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Moser's estimates for degenerate Kolmogorov equations with non-negative divergence lower order coefficients

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ABSTRACT

We prove L^∞_{loc} estimates for positive solutions to the following degenerate second order partial differential equation of Kolmogorov type with measurable coefficients of the form

$$\sum_{i,j=1}^{m_0} \partial_{x_i}\left(a_{ij}(x,t)\partial_{x_j}u(x,t)\right) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t) + \\ + \sum_{i=1}^{m_0} b_i(x,t)\partial_i u(x,t) - \sum_{i=1}^{m_0} \partial_{x_i}\left(a_i(x,t)u(x,t)\right) + c(x,t)u(x,t) = 0$$

where $(x,t) = (x_1,\dots,x_N,t) = z$ is a point of $\mathbb{R}^{N+1}$, and $1 \le m_0 \le N$. (a_{ij}) is a uniformly positive symmetric matrix with bounded measurable coefficients, (b_{ij}) is a constant matrix. We apply the Moser's iteration method to prove the local boundedness of the solution u under minimal integrability assumption on the coefficients.

1. Introduction

We consider second order partial differential operators of Kolmogorov–Fokker–Planck type of the form

$$\mathscr{L}u(x,t) := \sum_{i,j=1}^{m_0} \partial_{x_i}\left(a_{ij}(x,t)\partial_{x_j}u(x,t)\right) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t) + \\ + \sum_{i=1}^{m_0} b_i(x,t)\partial_i u(x,t) - \sum_{i=1}^{m_0} \partial_{x_j}\left(a_i(x,t)u(x,t)\right) + c(x,t)u(x,t) = 0, \tag{1.1}$$

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in some open set $\Omega \subseteq \mathbb{R}^{N+1}$. Here $z = (x,t) = (x_1, \dots, x_N, t)$ denotes a point of $\mathbb{R}^{N+1}$, and $1 \le m_0 \le N$. In the sequel we will use the following notation

$$A(x,t) = (a_{ij}(x,t))_{1 \le i,j \le N},$$

where a_{ij} is the coefficient appearing in (1.1) for $i,j = 1,\dots,m_0$, while $a_{ij} \equiv 0$ whenever $i > m_0$ or $j > m_0$. Eventually,

$$a(x,t) = (a_1(x,t),\dots,a_{m_0}(x,t),0,\dots,0), \qquad b(x,t) = (b_1(x,t),\dots,b_{m_0}(x,t),0,\dots,0)$$

$$Y = \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} - \partial_t. \tag{1.2}$$

Then the operator $\mathscr{L}$ takes the following compact form

$$\mathscr{L}u = \mathrm{div}\,(ADu) + Yu + \langle b, Du\rangle - \mathrm{div}\,(au) + cu.$$

Here and in the sequel

$$D = (\partial_{x_1},\dots,\partial_{x_N}), \qquad \langle\cdot,\cdot\rangle, \qquad \mathrm{div}, \tag{1.3}$$

denote the gradient, the inner product, and the divergence in $\mathbb{R}^N$, respectively. In general, solutions to $\mathscr{L}u = 0$ will be understood in the following weak sense.

Definition 1.1. Let Ω be an open subset of $\mathbb{R}^{N+1}$. A weak solution to $\mathscr{L}u = 0$ is a function u such that $u, D_{m_0}u, Yu \in L^2_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega -\langle ADu, D\varphi\rangle + \varphi Yu + \langle b, Du\rangle\varphi + \langle a, D\varphi\rangle u + cu\varphi = 0, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{1.4}$$

In the sequel, we will also consider weak sub-solutions to $\mathscr{L}u = 0$, namely function u such that $u, D_{m_0}u, Yu \in L^2_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega -\langle ADu, D\varphi\rangle + \varphi Yu + \langle b, Du\rangle\varphi + \langle a, D\varphi\rangle u + cu\varphi \ge 0, \quad \forall\varphi \in C_0^\infty(\Omega), \varphi \ge 0. \tag{1.5}$$

A function u is a super-solution of $\mathscr{L}u = 0$ if $-u$ is a sub-solution.

We note that if u is both a sub-solution and a super-solution of $\mathscr{L}u = 0$ then it is a solution, i.e. $\mathscr{L}u = 0$ holds. Indeed, for every given $\varphi \in C_0^\infty(\Omega)$, we may consider $\psi \in C_0^\infty(\Omega)$ such that $\psi \ge 0$ and $\psi - \varphi \ge 0$ in Ω. Therefore $\mathscr{L}u = 0$ follows by applying (1.5) to $\pm u$.

We assume the following structural condition on $\mathscr{L}$.

(H1) The matrix $(a_{ij}(x,t))_{i,j=1,\dots,m_0}$ is symmetric with real measurable entries. Moreover, $a_{ij}(x,t) = a_{ji}(x,t)$, $1 \le i,j \le m_0$, and there exists a positive constant λ such that

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(x,t)\xi_i\xi_j \le \lambda|\xi|^2,$$

for every $(x,t) \in \mathbb{R}^{N+1}$ and $\xi \in \mathbb{R}^{m_0}$. The matrix $B = (b_{ij})_{i,j=1,\dots,N}$ is constant.

Note that the operator $\mathscr{L}$ is uniformly parabolic when $m_0 = N$. In this note, we are mainly interested in the case $m_0 < N$, that is the strongly degenerate one. It is known that the first order part of $\mathscr{L}$ may provide it with strong regularity properties. To be more specific, let us consider the operator $\mathcal{K}$ defined as follows:

$$\mathcal{K}u(x,t) := \sum_{i=1}^{m_0} \partial_{x_i}^2 u(x,t) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t). \tag{1.6}$$

It is known that, if the matrix B satisfies a suitable assumption, then $\mathcal{K}$ is hypoelliptic. This means that, if u is a distributional solution to $\mathcal{K}u = f$ in some open set Ω of $\mathbb{R}^{N+1}$ and $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$ and it is a classic solution to the equation.

The hypoellipticity of $\mathcal{K}$ can be tested via the condition introduced by Hörmander in [11]:

$$\text{rank Lie}(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y)(x,t) = N+1, \quad \forall (x,t) \in \mathbb{R}^{N+1},$$

where $\text{Lie}(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y)(x,t)$ denotes the Lie algebra generated by the first order differential operators (vector fields) $(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y)$, computed at (x,t). We refer to E. Lanconelli and one of the authors [14] for a characterization of the hypoellipticity of $\mathcal{K}$ in terms of the matrix B.

(H2) The principal part $\mathcal{K}$ of $\mathscr{L}$ is hypoelliptic.

In Section 2, we recall a known structural condition on the matrix B equivalent to **(H2)**. We remark that if $\mathscr{L}$ is a uniformly parabolic operator (i.e. $m_0 = N$ and $B \equiv 0$), then **(H2)** is clearly satisfied. Indeed, the principal part of $\mathscr{L}$ simply is the heat operator, which is hypoelliptic and homogeneous with respect to the parabolic dilations $\delta_\lambda(x,t) = (\lambda x, \lambda^2 t)$. In the degenerate setting, $\mathcal{K}$ plays the same role that the heat operator plays in the family of the parabolic operators. For this reason, $\mathcal{K}$ will be referred to as *principal part of* $\mathscr{L}$.

The aim of this work is to prove L^∞_{loc} estimates for weak solutions to $\mathscr{L}u = 0$, by using the Moser's iteration method, under minimal assumptions on the integrability of the lower order coefficients $a_1, \dots, a_{m_0}, b_1, \dots, b_{m_0}, c$. The Moser's iterative scheme [16,17] has been applied to degenerate parabolic operator $\mathscr{L}$ with no lower order terms by Cinti, Pascucci and one of the authors in [20] and [6]. These results have been extended to operators with bounded first order coefficients by Lanconelli, Pascucci and one of the authors in [14] and [13], and to operators with first order coefficients belonging to some L^q space by Wang and Zhang [23].

Our study has been inspired by the article of Nazarov and Uralt'seva [18], who prove L^∞_{loc} estimates and Harnack inequalities for uniformly elliptic and parabolic operators in divergence form that are those with $m_0 = N$ according to our notation. The authors consider uniformly parabolic equations in $\mathbb{R}^{N+1}$

$$\mathscr{L}u = \text{div}\,(ADu) + \langle b, Du \rangle - \partial_t u = 0,$$

with $b_1, \dots, b_N \in L^q(\mathbb{R}^{N+1})$. They prove that the Moser's iteration can be accomplished provided that $\frac{N+2}{2} < q \le N+2$ relying on the condition $\text{div}\, b \ge 0$ to relax the integrability assumption on $b_1, \dots, b_{m_0}$. Here and in the sequel, the quantity $\text{div}\, b$ will be understood in the distributional sense

$$\int_\Omega \varphi(x,t) \text{div}\, b(x,t) dx\, dt = - \int_\Omega \langle b(x,t), \nabla \varphi(x,t) \rangle dx\, dt,$$

for every $\varphi \in C_0^\infty(\Omega)$. Of course, also the quantity $\text{div}\, a$ will be understood in the distributional sense.

When considering degenerate operators, a suitable dilation group $(\delta_r)_{r>0}$ in $\mathbb{R}^{N+1}$ replaces the usual parabolic dilation $\delta_r(x,t) = (rx, r^2 t)$, and the *parabolic dimension* $N+2$ of $\mathbb{R}^{N+1}$ is replaced by a bigger integer $Q+2$, which is called *homogeneous dimension* of $\mathbb{R}^{N+1}$ with respect to $(\delta_r)_{r>0}$. Our main result will be declared in terms of this quantity, that will be introduced in Section 2.

As far as it concerns degenerate operators, Wang and Zhang obtain in [23] the local boundedness and the Hölder continuity for weak solutions to $\mathscr{L}u = 0$ by assuming the condition $b_1, \dots, b_{m_0} \in L^q(\mathbb{R}^{N+1})$, with $q = Q+2$. Our assumption on the integrability of the lower order coefficients a_i, b_i, with $i = 1, \dots, m_0$ and c is stated as follows:

(H3) $a_i, b_i, c \in L^q_{\text{loc}}(\Omega)$, with $i = 1, \ldots, m_0$, for some $q > \frac{3}{4}(Q+2)$. Moreover,

$$\operatorname{div} a, \operatorname{div} b \geq 0 \qquad \text{in } \Omega.$$

A comparison of our result with that of Nazarov and Uralt'seva is in order. It would be natural to expect that the optimal lower bound for the exponent q is $\frac{Q+2}{2}$. Indeed, the difficulty in considering degenerate equations lies in the fact that a Caccioppoli inequality gives an *a priori* L^2 estimate for the derivatives $\partial_{x_1} u, \ldots, \partial_{x_{m_0}} u$ of the solution u, that are the derivatives with respect to the *non-degeneracy* directions of $\mathscr{L}$. Moreover, the standard Sobolev inequality cannot be used to obtain an improvement of the integrability of the solution as in the non-degenerate case. For this reason we rely on a representation formula for the solution u first used in [20]. Specifically, we represent a solution u to $\mathscr{L}u = 0$ in terms of the fundamental solution of $\mathcal{K}$. Indeed, if u is a solution to $\mathscr{L}u = 0$ in Ω, then we have

$$u(x,t) = \int_\Omega \Gamma(x,t,\xi,\tau) \mathcal{K}u(\xi,\tau) d\xi\, d\tau, \tag{1.7}$$

where Γ is the fundamental solution to $\mathcal{K}$ (see (2.19) and (2.20) in the sequel), and

$$\mathcal{K}u = (\mathcal{K} - \mathscr{L})\, u = \operatorname{div}\left((A_0 - A)Du\right) - \langle b, Du\rangle + \operatorname{div}(au) - cu, \tag{1.8}$$

where we denote

$$A_0 = \begin{pmatrix} \mathbb{I}_{m_0} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}, \tag{1.9}$$

where $\mathbb{I}_{m_0}$ is the identity matrix in $\mathbb{R}^{m_0}$, and $\mathbb{O}$ are zero matrices. This representation formula provides us with a Sobolev type inequality only for weak solutions to the equation $\mathscr{L}u = 0$. Specifically, we find that, for every $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$, there exists a positive constant $c_1\left(\| b \|_{L^q(\Omega)}, \Omega_1, \Omega_2\right)$ such that

$$\| u \|_{L^{2\alpha}(\Omega_1)} \leq c_1\left(\| a \|_{L^q(\Omega)}, \| b \|_{L^q(\Omega)}, \| c \|_{L^q(\Omega)}, \Omega_1, \Omega_2\right) \| D_{m_0} u \|_{L^2(\Omega_2)},$$

and, by considering u as a test function, we obtain the following Caccioppoli inequality

$$\| D_{m_0} u \|_{L^2(\Omega_2)} \leq c_2\left(\| a \|_{L^q(\Omega)}, \| b \|_{L^q(\Omega)}, \| c \|_{L^q(\Omega)}, \Omega_2, \Omega_3\right) \| u \|_{L^{2\beta}(\Omega_3)},$$

where

$$\alpha := \frac{q(Q+2)}{q(Q-2) + 2(Q+2)}, \qquad \beta := \frac{q}{q-1}. \tag{1.10}$$

As far as it concerns the Moser's iteration, the above inequalities are applied to a sequence of functions $u_k := u^{p_k}$, with $p_k \to +\infty$, in order to obtain an L^∞_{loc} bound for the solution u.

We note that, the Sobolev inequality is useful to the iteration whenever $\alpha > 1$, and this is true if, and only if $q > \frac{Q+2}{2}$. Moreover, the condition $q > \frac{Q+2}{2}$ is required by Nazarov and Uralt'seva in the proof of the Caccioppoli inequality for non-degenerate operators. Since in our work both Sobolev and Caccioppoli inequalities depend on the L^q norm of $a_1, \ldots, a_{m_0}, b_1, \ldots, b_{m_0}, c$, we require a more restrictive condition on q to improve the integrability of u. Specifically, if we combine the Sobolev and the Caccioppoli inequalities, we need to have $\alpha > \beta$, and this is true if, and only if $q > \frac{3}{4}(Q+2)$, as we require in Assumption (**H3**).

We next state our main result. As we shall see in Section 2, the natural geometry underlying the operator $\mathscr{L}$ is determined by a suitable homogeneous Lie group structure on $\mathbb{R}^{N+1}$. Our main result reflects this non-Euclidean background. Let "$\circ$" denote the Lie product on $\mathbb{R}^{N+1}$ defined in (2.17) and $\{\delta_r\}_{r>0}$ the family of dilations defined in (2.22). Let us consider the cylinder:

$$\mathcal{Q}_1 := \left\{(x,t) \in \mathbb{R}^N \times \mathbb{R} \ : \ |x| < 1, \ \ |t| < 1\right\}.$$

For every $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, we set

$$\mathcal{Q}_r(z_0) := z_0 \circ (\delta_r(\mathcal{Q}_1)) = \left\{z \in \mathbb{R}^{N+1} \ : \ z = z_0 \circ \delta_r(\zeta), \zeta \in \mathcal{Q}_1\right\}.$$

Theorem 1.2. *Let u be a non-negative weak solution to $\mathscr{L}u = 0$ in Ω. Let $z_0 \in \Omega$ and $r, \rho, \frac{1}{2} \le \rho < r \le 1$, be such that $\overline{\mathcal{Q}_r(z_0)} \subseteq \Omega$. Then there exist positive constants $C = C(p, \lambda)$ and $\gamma = \gamma(p, q)$ such that for every $p \neq 0$, it holds*

$$\sup_{\mathcal{Q}_\rho(z_0)} u^p \le \frac{C\left(1+ \| a \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| b \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| c \|_{L^q(\mathcal{Q}_r(z_0))}\right)^\gamma}{(r-\rho)^{9(Q+2)}} \int_{\mathcal{Q}_r(z_0)} u^p, \tag{1.11}$$

where $\gamma = \frac{2\alpha^2\beta}{\alpha-1}$, with α and β defined as in (1.10).

Remark 1.3. Estimate (1.11) is meaningful whenever the integral appearing on its right-hand side is finite. Note that (1.11) is an estimate of the *infimum* of u when $p < 0$. More precisely, we have that

$$\sup_{\mathcal{Q}_\rho(z_0)} u \le \frac{C^{\frac{1}{p}}\left(1+ \| a \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| b \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| c \|_{L^q(\mathcal{Q}_r(z_0))}\right)^{\frac{\gamma}{p}}}{(r-\rho)^{\frac{9(Q+2)}{p}}} \left(\int_{\mathcal{Q}_r(z_0)} u^p\right)^{\frac{1}{p}}, \qquad \forall p > 0, \tag{1.12}$$

$$\inf_{\mathcal{Q}_\rho(z_0)} u \ge \frac{C^{\frac{1}{p}}\left(1+ \| a \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| b \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| c \|_{L^q(\mathcal{Q}_r(z_0))}\right)^{\frac{\gamma}{p}}}{(r-\rho)^{\frac{9(Q+2)}{p}}} \left(\int_{\mathcal{Q}_r(z_0)} \frac{1}{u^{|p|}}\right)^{\frac{1}{p}}, \qquad \forall p < 0, \tag{1.13}$$

Corollary 1.4. *Let u be a weak solution to $\mathscr{L}u = 0$ in Ω. Then for every $p \ge 1$ we have*

$$\sup_{\mathcal{Q}_\rho(z_0)} |u|^p \le \frac{C\left(1+ \| a \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| b \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| c \|_{L^q(\mathcal{Q}_r(z_0))}\right)^\gamma}{(r-\rho)^{9(Q+2)}} \int_{\mathcal{Q}_r(z_0)} |u|^p. \tag{1.14}$$

Proposition 1.5. *Sub and super-solutions also verify estimate (1.11) for suitable values of p. More precisely, (1.11) holds for*

1. *$p \ge \frac{1}{2}$ or $p < 0$, if u is a non-negative weak sub-solution of (1.1);*
2. *$p \in]0, \frac{1}{2}[$, if u is a non-negative weak super-solution of (1.1).*

We conclude this introduction with some motivations for the study of operator $\mathscr{L}$ in the form (1.1). Degenerate equations of the form $\mathscr{L}u = 0$ naturally arise in the theory of stochastic processes, kinetic theory of gases and mathematical finance. For instance, if $(W_t)_{t\ge0}$ denotes a real Brownian motion, then the simplest non-trivial Kolmogorov operator

$$\frac{1}{2}\partial_{vv} + v\partial_x + \partial_t, \qquad t \ge 0,\ (v, x) \in \mathbb{R}^2$$

is the infinitesimal generator of the classical Langevin's stochastic equation that describes the position X and the velocity V of a particle in the phase space (cf. [15])

$$\begin{cases} dV_t &= dW_t, \\ dX_t &= V_t\, dt. \end{cases}$$

Notice that in this case we have $1 = m_0 < N = 2$.

Linear Fokker–Planck equations (cf. [7] and [22]), non-linear Boltzmann–Landau equations (cf. [15] and [5]) and non-linear equations for Lagrangian stochastic models commonly used in the simulation of turbulent flows (cf. [4]) can be written in the form

$$\sum_{i,j=1}^n \partial_{v_i}(a_{ij}\partial_{v_j} f) + \sum_{j=1}^n v_j \partial_{x_j} f + \partial_t f = 0, \qquad t \ge 0, v \in \mathbb{R}^n, x \in \mathbb{R}^n \tag{1.15}$$

with the coefficients $a_{ij} = a_{ij}(t, v, x, f)$ that may depend on the solution f through some integral expressions. It is clear that Eq. (1.15) is a particular case of $\mathscr{L}u = 0$ with $n = m_0 < d = 2n$ and

$$B = \begin{pmatrix} \mathbb{O}_n & \mathbb{O}_n \\ \mathbb{I}_n & \mathbb{O}_n \end{pmatrix}$$

where $\mathbb{I}_n$ and $\mathbb{O}_n$ denote the $(n \times n)$-identity matrix and the $(n \times n)$-zero matrix, respectively.

In mathematical finance, equations of the form $\mathscr{L}u = 0$ appear in various models for pricing of path-dependent derivatives such as Asian options (cf., for instance, [3,19]), stochastic velocity models (cf. [10,21]) and in theory of stochastic utility (cf. [1,2]).

This note is organized as follows. In Section 2 we recall some known facts about operators $\mathscr{L}$ and $\mathcal{K}$, and we give some preliminary results. In Section 3 we prove Theorem 3.1 and Proposition 3.2, which is an intermediate result (Caccioppoli type inequality for weak solutions to $\mathscr{L}u = 0$) needed for the bootstrap argument. Finally, in Section 4 we deal with the Moser's iterative method.

2. Preliminaries

In this Section we recall notation and results we need in order to deal with the non-Euclidean geometry underlying the operators $\mathscr{L}$ and $\mathcal{K}$. We refer to the articles [6] and [14] for a comprehensive treatment of this subject. The operator $\mathcal{K}$ is invariant with respect to a Lie product on $\mathbb{R}^{N+1}$. More precisely, we let

$$E(s) = \exp(-sB), \qquad s \in \mathbb{R}, \tag{2.16}$$

and we denote by $\ell_\zeta, \zeta \in \mathbb{R}^{N+1}$, the left translation $\ell_\zeta(z) = \zeta \circ z$ in the group law

$$(x,t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau), \qquad (x,t).(\xi,\tau) \in \mathbb{R}^{N+1}. \tag{2.17}$$

Thus we have

$$\mathcal{K} \circ \ell_\zeta = \ell_\zeta \circ \mathcal{K}.$$

This means that, if $v(x,t) = u\big((\xi,\tau) \circ (x,t)\big)$ and $g(x,t) = f\big((\xi,\tau) \circ (x,t)\big)$, we have

$$\mathcal{K}u = f \quad \iff \quad \mathcal{K}v = g.$$

We recall that, by [14] (Propositions 2.1 and 2.2), assumption **(H2)** is equivalent to assume that, for some basis on $\mathbb{R}^N$, the matrix B has the canonical form

$$B = \begin{pmatrix} * & * & \dots & * & * \\ B_1 & * & \dots & * & * \\ \mathbb{O} & B_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & * \end{pmatrix} \tag{2.18}$$

where every B_k is a $m_k \times m_{k-1}$ matrix of rank m_j, $j = 1, 2, \dots, \kappa$ with

$$m_0 \geq m_1 \geq \dots \geq m_\kappa \geq 1 \quad \text{and} \quad \sum_{j=0}^{\kappa} m_j = N$$

and the blocks denoted by "*" are arbitrary. In the sequel we shall assume that B has the canonical form (2.18).

We denote by $\Gamma(\cdot,\zeta)$ the fundamental solution of $\mathcal{K}$ in (1.6) with pole in $\zeta \in \mathbb{R}^{N+1}$. An explicit expression of $\Gamma(\cdot,\zeta)$ has first been constructed by Kolmogorov [12] for operators in the form (1.15), then by Hörmander in [11] under more general conditions

$$\Gamma(z,\zeta) = \Gamma(\zeta^{-1} \circ z, 0), \qquad \forall z,\zeta \in \mathbb{R}^{N+1},\ z \neq \zeta, \tag{2.19}$$

where

$$\Gamma((x,t),(0,0)) = \begin{cases} \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\langle C^{-1}(t)x, x\rangle - t\,\mathrm{tr}(B)\right), & \text{if } t > 0, \\ 0, & \text{if } t < 0, \end{cases} \tag{2.20}$$

and

$$C(t) = \int_0^t E(s)\, A_0\, E^T(s)\, ds,$$

where $E(\cdot)$ is the matrix defined in (2.16). Note that assumption **(H2)** implies that $C(t)$ is strictly positive for every $t > 0$ (see [14], Proposition A.1).

Among the operator $\mathcal{K}$ where the matrix B is of the form (2.18), the ones for which the $*$-blocks are equal to zero play a central role. Indeed, let us consider the principal part operator $\mathcal{K} = \Delta_{m_0} + Y_0$, where $Y_0 = \langle B_0 x, D\rangle - \partial_t$ and

$$B_0 = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \ldots & \mathbb{O} & \mathbb{O} \\ B_1 & \mathbb{O} & \ldots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & B_2 & \ldots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & B_\kappa & \mathbb{O} \end{pmatrix} \tag{2.21}$$

The operator $\mathcal{K}_0$ is invariant with respect to the dilations defined as

$$\delta_r = \mathrm{diag}(r\mathbb{I}_{m_0}, r^3\mathbb{I}_{m_1}, \ldots, r^{2\kappa+1}\mathbb{I}_{m_\kappa}, r^2), \qquad\qquad r > 0. \tag{2.22}$$

In order to explain the importance of this invariance property we introduce for every positive r the scaled operator

$$\mathcal{K}_r = r^2 \left(\delta_r \circ \mathcal{K} \circ \delta_{\frac{1}{r}}\right).$$

In order to explicitly write $\mathcal{K}_r$ we note that, if

$$B = \begin{pmatrix} B_{0,0} & B_{0,1} & \ldots & B_{0,\kappa-1} & B_{0,\kappa} \\ B_1 & B_{1,1} & \ldots & B_{\kappa-1,1} & B_{\kappa,1} \\ \mathbb{O} & B_2 & \ldots & B_{\kappa-1,2} & B_{\kappa,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & B_\kappa & B_{\kappa,\kappa} \end{pmatrix}, \tag{2.23}$$

where $B_{i,j}$ are the $m_i \times m_j$ blocks denoted by " $*$ " in (2.18), then we can rewrite $\mathcal{K}_r$ as follows

$$\mathcal{K}_r = \mathrm{div}\,(A_0 D) + Y_r, \tag{2.24}$$

where

$$Y_r := \langle B_r\, x, D\rangle - \partial_t \tag{2.25}$$

and $B_r := r^2\, D_r\, B\, D_{\frac{1}{r}}$, i.e.

$$B_r = \begin{pmatrix} r^2 B_{0,0} & r^4 B_{0,1} & \ldots & r^{2\kappa} B_{0,\kappa-1} & r^{2\kappa+2} B_{0,\kappa} \\ B_1 & r^2 B_{1,1} & \ldots & r^{2\kappa-2} B_{\kappa-1,1} & r^{2\kappa} B_{\kappa,1} \\ \mathbb{O} & B_2 & \ldots & r^{2\kappa-4} B_{\kappa-1,2} & r^{2\kappa-2} B_{\kappa,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & B_\kappa & r^2 B_{\kappa,\kappa} \end{pmatrix}.$$

Note that

$$B_r = B \qquad \text{for every } r > 0$$

if, and only if $B_{j,k} = \mathbb{O}$ with $j \le k$. In this case, if $v(x,t) = u\big(\delta_r(x,t)\big)$ and $g(x,t) = f\big(\delta_r(x,t)\big)$, then

$$\mathcal{K}u = f \qquad \Longleftrightarrow \qquad \mathcal{K}v = r^2 g.$$

Since K_0 is the blow-up limit of K_r, the dilation group $(\delta_r)_{r>0}$ plays a central role also for non-dilation invariant operators.

We next introduce a norm which is homogeneous of degree 1 with respect to the dilations $(\delta_r)_{r>0}$ and a corresponding quasi-distance which is invariant with respect to the translation group for the case of $*$-blocks equal to zero.

Definition 2.1. Let $\alpha_1, \dots, \alpha_N$ be the positive integers such that

$$\operatorname{diag}\left(r^{\alpha_1}, \dots, r^{\alpha_N}, r^2\right) = \delta_r.$$

If $\| z \| = 0$ we set $z = 0$ while, if $z \in \mathbb{R}^{N+1} \setminus \{0\}$ we define $\| z \| = r$ where r is the unique positive solution to the equation

$$\frac{x_1^2}{r^{2\alpha_1}} + \frac{x_2^2}{r^{2\alpha_2}} + \dots + \frac{x_N^2}{r^{2\alpha_N}} + \frac{t^2}{r^4} = 1.$$

We define the quasi-distance d by

$$d(z,w) = \| z^{-1} \circ w \|, \qquad z, w \in \mathbb{R}^{N+1}.$$

Remark 2.2. The Lebesgue measure is invariant with respect to the translation group associated to $\mathcal{K}$, since $\det E(t) = e^{t\,\mathrm{trace}\,B} = 1$, where $E(t)$ is the exponential matrix of Eq. (2.16). Moreover, since $\det \delta_r = r^{Q+2}$, we also have

$$\operatorname{meas}\left(\mathcal{Q}_r(z_0)\right) = r^{Q+2} \operatorname{meas}\left(\mathcal{Q}_1(z_0)\right), \qquad \forall\, r > 0, z_0 \in \mathbb{R}^{N+1},$$

where

$$Q = m_0 + 3m_1 + \dots + (2\kappa + 1) m_\kappa. \tag{2.26}$$

The natural number $Q + 2$ is usually called the homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to $(\delta_r)_{r>0}$.

Remark 2.3. The norm $\| \cdot \|$ is homogeneous of degree 1 with respect to $(\delta_r)_{r>0}$, that is

$$\| \delta_\rho(x,t) \| = \rho \, \| (x,t) \| \qquad \forall \rho > 0 \text{ and } (x,t) \in \mathbb{R}^{N+1}.$$

Actually in $\mathbb{R}^{N+1}$ all the norms, that are 1-homogeneous with respect to $(\delta_r)_{r>0}$, are equivalent. In particular, the norm introduced in Definition 2.1 is equivalent to the following one

$$\| (x,t) \|_1 = |x_1|^{\frac{1}{\alpha_1}} + \dots + |x_N|^{\frac{1}{\alpha_N}} + |t|^{\frac{1}{2}},$$

where the homogeneity with respect to $(\delta_r)_{r>0}$ can easily be showed. We prefer the norm of Definition 2.1 to $\| \cdot \|_1$ because its level sets (spheres) are smooth surfaces.

When $\mathcal{K}_0$ is dilation invariant with respect to $(\delta_r)_{r>0}$, also its fundamental solution $\varGamma_0$ is a homogeneous function of degree $-Q$, namely

$$\varGamma_0\left(\delta_r(z), 0\right) = r^{-Q}\, \varGamma_0\left(z, 0\right), \qquad \forall z \in \mathbb{R}^{N+1} \setminus \{0\},\ r > 0.$$

This property implies an L^p estimate for Newtonian potential (c. f. for instance [8]).

Proposition 2.4. *Let $\alpha \in]0, Q+2[$ and let $G \in C(\mathbb{R}^{N+1} \setminus \{0\})$ be a δ_λ-homogeneous function of degree $\alpha - Q - 2$. If $f \in L^p(\mathbb{R}^{N+1})$ for some $p \in]1, +\infty[$, then the function*

$$G_f(z) := \int_{\mathbb{R}^{N+1}} G(\zeta^{-1} \circ z) f(\zeta) d\zeta,$$

is defined almost everywhere and there exists a constant $c = c(Q, p)$ such that

$$\| G_f \|_{L^q(\mathbb{R}^{N+1})} \leq c \max_{\|z\|=1} |G(z)| \, \| f \|_{L^p(\mathbb{R}^{N+1})},$$

where q is defined by

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q+2}.$$

It is known that homogeneous operators provide a good approximation of the non-homogeneous ones. In order to be more specific, let us consider a homogeneous operator of the form

$$\mathcal{K}_0 = \text{div}\,(A_0 D) + \langle B_0 x, D \rangle - \partial_t,$$

where B_0 is the matrix in (2.21), and denote by Γ_0 the fundamental solution of $\mathcal{K}_0$. If Γ denotes the fundamental solution of $\mathcal{K}$ defined in (2.20), then, for every $M > 0$, there exists a positive constant c such that

$$\frac{1}{c} \Gamma_0 \leq \Gamma(z) \leq c\, \Gamma_0(z) \tag{2.27}$$

for every $z \in \mathbb{R}^{N+1}$ such that $\Gamma_0(z) \geq M$ (see [14], Theorem 3.1).

We define the Γtextit potential of the function $f \in L^1(\mathbb{R}^{N+1})$ as follows

$$\Gamma(f)(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta) f(\zeta) d\zeta, \qquad z \in \mathbb{R}^{N+1}. \tag{2.28}$$

We also remark that the potential $\Gamma(D_{m_0} f) : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{m_0}$ is well-defined for any $f \in L^p(\mathbb{R}^{N+1})$, at least in the distributional sense, that is

$$\Gamma(D_{m_0} f)(z) := - \int_{\mathbb{R}^{N+1}} D^{(\xi)}_{m_0} \Gamma(z, \xi)\, f(\xi)\, d\xi, \tag{2.29}$$

where $D^{(\xi)}_{m_0} \Gamma(x, t, \xi, \tau)$ is the gradient with respect to $\xi_1, \ldots, \xi_{m_0}$. Based on (2.27), in [6] are proved potential estimates for non-dilation invariant operators.

Theorem 2.5. *Let $f \in L^p(\mathcal{Q}_r)$. There exists a positive constant $c = c(T, B)$ such that*

$$\| \Gamma(f) \|_{L^{p**}(\mathcal{Q}_r)} \leq c \, \| f \|_{L^p(\mathcal{Q}_r)}, \tag{2.30}$$

$$\| \Gamma(D_{m_0} f) \|_{L^{p*}(\mathcal{Q}_r)} \leq c \, \| f \|_{L^p(\mathcal{Q}_r)}, \tag{2.31}$$

where $\frac{1}{p*} = \frac{1}{p} - \frac{1}{Q+2}$ and $\frac{1}{p**} = \frac{1}{p} - \frac{2}{Q+2}$.

We can use the fundamental solution Γ as a test function in the definition of sub and super-solution. The following result extends Lemma 2.5 in [20] and Lemma 3 in [6].

Lemma 2.6. *Let v be a non-negative weak sub-solution to $\mathscr{L} u = 0$ in Ω. For every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, and for almost every $z \in \mathbb{R}^{N+1}$, we have*

$$\int_\Omega -\langle ADv, D(\Gamma(z, \cdot)\varphi) \rangle + \Gamma(z, \cdot)\varphi Y v + \\ - \langle a, D(\Gamma(z, \cdot)\varphi) \rangle v - \langle b, D(\Gamma(z, \cdot)\varphi) \rangle v + c u \Gamma(z, \cdot)\varphi \geq 0.$$

An analogous result holds for weak super-solutions to $\mathscr{L} u = 0$.

Proof. We define the cut-off function $\chi_{\rho,r} \in C^\infty(\mathbb{R}^+)$

$$\chi_{\rho,r}(s) = \begin{cases} 0 & \text{if } s \geq r, \\ 1 & \text{if } 0 \leq s < \rho, \end{cases} \qquad |\chi'_{r,\rho}| \leq \frac{2}{r-\rho} \tag{2.32}$$

with $\frac{1}{2} \leq \rho < r \leq 1$. Moreover, for every $\varepsilon < 0$ we define

$$\psi_\varepsilon(x,t) = 1 - \chi_{\varepsilon,2\varepsilon}(\| (x,t) \|). \tag{2.33}$$

Because v is a weak sub-solution, then by (1.5) for every $\varepsilon > 0$ and $z \in \mathbb{R}^{N+1}$ we have

$$\begin{aligned} 0 \leq & \int_\Omega - [\langle ADv, D(\Gamma(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot)) \rangle + \Gamma(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot)\, Yv]\, d\zeta \\ & + \int_\Omega [\langle b, Dv\rangle\, \Gamma(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot) + \langle a, D(\Gamma(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot))\rangle v + cu\, \Gamma(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot)]\, d\zeta \\ = & - I_{1,\varepsilon}(z) + I_{2,\varepsilon}(z) - I_{3,\varepsilon}(z) + I_{4,\varepsilon}(z) + I_{5,\varepsilon}(z) \end{aligned}$$

where

$$\begin{aligned} I_{1,\varepsilon}(z) &= \int_\Omega \langle ADv, D\Gamma(z,\cdot) \rangle \varphi(\zeta)\psi_\varepsilon(z,\zeta)\, d\zeta \\ I_{2,\varepsilon}(z) &= \int_\Omega \Gamma(z,\cdot)\varphi(\zeta)\, \left(-\langle ADv, D\varphi(\zeta) \rangle + \varphi(\zeta) Yv \right)\, d\zeta \\ I_{3,\varepsilon}(z) &= \int_\Omega \langle ADv, D\psi_\varepsilon(z,\cdot) \rangle \varphi(\zeta)\Gamma(z,\cdot)\, d\zeta \\ I_{4,\varepsilon}(z) &= \int_\Omega \langle b, Dv\rangle\, \Gamma(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot)\, d\zeta + \int_\Omega \langle a, D(\Gamma(z,\cdot)\varphi)\rangle v\, d\zeta \\ I_{5,\varepsilon}(z) &= \int_\Omega cu\, \Gamma(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot)\, d\zeta \end{aligned}$$

Keeping in mind Theorem 2.5, it is clear that the integral which defines $I_{i,\varepsilon}(z),\ \ i = 1,2,3$ is a potential and it is convergent for almost every $z \in \mathbb{R}^{N+1}$. Thus, by a similar argument to the one used in [20] to prove Lemma 2.5 (pg. 403 – 404), we get that for almost every $z \in \mathbb{R}^{N+1}$

$$\begin{aligned} \lim_{\varepsilon \to 0^+} I_{1,\varepsilon}(z) &= \int_\Omega \langle ADv, D(\Gamma(z,\cdot))\rangle \varphi(\zeta)\, d\zeta \\ \lim_{\varepsilon \to 0^+} I_{2,\varepsilon}(z) &= \int_\Omega \Gamma(z,\cdot)\left(- \langle ADv, D\varphi(\zeta)\rangle + \varphi(\zeta) Yv\right)\, d\zeta \\ \lim_{\varepsilon \to 0^+} I_{3,\varepsilon}(z) &= 0. \end{aligned}$$

Let us consider the term $I_{4,\varepsilon}$. We integrate by parts and we consider assumption **(H3)**:

$$\begin{aligned} I_{4,\varepsilon} = & - \int_\Omega \operatorname{div} b\, \Gamma(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot) v\, d\zeta - \int_\Omega \langle b, D\left(\Gamma(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot)\right) \rangle\, v\, d\zeta \\ & - \int_\Omega \operatorname{div} a\, \Gamma(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot) v\, d\zeta - \int_\Omega \langle a, D\left(\Gamma(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot)\right) \rangle\, v\, d\zeta \\ \leq & - \int_\Omega \langle b, D\left(\Gamma(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot)\right) \rangle\, v\, d\zeta - \int_\Omega \langle a, D\left(\Gamma(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot)\right) \rangle\, v\, d\zeta \end{aligned}$$

We are left with the estimate of a potential and in order to do so we would like to use Theorem 2.5. Because $a_i, b_i \in L^q_{\text{loc}}(\Omega)$, with $i = 1, \ldots, m_0$ and $v \in L^2_{\text{loc}}(\Omega)$, we have that

$$|a|\, |\Gamma(z,\cdot)|\, |\varphi|\, |D_{m_0} v|\,,\ \ |b|\, |\Gamma(z,\cdot)|\, |\varphi|\, |D_{m_0} v| \in L^{2\alpha}_{\text{loc}}(\Omega)$$

where α is defined as in (1.10). This yields, for every $\varepsilon > 0$

$$|\langle\, a, D\,(\Gamma(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot))\,\rangle\, v| \leq |\langle\, a, D\,(\Gamma(z,\cdot)\varphi(\zeta))\,\rangle\, v| \in L^1_{\text{loc}}(\Omega),$$
$$|\langle\, b, D\,(\Gamma(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot))\,\rangle\, v| \leq |\langle\, b, D\,(\Gamma(z,\cdot)\varphi(\zeta))\,\rangle\, v| \in L^1_{\text{loc}}(\Omega).$$

Thus, by the Lebesgue convergence theorem, we get for a.e. $z \in \mathbb{R}^{N+1}$

$$\lim_{\varepsilon\to 0^+}\left[\int_\Omega -\langle\, b, D\,(\Gamma(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\zeta))\,\rangle\, v\, d\zeta \,-\, \int_\Omega \langle\, a, D\,(\Gamma(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\cdot))\,\rangle\, v\right] d\zeta =$$
$$= -\int_\Omega \langle\, b, D\,(\Gamma(z,\cdot)\varphi(\zeta))\,\rangle\, v - \int_\Omega \langle\, a, D\,(\Gamma(z,\cdot)\varphi(\zeta))\,\rangle\, v\, d\zeta.$$

Now, we are left with an estimate of the term $I_{5,\varepsilon}$, which is a Γ-potential such that

$$|c|\,|\Gamma(z,\cdot)|\,|\varphi|\,|v| \in L^{2\alpha}_{\text{loc}}(\Omega).$$

Thus, we have that

$$|cu\,\Gamma(z,\cdot)\varphi(\zeta)\psi_\varepsilon(z,\cdot)| \leq |cu\,\Gamma(z,\cdot)\varphi(\zeta)| \in L^1_{\text{loc}}(\Omega).$$

Then we can apply the Lebesgue convergence theorem and we get for a. e. $z \in \mathbb{R}^{N+1}$

$$\lim_{\varepsilon\to 0^+}\int_\Omega cv\Gamma(z,\cdot)\varphi(\zeta)\chi_\varepsilon(z,\zeta)\, d\zeta \;=\; \int_\Omega cv\Gamma(z,\cdot)\varphi(\zeta)\, d\zeta. \quad \square$$

3. Sobolev and Caccioppoli inequalities

In this Section we give proof of a Sobolev inequality and a Caccioppoli inequality for weak solutions to $\mathscr{L}u = 0$. We start considering the Sobolev inequality and we remark that it holds true for every $q > \frac{Q+2}{2}$.

Theorem 3.1 (*Sobolev Type Inequality for Sub-Solutions*)**.** *Let **(H1)**-**(H2)** hold. Let $a_1,\dots,a_{m_0}, b_1,\dots,b_{m_0}$, $c \in L^q_{\text{loc}}(\Omega)$, for some $q > (Q+2)/2$, and $\operatorname{div} a, \operatorname{div} b \geq 0$ in Ω. Let v be a non-negative weak sub-solution of $\mathscr{L}u = 0$ in $\mathcal{Q}_1$. Then there exists a constant $C = C(Q,\lambda) > 0$ such that $v \in L^{2\alpha}_{\text{loc}}(\mathcal{Q}_1)$, and the following statement holds*

$$\|\, v\,\|_{L^{2\alpha}(\mathcal{Q}_\rho(z_0))} \leq C\cdot\left(\|\, a\,\|_{L^q(\mathcal{Q}_r(z_0))} + \|\, b\,\|_{L^q(\mathcal{Q}_r(z_0))} + 1 + \frac{1}{r-\rho}\right)\|\, Dv\,\|_{L^2(\mathcal{Q}_r(z_0))} +$$
$$+\; C\cdot\left(\|\, c\,\|_{L^q(\mathcal{Q}_r(z_0))} + \frac{\rho+1}{\rho(r-\rho)}\right)\|\, v\,\|_{L^2(\mathcal{Q}_r(z_0))}$$

for every ρ, r with $\frac{1}{2} \leq \rho < r \leq 1$ and for every $z_0 \in \Omega$, where $\alpha = \alpha(q)$ is defined in (1.10).

Proof. Let v be a non-negative weak sub-solution to $\mathscr{L}u = 0$. We represent v in terms of the fundamental solution Γ. To this end, we consider the cut-off function $\chi_{\rho,r}$ defined in (2.32) for $\frac{1}{2} \leq \rho < r \leq 1$. Then we consider the following test function

$$\psi(x,t) = \chi_{\rho,r}(\|\,(x,t)\,\|) \tag{3.34}$$

and the following estimates hold true

$$|Y\psi| \leq \frac{c_0}{\rho(r-\rho)}, \qquad |\partial_{x_j}\psi| \leq \frac{c_1}{r-\rho} \quad \text{for } j = 1,\dots,m_0 \tag{3.35}$$

where c_0, c_1 are dimensional constants. For every $z \in \mathcal{Q}_\rho$, we have

$$\begin{aligned} v(z) &= v\psi(z) \\ &= \int_{\mathcal{Q}_r} \left[\langle A_0 D(v\psi), D\Gamma(z,\cdot)\rangle - \Gamma(z,\cdot) Y(v\psi)\right](\zeta) d(\zeta) \\ &= I_0(z) + I_1(z) + I_2(z) + I_3(z) \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} I_0(z) &= -\int_{\mathcal{Q}_r} \left[\langle a, D(\psi\Gamma(z,\cdot))\rangle v\right](\zeta) d\zeta \; - \int_{\mathcal{Q}_r} \left[\langle b, D(\psi\Gamma(z,\cdot))\rangle v\right](\zeta) d\zeta \; + \int_{\mathcal{Q}_r} \left[cv\Gamma(z,\cdot)\psi\right](\zeta) d\zeta \\ I_1(z) &= \int_{\mathcal{Q}_r} \left[\langle A_0 D\psi, D\Gamma(z,\cdot)\rangle v\right](\zeta) d\zeta \; - \int_{\mathcal{Q}_r} \left[\Gamma(z,\cdot) v Y\psi\right](\zeta) d\zeta = I_1' + I_1'', \\ I_2(z) &= \int_{\mathcal{Q}_r} \left[\langle (A_0 - A) Dv, D\Gamma(z,\cdot)\rangle \psi\right](\zeta) d\zeta \; - \int_{\mathcal{Q}_r} \left[\Gamma(z,\cdot)\langle ADv, D\psi\rangle\right](\zeta) d\zeta \\ I_3(z) &= \int_{\mathcal{Q}_r} \left[\langle ADv, D(\Gamma(z,\cdot)\psi)\rangle\right](\zeta) d\zeta \; - \int_{\mathcal{Q}_r} \left[(\Gamma(z,\cdot)\psi)\, Yv\right](\zeta) d\zeta \, + \\ &\quad + \int_{\mathcal{Q}_r} \left[\langle a, D(\Gamma(z,\cdot)\psi)\rangle v\right](\zeta) d\zeta \; + \int_{\mathcal{Q}_r} \left[\langle b, D(\Gamma(z,\cdot)\psi)\rangle v\right](\zeta) d\zeta - \int_{\mathcal{Q}_r} \left[cv\Gamma(z,\cdot)\psi\right](\zeta) d\zeta \end{aligned}$$

Since v is a non-negative weak sub-solution to $\mathscr{L}u = 0$, it follows from Lemma 2.6 that $I_3 \le 0$, then

$$0 \le v(z) \le I_0(z) + I_1(z) + I_2(z) \quad \text{for a.e. } z \in \mathcal{Q}_\rho.$$

To prove our claim is sufficient to estimate v by a sum of Γ-potentials.

We start by estimating I_0. In order to do so, we recall that

$$\langle a, Dv\rangle,\, \langle b, Dv\rangle,\, cv \in L^{2\frac{q}{q+2}} \quad \text{for } b \in L^q, \;\; q > \frac{Q+2}{2} \;\; \text{and } Dv \in L^2.$$

Thus by Theorem 2.5 we get

$$\Gamma * \langle a, Dv\rangle, \Gamma * \langle b, Dv\rangle, \Gamma * (cv) \in L^{2\alpha},$$

where $\alpha = \alpha(q)$ is defined in (1.10). When $q \le (Q+2)$ we have that $\alpha \le 2^{**}$. Moreover, thanks to estimate (2.30), we have

$$\begin{aligned} \| I_0(\zeta) \|_{L^{2\alpha}(\mathcal{Q}_\rho)} &\le \text{meas}(\mathcal{Q}_\rho)^{2/Q} \| I_0(\zeta) \|_{L^{2^{**}}(\mathcal{Q}_\rho)} \\ &= \text{meas}(\mathcal{Q}_\rho)^{2/Q} \| \Gamma * (\langle a, D_{m_0} v\rangle \psi) + \Gamma * (\langle b, D_{m_0} v\rangle \psi) + \Gamma * (cv\psi) \|_{L^{2^{**}}(\mathcal{Q}_\rho)} \\ &\le C \cdot \left(\| a \|_{L^q(\mathcal{Q}_\rho)} + \| b \|_{L^q(\mathcal{Q}_\rho)} \right) \| D_{m_0} v \|_{L^2(\mathcal{Q}_\rho)} + C \cdot \| c \|_{L^q(\mathcal{Q}_\rho)} \| v \|_{L^2(\mathcal{Q}_\rho)} . \end{aligned}$$

We prove an estimate for the term I_1. I_1' can be estimated by (2.31) of Theorem 2.5 as follows

$$\| I_1' \|_{L^{2\alpha}(\mathcal{Q}_\rho)} \le C \| I_1' \|_{L^{2^*}(\mathcal{Q}_\rho)} \le C \| v D_{m_0} \psi \|_{L^2(\mathbb{R}^{N+1})} \le \frac{C}{r-\rho} \| v \|_{L^2(\mathcal{Q}_\rho)},$$

where the last inequality follows from (3.35). To estimate I_1'' we use (2.30)

$$\begin{aligned} \| I_1'' \|_{L^{2\alpha}(\mathcal{Q}_\rho)} \le C \| I_1'' \|_{L^{2^*}(\mathcal{Q}_\rho)} &\le \text{meas}(\mathcal{Q}_\rho)^{2/Q} \| I_1'' \|_{L^{2^{**}}(\mathcal{Q}_\rho)} \\ &\le C \| vY\psi \|_{L^2(\mathbb{R}^{N+1})} \le \frac{C}{\rho(r-\rho)} \| v \|_{L^2(\mathcal{Q}_\rho)} . \end{aligned}$$

We can use the same technique to prove that

$$\| I_2 \|_{L^{2\alpha}(\mathcal{Q}_\rho)} \le C \left(1 + \frac{1}{r-\rho} \right) \| Dv \|_{L^2(\mathcal{Q}_\rho)},$$

for some constant $C = C(Q, \lambda)$.

A similar argument proves the thesis when v is a super-solution to $\mathscr{L}u = 0$. In this case we introduce the following auxiliary operator

$$\mathcal{K} = \operatorname{div}(A_0 D) + \widetilde{Y}, \qquad \widetilde{Y} \equiv -\langle x, BD\rangle - \partial_t . \tag{3.37}$$

Then we proceed analogously as in [20], Section 3, proof of Theorem 3.3. □

Finally, we give proof of a Caccioppoli inequality for weak solutions to $\mathscr{L}u = 0$.

Proposition 3.2. *Let (H1)-(H3) hold. Let u be a non-negative weak solution of $\mathscr{L}u = 0$ in $\mathcal{Q}_1$. Let $p \in \mathbb{R}$, $p \neq 0$, $p \neq 1/2$ and let r, ρ be such that $\frac{1}{2} \leq \rho < r \leq 1$. Then there exists a constant C such that*

$$\frac{1}{\lambda} \parallel Dv \parallel^2_{L^2(\mathcal{Q}_\rho)} \leq$$
$$\leq \left[\frac{C\,p}{2\lambda} \frac{1}{(r-\rho)^2} + \frac{C}{r-\rho} \left(1 + \parallel a \parallel_{L^q(\mathcal{Q}_r)} + \parallel b \parallel_{L^q(\mathcal{Q}_r)} \right) + \frac{p}{2} \parallel c \parallel_{L^q(\mathcal{Q}_r)} \right] \parallel v \parallel^2_{L^{2\beta}(\mathcal{Q}_r)},$$

where $\beta = \beta(q)$ is defined in (1.10).

Proof. We consider the case $p < 1$, $p \neq 0$, $p \neq 1/2$. First of all, we consider a uniformly positive weak solution u to $\mathscr{L}u = 0$, that is $u \geq u_0$ for some constant $u_0 > 0$. For every $\psi \in C_0^\infty(\mathcal{Q}_r)$ we consider the function $\varphi = u^{2p-1}\psi^2$. Note that $\varphi, D_{m_0}\varphi \in L^2(\mathcal{Q}_r)$, then we can use φ as a test function in (1.4):

$$0 = \int_{\mathcal{Q}_r} \left(-\langle ADu, D(u^{2p-1}\psi^2)\rangle + u^{2p-1}\psi^2 Yu + \langle a, D(u^{2p-1}\psi^2)\rangle u + \langle b, Du\rangle u^{2p-1}\psi^2 + cu^{2p}\psi^2 \right)$$

Let $v = u^p$. Since u is a weak solution to $\mathscr{L}u = 0$ and $u \geq u_0$, then $v, D_{m_0}v, Yv \in L^2(\mathcal{Q}_r)$:

$$0 = -\int_{\mathcal{Q}_r} \left(1 - \frac{1}{2p}\right) \langle ADv, Dv\rangle \psi^2 - \int_{\mathcal{Q}_r} \langle ADv, D\psi\rangle v\psi + \frac{1}{4}\int_{\mathcal{Q}_r} Y(v^2)\psi^2$$
$$- \int_{\mathcal{Q}_r} \operatorname{div} a\, v^2\psi^2 - \frac{1}{4}\int_{\mathcal{Q}_r} \langle a, D(v^2)\rangle\psi^2 + \frac{1}{4}\int_{\mathcal{Q}_r} \langle b, D(v^2)\rangle\psi^2 + \frac{p}{2}\int_{\mathcal{Q}_r} cv^2\psi^2.$$

Because of assumption **(H1)** and by definition (3.34) of the cut-off function ψ, we get the following inequality

$$\frac{1}{\lambda}\left(\frac{2p-1}{2p} + \varepsilon\right)\int_{\mathcal{Q}_\rho} |Dv|^2 \leq \tag{3.38}$$
$$\leq \frac{1}{4\varepsilon\lambda}\frac{C}{(r-\rho)^2}\int_{\mathcal{Q}_r} |v|^2 \boxed{-\int_{\mathcal{Q}_r} \operatorname{div} a\, v^2\psi^2 - \frac{1}{4}\int_{\mathcal{Q}_r} \langle a, D(v^2)\rangle\psi^2}_{A} +$$
$$+ \boxed{\frac{1}{4}\int_{\mathcal{Q}_r} \langle b, D(v^2)\rangle\psi^2}_{B} + \boxed{\frac{p}{2}\int_{\mathcal{Q}_r} cv^2\psi^2}_{C} + \boxed{\frac{1}{4}\int_{\mathcal{Q}_r} Y(v^2)\psi^2}_{D}$$

where ε is a positive constant coming from the application of the Young's inequality. In the following we are going to consider exponents $\alpha = \alpha(q)$ and $\beta = \beta(q)$ defined in (1.10). Now we need to estimate the boxed terms.

Let us consider the term A, by Assumption **(H3)** and a classic Hölder estimate we have that

$$\boxed{-\int_{\mathcal{Q}_r} \operatorname{div} a\, v^2\psi^2 - \frac{1}{4}\int_{\mathcal{Q}_r} \langle a, D(v^2)\rangle\psi^2}_{A} \leq -\frac{3}{4}\int_{\mathcal{Q}_r} \operatorname{div} a\, v^2\psi^2 + \frac{1}{2}\int_{\mathcal{Q}_r} |\langle a, D\psi\rangle|\,|\psi|\,v^2$$
$$\leq \frac{C}{r-\rho} \parallel a \parallel_{L^q(\mathcal{Q}_r)} \parallel v \parallel^2_{L^{2\beta}(\mathcal{Q}_r)} .$$

Let us consider the term B. Thus, by Assumption **(H3)** and a classic Hölder estimate we have that

$$\boxed{\frac{1}{4}\int_{\mathcal{Q}_r}\psi^2\langle b, D(v^2)\rangle}_B \leq -\frac{1}{4}\int_{\mathcal{Q}_r} v^2\psi^2 \mathrm{div}\, b + \frac{1}{2}\int_{\mathcal{Q}_r}|\langle b, D\psi\rangle||\psi|v^2 \leq \frac{C}{r-\rho}\parallel b\parallel_{L^q(\mathcal{Q}_r)}\parallel v\parallel^2_{L^{2\beta}(\mathcal{Q}_r)}.$$

Let us consider the linear term C. We estimate it via a classical Hölder estimate:

$$\boxed{\frac{p}{2}\int_{\mathcal{Q}_r} cv^2\psi^2}_C \leq \frac{p}{2}\parallel c\parallel_{L^q(\mathcal{Q}_r)}\parallel v\parallel^2_{L^{2\beta}(\mathcal{Q}_r)}.$$

As far as it concerns the term D, we begin considering the following equality:

$$\psi^2 Y(v^2) = Y(\psi^2 v^2) - 2v^2\psi Y\psi.$$

Since by the divergence theorem $D_1 = 0$ ($v^2\psi^2$ is null on the boundary of $\mathcal{Q}_r$), we get

$$\boxed{\frac{1}{4}\int_{\mathcal{Q}_r} Y(v^2)\psi^2}_D = D_1 + D_2 = \int_{\mathcal{Q}_r}\frac{1}{4}Y(v^2\psi^2) + \int_{\mathcal{Q}_r}\frac{v^2\psi}{2}Y\psi \leq \frac{C}{\rho(r-\rho)}\parallel v\parallel^2_{L^2(\mathcal{Q}_r)}.$$

Thus we have

$$\frac{1}{\lambda}\left(\frac{2p-1}{2p}+\varepsilon\right)\parallel Dv\parallel^2_{L^2(\mathcal{Q}_\rho)} \leq \left(\frac{c}{4\varepsilon\lambda}\frac{1}{(r-\rho)^2}+\frac{C}{\rho(r-\rho)}\right)\parallel v\parallel^2_{L^2(\mathcal{Q}_r)} + \\ + \frac{C}{r-\rho}\left(\parallel a\parallel_{L^q(\mathcal{Q}_r)} + \parallel b\parallel_{L^q(\mathcal{Q}_r)}\right)\parallel v\parallel^2_{L^{2\beta}(\mathcal{Q}_r)} + \frac{p}{2}\parallel c\parallel_{L^q(\mathcal{Q}_r)}\parallel v\parallel^2_{L^{2\beta}(\mathcal{Q}_r)}.$$

By choosing $\varepsilon = \frac{1}{2p}$ and considering that $\beta > 2$ we have that

$$\frac{1}{\lambda}\parallel Dv\parallel^2_{L^2(\mathcal{Q}_\rho)} \leq \\ \leq \left[\frac{C\,p}{2\lambda}\frac{1}{(r-\rho)^2} + \frac{C}{r-\rho}\left(1+\parallel a\parallel_{L^q(\mathcal{Q}_r)} + \parallel b\parallel_{L^q(\mathcal{Q}_r)}\right) + \frac{p}{2}\parallel c\parallel_{L^q(\mathcal{Q}_r)}\right]\parallel v\parallel^2_{L^{2\beta}(\mathcal{Q}_r)}. \tag{3.39}$$

The previous argument can be adapted to the case of a non-negative weak solution to $\mathscr{L}u = 0$. Indeed, we may consider the estimate (3.39) for the solution $u + \frac{1}{n}$, $n \in \mathbb{N}$,

$$\frac{1}{\lambda}\int_{\mathcal{Q}_\rho}\left|D\left(u+\frac{1}{n}\right)^p\right|^2 \leq \\ \leq \left[\frac{C\,p}{2\lambda}\frac{1}{(r-\rho)^2} + \frac{C}{r-\rho}\left(1+\parallel a\parallel_{L^q(\mathcal{Q}_r)} + \parallel b\parallel_{L^q(\mathcal{Q}_r)}\right) + \frac{p}{2}\parallel c\parallel_{L^q(\mathcal{Q}_r)}\right]\left(\int_{\mathcal{Q}_r}\left(u+\frac{1}{n}\right)^{2\beta}\right)^{\frac{1}{\beta}}.$$

We let n go to infinity. The passage to the limit in the first integral is allowed because

$$\left|D\left(u+\frac{1}{n}\right)^p\right| = p\left(u+\frac{1}{n}\right)^{p-1}|Du| \nearrow |Du^p|, \quad \forall p < 1,\ n\to\infty.$$

For the second integral we rely on the assumptions $u^p \in L^2(\mathcal{Q}_r)$ and $u^p \in L^{2\frac{q}{q-1}}(\mathcal{Q}_r)$.

Next, we consider the case $p \geq 1$. For any $n \in \mathbb{N}$, we define the function $g_{n,p}$ on $]0,+\infty[$ as follows

$$g_{n,p}(s) = \begin{cases} s^p, & \text{if } 0 < s \leq n, \\ n^p + pn^{p-1}(s-n), & \text{if } s > n, \end{cases}$$

then we let

$$v_{n,p} = g_{n,p}(u).$$

Note that

$$g_{n,p} \in C^1(\mathbb{R}^+), \quad g'_{n,p} \in L^\infty(\mathbb{R}^+).$$

Thus since u is a weak solution to $\mathscr{L}u = 0$, we have

$$v_{n,p} \in L^2_{\text{loc}}, \quad Dv_{n,p} \in L^2_{\text{loc}}, \quad Yv_{n,p} \in L^2_{\text{loc}}.$$

We also note that the function

$$g''_{n,p}(s) = \begin{cases} p(p-1)s^{p-2}, & \text{if } 0 < s < n \\ 0, & \text{if } s \geq n, \end{cases}$$

is the weak derivative of $g'_{n,p}$, then $Dg'_{n,p}(u) = g''_{n,p}(u)D(u)$ (for the detailed proof of this assertion, we refer to [9], Theorem 7.8). Hence, by considering

$$\varphi = g_{n,p}(u)\, g'_{n,p}(u)\, \psi^2, \quad \psi \in C_0^\infty(\mathcal{Q}_r)$$

as a test function in Eq. (1.4), we find

$$\begin{aligned} 0 &= \int_{\mathcal{Q}_1} -\langle ADu, D\varphi\rangle + \varphi Yu - \operatorname{div} a\, u\varphi - \langle a, Du\rangle\varphi + \langle b, Du\rangle\varphi + cu\varphi \\ &= \int_{\mathcal{Q}_1} - \left(g'_{n,p}(u)\right)^2 \psi^2 \langle ADu, Du\rangle - g''_{n,p}(u)\, g_{n,p}(u)\psi^2\langle ADu, Du\rangle - 2\psi\langle ADu, D\psi\rangle g_{n,p}(u)\, g'_{n,p}(u) + \\ &+ \int_{\mathcal{Q}_1} g_{n,p}(u)\, g'_{n,p}(u)\psi^2\, Yu - \operatorname{div} a\, u\, g_{n,p}(u)\, g'_{n,p}(u)\psi^2 - \langle a, Du\rangle\psi^2 g_{n,p}(u)\, g'_{n,p}(u) + \\ &+ \int_{\mathcal{Q}_1} \langle b, Du\rangle\psi^2 g_{n,p}(u)\, g'_{n,p}(u) + cug_{n,p}(u)\, g'_{n,p}(u)\psi^2. \end{aligned}$$

Since $v = g_{n,p}(u)$ we have that the following equality holds:

$$\begin{aligned} 0 = &\int_{\mathcal{Q}_r} -\psi^2\langle ADv_{n,p}, Dv_{n,p}\rangle - \boxed{g''_{n,p}(u)\, g_{n,p}(u)\psi^2\langle ADu, Du\rangle}_A - 2\psi\langle ADv_{n,p}, D\psi\rangle v_{n,p} + \\ &+ \int_{\mathcal{Q}_r} \frac{1}{2}\psi^2\, Y(v_{n,p}^2) + \boxed{\operatorname{div} a\left(\frac{1}{2}v_{n,p}^2\psi^2 - u\, g_{n,p}(u)\, g'_{n,p}(u)\psi^2\right) - \operatorname{div} b\, v_{n,p}^2\psi^2}_B \\ &+ \int_{\mathcal{Q}_r} \frac{1}{2}\langle a, D(\psi^2)\rangle v_{n,p}^2 - \langle b, D(\psi^2)\rangle v_{n,p}^2 + cug_{n,p}(u)\, g'_{n,p}(u)\psi^2. \end{aligned}$$

Since $g''_{n,p}(u) \geq 0$ we have that the boxed term A is non-negative. Moreover, by Assumption **(H3)** the boxed term B is non-positive. Thus, by considering Assumption **(H1)** and by choosing $\varepsilon = \frac{1}{2p}$ we have that

$$\frac{1}{\lambda}\int_{\mathcal{Q}_r} |Dv_{n,p}|^2 \leq \frac{C\,p}{2\lambda}\frac{1}{(r-\rho)^2}\int_{\mathcal{Q}_r} |v_{n,p}|^2 + \frac{1}{2}\langle a, D(\psi^2)\rangle v_{n,p}^2 - \langle b, D(\psi^2)\rangle v_{n,p}^2 + cug_{n,p}(u)\, g'_{n,p}(u)\psi^2$$

Since $0 < v_{n,p} \leq u^p$ and

$$|Dv_{n,p}| \uparrow |Du^p|, \qquad \text{as}\, n \to \infty,$$

we get from the above inequality

$$\frac{1}{\lambda}\int_{\mathcal{Q}_r} |Du^p|^2 \leq \frac{C\,p}{2\lambda}\frac{1}{(r-\rho)^2}\int_{\mathcal{Q}_r} |u^p|^2 + \frac{1}{2}\langle a, D(\psi^2)\rangle u^{2p} - \langle b, D(\psi^2)\rangle u^{2p} + cu^{2p}\psi^2$$

and we conclude the proof as in the previous case. $\square$

4. The Moser's iteration

In this Section we use the classical Moser's iteration scheme to prove Theorem 1.2. We begin with some preliminary remarks. First of all, we recall the following Lemma, whose proof can be found in [6], Lemma 6.

Lemma 4.1. *There exists a positive constant* $\overline{c} \in]0,1[$ *such that*

$$z \circ \mathcal{Q}_{\overline{c}r(r-\rho)} \subseteq \mathcal{Q}_r, \tag{4.40}$$

for every $0 < \rho < r \le 1$ *and* $z \in \mathcal{Q}_\rho$.

We are now in position to prove Theorem 1.2.

Proof of Theorem 1.2. It suffices to give proof in the case $z_0 = 0$, $r \in]0,1]$ and $0 < \rho < r$. Combining Theorems 3.1 and 3.2, we obtain the following estimate: if $s, \delta > 0$ verify the condition

$$|s - 1/2| \ge \delta,$$

then, for every ρ, r such that $\frac{1}{2} \le \rho < r \le 1$, there exists a positive constant $\widetilde{C}$ such that

$$\| u^s \|_{L^{2\alpha}(\mathcal{Q}_\rho)} \le \widetilde{C}\left(s, \lambda, \| a \|_{L^q(\mathcal{Q}_r)}, \| b \|_{L^q(\mathcal{Q}_r)}, \| c \|_{L^q(\mathcal{Q}_r)}\right) \| u^s \|_{L^{2\beta}(\mathcal{Q}_r)} \tag{4.41}$$

where

$$\begin{aligned}
\widetilde{C}\left(s, \lambda, \| a \|_{L^q(\mathcal{Q}_r)}, \| b \|_{L^q(\mathcal{Q}_r)}, \| c \|_{L^q(\mathcal{Q}_r)}\right) &= C(s,\lambda)\left(1 + \| a \|_{L^q(\mathcal{Q}_r)} + \| b \|_{L^q(\mathcal{Q}_r)}\right) \| c \|^{\frac{1}{2}}_{L^q(\mathcal{Q}_r)} + \\
&+ \frac{C(\lambda)\left(1 + \| a \|_{L^q(\mathcal{Q}_r)} + \| b \|_{L^q(\mathcal{Q}_r)}\right)^{\frac{3}{2}}}{(r-\rho)^{\frac{1}{2}}} + \frac{C}{(r-\rho)^{\frac{3}{2}}}\left(1 + \| a \|_{L^q(\mathcal{Q}_r)} + \| b \|_{L^q(\mathcal{Q}_r)}\right)^{\frac{1}{2}} + \\
&+ \frac{C(s)}{r-\rho}\left(1 + \| a \|_{L^q(\mathcal{Q}_r)} + \| b \|_{L^q(\mathcal{Q}_r)} + \lambda^{\frac{1}{2}} \| c \|^{\frac{1}{2}}_{L^q(\mathcal{Q}_r)}\right) + \frac{C(s)}{(r-\rho)^2}.
\end{aligned}$$

We remark that the previous constant $\widetilde{C}$ can be estimated as follows

$$\begin{aligned}
&\widetilde{C}(s, \lambda, \| a \|_{L^q(\mathcal{Q}_r)}, \| b \|_{L^q(\mathcal{Q}_r)}, \| c \|_{L^q(\mathcal{Q}_r)}) \le \\
&\qquad \le \frac{K(\lambda, s)\left(1 + \| a \|^2_{L^q(\mathcal{Q}_r)} + \| b \|^2_{L^q(\mathcal{Q}_r)} + \| c \|_{L^q(\mathcal{Q}_r)}\right)}{\left(\rho_n - \rho_{n+1}\right)^2}.
\end{aligned} \tag{4.42}$$

Fixed a suitable $\delta > 0$, we shall specify later on, and $p > 0$ we iterate inequality (4.41) by choosing

$$\rho_n = \rho + \frac{1}{2^n}(r-\rho), \qquad p_n = \alpha^n \frac{p}{2\beta}, \qquad n \in \mathbb{N} \cup \{0\}.$$

Then we set $v = u^{\frac{p}{2\beta}}$. If $p > 0$ is such that

$$|p\,\alpha^n - \beta| \ge 2\beta\delta, \qquad \forall n \in \mathbb{N} \cup \{0\}, \tag{4.43}$$

by (4.41) and estimate (4.42) we obtain the following inequality for every $n \in \mathbb{N} \cup \{0\}$

$$\| v^{\alpha^n} \|_{L^{2\alpha}(\mathcal{Q}_{\rho_{n+1}})} \le \frac{K(\lambda, p)\left(1 + \| a \|^2_{L^q(\mathcal{Q}_r)} + \| b \|^2_{L^q(\mathcal{Q}_r)} + \| c \|_{L^q(\mathcal{Q}_r)}\right)}{\left(\rho_n - \rho_{n+1}\right)^2} \| v^{\alpha^n} \|_{L^{2\beta}(\mathcal{Q}_{\rho_n})}. \tag{4.44}$$

Since

$$\| v^{\alpha^n} \|_{L^{2\alpha}} = \left(\| v \|_{L^{2\alpha^{n+1}}}\right)^{\alpha^n} \qquad \text{and} \qquad \| v^{\alpha^n} \|_{L^{2\beta}} = \left(\| v \|_{L^{2\alpha^n}}\right)^{\alpha^n}$$

we can rewrite Eq. (4.44) in the following form for every $n \in \mathbb{N} \cup \{0\}$

$$\| v \|_{L^{2\alpha^{n+1}}(\mathcal{Q}_{\rho_{n+1}})} \leq \left(\frac{K(\lambda, p) \left(1 + \| a \|^2_{L^q(\mathcal{Q}_r)} + \| b \|^2_{L^q(\mathcal{Q}_r)} + \| c \|_{L^q(\mathcal{Q}_r)}\right)}{(\rho_n - \rho_{n+1})^2} \right)^{\frac{1}{\alpha^n}} \| v \|_{L^{2\beta\,\alpha^n}(\mathcal{Q}_{\rho_n})} .$$

Iterating this inequality, we obtain

$$\| v \|_{L^{2\alpha^{n+1}}(\mathcal{Q}_{\rho_{n+1}})} \leq \prod_{j=0}^{n} \left(\frac{2^{2(j+1)}}{(r-\rho)^2} \right)^{\frac{1}{\alpha^j}} \cdot$$

$$\cdot \left(K(\lambda, p) \left(1 + \| a \|^2_{L^q(\mathcal{Q}_r)} + \| b \|^2_{L^q(\mathcal{Q}_r)} + \| c \|_{L^q(\mathcal{Q}_r)}\right)\right)^{\frac{1}{\alpha^j}} \| v \|_{L^{2\beta}(\mathcal{Q}_r)},$$

and letting n go to infinity, we get

$$\sup_{\mathcal{Q}_\rho} v \leq \frac{\widetilde{K}}{(r-\rho)^\mu} \| v \|_{L^{2\beta}(\mathcal{Q}_r)},$$

where $\mu = \frac{2\alpha}{\alpha - 1}$ and

$$\widetilde{K} = \prod_{j=0}^{n} \left(K(\lambda, p) \left(1 + \| a \|^2_{L^q(\mathcal{Q}_r)} + \| b \|^2_{L^q(\mathcal{Q}_r)} + \| c \|_{L^q(\mathcal{Q}_r)}\right)\right)^{\frac{1}{\alpha^j}}$$

is a finite constant dependent on δ. Thus, we have proved that

$$\sup_{\mathcal{Q}_\rho} u^p \leq \left(\frac{\widetilde{K}}{(r-\rho)^\mu} \right)^{2\beta} \int_{\mathcal{Q}_r} u^p, \tag{4.45}$$

for every p which verifies condition (4.43). Because

$$(Q+2) \leq 2\beta\mu < 9(Q+2)$$

we get estimate (1.11). We now make a suitable choice of $\delta > 0$, only dependent on the homogeneous dimension Q, in order to show that (4.43) holds for every positive p. We remark that, if p is a number of the form

$$p_m = \frac{\alpha^m(\alpha+1)}{2\beta}, \qquad m \in \mathbb{Z},$$

then (4.43) is satisfied with

$$\delta = \frac{|q - \frac{(Q+2)}{2}|}{(Q+2)^2}, \qquad \forall m \in \mathbb{Z}.$$

Therefore (4.45) holds for such a choice of p, with $\widetilde{K}$ only dependent on Q, λ and $\| a \|_{L^q(\mathcal{Q}_r)}$, $\| b \|_{L^q(\mathcal{Q}_r)}$, $\| c \|_{L^q(\mathcal{Q}_r)}$. On the other hand, if p is an arbitrary positive number, we consider $m \in \mathbb{Z}$ such that

$$p_m \leq p < p_{m+1}. \tag{4.46}$$

Hence, by (4.45) we have

$$\sup_{\mathcal{Q}_\rho} u \leq \left(\frac{\widetilde{K}}{(r-\rho)^\mu} \right)^{\frac{2\beta}{p_m}} \left(\int_{\mathcal{Q}_r} u^{p_m} \right)^{\frac{1}{p_m}} \leq \left(\frac{\widetilde{K}}{(r-\rho)^\mu} \right)^{\frac{2\beta}{p_m}} \left(\int_{\mathcal{Q}_r} u^{p} \right)^{\frac{1}{p}}$$

so that, by (4.46), we obtain

$$\sup_{\mathcal{Q}_\rho} u^p \leq \left(\frac{\widetilde{K}}{(r-\rho)^\mu} \right)^{2\alpha\beta} \int_{\mathcal{Q}_r} u^p$$

This concludes the proof of (1.11) for $p > 0$. We next consider $p < 0$. In this case, assuming that $u \geq u_0$ for some positive constant u_0, estimate (1.11) can be proved as in the case $p > 0$ or even more easily since condition (4.43) is satisfied for every $p < 0$. On the other hand, if u is a non-negative solution, it suffices to apply (1.11) to $u + \frac{1}{n}$, $n \in \mathbb{N}$, and let n go to infinity, by the monotone convergence theorem. □

As far as we are concerned with the proof of Corollary 1.4, it can be straightforwardly accomplished proceeding as in [20, Corollary 1.4]. Moreover, Proposition 1.5 can be obtained by the same argument used in the proof of Theorem 1.2. For this reason, we do not give here the proof of these two results.

We close this Section recalling that Theorem 1.2 also holds true in the sets

$$\mathcal{Q}_r^-((x_0,t_0)) := \mathcal{Q}_r((x_0,t_0)) \cap \{t < t_0\}, \tag{4.47}$$

in the case of non-negative exponents p. This result is analogous to [16], Theorem 3 (see also inequality (6^-) of Lemma 1 in [17]) and states that, in some sense, every point of $\overline{\mathcal{Q}_\rho^-}(z_0)$ can be considered as an interior point of $\mathcal{Q}_r^-(z_0)$, when $\rho < r$, even though it belongs to its topological boundary.

Proposition 4.2. *Let u be a non-negative weak sub-solution to $\mathscr{L}u = 0$ in Ω. Let $z_0 \in \Omega$ and $r, \rho, \frac{1}{2} \leq \rho < r \leq 1$, such that $\overline{\mathcal{Q}_\rho^-}(z_0) \subseteq \Omega$ and $p < 0$. Then, for every $p \geq 1$ there exist positive constants $C = C(p, \lambda)$ and $\gamma = \gamma(p, q)$ such that it holds*

$$\sup_{\mathcal{Q}_\rho^-(z_0)} u^p \leq \frac{C\left(1+ \| a \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| b \|^2_{L^q(\mathcal{Q}_r(z_0))} + \| c \|_{L^q(\mathcal{Q}_r(z_0))}\right)^\gamma}{(r-\rho)^{9(Q+2)}} \int_{\mathcal{Q}_r^-(z_0)} u^p, \tag{4.48}$$

where $\gamma = \frac{2\alpha^2\beta}{\alpha-1}$, with α and β defined in (1.10), provided that the integral is convergent.

The proof of the above Proposition can be straightforwardly accomplished proceeding as in Proposition 5.1 in [20], and therefore is omitted.

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