n$ and $\frac{n-2}{2} \leq a \leq n-2$. Using Corollary 3.11 in [9] we can produce an ideal $I$ in $R$, ht $I \geq 2$, such that $R / I$ has the requested Betti sequence.

Let us suppose that we have realized an algebra $R / I$ having the requested Betti sequence when $b-a=h$. We need to construct algebras $R / I$ with Betti sequence satisfying $\frac{n}{2}(h+1) \leq a \leq$ $(n-1)(h+1)-1$ and $s=n(h+1)$. Using Proposition 5.3 for $u_{i}=1$ for every $i$, we realize the Betti sequences satisfying $s=n(h+1)$ and $\frac{n}{2} h+n-1 \leq a \leq(n-1)(h+1)-1$. So, it remains to build the Betti sequences such that

$$
s=n(h+1) \text { and } \frac{n}{2}(h+1) \leq a \leq \frac{n}{2} h+n-2
$$

i.e. $\frac{n-2}{2} h+1 \leq s-b \leq \frac{n-2}{2}(h+1)$. We are interested on the integer $s-b$ since it is the degree of the components of vector $\gamma\left(M^{T}\right)$, where $M$ is the presentation matrix which we will use to realize these Betti sequences. Note that $h$ and $s-b$ determine all the Betti sequence. For $h=1$ we have realized every Betti sequence such that $1 \leq s-b \leq \frac{n-2}{2}$, using bidiagonal matrices. Moreover, by the inductive hypothesis, we have also realized every Betti sequence such that $b-a=h$ and $1 \leq$
$s-b \leq h \frac{n-2}{2}$, using again bidiagonal matrices. Now let $1 \leq t \leq \frac{n-2}{2}$ and let $M$ be a minimal presentation bidiagonal matrix realizing the Betti sequence such that $s=n, b-a=1$ and $s-b=t$. Let $N$ be a minimal presentation bidiagonal matrix realizing the Betti sequence such that $s=n h$, $b-a=h$ and $s-b=h \frac{n-2}{2}$. Applying Lemma 5.12 we get a matrix $M * N$ realizing the Betti sequence such that $s=n(h+1), b-a=h+1$ and $s-b=h \frac{n-2}{2}+t$.

Remark 5.16. Unfortunately our construction does not allow building all the sequences satisfying the conditions of Proposition 5.13. For instance the sequence ( $5,5,5,5 ; 8,8,8,8 ; 12$ ) cannot be built with the tools of Proposition 5.15. Nevertheless it is an essential Betti sequence. In fact, using Macaulay 2 , one can verify that the ideal $I=\left(f_{1}, f_{2}, f_{3}, f_{4}\right)$ with

$$
\begin{aligned}
f_{1}= & -x_{3} y_{4} y_{5} z_{4} z_{5}-y_{1} y_{4} y_{5} z_{4} z_{6}+x_{3} y_{4} y_{5} z_{1} z_{8}+y_{1} y_{4} y_{5} z_{2} z_{8} \\
f_{2}= & x_{1} x_{2} x_{3} z_{4} z_{5}+x_{1} x_{2} y_{1} z_{4} z_{6}+y_{1} y_{2} y_{3} z_{4} z_{7}+ \\
& -x_{1} x_{2} x_{3} z_{1} z_{8}-x_{1} x_{2} y_{1} z_{2} z_{8}-y_{1} y_{2} y_{3} z_{3} z_{8} \\
f_{3}= & x_{3} y_{2} y_{3} z_{3} z_{5}+y_{1} y_{2} y_{3} z_{3} z_{6}-x_{3} y_{2} y_{3} z_{1} z_{7}-y_{1} y_{2} y_{3} z_{2} z_{7} \\
f_{4}= & -x_{1} x_{2} x_{3} z_{3} z_{5}-x_{1} x_{2} y_{1} z_{3} z_{6}+x_{1} x_{2} x_{3} z_{1} z_{7}+ \\
& +x_{1} x_{2} y_{1} z_{2} z_{7}+x_{3} y_{4} y_{5} z_{4} z_{7}-x_{3} y_{4} y_{5} z_{3} z_{8}
\end{aligned}
$$

has height 2 and the above Betti sequence.

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