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On mixed Hessians and the Lefschetz properties

Rodrigo Gondim [a,1], Giuseppe Zappalà [b,*]

[a] Universidade Federal Rural de Pernambuco, av. Don Manoel de Medeiros s/n, Dois Irmãos - Recife - PE 52171-900, Brazil
[b] Dipartimento di Matematica e Informatica, Università degli Studi di Catania, Viale A. Doria 5, 95125 Catania, Italy

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ABSTRACT

We introduce a new type of Hessian matrix, that we call Mixed Hessian. The mixed Hessian is used to compute the rank of a multiplication map by a power of a linear form in a standard graded Artinian Gorenstein algebra. In particular we recover the main result of a paper by Maeno and Watanabe for identifying Strong Lefschetz elements, generalizing it also for Weak Lefschetz elements. This criterion is also used to give a new proof that Boolean algebras have the Strong Lefschetz Property. We also construct new examples of Artinian Gorenstein algebras presented by quadrics that does not satisfy the Weak Lefschetz Property; we construct minimal examples of such algebras and we give bounds, depending on the degree, for their existence. Artinian Gorenstein algebras presented by quadrics were conjectured to satisfy WLP in two papers by Migliore and Nagel, and in a previous paper we constructed the first counter-examples.

0. Introduction

The Hessian matrix of a form is the matrix of its second derivatives and its Hessian is the determinant of this matrix. The first instance of such object goes back to the seminal paper of Gauss [8]. In this context the Hessian describes curvature for surfaces given by an implicit function, see also Segre [27] for the n-dimensional analog. Complete hypersurfaces with zero Gaussian curvature are also called developable. We recall that for $X = V(f) \subset \mathbb{P}^N_{\mathbb{K}}$, a hypersurface defined over $\mathbb{K} = \mathbb{R}, \mathbb{C}$, we get $\mathrm{hess}_f = 0 \pmod f$ if and only if the hypersurface is developable, that is, the Gauss map is degenerated. In $\mathbb{P}^3$ only cones and the tangent surface of a curve are developable. While the cones have $\mathrm{hess}_f = 0$ the tangent surfaces have $\mathrm{hess}_f \neq 0$ (see [26, Chapter 7]).

* Corresponding author.
E-mail addresses: rodrigo.gondim@ufrpe.br (R. Gondim), zappalag@dmi.unict.it (G. Zappalà).
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Hesse claimed in [14] that for arbitrary N, $\text{hess}_f = 0$ if and only if $X = V(f) \subset \mathbb{P}^N$ is a cone. Gordan and Noether in the fundamental paper [11] showed that Hesse's claim is true for $N \leq 3$ and they produced series of counterexamples for $N \geq 4$. Moreover, these counterexamples can be characterized as the only hypersurfaces in $\mathbb{P}^4$ with vanishing hessian which are not cones. A modern proof of this fact can be found in [12] while a very detailed account on the subject appears in [26, Chapter 7]. The so called Gordan–Noether Theory is also treated in very different aspects in [4,6,19,12,13,31,26].

Hessians of higher degree were introduced in [22] and used to control the so called Strong Lefschetz property (SLP). This property for a graded Artinian Gorenstein algebra was inspired by the Hard Lefschetz Theorem on the cohomology of smooth projective complex varieties. In this paper we introduce the mixed Hessians, that generalize the Hessians of higher order, providing a generalization of the criterion for Strong Lefschetz elements also for Weak Lefschetz elements (see Theorem 2.4 and Corollary 2.5).

The Lefschetz properties have attracted a great deal of attention over the years, since they are *phenomena* connected with Commutative Algebra, Algebraic and Tropical Geometry and Combinatorics, see [28,29,16, 15,9,10]. The first result in the area, proved by Stanley [28] and independently by Watanabe, asserts that a complete intersection of monomials have the SLP. Here we (re)prove a special case of this result for quadratic complete intersections of monomials, also called Boolean algebras.

A standard graded $\mathbb{K}$-algebra is said to be presented by quadrics if it is isomorphic to the quotient of a polynomial ring over $\mathbb{K}$ by a homogeneous ideal generated by quadratic forms. These algebras are related with Koszul algebras and Gröbner basis, see for example [5]. In [10] we disprove a conjecture posed in [21] that Artinian Gorenstein algebras presented by quadrics have the WLP. Here we study in more detail the family introduced in [10] to give minimal examples for those algebras failing the WLP.

We now describe the contents of the paper in more detail. In the first section we recall the basic definitions and constructions of standard graded Artinian Gorenstein algebras, and we recall a combinatorial construction introduced in [10].

In the second section we introduce the mixed Hessians and prove the main result, Theorem 2.4, a generalization of the Hessian criterion to mixed Hessians, see also Corollary 2.5. In the third section we prove an inductive construction (see Proposition 3.3) whose Corollary is the very well known fact that Boolean algebras have the SLP (see Corollary 3.4).

The next section is devoted to recall a combinatorial construction introduced in [10], we associate a homogeneous simplicial complex to a standard graded Artinian Gorenstein algebra. A special family called Turan algebras have been used in [10] to produce counterexamples to the conjecture posed in [20,21]. The conjecture was that Artinian Gorenstein algebras presented by quadrics have the WLP.

In the last section we deal with algebras presented by quadrics of minimal codimension failing the WLP. For degree $d = 3$ we find the minimal example in codimension 8 (see Example 5.6). We also classify algebras associated to graphs with respect to WLP (see Proposition 5.9). Applying the inductive construction we get a lower bound for the codimension of algebras of odd degree to fail the WLP (see Corollary 5.12); this bound is relatively sharp. For even degrees we also give a bound for the failure of the WLP, Corollary 5.17.

1. Artinian Gorenstein algebras and the Lefschetz properties

1.1. Lefschetz properties

Let $\mathbb{K}$ be an infinite field and $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates.

If $A = R/I$ is an Artinian standard graded R-algebra, then A has a decomposition $A = \bigoplus_{i=0}^{d} A_i$, as a sum of finite dimensional $\mathbb{K}$-vector spaces with $A_d \neq 0$.

A form $F \in R_k$ induces a $\mathbb{K}$-vector space map $\mu_{i,F} : A_i \to A_{i+k}$, defined by $\mu_{i,F}(\alpha) = F\alpha$, for every $\alpha \in A_i$.

Definition 1.1. We say that A has the *Strong Lefschetz property* (in short SLP) if there exists a linear form $L \in R_1$ such that $\operatorname{rk} \mu_{i,L^k} = \min\{\dim_{\mathbb{K}} A_i, \dim_{\mathbb{K}} A_{i+k}\}$, for every i, k.

Definition 1.2. We say that A has the *Weak Lefschetz property* (in short WLP) if there exists a linear form $L \in R_1$ such that $\operatorname{rk} \mu_{i,L} = \min\{\dim_{\mathbb{K}} A_i, \dim_{\mathbb{K}} A_{i+1}\}$, for every i.

Definition 1.3. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and $A = R/I$ be an Artinian standard graded R-algebra, with $I_1 = 0$. The integer n is said to be the codimension of A. If $A_d \neq 0$ and $A_i = 0$ for all $i > d$, then d is called the maximal socle degree of A. The Hilbert vector of A is $h_A = \operatorname{Hilb}(A) = (1, h_1, h_2, \ldots, h_d)$, where $h_k = \dim A_k$. We say that h_A is unimodal if there exists k such that $1 \leq h_1 \leq \ldots \leq h_k \geq h_{k+1} \geq \ldots \geq h_d$.

Remark 1.4. We recall that an Artinian algebra $A = \bigoplus_{i=0}^{d} A_i$, $A_d \neq 0$, is a *Gorenstein algebra* if and only if $\dim_{\mathbb{K}} A_d = 1$ and the bilinear pairing

$$A_i \times A_{d-i} \to A_d$$

induced by the multiplication is non-degenerate for $0 \leq i \leq d$. So we have an isomorphism $A_i \simeq \operatorname{Hom}_{\mathbb{K}}(A_{d-i}, A_d)$ for $i = 0, \ldots, d$. In particular, $\dim_{\mathbb{K}} A_i = \dim_{\mathbb{K}} A_{d-i}$, for $i = 0, \ldots, d$. Moreover, for every $L \in R_1$, $\operatorname{rank} \mu_{i,L} = \operatorname{rank} \mu_{d-i-1,L}$, for $0 \leq i \leq d$.

Since A is generated in degree 0 as an R-module, if $\mu_{i,L}$ is surjective, then $\mu_{j,L}$ is surjective for every $j \geq i$. Therefore, for an Artinian Gorenstein algebra A, if $\mu_{i,L}$ is injective, then $\mu_{j,L}$ is injective for every $j \leq i$. Of course SLP implies WLP. Notice also that the WLP implies the unimodality of the Hilbert vector of A. Unimodality in the Gorenstein case implies that $\dim A_{k-1} \leq \dim A_k$ for all $k \leq \frac{d}{2}$. The converse of these implications are not true, (see Corollary 3.3 and Theorem 3.8 in [9]).

1.2. Macaulay–Matlis duality

From now on we assume that $\operatorname{char} \mathbb{K} = 0$. Let us regard the polynomial algebra R as a module over the algebra $Q = \mathbb{K}[X_1, \ldots, X_n]$ via the identification $X_i = \partial/\partial x_i$. If $f \in R$ we set

$$\operatorname{Ann}_Q(f) = \{p(X_1, \ldots, X_n) \in Q \mid p(\partial/\partial x_1, \ldots, \partial/\partial x_n) f = 0\}.$$

It is well known that $A = Q/I$ is a standard graded Artinian Gorenstein algebra if and only if there exists a form $f \in R$ such that $I = \operatorname{Ann}_Q(f)$ (for more details see, for instance, [22]).

In the sequel we always assume that $A = Q/I$, $I = \operatorname{Ann}_Q(f)$ and $I_1 = 0$.

When we deal with standard bigraded Artinian Gorenstein algebras $A = \bigoplus_{i=0}^{d} A_i$, $A_d \neq 0$, with $A_k = \bigoplus_{i=0}^{k} A_{(i,k-i)}$, $A_{(d_1,d_2)} \neq 0$ for some d_1, d_2 such that $d_1 + d_2 = d$, we call (d_1, d_2) the socle bidegree of A. Since $A_k^* \simeq A_{d-k}$ and since duality is compatible with direct sum, we get $A_{(i,j)}^* \simeq A_{(d_1-i, d_2-j)}$.

In this case given a presentation of $A = Q/\operatorname{Ann}_Q(f)$ with $R = \mathbb{K}[x, u]$ and $Q = \mathbb{K}[X, U]$ standard bigraded, we get $I = \operatorname{Ann}_Q(f)$ a bihomogeneous ideal. It is easy to see that the Macaulay dual of the defining ideal is $f \in R_{(d_1,d_2)}$ a bihomogeneous polynomial of total degree $d = d_1 + d_2$.

Definition 1.5. With the previous notation, all bihomogeneous polynomials of bidegree $(1, d-1)$ can be written in the form

$$f = x_1 g_1 + \ldots + x_n g_n,$$

where $g_i \in \mathbb{K}[u_1, \ldots, u_m]_{d-1}$. We say that f is of *square-free monomial type* if all g_i are square free monomials. The associated algebra, $A = Q/\operatorname{Ann}_Q(f)$, is bigraded, has socle bidegree $(1, d-1)$ and we assume that $I_1 = 0$, so $\operatorname{codim} A = m + n$.

2. Mixed Hessians and dual mixed Hessians

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and $Q = \mathbb{K}[\partial/\partial x_1, \ldots, \partial/\partial x_n]$. Let $f \in R_d$. Let $A = Q/\operatorname{Ann}(f)$ be a standard graded Artinian Gorenstein $\mathbb{K}$-algebra,

$$A = A_0 \oplus \ldots \oplus A_d, \;\; \dim_K A_d = 1.$$

Let $k \leq l$ be two integers, take $L \in A_1$ and let us consider the $\mathbb{K}$-vector space map

$$\mu_L : A_k \to A_l, \;\; \mu_L(\alpha) = L^{l-k}\alpha.$$

Let $\mathcal{B}_k = (\alpha_1, \ldots, \alpha_r)$ be a $\mathbb{K}$-linear basis of A_k and $\mathcal{B}_l = (\beta_1, \ldots, \beta_s)$ be a $\mathbb{K}$-linear basis of A_l.

Definition 2.1. We call the matrix

$$\operatorname{Hess}_f^{(k,l)} := [\alpha_i \beta_j(f)]$$

the mixed Hessian (matrix) of f of mixed order (k, l) with respect to the bases $\mathcal{B}_k$ and $\mathcal{B}_l$. Moreover, we define $\operatorname{Hess}_f^k = \operatorname{Hess}_f^{(k,k)}$, $\operatorname{hess}_f^k = \det(\operatorname{Hess}_f^k)$ and $\operatorname{hess}_f = \operatorname{hess}_f^1$.

Now we consider the unique generator $\vartheta \in A_d$, such that $\vartheta(f) = 1$. So we can define the dual basis in $\operatorname{Hom}_{\mathbb{K}}(A_l, A_d)$, $\mathcal{B}_l^* = (\beta_1^*, \ldots, \beta_s^*)$, in the following way

$$\beta_i^*(\beta_j) = \delta_{ij}\vartheta.$$

Since A is Gorenstein, the multiplication induces a non-degenerate bilinear map $A_l \times A_{d-l} \to A_d$, so we have an isomorphism $\varphi : A_{d-l} \to \operatorname{Hom}_K(A_l, A_d)$, defined by $\varphi(\gamma)(\beta) = \gamma\beta$. In particular we have $\varphi^{-1}(\beta_i^*) = \vartheta/\beta_i \in A_{d-l}$.

Definition 2.2. We call the matrix

$$\operatorname{Hess}_f^{(l^*,k)} := [(\vartheta/\beta_i)\alpha_j(f)]$$

the dual mixed Hessian (matrix) of f of mixed order (k, l) with respect to the bases $\mathcal{B}_k$ and $\mathcal{B}_l$.

Note that $\operatorname{Hess}_f^{(k,l)} \in (R_{d-l-k})^{s,r}$ and $\operatorname{Hess}_f^{(l^*,k)} \in (R_{l-k})^{s,r}$.

Remark 2.3. First of all, since we are interested only in the rank of such matrices, the dependence on the basis is not relevant.

Therefore, it is easy to see that $\operatorname{rk}\operatorname{Hess}_f^{(l^*,k)} = \operatorname{rk}\operatorname{Hess}_f^{(d-l,k)}$.

We observe that, under the natural assumption that $\operatorname{Ann}_Q(f)_1 = 0$, the notation hess_f is consistent with the classical definition of Hessian, by taking $B_1 = \{X_1, \ldots, X_n\}$, the standard basis of the embedding.

Moreover, the notation is also compatible with the Definition of higher order Hessians given in [22].

If A is bigraded, and if $B_k = \{\alpha_1, \ldots, \alpha_s\}$ and $B_l = \{\beta_1, \ldots, \beta_t\}$ are bases of the $\mathbb{K}$-vector spaces $A_{(k,l)}$ and $A_{(k',l')}$ respectively, we can also define $\operatorname{Hess}_f^{((k,l),(k',l'))} = (\alpha_i(\beta_j(f)))_{s\times t}$.

If $L = a_1\partial/\partial x_1 + \ldots + a_n\partial/\partial x_n$, we set $L^\perp = (a_1, \ldots, a_n)$. We regard it as a point in $\mathbb{A}^n$. For example if $F \in \mathbb{K}[x_1, \ldots, x_n]$, then $F(L^\perp) = F(a_1, \ldots, a_n)$.

Theorem 2.4. *With the previous notation, let M be the matrix associated to the map $\mu_L : A_k \to A_l$ with respect to the bases $\mathcal{B}_k$ and $\mathcal{B}_l$. Then*

$$M = (l-k)!\,\mathrm{Hess}_f^{(l^*,k)}(L^\perp).$$

Proof. First of all note that if $g \in R_d$ then $L^d(g) = d!g(L^\perp)$.

Let $M = (b_{ij})$. Then

$$L^{l-k}\alpha_j = \sum_{h=1}^{s} b_{hj}\beta_h.$$

Consequently

$$\beta_i^*(L^{l-k}\alpha_j) = \sum_{h=1}^{s} b_{hj}\beta_i^*(\beta_h) \Rightarrow (\vartheta/\beta_i)L^{l-k}\alpha_j = b_{ij}\vartheta \Rightarrow L^{l-k}(\vartheta/\beta_i)\alpha_j = b_{ij}\vartheta.$$

Now we evaluate in f

$$L^{l-k}(\vartheta/\beta_i)\alpha_j(f) = b_{ij}\vartheta(f) \Rightarrow (l-k)!(\vartheta/\beta_i)\alpha_j(f)(L^\perp) = b_{ij}. \quad \square$$

The previous results give us a generalization of [30, Theorem 4] and [22, Theorem 3.1].

Corollary 2.5 *(Hessian criteria for Strong and Weak Lefschetz elements). Let $A = Q/\operatorname{Ann}_Q(f)$ be a standard graded Artinian Gorenstein algebra of codimension n and socle degree d and let $L = a_1x_1 + \ldots + a_nx_n \in A_1$, such that $f(a_1, \ldots, a_n) \neq 0$. The map $\mu_{L^{l-k}} : A_k \to A_l$, for $k < l \leq \frac{d}{2}$, has maximal rank if and only if the (mixed) Hessian matrix $\mathrm{Hess}_f^{(k,d-l)}(a_1, \ldots, a_n)$ has maximal rank. In particular, we get the following:*

(1) **(Strong Lefschetz Hessian criterion, [30], [22])** *L is a strong Lefschetz element of A if and only if $\mathrm{hess}_f^k(a_1, \ldots, a_n) \neq 0$ for all $k = 1, \ldots, [d/2]$.*
(2) **(Weak Lefschetz Hessian criterion)** *$L \in A_1$ is a weak Lefschetz element of A if and only if either $d = 2q + 1$ is odd and $\mathrm{hess}_f^q(a_1, \ldots, a_n) \neq 0$ or $d = 2q$ is even and $\mathrm{Hess}_f^{(q-1,q)}(a_1, \ldots, a_n)$ has maximal rank.*

Proof. Let $\mu : A_k \to A_l$ be the map defined by the multiplication by L^{l-k}. By Theorem 2.4,

$$\mathrm{rk}\,\mu = \mathrm{rk}\,\mathrm{Hess}_f^{(l^*,k)}(L^\perp) = \mathrm{rk}\,\mathrm{Hess}_f^{(l^*,k)}(a_1, \ldots, a_n) = \mathrm{rk}\,\mathrm{Hess}_f^{(k,d-l)}(a_1, \ldots, a_n),$$

(see also Remark 2.3).

The other claims are a direct consequence of it. $\square$

3. An inductive construction

In this section we want to study the relations between the algebras $A = Q/\operatorname{Ann}_Q(f)$ and $\tilde{A} = \tilde{Q}/\operatorname{Ann}_{\tilde{Q}}(\tilde{f})$ with $f \in R = \mathbb{K}[x_1, \ldots, x_r]$ and $\tilde{f} = uf \in \tilde{R} = \mathbb{K}[x_1, \ldots, x_r, u]$. As a Corollary we prove that the Boolean algebras have the SLP. This result have been proved in a number of different ways, it was the genesis of the area with the work of R. Stanley and J. Watanabe.

Lemma 3.1. *Let $f \in R = \mathbb{K}[x_1, \dots, x_r]$ be a homogeneous polynomial of degree d and let $\tilde{f} = uf \in \tilde{R} = \mathbb{K}[x_1, \dots, x_r, u]$. Let Q and $\tilde{Q}$ be the rings of differential operators associated to R and $\tilde{R}$ respectively. Then*

$$\operatorname{Ann}_{\tilde{Q}}(\tilde{f}) = \operatorname{Ann}_Q(f)\tilde{Q} + U^2\tilde{Q} \subset \tilde{Q}.$$

In particular, if $A = Q/\operatorname{Ann}_Q(f)$ is presented by quadrics, then $\tilde{A} = \tilde{Q}/\operatorname{Ann}_{\tilde{Q}}(\tilde{f})$ is also presented by quadrics.

Proof. It is easy to see that if $\alpha \in \operatorname{Ann}_Q(f)$, then $\alpha \in \operatorname{Ann}_{\tilde{Q}}(\tilde{f})$, and also $U^2 \in \operatorname{Ann}_{\tilde{Q}}(\tilde{f})$, hence $I = \operatorname{Ann}_Q(f)\tilde{Q} + U^2\tilde{Q} \subset \operatorname{Ann}_{\tilde{Q}}(\tilde{f})$. To prove the equality, let $\overline{\alpha} \in \operatorname{Ann}_{\tilde{Q}}(\tilde{f})/I$, then we can write:

$$\overline{\alpha} = \overline{\beta} + U\overline{\gamma}.$$

Where $\beta, \gamma \in Q/I \subset \tilde{Q}/I$. Therefore:

$$\alpha(\tilde{f}) = \beta(f) + U\gamma(f) = 0.$$

This give us $\overline{\beta} = \overline{\gamma} = 0$, hence $\overline{\alpha} = 0$ and the result follows. $\square$

Lemma 3.2. *With the previous notation we have the following decomposition as $\mathbb{K}$-vector spaces:*

$$\tilde{A}_k = A_k \oplus A_{k-1}U.$$

Proof. Let $\{\beta_1, \dots, \beta_s\} \subset A_k$ be a $\mathbb{K}$-basis of A_k and let $\{\gamma_1, \dots, \gamma_l\} \subset A_{k-1}$ be a $\mathbb{K}$-basis of A_{k-1}. We claim that $\{\beta_1, \dots, \beta_s, U\gamma_1, \dots, U\gamma_l\} \subset \tilde{A}_k$ is a $\mathbb{K}$-basis of $\tilde{A}_k$.

(i) Linear independence. Suppose that

$$b_1\beta_1 + \ldots + b_s\beta_s + c_1U\gamma_1 + \ldots + c_lU\gamma_l = 0.$$

Hence, by Lemma 3.1 $b_1\beta_1 + \ldots + b_s\beta_s = 0$ implying $b_1 = \ldots = b_s = 0$, in the same way $c_1U\gamma_1 + \ldots + c_lU\gamma_l = 0$ implying $c_1 = \ldots = c_l = 0$.

(ii) Spanning. Let $\alpha \in \tilde{A}_k$, by Lemma 3.1, $\alpha = \beta + U\gamma$, with $\beta \in A_k$ and $\gamma \in A_{k-1}$. Therefore $\beta = b_1\beta_1 + \ldots + b_s\beta_s$ and $\gamma = c_1U\gamma_1 + \ldots + c_lU\gamma_l$ since $\{\beta_1, \dots, \beta_s\}$ is a $\mathbb{K}$-basis of A_k and $\{\gamma_1, \dots, \gamma_l\}$ is a $\mathbb{K}$-basis of A_{k-1}. $\square$

Proposition 3.3. *With the same notation, if A has the SLP, then $\tilde{A}$ has the SLP.*

Proof. By Lemma 3.1 and Lemma 3.2, we get:

$$\operatorname{Hess}^k_{\tilde{f}} = \begin{bmatrix} 0 & \operatorname{Hess}^{(k-1,k)}_f \\ \operatorname{Hess}^{(k,k-1)}_f & u \operatorname{Hess}^k_f \end{bmatrix}$$

By hypothesis and by Corollary 2.5, $\operatorname{hess}^k_f \neq 0$, hence one can apply the determinant of block matrix to get:

$$\operatorname{hess}^k_{\tilde{f}} = u^s \operatorname{hess}^k_f \det[0 - \operatorname{Hess}^{(k-1,k)}_f (u \operatorname{Hess}^k_f)^{-1} \operatorname{Hess}^{(k,k-1)}_f]$$

Multiplying by u we get:

$$\operatorname{hess}^k_{\tilde{f}} = \operatorname{hess}^k_f \det[-\operatorname{Hess}^{(k-1,k)}_f (\operatorname{Hess}^k_f)^{-1} \operatorname{Hess}^{(k,k-1)}_f]$$

By Theorem 2.4 we can interpret the multiplication $[\mathrm{Hess}_f^{(k-1,k)}(\mathrm{Hess}_f^k)^{-1}\mathrm{Hess}_f^{(k,k-1)}]$, up to a scalar multiple, as a composition of multiplication maps by a general linear form $L \in A_1$ in the following way:

$$\begin{array}{ccccccc} A_{k-1} & \to & A_{d-k} & \to & A_k & \to & A_{d-k+1} \\ \alpha & \mapsto & L^{d-2k+1}\alpha & \mapsto & L\alpha & \mapsto & L^{d-2k+2}\alpha \end{array}$$

In fact, $\mathrm{Hess}_f^{(k,k-1)}(L^\perp)$ is the matrix of the map $\mu_{L^{d-2k+1}} : A_{k-1} \to A_{d-k}$, $(\mathrm{Hess}_f^k)^{-1}(L^\perp)$ is the inverse of the matrix of the map $\mu_{L^{d-2k}} : A_k \to A_{d-k}$ and $\mathrm{Hess}_f^{(k-1,k)}(L^\perp)$ is the matrix of the map $\mu_{L^{d-2k+1}} : A_k \to A_{d-k+1}$.

Notice that the composition is the map $\mu_{L^{d-2k+2}} : A_{k-1} \to A_{d-(k-1)}$ and hence, by Theorem 2.4, its matrix is just $\mathrm{Hess}_f^{k-1}(L^\perp)$ whose determinant is non zero by hypothesis. □

A codimension n Boolean $\mathbb{K}$-algebra can be presented as the complete intersection

$$\mathbb{K}[x_1,\dots,x_n]/(x_1^2,\dots,x_n^2) \simeq \mathbb{K}[X_1,\dots,X_n]/\operatorname{Ann}(x_1\dots x_n).$$

It is a particular case of the algebras given by the annihilator of a monomial that have been treated by Stanley [28] and Watanabe [16]. This result motivated the entire area and has been reproved by using different methods in [25,18,16]. As a consequence of Proposition 3.3 we give a simple proof that the Boolean algebras have the SLP using Mixed Hessians.

Corollary 3.4. *Let $\mathbb{K}$ be a field of characteristic zero. Then, the complete intersection algebra $\mathbb{K}[x_1,\dots,x_n]/(x_1^2,\dots,x_n^2)$ has the SLP.*

Proof. By induction in $n = \mathrm{codim}(A)$, the result is trivial for $n = 1$. Suppose the result is true for an $n \geq 1$, then, for $f = x_1\dots x_n$ all the k-th Hessians satisfy $\mathrm{hess}_f^k \neq 0$. Let us call $A = Q/\operatorname{Ann}(f)$. To prove the result for $B = \mathbb{K}[x_1,\dots,x_n,u]/(x_1^2,\dots,x_n^2,u^2)$, we consider $g = uf = ux_1\dots x_n \in \mathbb{K}[x_1,\dots,x_n,u]$, by Proposition 3.3 the result follows.

See also [17] and [18], for other methods. □

4. A combinatorial construction

Definition 4.1. Let $V = \{u_1,\dots,u_m\}$ be a finite set. A simplicial complex Δ with vertex set V is a subset of the power set 2^V, such that for all $A \in \Delta$ and for all $B \subseteq A$ we have $B \in \Delta$. The members of Δ are referred as faces. Faces with the maximal dimension are called facets. If $A \in \Delta$ and $|A| = k$, it is called a $(k-1)$-face, or a face of dimension $k-1$. If all the facets have the same dimension d the complex is said to be homogeneous of (pure) dimension d. We say that Δ is a simplex if $\Delta = 2^V$.

In our context we identify the faces of a simplicial complex with square-free monomials in the variables $\{u_1,\dots,u_m\}$. Let $\mathbb{K}$ be any field and let $R = \mathbb{K}[u_1,\dots,u_m]$ be the polynomial ring. To any finite subset $F \subset \{u_1,\dots,u_m\}$ we associate the monomial $m_F = \prod_{u_i\in F} u_i$. In this way there is a natural bijection between the simplicial complex Δ and the set of the monomials m_F, where F is a facet of Δ.

Let Δ be a homogeneous simplicial complex of dimension $d-2$ whose facets are given by the monomials $g_i \in \mathbb{K}[u_1,\dots,u_m]_{d-1}$. Let $f \in \mathbb{K}[x_1,\dots,x_n,u_1,\dots,u_m]_{(1,d-1)}$ be the bihomogeneous form of monomial square free type associated to Δ, that is $f = f_\Delta = \sum_{i=1}^n x_i g_i$ (see Definition 1.5). The vertex set of Δ is also called the 0-skeleton and we write $V = \{u_1,\dots,u_m\}$. We identify the 1-skeleton with a simple graph $\Delta_1 = (V,E)$, hence the 1-faces are called edges. Since, by differentiation, $X_i(f) = g_i$, we can identify each facet g_i with the differential operator X_i. We denote by e_k the number of $(k-1)$-faces, hence $e_1 = m$ and $e_{d-1} = n$ and we put $e_0 := 1$ and $e_j := 0$ for $j \geq d-1$. Let $A = Q/\operatorname{Ann}(f_\Delta)$ be the associated algebra, we suppose that $I_1 = 0$.

Definition 4.2. Let Δ be a homogeneous simplicial complex of dimension $d-2$. We will call $A_\Delta = Q/\operatorname{Ann}(f_\Delta)$ the associated algebra to Δ.

Definition 4.3. Let Δ be a homogeneous simplicial complex of dimension $d-2$. We say that Δ is facet connected if for any pair of facets F, F' of Δ there exists a sequence of facets, $F_0 = F, F_1, \ldots, F_s = F'$ such that $F_i \cap F_{i+1}$ is a $(d-3)$-face. We say that Δ is a flag complex if every collection of pairwise adjacent vertices spans a simplex.

The definition of a flag complex Δ is equivalent to saying that for all complete subgraphs $H = K_l \subset \Delta_1$ for $l \geq 3$, there exists a $(l-1)$-face $F \in \Delta_l$ such that H is the first skeleton of F. In particular, if Δ is a flag complex, then Δ_1 does not contain any K_{d-1}.

Theorem 4.4. *Let Δ be a homogeneous simplicial complex of dimension $d-2 \geq 1$ and let A_Δ be the associated Artinian Gorenstein algebra. A is presented by quadrics if and only if Δ is a facet connected flag complex.*

Proof. This is Theorem 3.5 in [10]. $\square$

4.1. Turan algebras

Definition 4.5. Let $2 \leq a_1 \leq \ldots \leq a_{d-1}$ be integers. The Turan complex of order $a_1, \ldots, a_{d-1}$, $\Delta = \mathcal{TK}(a_1, \ldots, a_{d-1})$, is the homogeneous simplicial complex whose facet set is the Cartesian product $\pi = \prod_{i=1}^{d-1}\{1, 2, \ldots, a_i\}$. The associated algebra is called the Turan algebra of order $(a_1, \ldots, a_{d-1})$ and denoted by $TA(a_1, \ldots, a_{d-1})$.

Theorem 4.6. *Every Turan algebra $TA(a_1, \ldots, a_{d-1})$ is presented by quadrics. Its Hilbert vector is given by $h_k = s_{k-1} + s_{d-k-1}$ where $s_k = s_k(a_1, \ldots, a_{d-1})$ is the elementary symmetric polynomial of order k.*

Proof. This is Theorem 3.7 in [10]. $\square$

Lemma 4.7. *Let Δ be a simplicial complex of pure dimension $d-2$ and let $A_\Delta = Q/\operatorname{Ann} f_\Delta$ be the associated algebra. Then the map $\mu_L : A_{k-1} \to A_k$, for $k \leq \frac{d}{2}$, is injective for a general $L \in A_1$ if, and only if $\operatorname{rk}\operatorname{Hess}_f^{((1,k-2),(0,d-k))} = e_{d-k+1}$ and $\operatorname{rk}\operatorname{Hess}_f^{((0,k-1),(1,d-k-1))} = e_{k-1}$.*

Proof. Since $A_k = A_{(1,k-1)} \oplus A_{(0,k)}$, and, since by Theorem 4.4, $\dim A_{(0,k)} = e_k$ and

$$\dim A_{(1,k-1)} = \dim A_{(0,d-k)} = e_{d-k},$$

with a choice of bases consistent with the decomposition as direct sum, we have:

$$\operatorname{Hess}_f^{(k-1,d-k)} = \begin{bmatrix} 0 & \operatorname{Hess}_f^{((1,k-2),(0,d-k))} \\ \operatorname{Hess}_f^{((0,k-1),(1,d-k-1))} & \operatorname{Hess}_f^{((0,k-1),(0,d-k))} \end{bmatrix}_{(e_{d-k+1}+e_{k-1})\times(e_k+e_{d-k})},$$

where the matrices $\operatorname{Hess}_f^{((1,k-2),(0,d-k))}$ and $\operatorname{Hess}_f^{((0,k-1),(1,d-k-1))}$ have order $e_{d-k+1} \times e_{d-k}$ and $e_{k-1} \times e_k$ respectively.

The injectivity of $\mu_L : A_{k-1} \to A_k$ implies $e_{d-k+1}+e_{k-1} \leq e_{d-k}+e_k$ and $\operatorname{rk}\operatorname{Hess}_f^{(k-1,d-k)} = e_{d-k+1}+e_{k-1}$. By the shape of the matrix this maximal rank can be achieved if and only if $\operatorname{rk}\operatorname{Hess}_f^{((1,k-2),(0,d-k))} = e_{d-k+1}$ and $\operatorname{rk}\operatorname{Hess}_f^{((0,k-1),(1,d-k-1))} = e_{k-1}$.

Conversely, if $\operatorname{rk}\operatorname{Hess}_f^{((1,k-2),(0,d-k))} = e_{d-k+1}$ and $\operatorname{rk}\operatorname{Hess}_f^{((0,k-1),(1,d-k-1))} = e_{k-1}$, then $\operatorname{rk}\operatorname{Hess}_f^{(k-1,d-k)} = e_{d-k+1} + e_{k-1}$ yielding the desired result. □

Lemma 4.8. *Let $d \geq 3$ be an integer and consider the Turan complex*

$$\mathcal{TK}(2^{(d-1)}) := \mathcal{TK}(2, \ldots, 2)$$

of dimension $d-1$. Let f be the associated form. Then

$$\operatorname{rk}\operatorname{Hess}_f^{((1,0),(0,d-2))} < 2^{d-1}.$$

Proof. Let us write $\Delta = \mathcal{TK}(2^{(d-1)})$. First of all note that the rows of $\operatorname{Hess}_f^{((1,0),(0,d-2))}$ are indexed by the 2^{d-1} facets x_α of Δ and the columns are indexed by the $(d-2)$-faces F of Δ. A non zero element of $\operatorname{Hess}_f^{((1,0),(0,d-2))}$ is a degree one monomial representing the remaining vertex of the facet u_α that does not belongs to the $(d-2)$-face F. For instance, every column F has only two non zero elements, say u_{ij} and u_{kj} representing the remaining vertex of the two faces that contain F. Furthermore, the other non zero elements of the rows i and k are the same.

If we multiply the row indexed by x_α, with $\alpha = (j_1, \ldots, j_{d-1})$ where $j_i \in \{0,1\}$ by $(-1)^{j+1+\cdots+j_{d-1}}$, and by all the variables that do not figure in the row, then we get a matrix M such that every column j has only two non zero elements and they are opposite, say M_j and $-M_j$. Summing up the rows, the result follows. □

Lemma 4.9. *Let Δ be a pure simplicial complex of dimension $d-2$ with n facets and let $A_\Delta = Q/\operatorname{Ann} f_\Delta$ be the associated algebra. Let $v \in V(\Delta)$ be a vertex and denote $\Delta' = \Delta \setminus v$ be the complex obtained from Δ by deleting v, let n' be the number of facets of Δ' and let $A_{\Delta'} = Q/\operatorname{Ann} f_{\Delta'}$ the associated algebra. Then*

$$\operatorname{rk}\operatorname{Hess}_f^{((1,0),(0,d-2))} = n \Rightarrow \operatorname{rk}\operatorname{Hess}_{f'}^{((1,0),(0,d-2))} = n'.$$

Proof. Let us choose an ordered basis of $A_{(0,1)}$ such that the last n' vectors represent the faces containing v. Let us choose a basis of $A_{(0,d-2)}$ in such a way that the first vectors represent $d-2$ faces that does not contain v and the last vectors the faces that contain v. The matrix $\operatorname{Hess}_f^{((1,0),(0,d-2))}$ with respect to this basis is

$$\operatorname{Hess}_f^{((1,0),(0,d-2))} = \begin{bmatrix} * & * \\ 0 & \operatorname{Hess}_{f'}^{((1,0),(0,d-2))} \end{bmatrix}.$$

The zero sub-matrix occurs by our choice of ordered basis. In fact, if X_i represents a face not containing v, then $X_i(f)$ does not contain the variable v and since the first vector of $A_{(0,d-2)}$ contains v, the derivative is zero. The result easily follows. □

Definition 4.10. Let Δ be a simplicial complex of pure dimension. We say that a new complex Δ' is constructed from Δ attaching a leaf if we add one vertex and one facet, that is, $V_{\Delta'} = V_\Delta \cup \{v\}$ and $F_{\Delta'} = F_\Delta \cup \{F\}$ with $v \in F$.

The following Lemma will be useful in the sequel.

Lemma 4.11. *Let Δ be a $d-2 \geq 2$ dimensional simplicial complex and let A_Δ be the associated algebra. Suppose that A_Δ is presented by quadrics and $e_1 \leq e_2$ and $e_{d-1} \leq e_{d-2}$. Let Δ' be the simplicial complex constructed from Δ by attaching a leaf. Then the algebra $A' = A_{\Delta'}$ associated to Δ' is presented by quadrics. Moreover, if there is $L \in A_1$ such that $\mu_L : A_1 \to A_2$ is injective, then there is $L' \in A'$ such that $\mu_{L'} : A'_1 \to A'_2$ is also injective.*

Proof. It is easy to see that A' is presented by quadrics by Theorem 4.4. By Lemma 4.7, since $e_1 \leq e_2$ and $e_{d-1} \leq e_{d-2}$, we have that A satisfies the injective conjecture if, and only if $\operatorname{rk}\operatorname{Hess}_f^{((1,0),(0,d-2))} = e_{d-1}$ and $\operatorname{rk}\operatorname{Hess}_f^{((0,1),(1,d-3))} = e_1$. Since attaching a leaf we still have $e'_1 \leq e'_2$ and $e'_{d-1} \leq e'_{d-2}$ and since it does not alter the fact that the rank of the desired mixed Hessians is maximal, the result follows. $\square$

Theorem 4.12. *Let $A = TA(a_1, \ldots, a_{d-1})$ be the Turan algebra of order $(a_1, \ldots, a_{d-1})$ with $d \geq 3$ and $2 \leq a_1 \leq a_2 \leq \ldots \leq a_{d-1}$. Then for all $L \in A_1$ the map $\mu_L : A_1 \to A_2$ is not injective.*

Proof. For $a_1 = \ldots = a_{d-1} = 2$, the result follows by Lemma 4.8 and Lemma 4.7. For each $a_i > 2$ we can delete vertices until to obtain $\mathcal{TK}(2^{d-1})$ and by Lemma 4.9 and Lemma 4.7 the result follows. $\square$

5. Algebras presented by quadrics

The WLP works in codimension $n \leq 2$, it is an open problem in codimension $n = 3$ and there are algebras not satisfying it in codimension $n \geq 4$. Nevertheless, examples of Artinian algebras failing WLP were sporadic and the only systematic way to produce it were making the Hilbert vector non unimodal (see [1–3]). In recent times the first author, in [9], constructed families of algebras failing WLP. We recall the following result:

Theorem 5.1. *[9] For each pair $(N, d) \notin \{(3,3), (3,4), (4,4), (3,6)\}$ with $N \geq 3$ and with $d \geq 3$ there exist standard graded Artinian Gorenstein algebras $A = \oplus_{i=0}^{d} A_i$ of codimension $N+1$ and socle degree d, with a unimodal Hilbert vector that do not satisfy the WLP.*

On the other hand, for algebras presented by quadrics there was a conjecture posed in [20,21]:

Conjecture 5.2 *(Migliore–Nagel WLP Conjecture).* Any Artinian Gorenstein algebra presented by quadrics, over a field $\mathbb{K}$ of characteristic zero, has the Weak Lefschetz Property.

The conjecture has been disproved by us in [10, Cor. 3.8]. In this section we study this phenomena in more details. We look for minimal examples of algebras presented by quadrics failing WLP.

5.1. Artinian Gorenstein algebras with odd socle degree

Let A be a standard graded Artinian Gorenstein algebra with socle degree three, then $A = Q/\operatorname{Ann}_Q(f)$ with $f \in R$ a homogeneous polynomial of degree 3. Corollary 2.5 applied to this case tells us that A satisfies the WLP if and only if $\operatorname{hess}_f \neq 0$.

By a result due to Dimca–Papadima, see [7, Thm. 1], if f is not a reduced polynomial and $\tilde{f}$ is its radical, then $\operatorname{hess}_f = 0$ if and only if $\operatorname{hess}_{\tilde{f}} = 0$. For quadratic polynomials not defining a cone, $\operatorname{hess}_{\tilde{f}} \neq 0$, so we can restrict ourselves to reduced cubic polynomials. Furthermore, if $f = f_1 f_2$ and $\operatorname{hess}_f = 0$, then all the components of $X = V(f) \subset \mathbb{P}^n$ are developable, yielding $\operatorname{hess}_{f_i} \equiv 0 \pmod{f_i}$, in this case $f = l_1 l_2 l_3$ and X is an arrangement of hyperplanes passing through a $\mathbb{P}^{N-2}$, which is a cone as soon as $N \geq 2$. So, from now on, we can restrict ourselves to the case that f is an absolutely irreducible polynomial.

Let us recall Perazzo's construction which works like an atom for the constructions of forms with vanishing Hessian not defining a cone (see the Appendix of [9]).

Definition 5.3. A Perazzo polynomial is (up to a projective transformation) a form of type:

$$f = \sum_{i=1}^{s} x_i g_i(\underline{u}) + h(\underline{u}) \in \mathbb{K}[\underline{x}, \underline{u}]$$

with $g_i \in \mathbb{K}[\underline{u}]_{d-1}$ linearly independent and algebraically dependent and $h \in \mathbb{K}[\underline{u}]_d$.

Theorem 5.4. *[24,13] Perazzo hypersurfaces are not cones and have vanishing Hessian. Suppose that $N \leq 6$, and let $X = V(f) \subset \mathbb{P}^N$ be an irreducible cubic hypersurface which is not a cone and such that $\operatorname{hess}_f = 0$. Then, up to a projective transformation, f is a Perazzo polynomial.*

Corollary 5.5. *Let A be a standard graded Artinian Gorenstein $\mathbb{K}$-algebra of socle degree 3. If A is presented by quadrics and $\operatorname{codim} A \leq 7$, then A satisfies the WLP.*

Proof. Suppose that A does not have the WLP. Then by the Hessian criterion, Corollary 2.5, $\operatorname{hess}_f = 0$. For $N \leq 6$, by Theorem 5.4 and by [13, Thm. 5.2,5.3,5.4], if f is an irreducible cubic polynomial such that $\operatorname{hess}_f = 0$, then, up to a projective transformation, either $f = xu_1^2 + yu_1u_2 + zu_2^2 + h(\underline{u})$ or $N = 6$ and $f = x_0g_0(\underline{u}) + \ldots + x_3g_3(\underline{u}) + h(\underline{u})$. Let us suppose, without loss of generality, that x_4^2 occurs as a monomial only in g_2. In both cases, if we consider the associated standard graded Artinian Gorenstein one can verify directly that $X_2^3 \in \operatorname{Ann}_Q(f)$ is a minimal generator. □

The next example was treated from the geometric point of view in [13, p. 803, Example 6]. By Corollary 5.5 it is a counter-example of minimal codimension and minimal socle degree for the MN-conjecture.

Example 5.6. In $\mathbb{P}^7$ consider the cubic hypersurface $X = V(f) \subset \mathbb{P}^7$, given by

$$f = \begin{vmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \\ x_6 & x_7 & 0 \end{vmatrix} \in \mathbb{K}[x_0, \ldots, x_7].$$

As pointed out in [13, p. 803, Example 6], X represents a tangent section of the secant variety of the Segre variety $\operatorname{Seg}(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$. After a linear change of coordinates we can rewrite f as a (Perazzo) bigraded polynomial of monomial square free type:

$$f = x_1u_1u_2 + x_2u_2u_3 + x_3u_3u_4 + x_4u_4u_1 \in R = \mathbb{K}[x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4].$$

Notice that $f_1f_3 = u_1u_2u_3u_4 = f_2f_4$, hence by the Gordan–Noether criterion, $\operatorname{hess}_f = 0$. Let $A = Q/\operatorname{Ann}_Q(f)$ be the associated algebra, of codimension 8 and socle degree 3. By the Hessian Lefschetz criterion, Theorem 2.5, A does not satisfy the WLP. On the other hand, since its graph is a square, by Theorem 4.4 it is presented by quadrics. Indeed, one can verify that

$$I = (u_4^2, u_2u_4, x_2u_4, x_1u_4, u_3^2, u_1u_3, x_4u_3, x_1u_3, u_2^2, x_4u_2, x_3u_2, x_2u_2 - x_3u_4, x_1u_2 - x_4u_4,$$
$$u_1^2, x_4u_1 - x_3u_3, x_3u_1, x_2u_1, x_1u_1 - x_2u_3, x_4^2, x_3x_4, x_2x_4, x_1x_4, x_3^2, x_2x_3, x_1x_3, x_2^2, x_1x_2, x_1^2).$$

Example 5.7. Consider the algebras $A = Q/\operatorname{Ann}_Q(f)$ of codimension $r = 9, 11$. For $r = 9$, take $f = x_1u_1u_2 + x_2u_2u_3 + x_3u_3u_4 + x_4u_4u_1 + w^2u_1$ and for $r = 11$, take $f = x_1u_1u_2 + x_2u_2u_3 + x_3u_3u_4 + x_4u_4u_1 + x_5u_5u_1 + w^2u_1$.

For both we have, $f_1 f_3 = f_2 f_4$, hence by Gordan–Noether criterion and the Hessian criterion, A does not have the WLP. One can verify that in all the cases $\mathrm{Ann}_Q(f)$ is generated by quadrics.

Lemma 5.8. *Let $R = \mathbb{K}[u_1, \dots, u_m]$ be the polynomial ring. Let $G = (V, E)$ be a connected graph such that $V = \{u_1, \dots, u_m\}$ and E is given by square free quadratic monomials. If $|E| = |V|$, then G has a unique circuit C and furthermore:*

(1) *If $|C|$ is even, then* $\det \nabla G = 0$*;*
(2) *If $|C|$ is odd, then* $\det \nabla G \neq 0$

Proof. Recall that a graph (V, E) is a tree if it is connected and if $|V| - 1 = |E|$. First of all let us show that the gradient matrix of any tree has maximal rank. Let $T = (V', E')$ be a tree where $V' = \{u_1, \dots, u_m\}$ and $E' = \{g_1, \dots, g_{m-1}\}$. By induction on $m \geq 2$, the result is trivial for $m = 2$. Let us suppose that for any tree with $|V'| = m \geq 2$, the gradient matrix has maximal rank. Let $\tilde{T}$ be a tree with $|\tilde{T}| = m + 1$, $\tilde{T} = T \cup g_m$ where $g_m = u_j u_{m+1}$, hence

$$\nabla\tilde{T} = \begin{bmatrix} \nabla T & * \\ 0 & u_j \end{bmatrix}.$$

The claim follows.

Let $T \subset G$ be a generating tree of G, then $T = (V, E')$ with $|E'| = |V| - 1$, since $|V| = |E|$, G contains a unique circuit, say $C = \{u_1u_2, u_2u_3, \dots, u_{k-1}u_k, u_ku_1\}$ and let us suppose that $E = E' \cup u_ku_1$.

Since $G \setminus u_1u_k = T$ is a tree, $\nabla G = (\nabla T | \nabla g_m)$ where

$$\nabla T = \begin{bmatrix} u_2 & 0 & \dots & 0 & * \\ u_1 & u_3 & \dots & 0 & * \\ 0 & u_2 & \dots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & u_k & * \\ 0 & 0 & \dots & u_{k-1} & * \\ 0 & 0 & \dots & 0 & N \end{bmatrix} \text{ and } \nabla G = \begin{bmatrix} u_2 & 0 & \dots & 0 & * & u_k \\ u_1 & u_3 & \dots & 0 & * & 0 \\ 0 & u_2 & \dots & 0 & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & u_k & * & 0 \\ 0 & 0 & \dots & u_{k-1} & * & u_1 \\ 0 & 0 & \dots & 0 & N & 0 \end{bmatrix}.$$

Where in the last row 0 represents $0_{(m-k,1)}$ and N is a square matrix of order $m - k$ such that $\det(N) \neq 0$, since ∇T has maximal rank. Using the Laplace expansion, we get

$$\det(\nabla G) = \det(N) u_1 u_2 \dots u_k (1 + (-1)^{k-1}).$$

The result follows. $\square$

Proposition 5.9. *Let $f \in \mathbb{K}[x_1, \dots, x_n, u_1, \dots, u_m]_{(1,2)}$ be a bigraded cubic polynomial of monomial square free type and let $G = (E, V)$ be the associated graph. Then A is presented by quadrics if, and only if, G is connected and triangle free, in this case we have the following possibilities:*

(1) *If G is a tree, then A has the WLP;*
(2) *If G contains only one circuit C, then either*
   (a) *$|C|$ is even and A does not have the WLP or*
   (b) *$|C| \geq 5$ is odd and A has the WLP*
(3) *If G contains at least two circuits, then A does not have the WLP.*

Proof. The characterization of the graphs that represent algebras presented by quadrics follows from Theorem 4.4. We recall a very well known result that a set of n monomials in n variables are algebraically independent if and only if the incidence matrix has determinant different by zero (see, for instance, [23], Lemma 1.1). Since the incidence matrix of a graph of monomials is the gradient matrix evaluated in $u_1 = 1, \ldots, u_m = 1$, the first and the second cases easily follow. In the last case $n > m$ hence the g_i are algebraically dependent. The result follows from the Hessian criterion, Corollary 2.5 and the Gordan–Noether Theorem. $\square$

Corollary 5.10. *For all $r \geq 8$ there exist standard graded Artinian Gorenstein $\mathbb{K}$-algebras of socle degree 3 and $\operatorname{codim} A = r$, presented by quadrics, not satisfying the WLP.*

Proof. For the second one, if $r = 9, 11$ the result follows from Examples 5.6 and 5.7.

For all $r = 2q \geq 8$ start with the square and then attach leaves as in Lemma 4.11 and the result follows. For all $r = 2q + 1 \geq 13$ start with the hexagon together with the central diagonal and then attach leaves as in Lemma 4.11 and the result follows. $\square$

Lemma 5.11. *Let $f \in \mathbb{K}[x_1, \ldots, x_r]$ be a homogeneous polynomial of degree $\deg(f) = 2k - 1$ and such that $\operatorname{hess}_f^{k-1} = 0$ and let $\tilde{f} = uvf \in \mathbb{K}[x_1, \ldots, x_n, u, v]$. Then $\operatorname{hess}_{\tilde{f}}^k = 0$.*

Proof. Let $R = \mathbb{K}[x_1, \ldots, x_r]$ and $\tilde{R} = \mathbb{K}[x_1, \ldots, x_r, u, v]$ and let Q and $\tilde{Q}$ be the associated rings of differential operators. Let $A = Q/\operatorname{Ann}_Q(f)$ and let $\tilde{A} = \tilde{Q}/\operatorname{Ann}_{\tilde{Q}}(f)$. Applying twice Lemma 3.2, we get

$$\tilde{A}_k = A_{k-2}UV \oplus A_{k-1}U \oplus A_{k-1}V \oplus A_k.$$

Therefore,

$$\operatorname{Hess}_{\tilde{f}}^k = \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & \operatorname{Hess}_f^{k-1} & * \\ 0 & \operatorname{Hess}_f^{k-1} & 0 & * \\ * & * & * & 0 \end{bmatrix}.$$

Since $\dim A_k = \dim A_{k-1}$, using Laplace expansion on the second block row, it is easy to check that $\operatorname{hess}_f^{k-1} = 0 \Rightarrow \operatorname{hess}_{\tilde{f}}^k = 0$. $\square$

Corollary 5.12. *There exist standard graded Artinian Gorenstein algebras A presented by quadrics of socle degree $d = 2k + 1 \geq 3$ that do not satisfy the weak Lefschetz property if $\operatorname{codim} A \geq d + 5$.*

Proof. By Corollary 5.10, for all $r \geq 8$ exist $f_r \in R = \mathbb{K}[x_1, \ldots, x_r]_3$ such that $\operatorname{hess}_{f_r} = 0$ and A_r is presented by quadrics. Let

$$\tilde{f} = \tilde{f}_{r,k} = f_r u_1 \ldots u_{2k-2} \in \tilde{R} = \mathbb{K}[x_1, \ldots, x_r, u_1, \ldots, u_{2k-2}].$$

We have $\deg \tilde{f} = 2k + 1 \geq 3$. Let $\tilde{A} = \tilde{Q}/\operatorname{Ann}_{\tilde{Q}}(\tilde{f})$. Then

$$\operatorname{codim} \tilde{A} = 2k - 2 + r \geq 2k + 6 = d + 5.$$

By Lemma 3.1, since A_r is presented by quadrics, $\tilde{A}$ is also presented by quadrics.

By Lemma 3.2 and by induction on k, we get that the Hilbert vector of $\tilde{A}$ are maximal.

By induction on k and Lemma 5.11, $\operatorname{hess}^k(\tilde{f}) = 0$, hence, by the Strong Lefschetz Hessian criterion, Corollary 2.5, $\tilde{A}$ does not have the WLP. $\square$

5.2. Artinian Gorenstein algebras with even socle degree

Notice that Lemma 3.2 and Lemma 3.1 together with the Hessian criterion Corollary 2.5 and the inductive procedure allows us to produce for any socle degree $d \geq 3$ and $\operatorname{codim} A \geq d+5$ Artinian Gorenstein algebras presented by quadrics that do not satisfy the SLP, but this construction, in even socle degree is not enough to failure of the WLP.

Remark 5.13. Let $f \in \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]_{(1,3)}$ be a quartic bihomogeneous polynomial of bidegree $(1,3)$ of monomial square free type. By Lemma 5.8 and by Lemma 4.7, $\operatorname{rk} \operatorname{Hess}_f^{(1,2)}$ is maximal if and only if $\operatorname{rk} \operatorname{Hess}_f^{((1,0),(0,2))} = n$.

Example 5.14. Let A be the algebra associated to the complex $\mathcal{TK}(2,2,3) \setminus e$ where e is an edge having two incident faces, it has codimension 17 and it does not have the WLP. Indeed, $|V| = 7$, $|E| = 15$ and $|F| = 10$ but the matrix $\operatorname{Hess}_f^{((0,1),(2,0))}$ does not have maximal rank.

Corollary 5.15. *For all $r \geq 16$ there exist standard graded Artinian Gorenstein $\mathbb{K}$-algebras of socle degree 4 and $\operatorname{codim} A = r$, presented by quadrics and not satisfying the WLP.*

Proof. For codimension $r = 2q \geq 14$, start with the Turan algebra $TA(2,2,2)$ of codimension 14, which by Corollary 4.12 does not have the WLP and attach leaves as in Lemma 4.11 and the result follows. For codimension $r = 2q+1 \geq 17$, start with the Turan algebra $TA(2,2,3)$ of codimension 17, which by Corollary 4.12 does not have the WLP and attach leaves to conclude the result. $\square$

Lemma 5.16. *Let $A = Q/\operatorname{Ann}_Q(f)$ be a standard graded Artinian Gorenstein algebra of socle degree $2q$ with $f \in \mathbb{K}[x_1, \ldots, x_n]$ and let $\tilde{f} \in \mathbb{K}[x_1, \ldots, x_n, u, v]$ be $\tilde{f} = fuv$. If $\operatorname{rk} \operatorname{Hess}_f^{(q-1,q)} < \dim A_{q-1}$, then $\operatorname{rk} \operatorname{Hess}_{\tilde{f}}^{(q,q+1)} < \dim \tilde{A}_q$.*

Proof. Let $R = \mathbb{K}[x_1, \ldots, x_r]$ and $\tilde{R} = \mathbb{K}[x_1, \ldots, x_r, u, v]$ and let $Q, \tilde{Q}, A, \tilde{A}$ as usual. Applying twice Lemma 3.2, we get

$$\tilde{A}_k = A_{k-2}UV \oplus A_{k-1}U \oplus A_{k-1}V \oplus A_k.$$

Therefore,

$$\operatorname{Hess}_{\tilde{f}}^{(q,q+1)} = \begin{bmatrix} 0 & 0 & 0 & \operatorname{Hess}_f^{(q-2,q+1)} \\ 0 & 0 & \operatorname{Hess}_f^{(q-1,q)} & V \operatorname{Hess}_f^{(q-1,q+1)} \\ 0 & \operatorname{Hess}_f^{(q-1,q)} & 0 & U \operatorname{Hess}_f^{(q-1,q+1)} \\ \operatorname{Hess}_f^{(q,q-1)} & V \operatorname{Hess}_f^{q} & U \operatorname{Hess}_f^{q} & 0 \end{bmatrix}.$$

Multiplying the second block row by U and the third one by $-V$ and summing up we got a block row of type.

$$\begin{bmatrix} 0 & -V \operatorname{Hess}_f^{(q-1,q)} & U \operatorname{Hess}_f^{(q-1,q)} & 0 \end{bmatrix}.$$

Since, by hypothesis, $\operatorname{rk} \operatorname{Hess}_f^{(q-1,q)}$ is not maximal, the result follows. $\square$

Corollary 5.17. *For given integers, $n, d = 2k \geq 4$ such that $c \geq d+12$, there exist standard graded Artinian Gorenstein algebras presented by quadrics of socle degree d and codimension c.*

Proof. The result follows by induction in the same way as Corollary 5.12. □

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