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# Existence and uniqueness for a two-temperature energy-transport model for semiconductors ☆

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An existence and uniqueness result for a two-temperature energy-transport model is proved, in the one-dimensional steady-state case, considering a bounded domain and physically appropriate boundary conditions. The model arises in the description of heat effects in semiconductors, the two temperatures account for the electron and the lattice temperature.



## 1. Introduction

The effects of crystal heating have became crucial for the design of electronic devices with nanoscale dimensions, due to the possibility of having hot spots, that is, zones where the temperature of the lattice is very high, even close to the melting one. This has increased the interest on the analysis of thermal effects in semiconductors and prompted the formulation of improved models (see for example [4,14]). Recently more sophisticated energy-transport models, based on closure relations obtained by employing the maximum entropy principle, have been proposed, e.g. see [1,9–12,16], and used for simulating electron devices [2,15,17].

The main features of these models are to include an additional variable representing the lattice temperature and a relative equation for that. The scattering mechanisms force equilibrium between the electron and lattice temperature. In turn the latter tends to an equilibrium state with the environment. The simplest way to take into account such a physical effect is with a relaxation time approximation involving two relaxation times, one for the electron–phonon interaction and another for the phonon–environment interaction.

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From a mathematical point of view one has a standard energy-transport model augmented with a balance equation for the lattice temperature.

The first energy-transport model for semiconductors was introduced by Stratton [18], in 1962. The first mathematical results had to wait 35 years after the presentation of the model, and are due to Degond, Génieys and Jüngel [5,6] (for a comprehensive review see [7] along with [8] for the modeling aspect). They consider a general parabolic–elliptic system, arising in irreversible thermodynamics with thermal and electrical effects, which includes as a special case the energy-transport model. This general model is studied in a bounded multi-dimensional domain, with physics-based mixed Dirichlet–Neumann boundary conditions, under the restrictive hypothesis of uniform parabolicity and existence of a strictly positive energy.

A few years later, Chen, Hsiao and Li consider the same model, with unphysical no-flow boundary conditions, proving a stability theorem for small initial perturbations, without the last two restrictive assumptions [3].

An improvement on this result is due to Nishibata and Suzuki, who in a recent publication [13], for a one-dimensional energy-transport model, are able to prove existence and uniqueness, under physical boundary conditions, without assuming uniform parabolicity and existence of a strictly positive energy.

In this work we extend Nishibata and Suzuki's results to an augmented energy-transport model with two temperatures. We consider the (scaled) system:

$$\partial_t n + \nabla \cdot j = 0, \tag{1.1}$$

$$j = n\nabla\phi - \nabla(nT), \tag{1.2}$$

$$\partial_t \left(\frac{3}{2}nT\right) + \nabla \cdot \left(\frac{5}{2}Tj - \kappa_0 \nabla T\right) = j \cdot \nabla\phi - \frac{3}{2\tau}n(T - T_L), \tag{1.3}$$

$$\partial_t(\rho c_V T_L) + \nabla \cdot (-\kappa_L \nabla T_L) = \frac{3}{2\tau}n(T - T_L) - \frac{1}{\tau_L}(T_L - 1), \tag{1.4}$$

$$\nabla^2\phi = n - D, \tag{1.5}$$

with $x \in \Omega \subset \mathbb{R}^n$, $n$ electron density, $j$ current density, $T$ electron temperature, $T_L$ lattice temperature, $\phi$ electrostatic potential. $\kappa_0$ and $\kappa_L$ represent the thermal conductivity of electron and lattice heat flux, $\tau$ and $\tau_L$ are the relaxation times for electron and lattice energy. $D$ denotes the doping profile.

System (1.3) is the simplest energy-transport model with varying lattice temperature. For this system no analytical results are available in the literature. In this paper we present an existence and uniqueness result for the one-dimensional steady-state solutions of system (1.3) in a bounded domain with physically appropriate boundary conditions.

The plan of the paper is as follows. In section 2 the mathematical problem is presented and the main theorem is stated. In section 3 the proof of the existence of the solutions is given and the last section is devoted to the uniqueness result.

## 2. Mathematical model and main results

In this section we consider the steady-state one-dimensional case of system (1.3) with physics-based boundary conditions, and for the resulting problem we state an existence and uniqueness result.

The one-dimensional stationary case of system (1.3) reads

$$J_x = 0, \tag{2.1a}$$

$$J = n\phi_x - (nT)_x, \tag{2.1b}$$

$$\left(\frac{5}{2}TJ - \kappa_0 T_x\right)_x = J\phi_x - \frac{3}{2\tau}n(T - \theta), \tag{2.1c}$$

$$(-\kappa_L \theta_x)_x = \frac{3}{2\tau} n(T-\theta) - \frac{1}{\tau_L}(\theta - 1), \tag{2.1d}$$

$$\phi_{xx} = n - D, \tag{2.1e}$$

with $x \in \Omega \equiv (0,1)$. Here, the thermal conductivities $\kappa_0$ and $\kappa_L$ are assumed to be positive constants. Similarly the electron energy and the lattice thermal relaxation times, $\tau$ and $\tau_L$, are assumed to be small enough positive constants. Finally, we assume that the doping profile, $D(x)$ is an assigned bounded, continuous and positive function on $\bar{\Omega}$, such that

$$\inf_{x \in \bar{\Omega}} D(x) > 0. \tag{2.2}$$

For system (2.1), we consider standard Ohmic contacts at the boundaries. This translates into neutrality of charge and equilibrium conditions for the electron gas at the boundary, that is, the electron density equals the doping density, the electron temperature equals the lattice temperature, and the electric potential equals the applied voltage. We still need boundary conditions for the lattice temperature. We impose a dissipative condition at the contacts by Robin type relations which express the tendency of the lattice temperature to reach an equilibrium with the environment. Thus, we consider the boundary conditions

$$n(0) = n_l, \quad n(1) = n_r, \tag{2.3a}$$

$$T(0) = \theta(0), \quad T(1) = \theta(1), \tag{2.3b}$$

$$\kappa_L \theta_x(0) = \frac{1}{R}(\theta(0) - 1), \quad -\kappa_L \theta_x(1) = \frac{1}{R}(\theta(1) - 1), \tag{2.3c}$$

$$\phi(0) = 0, \quad \phi(1) = \phi_r. \tag{2.3d}$$

For the boundary conditions, we assume that the thermal resistivity $R$ is a positive constant along with

$$n_l > 0, \quad n_r > 0. \tag{2.4}$$

We introduce the parameter

$$\delta = |n_r - n_l| + |\phi_r|, \tag{2.5}$$

which we assume to be small.

The number of equations can be reduced by computing explicitly the electric field from equation (2.1b),

$$\phi_x = \frac{T}{n} n_x + T_x + \frac{J}{n}. \tag{2.6}$$

Using this expression in (2.1c) and (2.1e), we find two coupled parabolic equations for $n$ and $T$:

$$-\kappa_0 T_{xx} + \frac{3}{2} J T_x = \frac{JT}{n} n_x + \frac{J^2}{n} - \frac{3}{2\tau} n(T-\theta), \tag{2.7}$$

$$\left(\frac{T}{n} n_x\right)_x + T_{xx} - \frac{J}{n^2} n_x - n = -D. \tag{2.8}$$

Furthermore, we can integrate the relation (2.6) to get a voltage–current relation:

$$\phi_r = \left(\int_0^1 \frac{1}{n}\,dx\right) J + T(1) - T(0) + T(1)\log n_r - T(0)\log n_l - \int_0^1 T_x(\log n)\,dx. \tag{2.9}$$

This relation can be used to express the constant $J$ in terms of $n$, $T$ and the boundary conditions as

$$J = \left(\int_0^1 \frac{1}{n}\,dx\right)^{-1} \left(\mathcal{L}[T] + \int_0^1 T_x \log n\,dx\right), \tag{2.10}$$

with

$$\mathcal{L}[T] := \phi_r - T(1) + T(0) - T(1)\log n_r + T(0)\log n_l.$$

Finally, we can compute $\phi$ in terms of $n$ and of the doping profile by the explicit formula:

$$\phi = \phi_r x - \int_0^x dx''' \int_x^1 dx'' \int_{x'''}^{x''} (n(x') - D(x'))\,dx'. \tag{2.11}$$

In conclusion, system (2.1) can be replaced with

$$\left(\frac{T}{n} n_x\right)_x + T_{xx} - \frac{J}{n^2} n_x - n = -D, \tag{2.12a}$$

$$-\kappa_0 T_{xx} + \frac{3}{2} J T_x = \frac{JT}{n} n_x + \frac{J^2}{n} - \frac{3}{2\tau} n(T-\theta), \tag{2.12b}$$

$$-\kappa_L \theta_{xx} = \frac{3}{2\tau} n(T-\theta) - \frac{1}{\tau_L}(\theta - 1), \tag{2.12c}$$

furnished with boundary conditions (2.3a)–(2.3c), with $J$ given by (2.10) and $\phi$ given by (2.11).

Before stating the main result, we introduce some notations. For a nonnegative integer $l \geq 0$, $H^l$ denotes the usual Sobolev space in the $L^2$ sense, equipped with the norm $\|\cdot\|_l$, in particular, $\|\cdot\| := \|\cdot\|_0$. For a nonnegative integer $k$, $\mathcal{B}^k(\bar{\Omega})$ denotes the space of the functions whose derivatives up to $k$-th order are continuous and bounded over $\bar{\Omega}$, equipped with the norm

$$|f|_k := \sum_{i=0}^{k} \sup_{x\in\bar{\Omega}} |\partial_x^i f(x)|.$$

Throughout the rest of this paper, $C$ always denotes a generic positive constant. If the generic constant $C$ depends on some parameters $\alpha, \beta, \ldots$, we write $C[\alpha, \beta, \ldots]$.

The main result of this paper is the following existence and uniqueness theorem.

**Theorem 2.1** *(Existence and uniqueness). Let the doping profile and the boundary data satisfy conditions (2.2) and (2.4). For any $n_l > 0$, there exist positive constants $\delta_0$, $\tau_0$, $\sigma_0$, $\eta_0$, $\mu_0$ such that if $\delta \leq \delta_0$, $0 < \tau \leq \tau_0$, $0 < \tau_L \leq \sigma_0$, then the boundary value problem (2.1)–(2.3) has a unique solution $(n, J, T, \theta, \phi)$ in $\mathcal{B}^2(\bar{\Omega}) \times \mathcal{B}^2(\bar{\Omega}) \times \mathcal{B}^3(\bar{\Omega}) \times \mathcal{B}^3(\bar{\Omega}) \times \mathcal{B}^2(\bar{\Omega})$ satisfying $|T - \theta|_0 \leq \eta_0$, $|\theta - 1|_0 \leq \mu_0$.*

The proof of the above theorem is split in two parts, existence and uniqueness.

## 3. Proof of the existence result

The existence result is comprised in the following Lemma.

**Lemma 3.1.** *Let the doping profile and the boundary data satisfy conditions (2.2) and (2.4). For any $n_l > 0$, there exist positive constants $\delta_0$, $\tau_0$, $\sigma_0$, such that if $\delta \le \delta_0$, $0 < \tau \le \tau_0$, $0 < \tau_L \le \sigma_0$, then the boundary value problem (2.1)–(2.3) has a solution $(n, J, T, \theta, \phi)$ in $\mathcal{B}^2(\bar\Omega) \times \mathcal{B}^2(\bar\Omega) \times \mathcal{B}^3(\bar\Omega) \times \mathcal{B}^3(\bar\Omega) \times \mathcal{B}^2(\bar\Omega)$ satisfying the conditions*

$$\frac{1}{2} n_m \le n \le 2 n_M, \tag{3.1}$$

$$|T - \theta|_0 \le C\delta, \tag{3.2}$$

$$|\theta - 1|_0 \le C\delta, \tag{3.3}$$

*and*

$$\inf_{x\in\Omega} T(x) > 0, \tag{3.4}$$

*where*

$$n_m = \min\left\{ n_l, n_r, \inf_{x\in\Omega} D(x) \right\}, \quad n_M = \max\left\{ n_l, n_r, \sup_{x\in\Omega} D(x) \right\},$$

*and the constant $C$ is independent of $\delta_0$, $\tau_0$, $\sigma_0$ and $\delta$.*

**Proof.** The existence proof is based on the equivalent formulation (2.1) of (1.3). We consider $\tilde n$, $\tilde T$, $\tilde\theta$ in the space

$$W := \left\{ (f, g, h) \in H^2(\Omega) \;\middle|\; \begin{array}{l} \underline{n} \le f \le \overline{n}, \quad \|f_x\| \le \tilde C_0, \quad \|f_{xx}\| \le \tilde C_1, \\ \|g - h\|_1 \le \tilde C_2 \delta, \quad \|g_{xx}\| \le \tilde C_3, \\ \|h - 1\|_1 \le \tilde C_4 \delta, \quad \|h_{xx}\| \le \tilde C_5 \end{array} \right\},$$

where the constants $\underline{n}$, $\overline{n}$, $\tilde C_i$, $i = 0, 1, \dots, 5$, will be chosen later.

We introduce a map $\mathcal{F}$ defined in $W$, which maps $(\tilde n, \tilde T, \tilde\theta) \mapsto (n, T, \theta)$ as follows:

*(i)* $\theta$ is the solution to the linearized elliptic problem

$$-\kappa_L \theta_{xx} = \frac{3}{2\tau} \tilde n (\tilde T - \tilde\theta) - \frac{1}{\tau_L}(\theta - 1), \tag{3.5}$$

with Robin boundary conditions

$$\kappa_L \theta_x(0) = \frac{1}{R}(\theta(0) - 1), \quad -\kappa_L \theta_x(1) = \frac{1}{R}(\theta(1) - 1). \tag{3.6}$$

*(ii)* $T$ is the solution to the linearized elliptic problem

$$-\kappa_0 T_{xx} + \frac{3}{2} \tilde J T_x = \frac{\tilde J \tilde T}{\tilde n} \tilde n_x + \frac{\tilde J^2}{\tilde n} - \frac{3}{2\tau} \tilde n (T - \theta), \tag{3.7}$$

with Dirichlet boundary conditions

$$T(0) = \theta(0), \quad T(1) = \theta(1). \tag{3.8}$$

In (3.7), we have denoted by $\tilde{J}$ the expression of $J$ in (2.10) evaluated for $\tilde{n}$, $\tilde{T}$.

*(iii)* $n$ is the solution to the linearized problem

$$\left(\frac{T}{\tilde{n}} n_x\right)_x + T_{xx} - \frac{\tilde{J}}{\tilde{n}^2} n_x - n = -D, \tag{3.9}$$

with Dirichlet boundary conditions

$$n(0) = n_l, \quad n(1) = n_r. \tag{3.10}$$

*Well-posedness of the map $\mathcal{F}$.* The functions $\theta$ and $T$, in this order, are defined by two linear uniformly elliptic problems, (3.5)–(3.6) and (3.7)–(3.8), for which it is guaranteed existence and uniqueness of solutions. Thus, the map $\mathcal{F}$ is well-defined, provided the third problem for $n$, (3.9)–(3.10), is uniformly elliptic, that is,

$$\inf_{x \in \bar{\Omega}} T(x) > 0. \tag{3.11}$$

In order to prove this estimate, by virtue of Sobolev embedding theorems, we prove that we can choose $\delta$ small enough, so that $\|\theta - 1\|_1$ and $\|T - \theta\|_1$ are also small enough.

First, we estimate $\|\theta - 1\|_1$. We multiply equation (3.5) by $\theta - 1$ and integrate over $[0, 1]$. Integrating by part, and using the Robin boundary conditions, we obtain:

$$\begin{aligned} &\frac{1}{R}\left[(\theta(1) - 1)^2 + (\theta(0) - 1)^2\right] + \int_0^1 \kappa_L (\theta - 1)_x^2 \, dx \\ &+ \int_0^1 \frac{1}{\tau_L}(\theta - 1)^2 \, dx = \int_0^1 \frac{3\tilde{n}}{2\tau}(\tilde{T} - \tilde{\theta})(\theta - 1) \, dx \\ &\le \frac{3\overline{n}}{2\tau}\tilde{C}_2 \delta \|\theta - 1\| \le \delta^2 + \left(\frac{3\overline{n}}{4\tau}\tilde{C}_2\right)^2 \|\theta - 1\|^2. \end{aligned}$$

We take $\tau_L$ sufficiently small so that $\frac{1}{2\tau_L} \ge \left(\frac{3\overline{n}}{4\tau}\tilde{C}_2\right)^2$, that is,

$$\tau_L \le \left(\frac{8}{9\overline{n}^2 \tilde{C}_2^2}\right)\tau^2. \tag{3.12}$$

We find

$$\kappa_L \|(\theta - 1)_x\|^2 + \frac{1}{2\tau_L}\|\theta - 1\|^2 \le \delta^2,$$

which implies

$$\|\theta_x\| + \frac{1}{\sqrt{\tau_L}}\|\theta - 1\| \le C_4[\kappa_L]\delta, \tag{3.13}$$

with

$$C_4[\kappa_L] = \left(2 + \frac{1}{\kappa_L}\right)^{1/2}.$$

Here, we have used the inequality

$$\frac{c}{1+c}(a+b)^2 \le ca^2 + b^2, \quad \text{for all } a, b, c > 0.$$

Next, we estimate $\|T-\theta\|_1$. From (2.10) we find

$$|\tilde{J}| \le \overline{n}\left(\left|\mathcal{L}[\tilde{T}]\right| + C[\underline{n},\overline{n}]\|\tilde{T}_x\|\right) \le \overline{n}\left(\left|\mathcal{L}[\tilde{T}]\right| + C[\underline{n},\overline{n}](\tilde{C}_2 + \tilde{C}_4)\delta\right).$$

For the first term, using the boundary conditions (2.3b), we have

$$\begin{aligned}|\mathcal{L}[\tilde{T}]| &= |\phi_r - \theta(1) + \theta(0) - \theta(1)\log n_r + \theta(0)\log n_l| \\ &\le |\phi_r| + |\theta(1)-1||1+\log n_r| + |\theta(0)-1||1+\log n_l| + |\log n_r - \log n_l| \\ &\le |\phi_r| + C[\underline{n},\overline{n}]\left(|\theta-1|_0 + |n_r - n_l|\right) \le C[\underline{n},\overline{n}]\delta.\end{aligned}$$

It follows that

$$|\tilde{J}| \le C[\underline{n},\overline{n},\tilde{C}_2,\tilde{C}_4]\delta. \tag{3.14}$$

We multiply equation (3.7) by $T-\theta$ and integrate over $[0,1]$. Integrating by parts, and using the identity

$$\int_0^1 \frac{3}{2}\tilde{J}(T-\theta)_x(T-\theta)\,dx = 0,$$

we find

$$\begin{aligned}&\int_0^1 \kappa_0(T-\theta)_x^2\,dx + \int_0^1 \frac{3\tilde{n}}{2\tau}(T-\theta)^2\,dx \\ &\quad = -\int_0^1 \kappa_0\theta_x(T-\theta)_x\,dx - \int_0^1 \left(\frac{3}{2}\tilde{J}\theta_x - \frac{\tilde{J}\tilde{T}}{\tilde{n}}\tilde{n}_x - \frac{\tilde{J}^2}{\tilde{n}}\right)(T-\theta)\,dx \\ &\quad \le \frac{1}{2}\int_0^1 \kappa_0(T-\theta)_x^2\,dx + \frac{\kappa_0}{2}C_4[\kappa_L]^2\delta^2 + \left\|\frac{3}{2}\tilde{J}\theta_x - \frac{\tilde{J}\tilde{T}}{\tilde{n}}\tilde{n}_x - \frac{\tilde{J}^2}{\tilde{n}}\right\|\|T-\theta\| \\ &\quad \le \frac{1}{2}\int_0^1 \kappa_0(T-\theta)_x^2\,dx + \left(1 + \frac{\kappa_0}{2}C_4[\kappa_L]^2\right)\delta^2 + C[\underline{n},\overline{n},\tilde{C}_0,\tilde{C}_2,\tilde{C}_4]\|T-\theta\|^2,\end{aligned}$$

$$\left\|\frac{3}{2}\tilde{J}\theta_x - \frac{\tilde{J}\tilde{T}}{\tilde{n}}\tilde{n}_x - \frac{\tilde{J}^2}{\tilde{n}}\right\| \le C[\underline{n},\overline{n},\tilde{C}_2,\tilde{C}_4]\delta\left(\frac{3}{2}C_4\delta + \frac{1}{\underline{n}}\tilde{C}_0 + \frac{1}{\underline{n}}C[\underline{n},\overline{n},\tilde{C}_2,\tilde{C}_4]\delta\right)$$

Then we get

$$\begin{aligned}&\frac{\kappa_0}{2}\|(T-\theta)_x\|^2 + \frac{3\underline{n}}{2\tau}\|T-\theta\|^2 \\ &\qquad \le \left(1 + \frac{\kappa_0}{2}C_4[\kappa_L]^2\right)\delta^2 + C[\underline{n},\overline{n},\tilde{C}_0,\tilde{C}_2,\tilde{C}_4]\|T-\theta\|^2.\end{aligned}$$

We choose $\tau$ small enough, so that $\frac{3\underline{n}}{4\tau} \geq C[\underline{n}, \overline{n}, \tilde{C}_0, \tilde{C}_2, \tilde{C}_4]$, that is,

$$\tau \leq \frac{3\underline{n}}{4C[\underline{n}, \overline{n}, \tilde{C}_0, \tilde{C}_2, \tilde{C}_4]}. \tag{3.15}$$

Then, we find

$$\frac{\kappa_0}{2}\|(T-\theta)_x\|^2 + \frac{3\underline{n}}{4\tau}\|T-\theta\|^2 \leq \left(1 + \frac{\kappa_0}{2} C_4[\kappa_L]^2\right)\delta^2,$$

which yields the estimate

$$\|(T-\theta)_x\| + \frac{1}{\sqrt{\tau}}\|T-\theta\| \leq C_2[\underline{n}, \kappa_0, \kappa_L]\delta, \tag{3.16}$$

with

$$C_2[\underline{n}, \kappa_0, \kappa_L] = \left(\frac{3\underline{n} + 2\kappa_0}{3\underline{n}\kappa_0}(2 + \kappa_0 C_4[\kappa_L]^2)\right)^{1/2}.$$

Using (3.13) and (3.16), together with the Sobolev embedding inequalities, we find

$$|T - 1|_0 \leq C(\|T-\theta\|_1 + \|\theta - 1\|_1) \leq C\delta,$$

which implies $T \geq 1 - C\delta$. By choosing $\delta$ small enough we find (3.11), which implies the uniform ellipticity of (3.9). It follows that the solution $(n, T, \Theta)$ is $\mathcal{B}^0(\bar{\Omega})$, and, considering the regularity of the coefficient in (3.5), (3.7) and (3.9), the map $\mathcal{F}$ is well-posed from $W$ to $\mathcal{B}^2(\bar{\Omega}) \times \mathcal{B}^3(\bar{\Omega}) \times \mathcal{B}^3(\bar{\Omega})$.

*Higher order estimates for problem (i) and (ii).* First we estimate $\theta_{xx}$. Multiplying equation (3.5) by $-\theta_{xx}$, integrating over $\Omega$, and using Robin boundary conditions (3.6), we get

$$\int_0^1 \kappa_L \theta_{xx}^2 \, dx + \frac{1}{\tau_L}\int_0^1 \theta_x^2 \, dx \leq \frac{3}{2\tau}\int_0^1 [\tilde{n}(\tilde{T} - \tilde{\theta})]_x \theta_x \, dx$$
$$\leq \frac{3}{2\tau}(\overline{n} + \tilde{C}_0)\tilde{C}_2\delta\|\theta_x\| \leq \left(1 + \frac{\tilde{C}_0}{\overline{n}}\right)^2 \delta^2 + \left(\frac{3\overline{n}}{4\tau}\tilde{C}_2\right)^2 \|\theta_x\|^2.$$

Then, choosing $\tau_L$ small enough, as in (3.12), we find

$$\|\theta_{xx}\| + \frac{1}{\sqrt{\tau_L}}\|\theta_x\| \leq C_5[\overline{n}, \tilde{C}_0]\delta, \tag{3.17}$$

with

$$C_5[\overline{n}, \tilde{C}_0] = \left(1 + \frac{\tilde{C}_0}{\overline{n}}\right) C_4[\kappa_L].$$

Next, we multiply equation (3.7) by $-\dfrac{T_{xx}}{\tilde{n}}$ and integrate the resulting equation over $\Omega$:

$$\int_0^1 \frac{\kappa_0}{\tilde{n}} T_{xx}^2 \, dx + \frac{3}{2\tau}\int_0^1 T_x^2 \, dx$$

$$
\begin{aligned}
&= \frac{3}{2\tau}\int_0^1 \theta_x T_x\,dx + \int_0^1 \left\{ \frac{3}{2}\tilde{J}T_x - \frac{\tilde{J}\tilde{T}}{\tilde{n}}\tilde{n}_x - \frac{\tilde{J}^2}{\tilde{n}} \right\} \frac{T_{xx}}{\tilde{n}}\,dx \\
&\le \frac{3}{4\tau}\int_0^1 T_x^2\,dx + \frac{3}{4\tau}\|\theta_x\|^2 + \frac{3}{2\underline{n}} C[\underline{n},\overline{n},\tilde{C}_2,\tilde{C}_4]\delta\|T_x\|_1^2 \\
&\quad + \mu\|T_{xx}\|^2 + C[\mu,\underline{n},\overline{n},\tilde{C}_0,\tilde{C}_2,\tilde{C}_4]\delta^2.
\end{aligned}
$$

Using the estimate (3.17) and the smallness assumptions (3.12) and (3.15), we find

$$
\frac{3}{4\tau}\|\theta_x\|^2 \le \frac{3\tau_L}{4\tau} C_5[\overline{n},\tilde{C}_0,\tilde{C}_2]^2\delta^2 \le \left(\frac{\underline{n}}{2\overline{n}^2\tilde{C}_2^2}\right) \frac{C_5[\overline{n},\tilde{C}_0,\tilde{C}_2]^2}{C[\underline{n},\overline{n},\tilde{C}_0,\tilde{C}_2,\tilde{C}_4]}\delta^2.
$$

Thus

$$
\begin{aligned}
&\int_0^1 \frac{\kappa_0}{\tilde{n}} T_{xx}^2\,dx + \frac{3}{4\tau}\int_0^1 T_x^2\,dx \\
&\quad \le \left(\mu + C[\underline{n},\overline{n},\tilde{C}_2,\tilde{C}_4]\delta\right)\|T_x\|_1^2 + C[\mu,\underline{n},\overline{n},\tilde{C}_0,\tilde{C}_2,\tilde{C}_4]\delta^2,
\end{aligned}
$$

wherefrom, choosing $\mu$ and $\delta$ small enough,

$$
\|T_{xx}\| + \frac{1}{\sqrt{\tau}}\|T_x\| \le C_3[\underline{n},\overline{n},\tilde{C}_0,\tilde{C}_2,\tilde{C}_4]\delta. \tag{3.18}
$$

Finally, we need to estimate $\|T_{xxx}\|$, in order to have a bound on $|T_{xx}|_0$. We divide (3.7) by $\tilde{n}$, take the $x$-derivative, and multiply the resulting equation by $-T_{xxx}$. Integrating over $\Omega$ we get

$$
\begin{aligned}
\int_0^1 \frac{\kappa_0}{\tilde{n}} T_{xxx}^2\,dx &= \frac{3}{2\tau}\int_0^1 (T_x - \theta_x)T_{xxx}\,dx \\
&+ \int_0^1 \left\{ \frac{\kappa_0\tilde{n}_x}{\tilde{n}^2}T_{xx} + \left(\frac{3}{2}\frac{\tilde{J}}{\tilde{n}}T_x - \frac{\tilde{J}\tilde{T}}{\tilde{n}^2}\tilde{n}_x - \frac{\tilde{J}^2}{\tilde{n}^2}\right)_x \right\} T_{xxx}.
\end{aligned}
$$

Using estimate (3.16), we find

$$
\begin{aligned}
\frac{3}{2\tau}\int_0^1 (T_x - \theta_x)T_{xxx}\,dx &\le \frac{3}{2\tau}\|T_x - \theta_x\|\|T_{xxx}\| \\
&\le \frac{\kappa_0}{\overline{n}}\frac{3\overline{n}C_2}{2\kappa_0\tau}\delta\|T_{xxx}\| \le \frac{9\overline{n}C_2^2}{4\kappa_0\tau^2}\delta^2 + \frac{\kappa_0}{4\overline{n}}\|T_{xxx}\|^2.
\end{aligned}
$$

Also, we find

$$
\begin{aligned}
&\int_0^1 \left\{ \frac{\kappa_0\tilde{n}_x}{\tilde{n}^2}T_{xx} + \left(\frac{3}{2}\frac{\tilde{J}}{\tilde{n}}T_x - \frac{\tilde{J}\tilde{T}}{\tilde{n}^2}\tilde{n}_x - \frac{\tilde{J}^2}{\tilde{n}^2}\right)_x \right\} T_{xxx} \\
&\quad \le \frac{\kappa_0}{4\overline{n}}\|T_{xxx}\|^2 + C[\underline{n},\overline{n},\tilde{C}_0,\tilde{C}_1,\tilde{C}_2,\tilde{C}_4]\delta^2.
\end{aligned}
$$

In conclusion, we have

$$\|T_{xxx}\| \le C[\underline{n}, \overline{n}, \tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \tilde{C}_4]\delta. \tag{3.19}$$

*Estimates for problem (iii).* Let us introduce the boundary extension function

$$n_B(x) = n_l\,(1-x) + n_r\,x. \tag{3.20}$$

Multiplying (3.9) by $n_B - n$ and integrating by part, one obtains

$$\begin{aligned}
&\int_0^1 \frac{T}{\tilde{n}}(n-n_B)_x^2\,dx + \int_0^1 (n-n_B)^2\,dx \\
&= -\int_0^1 \frac{\tilde{J}}{\tilde{n}^2}(n-n_B)_x(n-n_B)\,dx + \int_0^1 \left(\frac{\tilde{J}}{\tilde{n}^2}(n_B)_x + n_B - D\right)(n-n_B)\,dx \\
&- \int_0^1 \left(\frac{T}{\tilde{n}}(n_B)_x + T_x\right)(n-n_B)_x\,dx \\
&\le \left(\varepsilon + C[\underline{n}, \overline{n}, \tilde{C}_2, \tilde{C}_4]\delta\right)\|n-n_B\|_1^2 + C[\varepsilon, \overline{D}, \underline{n}, \overline{n}, C_2, C_4],
\end{aligned}$$

where $\varepsilon$ is a positive constant to be chosen and $\overline{D} = \sup_\Omega D$. Above, we have used $|(n_B)_x| = |n_r - n_l| \le \delta$. By choosing $\varepsilon$ and $\delta$ small enough, we get

$$\|n-n_B\|_1 \le C[\underline{n}, \overline{n}, \tilde{C}_2, \tilde{C}_4, C_2, C_4], \tag{3.21}$$

which leads to

$$\|n_x\| \le C_0[\underline{n}, \overline{n}, \tilde{C}_2, \tilde{C}_4, C_2, C_4]. \tag{3.22}$$

Starting again from (3.9), and solving for $n_{xx}$, we get

$$n_{xx} = -\frac{\tilde{n}}{T}\left(\left(\frac{T}{\tilde{n}}\right)_x n_x + T_{xx} + \frac{\tilde{J}}{\tilde{n}^2}n_x - n + D\right),$$

wherefrom

$$\|n_{xx}\| \le C_1[\underline{n}, \overline{n}, \tilde{C}_0, \tilde{C}_2, \tilde{C}_4, C_0, C_2, C_4]. \tag{3.23}$$

Finally, we need to find bounds for $n$. Applying the pointwise maximum principle to the problem *(iii)*, which is uniformly elliptic because of (3.11), we find the interior bounds

$$\inf_{x\in\Omega}\left(D(x) + T_{xx}(x)\right) \le n \le \sup_{x\in\Omega}\left(D(x) + T_{xx}(x)\right). \tag{3.24}$$

By Sobolev inequality we can estimate $T_{xx}$ by means of (3.18), (3.19). By choosing $\delta$ small enough, we find $|T_{xx}(x)| \le \frac{1}{2}\inf_{x\in\Omega} D(x)$, so that

$$\frac{1}{2}\inf_{x\in\Omega} D(x) \le n \le \sup_{x\in\Omega} D(x) + \frac{1}{2}\inf_{x\in\Omega} D(x) \le 2\sup_{x\in\Omega} D(x). \tag{3.25}$$

Taking into account the boundary conditions we find the bound (3.1).

*Existence of a fixed point and solution of the boundary-value problem.* First we show that the constants $\underline{n}$, $\overline{n}$, $\tilde{C}_i$, $i = 0, 1, \ldots, 5$, can be chosen so that $\mathcal{F}$ maps $W$ in $W$. We set $\underline{n} = \frac{1}{2}n_m$, $\overline{n} = 2n_M$, and we choose $\tilde{C}_2 = C_2[\underline{n}, \kappa_0, \kappa_L]$, $\tilde{C}_4 = C_4[\kappa_L]$. Then the other constants are consequently determined in the following order: $\tilde{C}_0 = C_0$, $\tilde{C}_1 = C_1$, $\tilde{C}_3 = C_3$, $\tilde{C}_5 = C_5$. Once the constant $C_i$, $i = 0, 1, \ldots, 5$, have been chosen, the upper bounds $\tau_0$, $\sigma_0$ for the relaxation times $\tau$, $\sigma$ are given by (3.15) and (3.12).

Next, we show that the map $\mathcal{F}$ has a fixed point. The set $W$ is a compact convex subset of $\left(\mathcal{B}^1(\bar{\Omega})\right)^3$. Moreover, the map $\mathcal{F}$ is continuous in the $\mathcal{B}^1(\bar{\Omega})$-norm. Then the Schauder's fixed-point theorem implies the existence of a fixed point.

Let $(n, T, \theta)$ be a fixed point for the map $\mathcal{F}$. Then we can define the constant flux $J$ by (2.10), and the electric $\phi$ by (2.11). In this way we obtain a solution to the boundary value problem (2.1)–(2.3), satisfying the required estimates, and the Lemma is proved. □

## 4. Proof of the uniqueness result

In order to prove the uniqueness of the solution ensured by the previous section, we need some preliminary a priori estimates.

**Lemma 4.1.** *Let $(n, J, T, \theta, \phi)$ be a solution in $\mathcal{B}^2(\bar{\Omega}) \times \mathcal{B}^2(\bar{\Omega}) \times \mathcal{B}^3(\bar{\Omega}) \times \mathcal{B}^3(\bar{\Omega}) \times \mathcal{B}^2(\bar{\Omega})$ to the problem (2.1)–(2.3) satisfying the conditions*

$$\underline{n} \leq n \leq \overline{n}, \quad T > 0.$$

*Then, for arbitrary constant $n_l > 0$, there exist positive constants $\delta_0$, $\tau_0$, $\sigma_0$, $\eta_0$, $\mu_0$ such that if $\delta \leq \delta_0$, $0 < \tau \leq \tau_0$, $0 < \tau_L \leq \sigma_0$, $|T - \theta|_0 \leq \eta_0$, $|\theta - 1|_0 \leq \mu_0$, then the solution satisfies the formula (2.10) and the estimates:*

$$|J| \leq C\delta, \tag{4.1a}$$

$$T \geq c > 0, \tag{4.1b}$$

$$|(n, \phi)|_2 \leq C, \tag{4.1c}$$

$$\frac{1}{\tau}\|T - \theta\|_1 + \|T_{xx}\| + \|T_{xxx}\| \leq C\delta, \tag{4.1d}$$

$$\frac{1}{\tau_L}\|\theta - 1\| + \frac{1}{\sqrt{\tau_L}}\|\theta_x\| + \|\theta_{xx}\| \leq C\delta, \tag{4.1e}$$

*where $C$ and $c$ are positive constants independent of $\tau$, $\tau_L$ and $\delta$.*

**Proof.** Integrating equation (2.1b) divided by $n$, we see that $J$ must satisfy (2.9), thus (2.10). Using (2.10), and the bounds on $n$, we find

$$|J| \leq \overline{n}\left(|\mathcal{L}[T]| + C[\underline{n}, \overline{n}]\|T_x\|\right).$$

Moreover, taking into account the boundness of $|T - \theta|$ and $|\theta - 1|$, we have

$$\begin{aligned}|\mathcal{L}[T]| &\leq |\phi_r| + T(1)|\log n_r - \log n_l| + |1 + \log n_l| \left| \int_0^1 T_x\, dx \right| \\ &\leq C(\delta + \|T_x\|),\end{aligned}$$

which, combined with the previous estimate, yields

$$|J| \leq C[\underline{n}, \overline{n}]\left(\delta + \|T_x\|\right). \tag{4.2}$$

To prove (4.1a) we need to estimate $\|T_x\|$. To do so, we estimate $\|(T-\theta)_x\|$ and $\|\theta_x\|$. Multiplying (2.1d) by $\theta - 1$ and integrating over $\Omega$, after integration by part, we find

$$\frac{1}{R}\left[(\theta(1)-1)^2 + (\theta(0)-1)^2\right] + \kappa_L \int_0^1 \theta_x^2\, dx + \frac{1}{\tau_L}\int_0^1 (\theta-1)^2\, dx$$
$$= \int_0^1 \frac{3}{2\tau} n(T-\theta)(\theta-1)\, dx \leq \|T-\theta\|^2 + \left(\frac{3\overline{n}}{\tau}\right)^2 \|\theta - 1\|^2.$$

Choosing $\tau_L$ sufficiently small, that is,

$$\frac{1}{2\tau_L} \geq \left(\frac{3\overline{n}}{\tau}\right)^2, \tag{4.3}$$

we find the estimate

$$\kappa_L\|\theta_x\|^2 + \frac{1}{2\tau_L}\|\theta-1\|^2 \leq \|T-\theta\|^2,$$

which implies

$$\|\theta_x\| + \frac{1}{\sqrt{\tau_L}}\|\theta - 1\| \leq C_4[\kappa_L]\|T-\theta\|. \tag{4.4}$$

Next, multiplying (2.12b) by $T-\theta$ and integrating over $\Omega$, after integration by part we obtain

$$\kappa_0 \int_0^1 (T-\theta)_x^2\, dx + \int_0^1 \frac{3}{2\tau} n (T-\theta)^2\, dx = -\kappa_0 \int_0^1 \theta_x (T-\theta)_x\, dx$$
$$+ \int_0^1 J(\log n)_x (T-\theta)\, dx + \int_0^1 J(\log n)_x (T-1)(T-\theta)\, dx$$
$$- \int_0^1 \frac{3}{2} J T_x (T-\theta)\, dx + \int_0^1 \frac{J^2}{n}(T-\theta)\, dx. \tag{4.5}$$

We estimate separately the five terms on the right-hand side. For the first term, using (4.4), we find

$$-\kappa_0 \int_0^1 \theta_x (T-\theta)_x\, dx \leq \kappa_0 C_4[\kappa_L]\|T-\theta\|\|(T-\theta)_x\|$$
$$\leq \mu\|(T-\theta)_x\|^2 + C[\mu, \kappa_0, \kappa_L]\|T-\theta\|^2, \tag{4.6}$$

where $\mu$ is a positive constant to be chosen later.

Integrating by part the second term, and using (2.10), we find

$$
\begin{aligned}
&\int_0^1 J(\log n)_x (T-\theta)\,dx \\
&= -\left(\int_0^1 \frac{1}{n}\,dx\right)^{-1} \left(\mathcal{L}[T] + \int_0^1 T_x \log n\,dx\right) \int_0^1 (T_x-\theta_x)\log n\,dx \\
&= -\left(\int_0^1 \frac{1}{n}\,dx\right)^{-1} \left(\mathcal{L}[T] + \int_0^1 \theta_x \log n\,dx\right) \int_0^1 (T_x-\theta_x)\log n\,dx \\
&\quad - \left(\int_0^1 \frac{1}{n}\,dx\right)^{-1} \left(\int_0^1 (T_x-\theta_x)\log n\,dx\right)^2 \\
&\le C(|\mathcal{L}[T]| + \|\theta_x\|)\|(T-\theta)_x\|.
\end{aligned}
$$

We can estimate $|\mathcal{L}[T]$ differently from what done above:

$$
\begin{aligned}
|\mathcal{L}[T]| &= |\phi_r - (1+\log n_r)T(1) + (1+\log n_l)T(0)| \\
&= |\phi_r - (1+\log n_r)\theta(1) + (1+\log n_l)\theta(0)| \\
&\le |\phi_r| + |\log n_r - \log n_l| \\
&\quad + |1+\log n_r|\,|\theta(1)-1| + |1+\log n_l|\,|\theta(0)-1| \\
&\le C[\underline{n},\overline{n}](\delta + |\theta-1|_0).
\end{aligned}
$$

Then, using Sobolev inequality, estimate (4.4) and the weighted Young inequality, we get

$$
\begin{aligned}
\int_0^1 J(\log n)_x(T-\theta)\,dx &\le C\big(\delta + \|T-\theta\|\big)\|(T-\theta)_x\| \\
&\le \mu\|(T-\theta)_x\|^2 + C[\mu,\underline{n},\overline{n}]\big(\delta^2 + \|T-\theta\|^2\big),
\end{aligned} \tag{4.7}
$$

where $\mu$ is an arbitrary positive constant to be chosen later.

For the third term, integrating by part and using (4.2) and (4.4), we find

$$
\begin{aligned}
J\int_0^1 (\log n)_x (T-1)(T-\theta)\,dx &= -J\int_0^1 \log n\Big\{(T-1)(T-\theta)\Big\}_x dx \\
&\le C(\delta + \|T_x\|)\Big\{\|T-1\|\|(T-\theta)_x\| + \|T-\theta\|\|T_x\|\Big\} \\
&\le C(\delta + \|T-\theta\| + \|\theta-1\|)\|T-\theta\|_1^2.
\end{aligned} \tag{4.8}
$$

The fourth term can be estimated similarly to the previous term, obtaining

$$
\begin{aligned}
-\int_0^1 \frac{3}{2} J T_x (T-\theta)\,dx &\le C(\delta + \|T_x\|)\|T_x\|\|T-\theta\| \\
&\le C(\delta + \|T-\theta\|)\|T-\theta\|_1^2.
\end{aligned} \tag{4.9}
$$

Finally, for the last term, one obtains

$$\int_0^1 \frac{J^2}{n}(T-\theta)\,dx \le C(\delta+\|T_x\|)^2\|T-\theta\| \le C\|T-\theta\|(\delta^2+\|T-\theta\|_1^2). \tag{4.10}$$

Using the estimates (4.6)–(4.10) in (4.5), we find

$$\kappa_0\int_0^1 (T-\theta)_x^2\,dx + \int_0^1 \frac{3}{2\tau}n(T-\theta)^2\,dx \le C(\mu+\delta+\|T-\theta\|+\|\theta-1\|)\|(T-\theta)_x\|^2 + C(1+\delta+\|T-\theta\|+\|\theta-1\|)\|T-\theta\|^2 + C(1+\|T-\theta\|)\delta^2.$$

Since the quantities $\|T-\theta\|$, $\|\theta-1\|$ are controlled by $|T-\theta|_0 \le \eta_0$, $|\theta-1|_0 \le \mu_0$, respectively, after choosing $\delta$, $\eta_0$, $\mu_0$ and $\mu$ small enough we can control the term in $\|(T-\theta)_x\|^2$. Then, choosing $\tau$ small enough, we can control the term in $\|T-\theta\|^2$, and we get the estimate

$$\|(T-\theta)_x\| + \frac{1}{\sqrt{\tau}}\|T-\theta\| \le C\delta. \tag{4.11}$$

Combining with (4.4), one has

$$\|\theta_x\| + \frac{1}{\sqrt{\tau_L}}\|\theta-1\| \le C\delta. \tag{4.12}$$

Using (4.11) and (4.12) in (4.2), we get (4.1a). The same estimates ensure that $|T-1| \le |T-\theta| + |\theta-1|$ is small enough for $\delta$ small enough, thus implying (4.1b).

The part of estimate (4.1c) involving $\phi$, follows immediately from (2.11). The estimates for $n$, $T$ and $\theta$ are derived as follows. Similarly to the derivation of (3.22) we find $\|n_x\| \le C$ by using (4.11) and (4.12). Estimates for $\theta_{xx}$, $T_{xx}$, $n_{xx}$ can be obtained as done for (3.17), (3.18) and (3.23), leading to

$$\|\theta_{xx}\| + \frac{1}{\sqrt{\tau_L}}\|\theta_x\| \le C\delta, \quad \|T_{xx}\| + \frac{1}{\sqrt{\tau}}\|T_x\| \le C\delta, \quad \|n_{xx}\| \le C.$$

In a similar way, proceeding as in (3.19), we find $\|T_{xxx}\| \le C\delta$. The $\mathcal{B}^1$-bounds for $n$ in (4.1c) can be derived by the previous estimates for $\|n\|_2$, and we find $|n_{xx}|_0 \le C$ by solving equation (3.9) for $n_{xx}$. Finally, solving equation (3.7) for $(T-\theta)/\tau$ and equation (3.5) for $(\theta-1)/\tau_L$, we find $\|T-\theta\|_1/\tau \le C\delta$ and $\|\theta-1\|/\tau_L \le C\delta$, and the Lemma is proved. □

Now we are in the condition to prove the uniqueness part of the main theorem.

**Lemma 4.2.** *Under the same assumptions of the previous Lemma 4.1, the solution $(n, J, T, \theta, \phi)$ in $\mathcal{B}^2(\bar\Omega) \times \mathcal{B}^2(\bar\Omega) \times \mathcal{B}^3(\bar\Omega) \times \mathcal{B}^3(\bar\Omega) \times \mathcal{B}^2(\bar\Omega)$ to the problem (2.1)–(2.3) is unique.*

**Proof.** Let $(n_1, J_1, T_1, \theta_1, \phi_1)$ and $(n_2, J_2, T_2, \theta_2, \phi_2)$ be two solutions to the problem (2.1)–(2.3). We define

$$p := \theta_1 - \theta_2, \quad q := T_1 - T_2, \quad r := \log n_1 - \log n_2.$$

Using the results of Lemma 4.1, the mean value theorem and Poincaré inequality, we find

$$
\begin{aligned}
|J_1 - J_2| &\le C \left| \mathcal{L}[T_1] + \int_0^1 T_{1x} \log n_1 \, dx \right| \int_0^1 \left| \frac{1}{n_2} - \frac{1}{n_1} \right| dx \\
&\quad + C \left( |\mathcal{L}[T_1] - \mathcal{L}[T_2]| + \int_0^1 |T_{1x} \log n_1 - T_{2x} \log n_2| \, dx \right) \\
&\le C(\delta \|r_x\| + \|q\|_1).
\end{aligned} \tag{4.13}
$$

We can obtain an equation for $p$:

$$
(-\kappa_L p_x)_x + \frac{1}{\tau_L} p = \frac{3}{2\tau} n_1 (T_1 - \theta_1) - \frac{3}{2\tau} n_2 (T_2 - \theta_2). \tag{4.14}
$$

Multiplying by $p$ and integrating over the domain $\Omega$, using Lemma 4.1, we obtain

$$
\begin{aligned}
&\int_0^1 \kappa_L p_x^2 \, dx + \frac{1}{R}(p(1)^2 + p(0)^2) + \frac{1}{\tau_L} \int_0^1 p^2 \, dx \\
&\quad = \int_0^1 \frac{3}{2\tau} \left( n_1(q-p) + (n_1 - n_2)(T_2 - \theta_2) \right) p \, dx \\
&\quad \le C \frac{1}{\tau} (\|q - p\| + \delta \|r_x\|) \|p\|.
\end{aligned}
$$

In conclusion, choosing $\tau_L$ small enough, we find

$$
\|p\|_1 \le C(\|q - p\| + \delta \|r_x\|). \tag{4.15}
$$

For the difference $q$ we get:

$$
\begin{aligned}
&-\kappa_0 q_{xx} + \frac{3}{2} J_1 T_{1x} - \frac{3}{2} J_2 T_{2x} - J_1 T_1 (\log n_1)_x + J_2 T_2 (\log n_2)_x \\
&\quad = \frac{J_1^2}{n_1} - \frac{J_2^2}{n_2} - \frac{3}{2\tau} \left( n_1(q-p) + (n_1 - n_2)(T_2 - \theta_2) \right).
\end{aligned} \tag{4.16}
$$

Multiplying by $q - p$ and integrating over $\Omega$ we find

$$
\begin{aligned}
&\int_0^1 \kappa_0 (q_x - p_x)^2 \, dx + \int_0^1 \frac{3}{2\tau} n_1 (q-p)^2 \, dx = -\int_0^1 \kappa_0 p_x (q_x - p_x) \, dx \\
&\quad + \int_0^1 \left\{ J_1 T_1 (\log n_1)_x - J_2 T_2 (\log n_2)_x - \frac{3}{2} J_1 T_{1x} + \frac{3}{2} J_2 T_{2x} \right\} (q-p) \, dx \\
&\quad + \int_0^1 \left\{ \frac{J_1^2}{n_1} - \frac{J_2^2}{n_2} - \frac{3}{2\tau} (n_1 - n_2)(T_2 - \theta_2) \right\} (q-p) \, dx.
\end{aligned}
$$

Using Lemma 4.1, and the inequalities (4.13), (4.15), we estimate:

$$
\begin{aligned}
-\int_0^1 \kappa_0 p_x (q_x - p_x)\, dx &\le \frac{1}{2}\kappa_0 \|q_x - p_x\|^2 + \frac{1}{2}\kappa_0 \|p_x\|^2 \\
&\le \frac{1}{2}\kappa_0 \|q_x - p_x\|^2 + C(\|q-p\|^2 + \delta^2 \|r_x\|^2),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \left\{ J_1 T_1 (\log n_1)_x - J_2 T_2 (\log n_2)_x - \frac{3}{2} J_1 T_{1x} + \frac{3}{2} J_2 T_{2x} \right\} (q-p) \\
&\qquad \le C(\|q-p\| + \delta \|r_x\|)\|q-p\| \le C(\|q-p\|^2 + \delta^2 \|r_x\|^2),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \left\{ \frac{J_1^2}{n_1} - \frac{J_2^2}{n_2} - \frac{3}{2\tau}(n_1 - n_2)(T_2 - \theta_2) \right\} (q-p)\, dx \\
&\qquad \le C\delta(\|q-p\| + \|r_x\|)\|q-p\| \le C(\|q-p\|^2 + \delta^2 \|r_x\|^2).
\end{aligned}
$$

Choosing $\tau$ small enough, we find

$$
\|q-p\|_1 \le \delta C \|r_x\|. \tag{4.17}
$$

To get an equation for the variable $r$, we start from (2.1b), which can be written in the form

$$
T(\log n)_x + T_x = \phi_x - \frac{J}{n}.
$$

We obtain

$$
T_1 r_x + q(\log n_2)_x + q_x = (\phi_1 - \phi_2)_x - \left( \frac{J_1}{n_1} - \frac{J_2}{n_2} \right). \tag{4.18}
$$

We multiplying by $r_x$ and integrate over $\Omega$. After integration by parts, and using (2.1e), we find

$$
\begin{aligned}
&\int_0^1 T_1 r_x^2\, dx + \int_0^1 (n_1 - n_2) r\, dx \\
&\qquad = -\int_0^1 \left\{ q(\log n_2)_x + q_x + \left( \frac{J_1}{n_1} - \frac{J_2}{n_2} \right) \right\} r_x\, dx.
\end{aligned}
$$

Using (4.13), (4.15) and (4.17), we find

$$
\int_0^1 T_1 r_x^2\, dx + \int_0^1 (n_1 - n_2) r\, dx \le \delta C \|r_x\|^2. \tag{4.19}
$$

Observing that $(\log n_1 - \log n_2)(n_1 - n_2) \ge 0$, and using the bound for $T$ in Lemma 4.1, we end up with $\|r_x\| \le 0$, that is, $\|r_x\| = 0$. This, in turn, implies $\|q-p\|_1 = 0$, $\|p\|_1 = 0$. It follows $n_1 = n_2$, $T_1 = T_2$, $\theta_1 = \theta_2$, which yields $J_1 = J_2$ and $\phi_1 = \phi_2$. □

The proof of the main Theorem 2.1 follows from the existence Lemma 3.1 and the uniqueness Lemma 4.2.

## References

[1] G. Alì, G. Mascali, V. Romano, R.C. Torcasio, A hydrodynamic model for covalent semiconductors with a generalized energy dispersion relation, European J. Appl. Math. 25 (2) (2014) 255–275.
[2] V.D. Camiola, G. Mascali, V. Romano, Numerical simulation of a double-gate MOSFET with a subband model for semiconductors based on the maximum entropy principle, Contin. Mech. Thermodyn. 24 (4–6) (2012) 417–436.
[3] L. Chen, L. Hsiao, Y. Li, Large time behavior and energy relaxation time limit of the solutions to an energy transport model in semiconductors, J. Math. Anal. Appl. 312 (2005) 596–619.
[4] C.-H. Choi, J.-H. Chun, R.W. Dutton, Electrothermal characteristics of strained-Si MOSFETs in high-current operation, IEEE Trans. Electron Devices 51 (2004) 1928–1931.
[5] P. Degond, S. Génieys, A. Jüngel, A system of parabolic equations in nonequilibrium thermodynamics including thermal and electrical effects, J. Math. Pures Appl. 76 (1997) 991–1015.
[6] P. Degond, S. Génieys, A. Jüngel, A steady-state system in non-equilibrium thermodynamics including thermal and electrical effects, Math. Methods Appl. Sci. 21 (1998) 1399–1413.
[7] A. Jüngel, Quasi-hydrodynamic Semiconductor Equations, Birkhäuser, Basel, 2001.
[8] A. Jüngel, Transport Equations for Semiconductors, Springer, Berlin, Heidelberg, 2009.
[9] G. Mascali, A hydrodynamic model for silicon semiconductors including crystal heating, European J. Appl. Math. 26 (2015) 477–496.
[10] O. Muscato, V. Di Stefano, Hydrodynamic modeling of the electro-thermal transport in silicon semiconductors, J. Phys. A: Math. Theor. 44 (10) (2011) 105501–105527.
[11] O. Muscato, V. Di Stefano, An energy transport model describing heat generation and conduction in silicon semiconductors, J. Stat. Phys. 144 (1) (2011) 171–197.
[12] O. Muscato, V. Di Stefano, Electro-thermal behaviour of a sub-micron silicon diode, Semicond. Sci. Technol. 28 (2013) 025021–025031.
[13] S. Nishibata, M. Suzuki, Hierarchy of Semiconductor Equations: Relaxation Limits with Initial Layers for Large Initial Data, The Mathematical Society of Japan, Tokio, 2011.
[14] E. Pop, K. Banerjee, P. Sverdrup, R. Dutton, K. Goodson, Localized heating effects and scaling of sub-0.18 micron CMOS devices, in: International Electron Devices Meeting, 2001.
[15] V. Romano, A. Rusakov, 2d numerical simulations of an electron–phonon hydrodynamical model based on the maximum entropy principle, Comput. Methods Appl. Mech. Engrg. 199 (2010) 2741–2751.
[16] V. Romano, M. Zwierz, Electron–phonon hydrodynamical model for semiconductors, Z. Angew. Math. Phys. 61 (2010) 1111–1131.
[17] G. Stracquadanio, V. Romano, G. Nicosia, Semiconductor device design using the BIMADS algorithm, J. Comput. Phys. 242 (2013) 304–320.
[18] R. Stratton, Diffusion of hot and cold electrons in semiconductor barriers, Phys. Rev. 126 (1962) 2002–2013.