

# A TOPOLOGICAL VERSION OF HILBERT'S NULLSTELLENSATZ

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ABSTRACT. We prove that the space of radical ideals of a ring  $R$ , endowed with the hull-kernel topology, is a spectral space, and that it is canonically homeomorphic to the space of the non-empty Zariski closed subspaces of  $\mathbf{Spec}(R)$ , endowed with a Zariski-like topology.

## 1. INTRODUCTION AND PRELIMINARIES

Hilbert's Nullstellensatz establishes a fundamental relationship between geometry and algebra, relating algebraic sets in affine spaces to radical ideals in polynomial rings over algebraically closed fields. On the other hand, for any ring  $R$ , the set of radical ideals of  $R$  can be thought as a set of representatives of the closed sets of  $X := \mathbf{Spec}(R)$ , in the sense that the map  $\mathcal{J}$ , sending a closed set  $C$  of  $X$  to the radical ideal  $\mathcal{J}(C) := \bigcap \{P \mid P \in C\}$ , is a natural order-reversing bijection, having as inverse the map  $\mathbf{V}$  defined by sending a radical ideal  $H$  of  $R$  to the Zariski closed subspace  $\mathbf{V}(H) := \{P \in \mathbf{Spec}(R) \mid H \subseteq P\}$  of  $X$ .

In the present paper, we will put into a topological perspective the relationship between the geometry of  $\mathbf{Spec}(R)$  and ideal theory of  $R$ , shedding new light onto the Nullstellensatz-type correspondence established by the maps  $\mathcal{J}$  and  $\mathbf{V}$ .

Precisely, we consider  $\mathbf{Rd}(R) := \{H \text{ ideal of } R \mid H = \text{rad}(H) \subsetneq R\}$  endowed with the so called *hull-kernel topology*, that is the topology defined by taking, as a subbasis of open sets, the collection of all the subsets of the form  $\{H \in \mathbf{Rd}(R) \mid x_1, \dots, x_n \notin H\}$ , for  $x_1, \dots, x_n$  varying in the ring  $R$ . In this situation, we show that  $\mathbf{Rd}(R)^{\text{hk}}$  (i.e.,  $\mathbf{Rd}(R)$  with hull-kernel topology) is a spectral space (after Hochster [12]), using a general approach described below. On the other hand, we introduce a natural topology, called the *Zariski topology*, on the space  $\mathcal{X}'(R)$  of all the nonempty closed subspaces of the spectral space  $\mathbf{Spec}(R)$ , by declaring as a basis of open sets the collection of the sets of the form

$$\mathcal{U}'(\Omega) := \{C \in \mathcal{X}'(R) \mid C \cap \Omega = \emptyset\},$$

where  $\Omega$  runs in the family of all quasi-compact open subspaces of  $\mathbf{Spec}(R)$ .

In such a way,  $\mathcal{X}'(R)$  becomes a  $T_0$  topological space which can be considered as a natural order-reversing topological extension of  $\mathbf{Spec}(R)$ . More precisely, if

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$\mathrm{Spec}(R)$  is endowed with the inverse topology (as defined by Hochster; the definition will be recalled later), then the natural map  $\varphi' : \mathrm{Spec}(R) \rightarrow \mathcal{X}'(R)$ ,  $P \mapsto \mathrm{Cl}(\{P\})$ , turns out to be a topological embedding, where  $\mathrm{Cl}(\{P\})$  denotes the Zariski closure in  $\mathrm{Spec}(R)$  of the singleton  $\{P\}$ , i.e.,  $\mathrm{Cl}(\{P\}) = \mathfrak{V}(P)$ .

Among the main results of the present paper, we show that the topological space  $\mathcal{X}'(R)$ , endowed with the Zariski topology (denoted by  $\mathcal{X}'(R)^{\mathrm{zar}}$ ), is a spectral space. By linking algebraic and topological properties, we show that  $\mathcal{J}$  establishes a homeomorphism between  $\mathcal{X}'(R)^{\mathrm{inv}}$  (that is,  $\mathcal{X}'(R)$  endowed with the inverse topology) and  $\mathrm{Rd}(R)^{\mathrm{hk}}$  (Theorem 4.1).

The topological properties that we prove concerning the space  $\mathrm{Rd}(R)$  are obtained as particular cases of a more general construction. Indeed, given a  $R$ -module  $M$ , we define in a standard way the hull-kernel topology on the set  $\mathrm{SMod}(M|R)$  of all  $R$ -submodules of  $M$ , and we prove that this topological space is a spectral space, by using a characterization based on ultrafilters. Then, we focus on the subspace  $\mathrm{Spec}_R(M)$  of  $\mathrm{SMod}(M|R)$  given by the prime  $R$ -submodules of  $M$  (definition recalled later), and we show that  $\mathrm{Spec}_R(M)$  is spectral if and only if it is quasi-compact; this happens, for example, when  $M$  is finitely generated. Among other facts, we investigate whether some distinguished subspaces of  $\mathrm{SMod}(M|R)$  are closed, with respect to the constructible topology. We show that this happens to the space  $\mathrm{SMod}^c(M|R) := \{N \in \mathrm{SMod}(M|R) \mid N = N^c\}$ , where  $c : \mathrm{SMod}(R|M) \rightarrow \mathrm{SMod}(R|M)$ ,  $N \mapsto N^c$ , is a closure operation of finite type; in particular, it is a spectral space, with the hull-kernel topology. Thus, keeping in mind that the set of all ideals of  $R$ , denoted by  $\mathrm{Id}(R)$ , coincides with the spectral space  $\mathrm{SMod}(R|R)$  and that the mapping  $\mathrm{rad} : \mathrm{Id}(R) \rightarrow \mathrm{Id}(R)$  (sending an ideal  $I$  of  $R$  to its radical) is a closure operation of finite type, we deduce that  $\mathrm{Rd}(R)$  (with the hull-kernel topology) is a spectral space. Furthermore, we show that the Krull dimension of this spectral space can be evaluated by the formula  $\dim(\mathrm{Rd}(R)) = |\mathrm{Spec}(R)| - 1 \geq \dim(\mathrm{Spec}(R))$ .

In the following, we will freely use some well known facts on spectral spaces [12]. However, for convenience of the reader we recall now briefly some basic definitions and background material.

**1.1. Spectral spaces.** Let  $X$  be a topological space. According to [12],  $X$  is called a *spectral space* if there exists a ring  $R$  such that  $\mathrm{Spec}(R)$ , with the Zariski topology, is homeomorphic to  $X$ . Spectral spaces can be characterized in a purely topological way: a topological space  $X$  is spectral if and only if  $X$  is  $T_0$  (this means that for every pair of distinct points of  $X$ , at least one of them has an open neighborhood not containing the other), quasi-compact (i.e., any open cover of  $X$  admits a finite subcover), admits a basis of quasi-compact open subspaces that is closed under finite intersections, and every irreducible closed subspace  $C$  of  $X$  has a unique generic point (i.e., there exists a unique point  $x_C \in C$  such that  $C$  coincides with the closure of this point) [12, Proposition 4].

**1.2. The inverse topology on a spectral space.** Let  $X$  be a topological space and let  $Y$  be any subset of  $X$ . We denote by  $\mathrm{Cl}(Y)$  the closure of  $Y$  in the topological space  $X$ . Recall that the topology on  $X$  induces a natural preorder  $\leq$  on  $X$ , defined by setting  $x \leq y$  if  $y \in \mathrm{Cl}(\{x\})$ . It is straightforward to see that  $\leq$  is a partial order if and only if  $X$  is a  $T_0$  space (e.g., this holds when  $X$  is spectral). The set  $Y^{\mathrm{gen}} := \{x \in X \mid y \in \mathrm{Cl}(\{x\}), \text{ for some } y \in Y\}$  is called *closure under generizations* of  $Y$ . Similarly, using the opposite order, the set  $Y^{\mathrm{sp}} := \{x \in X \mid$

$x \in \mathbf{Cl}(\{y\})$ , for some  $y \in Y$  is called *closure under specializations of  $Y$* . We say that  $Y$  is *closed under generizations* (respectively, *closed under specializations*) if  $Y = Y^{\text{gen}}$  (respectively,  $Y = Y^{\text{sp}}$ ). For any two elements  $x, y$  in a spectral space  $X$ , we have:

$$x \leq y \iff \{x\}^{\text{gen}} \subseteq \{y\}^{\text{gen}} \iff \{x\}^{\text{sp}} \supseteq \{y\}^{\text{sp}}.$$

Suppose that  $X$  is a spectral space, then  $X$  can be endowed with another topology, introduced by Hochster [12, Proposition 8], whose basis of closed sets is the collection of all the quasi-compact open subspaces of  $X$ . This topology is called *the inverse topology on  $X$*  (called also *the  $O$ -topology in [21]*; see also [11]). For a subset  $Y$  of  $X$ , let  $\mathbf{Cl}^{\text{inv}}(Y)$  be the closure of  $Y$ , in the inverse topology of  $X$ ; we denote by  $X^{\text{inv}}$  the set  $X$ , equipped with the inverse topology. The name given to this new topology is due to the fact that, given  $x, y \in X$ ,  $x \in \mathbf{Cl}^{\text{inv}}(\{y\})$  if and only if  $y \in \mathbf{Cl}(\{x\})$ , i.e., the partial order induced by the inverse topology is the opposite order of the partial order induced by the given spectral topology [12, Proposition 8].

By definition, for any subset  $Y$  of  $X$ , we have

$$\mathbf{Cl}^{\text{inv}}(Y) := \bigcap \{U \mid U \text{ open and quasi-compact in } X, U \supseteq Y\}.$$

In particular, keeping in mind that the inverse topology reverses the order of the given spectral topology, it follows that the closure under generizations  $\{x\}^{\text{gen}}$  of a singleton is closed in the inverse topology of  $X$ , since

$$\{x\}^{\text{gen}} = \mathbf{Cl}^{\text{inv}}(\{x\}) = \bigcap \{U \mid U \subseteq X \text{ quasi-compact and open, } x \in U\}$$

[12, Proposition 8]. On the other hand, it is trivial, by the definition, that the closure under specializations of a singleton  $\{x\}^{\text{sp}}$  is closed in the given topology of  $X$ , since  $\{x\}^{\text{sp}} = \mathbf{Cl}(\{x\})$ .

For recent developments in the use of the inverse topology in Commutative Algebra and spaces of valuation domains see, for example, [20].

**1.3. The constructible topology on a spectral space.** Let  $X$  be a spectral space. As it is well known, the topology of  $X$  is Hausdorff if and only if  $X$  is zero-dimensional. Following [9], there is a natural way to refine the topology of  $X$  in order to make  $X$  an Hausdorff space without losing compactness. Precisely, define *the constructible topology on  $X$*  to be the coarsest topology for which the quasi-compact open subspaces of  $X$  form a collection of clopen sets. In this way,  $X$  becomes a totally disconnected Hausdorff spectral space. Let  $X^{\text{cons}}$  denote the set  $X$  endowed with the constructible topology. By [12, Proposition 9], any closed subset of  $X^{\text{cons}}$  is a spectral subspace of  $X$  (with respect to the original spectral topology). Thus, in particular, any quasi-compact open subspace  $\Omega$  of  $X$  is spectral, since  $\Omega$  is clopen in the constructible topology, by definition. It is, in general, not so easy to describe the closed sets of  $X^{\text{cons}}$ . The following results provides both a criterion to characterize when a topological space  $X$  is spectral and to characterize the closed sets of  $X^{\text{cons}}$ . This result is based on the use of ultrafilters. For background material on this topic and application of ultrafilters to Commutative Ring Theory see, for example, [16] and [22].

**Theorem 1.1.** [5, Corollary 3.3] *Let  $X$  be a topological space.*

- (1) *The following conditions are equivalent.*
  - (i)  *$X$  is a spectral space.*

- (ii) *There exists a subbasis  $\mathcal{S}$  of  $X$  such that, for any ultrafilter  $\mathcal{U}$  on  $X$ , the set*

$$X(\mathcal{U}) := \{x \in X \mid [\forall S \in \mathcal{S}, \text{ the following holds: } x \in S \Leftrightarrow S \in \mathcal{U}]\}$$

*is nonempty.*

- (2) *If the previous equivalent conditions hold and  $\mathcal{S}$  is as in (ii), then a subset  $Y$  of  $X$  is closed, with respect to the constructible topology, if and only if for any ultrafilter  $\mathcal{V}$  on  $Y$  we have*

$$Y(\mathcal{V}) := \{x \in X \mid [\forall S \in \mathcal{S} \text{ the following holds: } x \in S \Leftrightarrow S \cap Y \in \mathcal{V}]\} \subseteq Y.$$

**Corollary 1.2.** *Let  $X$  be a topological space satisfying the equivalent conditions of Theorem 1.1(1), and let  $\mathcal{S}$  be as in Theorem 1.1(1,ii). Then  $\mathcal{S}$  is a subbasis of quasi-compact open subspaces of  $X$ .*

*Proof.* By [5, Corollary 2.9, Propositions 2.11 and 3.2],  $\mathcal{S}$  is a collection of clopen sets with respect to the constructible topology on the spectral space  $X$ . In the constructible topology, every clopen set is quasi-compact with respect to the given spectral topology. The claim follows.  $\square$

## 2. SPECTRAL SPACES OF IDEALS AND MODULES

The main purpose of the present section is to apply the general construction of the space of inverse-closed subspaces of the prime spectrum of a ring, considered in the previous section, to obtain a topological version of Hilbert's Nullstellensatz.

Let  $R$  be a ring and  $M$  be an  $R$ -module. On the set  $\mathbf{SMod}(M|R)$  of  $R$ -submodules of  $M$  we can define an *hull-kernel topology* having, as a subbasis for the closed sets, the subsets of the form

$$\mathbf{V}(x_1, x_2, \dots, x_m) := \{N \in \mathbf{SMod}(M|R) \mid x_1, x_2, \dots, x_m \in N\},$$

where  $x_1, x_2, \dots, x_m$  varies among all finite subsets of  $M$ . Moreover, let

$$\mathbf{D}(x_1, x_2, \dots, x_m) := \mathbf{SMod}(M|R) \setminus \mathbf{V}(x_1, x_2, \dots, x_m).$$

Note that the hull-kernel topology is clearly  $T_0$  and, by definition, the order induced by this topology on  $\mathbf{SMod}(M|R)$  coincides with the order provided by the set-theoretic inclusion  $\subseteq$ .

**Proposition 2.1.** *For any ring  $R$  and for any  $R$ -module  $M$ ,  $\mathbf{SMod}(M|R)$  is a spectral space. Moreover, the collection of sets  $\mathcal{S} := \{\mathbf{D}(x_1, \dots, x_n) \mid x_1, \dots, x_n \in M\}$  is a subbasis of quasi-compact open subspaces of  $\mathbf{SMod}(M|R)$ .*

*Proof.* Let  $\mathcal{U}$  be an ultrafilter on  $\mathbf{SMod}(M|R)$ , and set  $N_{\mathcal{U}} := \{y \in M \mid \mathbf{V}(y) \in \mathcal{U}\}$ .

If  $y_1, y_2, y \in N_{\mathcal{U}}$  and  $r \in R$ , then  $\mathbf{V}(y_1)$ ,  $\mathbf{V}(y_2)$  and  $\mathbf{V}(y)$  are in  $\mathcal{U}$ . Since  $\mathbf{V}(y_1 - y_2) \supseteq \mathbf{V}(y_1) \cap \mathbf{V}(y_2)$  and  $\mathbf{V}(xr) \supseteq \mathbf{V}(y)$ , by definition of ultrafilter we have  $\mathbf{V}(y_1 - y_2) \in N_{\mathcal{U}}$  and  $\mathbf{V}(ry) \in N_{\mathcal{U}}$ , i.e.,  $y_1 - y_2, ry \in N_{\mathcal{U}}$ . Therefore,  $N_{\mathcal{U}}$  is a  $R$ -submodule of  $M$ .

From the definition, it follows easily that:

$$N_{\mathcal{U}} \in \mathbf{SMod}(M|R)(\mathcal{U}) := \{N \in \mathbf{SMod}(M|R) \mid [\forall \Omega \in \mathcal{S}, N \in \Omega \iff \Omega \in \mathcal{U}]\}.$$

Hence, by [5, Corollary 3.3],  $\mathbf{SMod}(M|R)$  is a spectral space. The last statement follows from Corollary 1.2.  $\square$

As particular cases of the spectral space of the submodules of a given module, we can consider the following distinguished cases.

(a) Given any ring  $R$ , let

$$\begin{aligned}\mathrm{Id}(R) &:= \mathrm{SMod}(R|R), \\ \mathrm{Id}_\bullet(R) &:= \mathrm{Id}(R) \setminus \{R\},\end{aligned}$$

where  $\mathrm{Id}(R)$  (respectively,  $\mathrm{Id}_\bullet(R)$ ) is the set of all ideals (respectively, the set of all proper ideals).

(b) Given any integral domain  $D$  with quotient field  $K$ , let

$$\overline{\mathcal{F}}(D) := \mathrm{SMod}(K|D) = \{E \mid E \text{ is a } D\text{-submodule of } K\}.$$

**Corollary 2.2.** *Let  $R$  be a ring and let  $D$  be an integral domain with quotient field  $K$ ,  $D \neq K$ .*

- (1) *The set  $\mathrm{Id}(R)$  (respectively,  $\mathrm{Id}_\bullet(R)$ ), endowed with the hull-kernel topology, is a spectral space.*
- (2) *Let  $\mathrm{Rd}(R)$  be the set of proper radical ideals of  $R$  and consider the following topological embeddings with respect to the hull-kernel topology, induced from  $\mathrm{Id}(R)$ ,*

$$\mathrm{Spec}(R) \subseteq \mathrm{Rd}(R) \subseteq \mathrm{Id}_\bullet(R) \subseteq \mathrm{Id}(R).$$

*Then, the hull-kernel topology induced on  $\mathrm{Spec}(R)$  coincides with the Zariski topology.*

- (3) *The space  $\overline{\mathcal{F}}(D)$  endowed with the hull-kernel topology, is a spectral space.*
- (4) *The space  $\mathcal{F}(D)$  of all fractional ideals of  $D$ , endowed with the hull-kernel topology, is not a spectral space.*

*Proof.* (1) and (3). The statements for  $\mathrm{Id}(R)$  for  $\overline{\mathcal{F}}(D)$  are direct consequences of Proposition 2.1. The claim for  $\mathrm{Id}_\bullet(R)$  follows if we show that  $N_{\mathcal{U}} \neq R$ , when  $\mathcal{U}$  is an ultrafilter of  $\mathrm{Id}_\bullet(R)$ . If  $N_{\mathcal{U}} = R$  then  $1 \in N_{\mathcal{U}}$ , i.e.,  $\mathbf{D}(1) \cap \mathrm{Id}_\bullet(R) \in \mathcal{U}$ . Since  $\mathbf{D}(1) \cap \mathrm{Id}_\bullet(R) = \emptyset$ , we reach a contradiction. Hence,  $N_{\mathcal{U}} \neq R$ .

(2) is straightforward.

(4) If  $\mathcal{F}(D)$  were a spectral space, then it would have proper maximal elements. If  $E$  is one of these, then there is an element  $x \in E \setminus K$  (since  $K$  is not a fractional ideal of  $D$  if  $D \neq K$ ) and so  $E + xD$  is a fractional ideal properly containing  $E$ , against the hypothesized maximality.  $\square$

**Remark 2.3.** Since we have proved that  $\mathrm{Id}_\bullet(R)$  is a spectral space (Corollary 2.2(1)), it is then natural to ask in general if similar cases might occur:

- (Q.1) Is  $\mathrm{SMod}_\bullet(M|R) := \mathrm{SMod}(M|R) \setminus \{(0)\}$  (with the hull-kernel topology) a spectral space?
- (Q.2) Is  $\mathrm{SMod}_\bullet(M|R) := \mathrm{SMod}(M|R) \setminus \{M\}$  (with the hull-kernel topology) a spectral space?

The answer to both question is negative: we shall see in Remark 3.7 a counterexample to question (Q.1), while the problem of question (Q.2) will be completely settled in the following Proposition 2.4.

**Proposition 2.4.** *Let  $M$  be a  $R$ -module. Then,  $\mathrm{SMod}_\bullet(M|R) := \mathrm{SMod}(M|R) \setminus \{M\}$  is a spectral space, endowed with the hull-kernel subspace topology, if and only if  $M$  is finitely generated.*

*Proof.* Consider the subbasis of open sets  $\mathcal{S} := \{\mathbf{D}(x_1, \dots, x_n) \mid x_1, \dots, x_n \in M\}$  of  $X := \mathbf{SMod}_\bullet(M|R)$  and assume first that  $M$  is finitely generated. If  $\mathcal{U}$  is an ultrafilter on  $X$ , recall that the subset  $N_{\mathcal{U}} := \{y \in M \mid \mathbf{V}(y) \cap X \in \mathcal{U}\}$  is a  $R$ -submodule of  $M$ , by the proof of Proposition 2.1. In the notation of Theorem 1.1, if we show that  $N_{\mathcal{U}}$  is a proper submodule of  $M$ , it will follow immediately that  $N_{\mathcal{U}} \in X(\mathcal{U})$ , thus  $X$  will be spectral. Let  $F$  be a finite set of generators for  $M$ . If  $N_{\mathcal{U}} = M$  then, by definition,  $\mathbf{V}(F) \cap X \in \mathcal{U}$  and, since the empty set is not a member of any ultrafilter, we can pick a submodule  $N \in \mathbf{V}(F) \cap X$ . But  $N \in \mathbf{V}(F)$  implies  $M = \langle F \rangle = N$ , a contradiction. Then  $N_{\mathcal{U}} \neq M$  and thus the first part of the proof is complete.

Conversely, assume that  $M$  is not finitely generated, and note that the family of subsets  $\{\mathbf{D}(x) \mid x \in M\}$  is obviously an open cover of  $X$ . Of course, for any finite subset  $F$  of  $M$ , the collection of open sets  $\{\mathbf{D}(x) \mid x \in F\}$  is not a subcover of  $X$ , since the finitely generated submodule  $N := \langle F \rangle$  of  $M$  is proper, by assumption, and thus  $N \in X \setminus \bigcup \{\mathbf{D}(x) \mid x \in F\}$ . This shows that, if  $M$  is not finitely generated, then  $X$  is not quasi-compact and, a fortiori, is not spectral.  $\square$

**Remark 2.5.** In Corollary 2.2, we considered the space of ideals of a ring  $R$  as a special case of the space of  $R$ -submodules of a  $R$ -module  $M$ . It is possible, however, to reverse this relation, in the following way.

With the same proof of Proposition 2.1, we can first show that, given two ideals  $I$  and  $J$  with  $J \subseteq I$ , the set  $\mathbf{Id}((I, J)|R) := \{H \in \mathbf{Id}(R) \mid J \subseteq H \subseteq I\}$  is a spectral space, with  $\mathbf{Id}(R)$  being the special case with  $J = (0)$  and  $I = R$ . Consider now an  $R$ -module  $M$ : then,  $M$  is an ideal of the idealization ring  $\mathcal{R} := R \times M$  [13, Section 25]. In this case, we have that  $\mathbf{Id}((M, (0))|\mathcal{R})$  coincides with  $\mathbf{SMod}(M)$  and so, from this fact, we can deduce that  $\mathbf{SMod}(M)$  is a spectral space.

In the next proposition we show that the construction of the spectral space  $\mathbf{SMod}(M|R)$  is functorial. Recall that a map  $f : X \rightarrow Y$  of spectral spaces is called a *spectral map* provided that, for any open and quasi-compact subspace  $\Omega$  of  $Y$ , the set  $f^{-1}(\Omega)$  is open and quasi-compact. In particular, any spectral map of spectral spaces is continuous.

**Proposition 2.6.** *Let  $R$  be a ring. For every  $R$ -module homomorphism  $f : M \rightarrow N$ , set  $\mathbf{SMod}(f) : \mathbf{SMod}(N|R) \rightarrow \mathbf{SMod}(M|R)$ , defined by  $\mathbf{SMod}(f)(L) := f^{-1}(L)$ , for each  $L \in \mathbf{SMod}(N|R)$ . The assignment  $M \mapsto \mathbf{SMod}(M|R)$ ,  $f \mapsto \mathbf{SMod}(f)$  gives rise to a contravariant functor  $\mathbf{SMod}$  from the category of  $R$ -modules and  $R$ -linear maps to the category of spectral spaces and spectral maps.*

*Proof.* By Proposition 2.1,  $\mathbf{SMod}(M|R)$  and  $\mathbf{SMod}(N|R)$  are spectral spaces. In order to show that  $\mathbf{SMod}(f)$  is continuous and spectral, it is enough to note that, for each finite subset  $\{x_1, x_2, \dots, x_m\}$  of  $K$ ,

$$\mathbf{SMod}(f)^{-1}(\mathbf{V}(x_1, x_2, \dots, x_m)) = \mathbf{V}(f(x_1), f(x_2), \dots, f(x_m)).$$

Moreover, it is clear that  $\mathbf{SMod}(g \circ f) = \mathbf{SMod}(f) \circ \mathbf{SMod}(g)$ , so that  $\mathbf{SMod}$  is a (contravariant) functor.  $\square$

For example, let  $D$  be an integral domain with quotient field  $K$  and let  $j : D \hookrightarrow K$  be the natural embedding. Then, the map  $\mathbf{SMod}(j) : \mathbf{SMod}(K|D) = \overline{\mathcal{F}}(D) \rightarrow \mathbf{SMod}(D|D) = \mathbf{Id}(D)$ , defined by  $E \mapsto E \cap D$ , is a spectral retraction (between

spectral spaces endowed with the hull-kernel topology). In fact, if  $i : \text{Id}(D) \hookrightarrow \overline{\mathcal{F}}(D)$  is the natural (spectral) embedding, then  $\text{SMod}(j) \circ i$  is the identity of  $\text{Id}(D)$ .

### 3. THE PRIME SPECTRUM OF A MODULE

Recall that a *prime submodule* of a  $R$ -module  $M$  is a submodule  $P \neq M$  such that, whenever  $am \in P$  for some  $a \in R$ ,  $m \in M$ , we have  $m \in P$  or  $aM \subseteq P$  (see, for example, [17]). Denote by  $\text{Spec}_R(M)$  the set of prime submodules of  $M$ . Note that  $\text{Spec}_R(M)$  may be empty (e.g., if  $R$  is a domain,  $K$  its quotient field and  $M = K/R$ ) and that when  $M = R$  it coincides with the prime spectrum of  $R$ .

**Proposition 3.1.** *Let  $M$  be a  $R$ -module and endow  $\text{SMod}(M|R)$  with the hull-kernel topology.*

- (1)  $\text{Spec}_R(M) \cup \{M\}$  is a spectral subspace of  $\text{SMod}(M|R)$ .
- (2)  $\text{Spec}_R(M)$  is a spectral space if and only if it is quasi-compact.
- (3) If  $M$  is finitely generated, then  $\text{Spec}_R(M)$  is a spectral space.

*Proof.* (1) Let  $\mathcal{U}$  be an ultrafilter on  $\text{Spec}_R(M)$ ; like in the proof of Proposition 2.1, it is enough to show that the set  $N_{\mathcal{U}} := \{x \in M \mid \mathbf{V}(x) \cap \text{Spec}_R(M) \in \mathcal{U}\}$  is a prime submodule of  $M$ , if it is different from  $M$ . To shorten the notation, set  $\overline{\mathbf{S}} := \text{Spec}_R(M) \cup \{M\}$ ,  $\mathbf{S} := \text{Spec}_R(M)$ ,  $\mathbf{V}_{\mathbf{S}}(x) := \mathbf{V}(x) \cap \text{Spec}_R(M)$  and  $\mathbf{D}_{\mathbf{S}}(x) := \text{Spec}_R(M) \setminus \mathbf{V}_{\mathbf{S}}(x)$ .

The proof of Proposition 2.1 shows that  $N_{\mathcal{U}}$  is a submodule of  $M$ . Suppose now that  $a \in R$ ,  $m \in M$ ,  $am \in N_{\mathcal{U}}$ , and that  $m \notin N_{\mathcal{U}}$ , so  $N_{\mathcal{U}} \neq M$ . By definition of a prime submodule, it follows easily that  $T := \mathbf{V}_{\mathbf{S}}(am) \cap \mathbf{D}_{\mathbf{S}}(m) \subseteq \mathbf{V}_{\mathbf{S}}(ax)$ , for any  $x \in M$ . Now, keeping in mind that  $m \notin N_{\mathcal{U}}$ ,  $am \in N_{\mathcal{U}}$  and that  $\mathcal{U}$  is an ultrafilter on  $\text{Spec}_R(M)$ , it follows that  $T \in \mathcal{U}$  and, a fortiori,  $\mathbf{V}_{\mathbf{S}}(ax) \in \mathcal{U}$ , for any  $x \in M$ , that is,  $xM \subseteq N_{\mathcal{U}}$ . In other words,  $N_{\mathcal{U}}$  is a prime submodule of  $M$ .

(2) If  $\mathbf{S} = \text{Spec}_R(M)$  is a spectral space then it is clearly quasi-compact. Conversely, keeping in mind that  $\{M\}$  is the unique closed point in  $\overline{\mathbf{S}}$ , we have that  $\mathbf{S}$  is open and quasi-compact in the spectral space  $\overline{\mathbf{S}}$ , and hence it is spectral.

(3) Let  $\mathcal{U}$  and  $N_{\mathcal{U}}$  be as in part (1). We need to prove that, if  $M$  is finitely generated, then  $N_{\mathcal{U}} \neq M$ . In fact, let  $M = \langle x_1, x_2, \dots, x_n \rangle$ , if  $N_{\mathcal{U}} = M$ , then, by definition, the set  $\bigcap_{i=1}^n \mathbf{V}(x_i) \in \mathcal{U}$ . Thus, we can pick a prime submodule  $P \in \bigcap_{i=1}^n \mathbf{V}(x_i)$ , that is,  $M = \langle x_1, \dots, x_n \rangle \subseteq P$ , reaching a contradiction. This proves that, if  $M$  is finitely generated, then  $\text{Spec}_R(M)$  is a closed set of  $\overline{\mathbf{S}}$ , with respect to the constructible topology, by Theorem 1.1(2). In particular,  $\text{Spec}_R(M)$  is quasi-compact, when endowed with the hull-kernel topology. The conclusion is then a consequence of part (2).  $\square$

**Remark 3.2.** (1) The condition that  $M$  is finitely generated is not necessary for  $\text{Spec}_R(M)$  to be spectral. For example, if  $R = D$  is an integral domain and  $M = K$  is its quotient field, then  $\text{Spec}_D(K) = \{(0)\}$ , which is compact and spectral. However,  $K$  is not finitely generated over  $D$  if  $D \neq K$ .

- (2)  $\text{Spec}_R(M)$  can indeed be non quasi-compact: let  $R$  be any ring,  $P \in \text{Spec}(R)$ , and let  $M = \bigoplus_{\alpha \in \mathcal{A}} e_{\alpha} R$  be a non-finitely generated free module over  $R$ . We always have  $\text{Spec}_R(M) \subseteq \bigcup_{\alpha \in \mathcal{A}} \mathbf{D}(e_{\alpha})$ . If  $\text{Spec}_R(M)$  were quasi-compact, there would be  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{A}$  such that  $\text{Spec}_R(M) \subseteq \mathbf{D}(e_{\alpha_1}) \cup \mathbf{D}(e_{\alpha_2}) \cup \dots \cup \mathbf{D}(e_{\alpha_n})$ , and so there would be no prime submodule containing all  $e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_n}$ . Since  $\mathcal{A}$  is infinite, there is an element

$\beta \in \mathcal{A}$  such that  $\beta \neq \alpha_i$  for every  $i$ ,  $1 \leq i \leq n$ . Define a submodule  $N$  of  $M$  as follows:

$$N := \bigoplus_{\alpha \in \mathcal{A}} e_\alpha N_\alpha, \text{ where } N_\alpha = R \text{ if } \alpha \neq \beta \text{ and } N_\beta = P.$$

We have  $M/N \simeq R/P$ , so that  $N$  is a prime submodule of  $M$ . However,  $N$  contains  $e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_n}$ , against our hypothesis. Therefore,  $\mathbf{Spec}_R(M)$  is not quasi-compact.

- (3) In [17], the set  $\mathbf{Spec}_R(M)$  (indicated with  $\mathbf{Spec}(M)$ ) was endowed with a topology  $\tau$  (which the author calls *Zariski topology*) whose closed sets are those in the form  $V(N) := \{P \in \mathbf{Spec}_R(M) \mid (P : M) \subseteq (N : M)\}$ , as  $N$  ranges among the submodules of  $M$ . This topology is in general weaker than the topology introduced in the present paper, and it is  $T_0$  if and only if the map  $\psi : \mathbf{Spec}_R(M) \rightarrow \mathbf{Spec}(R)$ , defined by  $P \mapsto (P : M)$ , is injective. In [17], it was also shown that, if  $\psi$  is injective and its image is the closed subspace  $V(\text{ann}(M))$ , then it is an homeomorphism on its image (so that, in particular,  $\mathbf{Spec}_R(M)$  endowed with the topology  $\tau$  is spectral). Even when  $\tau$  is  $T_0$ , however, this topology does not always coincide with the hull-kernel topology. Indeed, let  $R := \mathbb{Z}$ ,  $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$  and let  $M := \mathbb{Z}_2 \oplus \mathbb{Q}$ . We have  $\mathbf{Spec}_R(M) = \{P, Q\}$ , where  $P := \mathbb{Z}_2 \oplus (0)$  and  $Q := (0) \oplus \mathbb{Q}$ ; hence both  $P$  and  $Q$  are closed points in the hull-kernel topology of  $\mathbf{Spec}_R(M)$ . On the other hand, both  $V(P)$  and  $V(Q)$  are irreducible closed subsets in the topology  $\tau$  [17, Corollary 5.3]. However,  $(P : M) = 2\mathbb{Z}$  and  $(Q : M) = (0)$ , so  $V(P) = \{P\}$  and  $V(Q) = \{P, Q\}$ . It follows that  $\mathbf{Spec}_R(M)$  is  $T_0$  in the Zariski topology, but  $Q$  is not a closed point.

Denote by  $\mathbf{Overr}(D)$  the set of all overrings of the integral domain  $D$ . We observe that  $\mathbf{Overr}(D)$  is a subset of  $\overline{\mathcal{F}}(D)$  (in fact, it is a subset of  $\overline{\mathbf{F}}(D) := \overline{\mathcal{F}}(D) \setminus \{(0)\}$ , the set of all nonzero  $D$ -submodules of  $K$ ). On the other hand, the set  $\mathbf{Overr}(D)$  can be endowed with a topology, called the *Zariski topology*, having as basic open sets the subsets of the type  $\mathbf{B}(F) := \mathbf{Overr}(D[F]) = \{T \in \mathbf{Overr}(D) \mid F \subseteq T\}$ , where  $F$  is varying among the finite subsets of  $K$ . If we denote by  $\mathbf{Overr}(D)^{\text{zar}}$  the topological space  $\mathbf{Overr}(D)$  with the Zariski topology and  $\overline{\mathcal{F}}(D)^{\text{hk}}$  (respectively,  $\overline{\mathbf{F}}(D)^{\text{hk}}$ ) the space  $\overline{\mathcal{F}}(D)$  (respectively,  $\overline{\mathbf{F}}(D)$ ) with the hull-kernel topology (respectively, topology induced from the hull-kernel topology of  $\overline{\mathcal{F}}(D)$ ) then the inclusion maps  $\mathbf{Overr}(D) \subseteq \overline{\mathbf{F}}(D)$  and  $\mathbf{Overr}(D) \subseteq \overline{\mathcal{F}}(D)$  are not continuous. In fact, the quotient field  $K$  is the generic point of  $\mathbf{Overr}(D)^{\text{zar}}$  but it is a closed point for  $\overline{\mathbf{F}}(D)^{\text{hk}}$  (and for  $\overline{\mathcal{F}}(D)^{\text{hk}}$ ).

Recall that  $\mathbf{Overr}(D)^{\text{zar}}$  is a spectral space [5, Proposition 3.5(2)] and denote by  $\mathbf{Overr}(D)^{\text{inv}}$  (respectively,  $\mathbf{Overr}(D)^{\text{hk}}$ ) the set  $\mathbf{Overr}(D)$  with the inverse topology (respectively, with the hull-kernel topology, induced from  $\overline{\mathcal{F}}(D)^{\text{hk}}$ ).

**Proposition 3.3.** *For any domain  $D$ ,  $\mathbf{Overr}(D)^{\text{hk}}$  coincides with  $\mathbf{Overr}(D)^{\text{inv}}$ .*

*Proof.* By definition of the inverse topology, a basis for the closed sets of  $\mathbf{Overr}(D)^{\text{inv}}$  is given by the quasi-compact open subspaces of  $\mathbf{Overr}(D)^{\text{zar}}$ , i.e., by the finite unions of the subsets  $\mathbf{B}(F)$ , where  $F$  is varying among the finite subsets of  $K$ . On the other hand, by definition,  $\mathbf{Overr}(D[F]) = \mathbf{V}(F)$ . Moreover, if  $G$  is any subset of  $K$ , then  $\mathbf{V}(G) = \bigcap \{\mathbf{V}(F) \mid F \subseteq G \text{ and } F \text{ is finite}\}$ , so that  $\{\mathbf{V}(F) \mid F \text{ is finite subset of } K\}$  is a basis for the closed sets of the topological space  $\mathbf{Overr}(D)^{\text{hk}}$ . Therefore, we conclude that  $\mathbf{Overr}(D)^{\text{hk}} = \mathbf{Overr}(D)^{\text{inv}}$ .  $\square$



Given a ring  $R$ , on any  $R$ -module  $M$ , a *closure operation* on  $\mathbf{SMod}(M|R)$  is a map  $(-)^c : \mathbf{SMod}(M|R) \rightarrow \mathbf{SMod}(M|R)$  that is extensive (i.e.,  $N \subseteq N^c$ ), order-preserving (i.e.,  $N_1 \subseteq N_2$  implies  $N_1^c \subseteq N_2^c$ ) and idempotent (i.e.,  $(N^c)^c = N^c$ ). We also say that  $c$  is of *finite type* if, for any  $N \in \mathbf{SMod}(M|R)$ ,  $N^c = \bigcup \{L^c \mid L \subseteq N, L \in \mathbf{SMod}(M|R), L \text{ is finitely generated}\}$ . For a deeper insight on this topic see, for example, [1], [3], [4], [10], and [23].

**Proposition 3.4.** *Let  $M$  be an  $R$ -module and  $c$  be a closure operation of finite type on  $\mathbf{SMod}(M|R)$ . The set  $\mathbf{SMod}^c(M|R) := \{N \in \mathbf{SMod}(M|R) \mid N = N^c\}$  is a spectral space. Moreover,  $\mathbf{SMod}^c(M|R)$  is closed in  $\mathbf{SMod}(M|R)$ , endowed with the constructible topology.*

*Proof.* With the same notation of the proof of Proposition 2.1, to prove the first statement we only need to show that, if  $\mathcal{U}$  is an ultrafilter on  $\mathbf{SMod}^c(M|R)$ ,  $N_{\mathcal{U}}$  is also in  $\mathbf{SMod}^c(M|R)$ .

Let  $x \in (N_{\mathcal{U}})^c$ . Since  $c$  is of finite type, there is a finitely generated  $R$ -module  $L \subseteq N_{\mathcal{U}}$  such that  $x \in L^c$ . In particular,  $x \in H^c$  for all  $H \supseteq L$ , i.e., for all  $H \in \mathbf{V}(L)$ ; therefore,  $\mathbf{V}(L) \cap \mathbf{SMod}^c(M|R) \subseteq \mathbf{V}(x) \cap \mathbf{SMod}^c(M|R)$ . If  $L = \ell_1 R + \ell_2 R + \cdots + \ell_n R$ , then  $\mathbf{V}(L) = \mathbf{V}(\ell_1) \cap \mathbf{V}(\ell_2) \cap \cdots \cap \mathbf{V}(\ell_n)$ . Since each  $\mathbf{V}(\ell_i) \cap \mathbf{SMod}^c(M|R)$  is in  $\mathcal{U}$  (by definition of  $N_{\mathcal{U}}$ ), then  $\mathbf{V}(L) \cap \mathbf{SMod}^c(M|R) \in \mathcal{U}$ . Hence,  $\mathbf{V}(x) \cap \mathbf{SMod}^c(M|R) \in \mathcal{U}$ , i.e.,  $x \in N_{\mathcal{U}}$ . Thus,  $N_{\mathcal{U}} = (N_{\mathcal{U}})^c$  and  $\mathbf{SMod}^c(M|R)$  is a spectral space.

Finally, from Theorem 1.1(2) we deduce that  $\mathbf{SMod}^c(M|R)$  is a closed subspace of  $\mathbf{SMod}(M|R)$ , endowed with the constructible topology.  $\square$

**Corollary 3.5.** *Let  $D$  be an integral domain and  $\star$  be a semistar operation of finite type on  $D$  (for background material on semistar operations see, for instance, [3, 8, 19]). Then, the subspaces*

$$\overline{\mathcal{F}}(D)^{\star} := \{E \in \overline{\mathcal{F}}(D) \mid E^{\star} = E\} \quad \text{and} \quad \mathbf{Overr}^{\star}(D) := \{T \in \mathbf{Overr}(D) \mid T = T^{\star}\}$$

of  $\overline{\mathcal{F}}(D)^{hk}$  are spectral spaces.

*Proof.* By applying Proposition 3.4 and the proof of [5, Proposition 3.5] we note that  $\overline{\mathcal{F}}(D)^{\star}$  and  $\mathbf{Overr}^{\star}(D)$  are closed in  $\overline{\mathcal{F}}(D)$ , endowed with the constructible topology. Then, the conclusion follows by [12, Proposition 9].  $\square$

**Corollary 3.6.** *Let  $c$  be a closure operation of finite type on a ring  $R$ . Then,  $\mathbf{Id}^c(R) := \mathbf{SMod}^c(R|R)$  (respectively,  $\mathbf{Id}^{\bullet c}(R) := \mathbf{SMod}^c(R|R) \setminus \{R\}$ ), endowed with the hull-kernel topology, is a spectral space.*

*Proof.* The statements follow from Proposition 3.4 and its proof, using the same argument of the proof of Corollary 2.2(1).  $\square$

**Remark 3.7.** If  $c$  is a closure operation of finite type on an  $R$ -module  $M$ , we can always consider a canonical surjective map  $\psi_c : \mathbf{SMod}(M|R) \rightarrow \mathbf{SMod}^c(M|R)$ , by setting  $\psi_c(N) := N^c$ , for each  $N \in \mathbf{SMod}(M|R)$ . However,  $\psi_c$  is only rarely continuous (with respect to the hull-kernel topology). For example, let  $M = R$  be any infinite ring such that the intersection of all nonzero ideals is  $(0)$  (such a ring is, for example, an integral domain that is not a field). Set  $(0)^c := (0)$ , and set  $I^c$  to be equal to  $R$  if  $I \neq (0)$ . Therefore,  $\mathbf{SMod}^c(R|R) = \mathbf{Id}^c(R) = \{(0), R\}$ . Note that  $\psi_c^{-1}(R) = \{I \mid I \neq (0)\} = \mathbf{Id}(R) \setminus \{(0)\}$ . Since  $R$  is a closed point in  $\mathbf{SMod}(R|R) = \mathbf{Id}(R)$  (endowed with the hull-kernel topology) and  $R = R^c$ , then  $R$

is a closed point in  $\mathbf{SMod}^c(R|R) = \mathbf{Id}^c(R)$  (endowed with the hull-kernel topology). If  $\psi_c$  were continuous,  $\psi_c^{-1}(R) = \mathbf{Id}(R) \setminus \{(0)\}$  would be closed and thus (being a closed subset of a spectral space) it would be a spectral space itself. However,  $\mathbf{Id}(R) \setminus \{(0)\}$  cannot be a spectral space, when endowed with the hull-kernel topology induced from  $\mathbf{Id}(R)$ , since  $\mathbf{Id}(R) \setminus \{(0)\}$  is not quasi-compact. Indeed, by assumption, the intersection of all nonzero ideals of  $R$  is  $(0)$ , and thus the collection of sets  $\{\mathbf{D}(x) \setminus \{(0)\} \mid x \neq 0\}$  provides an infinite open cover of  $\mathbf{Id}(R) \setminus \{(0)\}$  without finite subcovers.

As a particular case of the Proposition 3.4 and Corollary 3.6, we have the following.

**Corollary 3.8.** *Let  $R$  be a ring. The sets  $\mathbf{Rd}(R)$  and  $\mathbf{Rd}(R) \cup \{R\}$ , endowed with the hull-kernel topology, are spectral spaces.*

*Proof.* As usual, let  $\mathbf{rad}(I)$  denote the radical of an ideal  $I$  of  $R$ . If  $x \in \mathbf{rad}(I)$ , then  $x \in \mathbf{rad}(x^n)$  for some  $x^n \in I$ , so  $\mathbf{rad}$  is a closure operation of finite type in  $\mathbf{Id}(R)$ , i.e.,  $\mathbf{Rd}(R) \cup \{R\} = \mathbf{Id}^c(R)$ , where  $c = \mathbf{rad}$ . The conclusion is now a consequence of Corollary 3.6.  $\square$

#### 4. A TOPOLOGICAL VERSION OF HILBERT'S NULLSTELLENSATZ

Let now  $X$  be a spectral space and let  $\mathbf{Cl}(Y)$  denote the closure of a subspace  $Y$  in the given topology of  $X$ . Let  $\mathcal{X}'(X)$  be the space of nonempty closed sets of  $X$ , and endow it with a topology whose subbasic open sets are the family of sets

$$\mathcal{U}'(\Omega) := \{Y \in \mathcal{X}'(X) \mid Y \cap \Omega = \emptyset\},$$

as  $\Omega$  ranges among the quasi-compact open subspaces of  $X$ . Note that the family of sets of the type  $\mathcal{U}'(\Omega)$  forms a basis, since  $\mathcal{U}'(\Omega_1) \cap \mathcal{U}'(\Omega_2) = \mathcal{U}'(\Omega_1 \cup \Omega_2)$ . We call this topology the *Zariski topology* of the space  $\mathcal{X}'(X)$ . The notation used here is chosen in analogy and for coherence with the construction of the space  $\mathcal{X}(X)$ , which is sketched in [6] and elaborated upon in [7].

Note that there is a canonical injective map  $\varphi' : X^{\text{inv}} \rightarrow \mathcal{X}'(X)^{\text{zar}}$ , defined by  $\varphi'(x) := \{x\}^{\text{sp}}$ , which is a topological embedding. Indeed,  $\varphi'$  is continuous since

$$\varphi'^{-1}(\mathcal{U}'(\Omega)) = \{x \in X^{\text{sp}} \mid \{x\}^{\text{sp}} \cap \Omega = \emptyset\} = X \setminus \Omega,$$

which is, by definition, a subbasic open set of  $X^{\text{inv}}$ . Moreover, since the family of the sets of the type  $X \setminus \Omega$ , for  $\Omega$  ranging among the quasi-compact open subspaces of  $X$ , forms a subbasis of  $X^{\text{inv}}$ , the calculation above shows that  $\varphi'(X \setminus \Omega) = \mathcal{U}'(\Omega) \cap \varphi'(X)$ , and thus the map  $\varphi'$  is a topological embedding.

Now, we are in condition to state a “topological version” of the Hilbert Nullstellensatz.

**Theorem 4.1.** *Let  $R$  be a ring and let  $\mathcal{X}'(R) := \mathcal{X}'(\mathbf{Spec}(R))$  be the topological space of the non-empty Zariski closed subspaces of  $\mathbf{Spec}(R)$ , endowed with the Zariski topology. Let  $\mathbf{Rd}(R)$  be the spectral space of all proper radical ideals of  $R$  with the inverse topology. Then, the map*

$$\begin{aligned} \mathcal{J} : \mathcal{X}'(R)^{\text{zar}} &\rightarrow \mathbf{Rd}(R)^{\text{inv}} \\ C &\mapsto \bigcap \{P \in \mathbf{Spec}(R) \mid P \in C\} \end{aligned}$$

*is a homeomorphism. In particular,  $\mathcal{X}'(R)$  is a spectral space. Moreover, the same map  $\mathcal{J}$  defines a homeomorphism between  $\mathcal{X}'(R)^{\text{inv}}$  and  $\mathbf{Rd}(R)^{\text{hk}}$ .*

*Proof.* For each  $x_1, \dots, x_n \in R$ , let  $\Delta(x_1, \dots, x_n) := \{H \in \text{Rd}(R) \mid (x_1, \dots, x_n) \not\subseteq H\} = \mathbf{D}(x_1, \dots, x_n) \cap \text{Rd}(R)$  be a subbasic open set of  $\text{Rd}(R)$  and let  $\mathbf{D}(x_1, \dots, x_n) := \{P \in \text{Spec}(R) \mid x \notin P\}$  be a subbasic open set of  $\text{Spec}(R)$ . By the definition of the hull-kernel topology, by Corollaries 1.2, 3.8 and Proposition 2.1, it follows that  $\mathcal{B} := \{\Delta(x_1, \dots, x_n) \mid x_1, \dots, x_n \in R\}$  is a collection of quasi-compact open subspaces of  $\text{Rd}(R)^{\text{hk}}$ , that is, it is a subbasis of closed sets of  $\text{Rd}(R)^{\text{inv}}$ . Set  $\mathcal{X}' := \mathcal{X}'(R)$ . Then,

$$\begin{aligned} \mathcal{J}^{-1}(\Delta(x_1, \dots, x_n)) &= \{C \in \mathcal{X}' \mid \mathcal{J}(C) \in \Delta(x_1, \dots, x_n)\} = \\ &= \{C \in \mathcal{X}' \mid (x_1, \dots, x_n) \not\subseteq \mathcal{J}(C)\} = \\ &= \{C \in \mathcal{X}' \mid (x_1, \dots, x_n) \not\subseteq \bigcap \{P \in \text{Spec}(R) \mid P \in C\}\} = \\ &= \{C \in \mathcal{X}' \mid x_i \notin P \text{ for some } P \in C \text{ and some } i\} = \\ &= \{C \in \mathcal{X}' \mid C \cap \mathbf{D}(x_1, \dots, x_n) \neq \emptyset\} = \\ &= \mathcal{X}' \setminus \mathcal{U}'(\mathbf{D}(x_1, \dots, x_n)) \end{aligned}$$

which is, by definition, a closed set of  $\mathcal{X}'$ . Hence,  $\mathcal{J}$  is continuous (when  $\text{Rd}(R)$  is equipped with the inverse topology). In order to show that it is a closed map, it is enough to note that  $\{\mathcal{X}' \setminus \mathcal{U}'(\mathbf{D}(x_1, \dots, x_n)) \mid x_1, \dots, x_n \in R\}$  is a basis of closed sets of  $\mathcal{X}'$  and that, by Hilbert Nullstellensatz,  $\mathcal{J}$  is bijective; hence  $\mathcal{J}(\mathcal{X}' \setminus \mathcal{U}'(\mathbf{D}(x_1, \dots, x_n))) = \Delta(x_1, \dots, x_n)$  is closed in  $\text{Rd}(R)^{\text{inv}}$ . Thus,  $\mathcal{J}$  is a homeomorphism.

The last claim follows directly from Hochster's duality, that is, more explicitly, from the fact that  $(\text{Rd}(R)^{\text{inv}})^{\text{inv}}$  coincides with  $\text{Rd}(R)^{\text{hk}}$ .  $\square$

In the following, if  $X$  is a topological space, we will denote by  $\dim(X)$  (respectively,  $|X|$ ) the dimension (respectively, the cardinality) of  $X$ .

**Proposition 4.2.** *Let  $R$  be a ring and let  $\text{Rd}(R)$  be the space of all proper radical ideals of  $R$ , endowed with the hull-kernel topology. Then*

$$\dim(\text{Rd}(R)) = |\text{Spec}(R)| - 1 \geq \dim(\text{Spec}(R)).$$

*Moreover, if  $\text{Spec}(R)$  is linearly ordered, then  $\dim(\text{Rd}(R)) = \dim(\text{Spec}(R))$ .*

*Proof.* Let  $X$  be a nonempty finite subset of  $\text{Spec}(R)$ , with  $|X| = n$ . Let  $P_n$  be a minimal element of  $X$  and, by induction, let  $P_i$  be a minimal element of  $X \setminus \{P_n, \dots, P_{i+1}\}$ , for  $1 \leq i \leq n-1$ . Consider the radical ideals  $H_i := \bigcap_{\ell=1}^i P_\ell$ , for  $i = 1, 2, \dots, n$ . By construction, we have  $P_i \not\subseteq P_1, \dots, P_{i-1}$ , for  $i = 2, \dots, n$ , that is  $P_i \not\subseteq H_{i-1}$ . Thus, we get a strictly increasing chain of radical ideals of  $R$

$$H_n \subsetneq H_{n-1} \subsetneq \dots \subsetneq H_1 := P_1.$$

Since the order induced by the hull-kernel topology is the set-theoretic inclusion, this chain corresponds to a chain of length  $n-1$  of irreducible closed subspaces of  $\text{Rd}(R)$ . Thus, when  $\text{Spec}(R)$  is infinite, we can get, by applying the previous argument, chains of irreducible closed subsets of  $\text{Rd}(R)$  of arbitrary length. Thus, in this case, the equality  $\dim(\text{Rd}(R)) = |\text{Spec}(R)| - 1$  is proved. Assume now that  $\text{Spec}(R)$  is finite. By applying the first part of the proof to  $X := \text{Spec}(R)$  we deduce immediately that  $|\text{Spec}(R)| - 1 \leq \dim(\text{Rd}(R))$ . Conversely, a chain of length  $t$  of irreducible closed subspaces of  $\text{Rd}(R)$  corresponds to a chain of radical ideals

$$L_0 \subsetneq L_1 \subsetneq \dots \subsetneq L_t$$

and it provides the following chain of closed sets

$$\mathbf{V}(L_t) \subsetneq \mathbf{V}(L_{t-1}) \subsetneq \dots \subsetneq \mathbf{V}(L_0)$$

of  $\text{Spec}(R)$ . Since  $\text{Spec}(R)$  is finite,  $V(L_0)$  has at most  $|\text{Spec}(R)|$  elements. Since all inclusions are proper, it follows that  $t \leq |\text{Spec}(R)| - 1$ . The first part of the proof is now complete. The last statement follows immediately by noting that  $\text{Rd}(R) = \text{Spec}(R)$  if and only if  $\text{Spec}(R)$  is linearly ordered.  $\square$

Topologies on the family of the closed subsets of a topological space were introduced and intensively studied since the beginning of 20th century, with applications to uniform spaces, Functional Analysis, Game Theory, etc. [2, 14, 15, 18]. In this circle of ideas, one of the first contributions was made by L. Vietoris in [24]. We briefly recall his construction. Let  $X$  be any topological space and let, as before,  $\mathcal{X}'(X)$  denote the collection of all the nonempty closed subspaces of  $X$  (called also the *hyperspace of  $X$* ). For any open subspace  $U$  of  $X$  set

$$U^+ := \{C \in \mathcal{X}'(X) \mid C \subseteq U\} \quad U^- := \{C \in \mathcal{X}'(X) \mid C \cap U \neq \emptyset\}$$

The *upper Vietoris topology on  $\mathcal{X}'(X)$*  (respectively, *lower Vietoris topology*) is the topology on  $\mathcal{X}'(X)$  having as a basis (respectively, subbasis) of open sets the collection  $\mathcal{V}^+ := \{U^+ \mid U \text{ open in } X\}$  (respectively,  $\mathcal{V}^- := \{U^- \mid U \text{ open in } X\}$ ).

We now unveil a relation between the lower Vietoris topology and the Zariski topology  $\mathcal{X}'(X)$ , considered at the beginning of the present section.

**Proposition 4.3.** *Let  $X$  be a spectral space. Then, the inverse topology of the spectral space  $\mathcal{X}'(X)^{\text{zar}}$  and the lower Vietoris topology on  $\mathcal{X}'(X)$  are the same.*

*Proof.* Note first that, for any spectral space  $\mathcal{X}$ , if  $\mathcal{B}$  is a basis of quasi-compact open subspaces of  $\mathcal{X}$  (such a  $\mathcal{B}$  exists, by definition of a spectral space), then  $\mathcal{B}^{\text{inv}} := \{\mathcal{X} \setminus B \mid B \in \mathcal{B}\}$  is a basis of open sets for  $\mathcal{X}^{\text{inv}}$ .

Starting from the given spectral space  $X$ , with the notation introduced at the beginning of the present section, for any open and quasi-compact subspace  $\Omega$  of  $X$ , we observe that the set  $\mathcal{U}'(\Omega)$  is quasi-compact, as a subspace of  $\mathcal{X}'(X)^{\text{zar}}$ . Indeed, note that  $X \setminus \Omega \in \mathcal{U}'(\Omega)$  and that, if  $\mathcal{U}'(\Omega) \subseteq \bigcup_{i \in I} \mathcal{U}'(\Omega_i)$ , with  $\Omega_i \subseteq X$  open and quasi-compact, then  $X \setminus \Omega \in \mathcal{U}'(\Omega_i)$ , for some  $i$ , that is  $\Omega_i \subseteq \Omega$ . Thus, a fortiori,  $\mathcal{U}'(\Omega) \subseteq \mathcal{U}'(\Omega_i)$ . This shows that the basis

$$\mathcal{B} := \{\mathcal{U}'(\Omega) \mid \Omega \text{ quasi-compact open in } X\}$$

consists of quasi-compact open subspaces of  $\mathcal{X}'(X)^{\text{zar}}$ , and thus  $\mathcal{B}^{\text{inv}}$  is a basis of open sets for  $\mathcal{X}'(X)^{\text{inv}}$ . Since, by definition, the typical element in  $\mathcal{B}^{\text{inv}}$  is a set of closed subspaces hitting a fixed quasi-compact open subspace of  $X$ , it follows immediately that the inverse topology of  $\mathcal{X}'(X)^{\text{zar}}$  is coarser than (or equal to) the lower Vietoris topology.

Conversely, let  $U$  be any open set of  $X$  and take a point  $C \in U^- := \{F \in \mathcal{X}'(X) \mid F \cap U \neq \emptyset\}$ . If  $x \in C \cap U$ , there is a quasi-compact open subspace  $V$  of  $X$  such that  $V \subseteq U$  and  $x \in C \cap V$ , since the collection of all the quasi-compact open subspaces of a spectral space forms a basis. Thus,  $C \in V^- = \mathcal{X}'(X) \setminus \mathcal{U}'(V) \subseteq U^-$ . This shows that  $U^-$  is open, in the inverse topology of  $\mathcal{X}'(X)^{\text{zar}}$ . The proof is now complete.  $\square$

**Remark 4.4. (a)** The previous proposition shows that, given a spectral space  $X$ , the lower Vietoris topology on  $\mathcal{X}'(X)$  is always spectral. However, the same property can fail to hold for the upper Vietoris topology. To see this, let  $D$  be any integral domain with Jacobson radical  $\mathfrak{J} \neq (0)$ , let  $X := \text{Spec}(D)$ , let  $Y := V(\mathfrak{J}) \in \mathcal{X}'(X)$ , and let  $\Omega \subseteq \mathcal{X}'(X)$  be any open neighborhood of  $Y$ , with respect

to the upper Vietoris topology. Without loss of generality, we can assume that  $\Omega = D(I)^+$ , for some ideal  $I$  of  $D$ . Since each maximal ideal  $M$  of  $D$  belongs to  $Y$ , we have  $I \not\subseteq M$ , for each  $M \in \mathbf{Max}(D)$  and thus  $I = D$ , that is,  $\Omega = \mathcal{X}'(X)$ . This proves that the unique open neighborhood of  $Y$  is  $\mathcal{X}'(X)$  and trivially the same holds for the point  $X \in \mathcal{X}'(X)$ , with  $Y \neq X$  since  $\mathfrak{J} \neq (0)$ . This shows that  $\mathcal{X}'(X)$ , equipped with the upper Vietoris topology, does not satisfy the  $T_0$  axiom and, a fortiori, it is not spectral.

Note also that the previous example shows that the inverse topology of the spectral space  $\mathcal{X}'(X)$ , endowed with the lower Vietoris topology, is not the upper Vietoris topology on  $\mathcal{X}'(X)$ .

(b) Following the idea of intertwining algebra and topology, it is possible to give an alternate proof of Proposition 4.3 based on Theorem 4.1.

Let  $X = \mathbf{Spec}(R)$ , and let  $\mathcal{J}_0$  be the map  $\mathcal{J}$  defined in the statement of Theorem 4.1, but considered as a map from  $\mathcal{X}'(R)^{\text{lov}}$  (i.e., the space  $\mathcal{X}'(R)$  equipped with the lower Vietoris topology) to  $\mathbf{Rd}(R)^{\text{hk}}$  (i.e., the space  $\mathbf{Rd}(R)$  equipped with the hull-kernel topology). Obviously,  $\mathcal{J}_0$  is bijective.

A subbasis of the space  $\mathbf{Rd}(R)^{\text{hk}}$  is composed by the sets of the form  $\mathbf{D}(I) = \{H \in \mathbf{Rd}(R) \mid I \not\subseteq H\}$ , as  $I$  ranges among the ideals of  $R$ , while a subbasis of  $\mathcal{X}'(R)^{\text{lov}}$  is composed of the sets of the form  $\mathbf{D}(I)^- = \{F \in \mathcal{X}'(R) \mid F \cap \mathbf{D}(I) \neq \emptyset\}$ , since the open sets of  $\mathbf{Spec}(R)$  are of the form  $\mathbf{D}(I)$ . However,

$$\begin{aligned} \mathcal{J}_0^{-1}(\mathbf{D}(I)) &= \{F \in \mathcal{X}'(R)^{\text{lov}} \mid I \not\subseteq P \text{ for some prime ideal } P \in F\} = \\ &= \{F \mid F \not\subseteq \mathbf{V}(I)\} = \\ &= \{F \mid F \cap \mathbf{D}(I) \neq \emptyset\} = \mathbf{D}(I)^-, \end{aligned}$$

and thus  $\mathcal{J}_0$  is a homeomorphism.

We thus have a chain of maps

$$\mathcal{X}'(R)^{\text{lov}} \xrightarrow{\mathcal{J}_0} \mathbf{Rd}(R)^{\text{hk}} \xrightarrow{\text{id}} ((\mathbf{Rd}(R)^{\text{hk}})^{\text{inv}})^{\text{inv}} \xrightarrow{(\mathcal{J}^{-1})^{\text{inv}}} \mathcal{X}'(R)^{\text{inv}},$$

where  $\text{id}$  is the identity on the set  $\mathbf{Rd}(R)$  and  $(\mathcal{J}^{-1})^{\text{inv}}$  indicates the map  $\mathcal{J}^{-1}$  in the inverse topology. By Hochster's duality,  $\text{id}$  is a homeomorphism, while  $(\mathcal{J}^{-1})^{\text{inv}}$  and  $\mathcal{J}_0$  are homeomorphism, respectively, by Theorem 4.1 and the above reasoning. Since the composition  $(\mathcal{J}^{-1})^{\text{inv}} \circ \text{id} \circ \mathcal{J}_0$  is clearly the identity on the set  $\mathcal{X}'(R)$ , we conclude that the lower Vietoris topology and the inverse topology on  $\mathcal{X}'(R)$  are identical, as claimed.

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