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Monatshefte für Mathematik

ISSN 0026-9255

Monatsh Math
DOI 10.1007/s00605-020-01403-w

ONLINE FIRST

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Monatshefte für Mathematik
https://doi.org/10.1007/s00605-020-01403-w

On weakening tightness to weak tightness

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Received: 12 January 2019 / Accepted: 11 March 2020
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Abstract

The weak tightness $wt(X)$ of a space X was introduced in Carlson (Topol Appl 249:103–111, 2018) with the property $wt(X) \leq t(X)$. We investigate several well-known results concerning $t(X)$ and consider whether they extend to the weak tightness setting. First we give an example of a non-sequential compactum X such that $wt(X) = \aleph_0 < t(X)$ under $2^{\aleph_0} = 2^{\aleph_1}$. In particular, this demonstrates the celebrated Balogh's (Proc Am Math Soc 105(3):755–764, 1989) Theorem does not hold in general if countably tight is replaced with weakly countably tight. Second, we introduce the notion of an S-free sequence and show that if X is a homogeneous compactum then $|X| \leq 2^{wt(X)\pi\chi(X)}$. This refines a theorem of de la Vega (Topol Appl 153:2118–2123, 2006). In the case where the cardinal invariants involved are countable, this also represents a variation of a theorem of Juhász and van Mill (Proc Am Math Soc 146(1):429–437, 2018). In this connection we also show $w(X) \leq 2^{wt(X)}$ for a homogeneous compactum. Third, we show that if X is a T_1 space, $wt(X) \leq \kappa$, X is κ^+-compact, and $\psi(\overline{D}, X) \leq 2^\kappa$ for any $D \subseteq X$ satisfying $|D| \leq 2^\kappa$, then (a) $d(X) \leq 2^\kappa$ and (b) X has at most 2^κ-many G_κ-points. This is a variation of another theorem of Balogh (Topol Proc 27:9–14, 2003). Finally, we show that if X is a regular space, $\kappa = L(X)wt(X)$, and λ is a caliber of X satisfying $\kappa < \lambda \leq (2^\kappa)^+$, then $d(X) \leq 2^\kappa$. This extends of theorem of Arhangel′skiĭ (Topol Appl 104:13–26, 2000).

Keywords Cardinality bounds · Cardinal invariants · Countably tight space · Homogeneous space · Weak tightness

Communicated by S.-D. Friedman.

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Published online: 16 March 2020

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Mathematics Subject Classification 54A25 · 54B10

1 Introduction

The cardinal function $t(X)$, the *tightness* of a topological space X, is the least infinite cardinal κ such that whenever $A \subseteq X$ and $x \in \overline{A}$ then there exists $B \subseteq A$ such that $|B| \leq \kappa$ and $x \in \overline{B}$. Motivated by results of Juhász and van Mill in [13], the cardinal function $wt(X)$, the *weak tightness of* X, was introduced in [9]. (See Definition 2.1 below). In [9] it was shown that $|X| \leq 2^{L(X)wt(X)\psi(X)}$ for any Hausdorff space X. As $wt(X) \leq t(X)$ for any space X, this improved the Arhangel′skiĭ-Šapirovskiĭ cardinality bound for Hausdorff spaces [1,15].

In this study we explore other fundamental cardinal function results involving tightness and consider whether $t(X)$ can be replaced with $wt(X)$. In 1989 Balogh [4] answered the Moore–Mrówca Problem by showing that every countably tight compactum is sequential under the Proper Forcing Axiom (PFA). In 2.3, we give an example demonstrating that countably tight cannot be replaced with weakly countably tight in Balogh's Theorem. It is also a consistent example of a compact group for which $wt(X) = \aleph_0 < t(X)$.

In [10], de la Vega answered a long-standing question of Arhangel′skiĭ by showing that $|X| \leq 2^{t(X)}$ for a homogeneous compactum X. Recall that a space X is *homogeneous* if for all $x, y \in X$ there exists a homeomorphism $h : X \to Y$ such that $h(x) = y$. Roughly, a space is homogeneous if all points in the space share identical topological properties. In [13], Juhász and van Mill introduced new techniques and gave a certain improvement of de la Vega's Theorem in the case where X is a countable union of countably tight subspaces. In Theorem 3.7, we prove that $|X| \leq 2^{wt(X)\pi\chi(X)}$ for any homogeneous compactum. In the case where the cardinal invariants involved are countable, Theorem 3.7 represents a variation of Theorem 4.1 in [13]. As $wt(X) \leq t(X)$ for any space and $\pi\chi(X) \leq t(X)$ for any compact space, our result gives a general improvement of de la Vega's Theorem. To this end we introduce the notion of an S-free sequence in Definition 3.1. In compact spaces these sequences play a role for $wt(X)$ similar to the role free sequences play for $t(X)$.

S-free sequences are also used to replace $t(X)$ with $wt(X)$ in a second theorem of Balogh [5]. In Theorem 4.1 a closing-off argument is used to show that if X is T_1, κ is a cardinal, $wt(X) \leq \kappa$, X is κ^+-compact, and $\psi(\overline{D}, X) \leq 2^\kappa$ for any $D \subseteq X$ satisfying $|D| \leq 2^\kappa$, then $d(X) \leq 2^\kappa$ and there are at most 2^κ-many G_κ-points. This result is related to Lemma 3.2 in [13] and produces an alternative proof that $|X| \leq 2^{L(X)wt(X)\psi(X)}$ for a Hausdorff space X.

In [3], Arhangel′skiĭ considered the notion of a caliber and showed that if X is a Lindelöf, regular space, $t(X) = \aleph_0$, and $\aleph_1$ is a caliber of X, then $d(X) \leq \mathfrak{c}$. Recall a cardinal κ is a *caliber* of a space X if every family of open sets of cardinality κ has a subfamily of cardinality κ with non-empty intersection. We show in Theorem 5.3 that in this result $t(X) = \aleph_0$ can be replaced with $wt(X) = \aleph_0$. It also extends to the uncountable case where the cardinal $\kappa = L(X)wt(X)$ is used.

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Finally, we mention that a rather easy modification of the argument used in Theorem 2.8 of [9] proves $|X| \leq 2^{aL_c(X)wt(X)\psi_c(X)}$ for every Hausdorff space X. This generalizes to the weak tightness setting the analogous inequality established in [6].

By *compactum* we mean a compact, Hausdorff space. For all undefined notions we refer the reader to [11,12].

2 An example and Balogh's theorem

The weak tightness $wt(X)$ [9] of a space X is defined as follows.

Definition 2.1 Let X be a space. Given a cardinal κ and $A \subseteq X$, the *κ-closure* of A is defined as $cl_\kappa A = \bigcup_{B \in [A]^{\leq\kappa}} \overline{B}$. The *weak tightness* $wt(X)$ of X is the least infinite cardinal κ for which there is a cover $\mathcal{C}$ of X such that $|\mathcal{C}| \leq 2^\kappa$ and for all $C \in \mathcal{C}$, (a) $t(C) \leq \kappa$ and (b) $X = cl_{2^\kappa} C$. We say that X is *weakly countably tight* if $wt(X) = \aleph_0$.

The condition (b) above can be difficult to work with. Instead a convenient tool for demonstrating that the weak tightness of a space X is at most κ was given in [9]. This replaces (b) with the more tractable condition that C is dense in X under mild assumptions involving $t(X)$ or $\pi\chi(X)$.

Proposition 2.2 ([9], Lemma 2.10) *Let X be a space, κ a cardinal, and $\mathcal{C}$ a cover of X such that $|\mathcal{C}| \leq 2^\kappa$, and for all $C \in \mathcal{C}$, $t(C) \leq \kappa$ and C is dense in X. If $t(X) \leq 2^\kappa$ or $\pi\chi(X) \leq 2^\kappa$ then $wt(X) \leq \kappa$.*

A σ-compact space X for which $wt(X) < t(X)$ is provided in [9]. In this section we will see that at least consistently there is a compact example.

In 1989 Balogh [4] answered the Moore–Mrówca Problem by proving under the Proper Forcing Axiom (PFA) that every compactum of countable tightness is sequential. Since PFA implies $2^{\aleph_0} = 2^{\aleph_1}$, the following example demonstrates that in that theorem countable tightness cannot be replaced with weak countable tightness. It is also, more generally, a consistent example of a compact group X such that $wt(X) = \aleph_0 < t(X)$.

Example 2.3 Assume $2^{\aleph_0} = 2^{\aleph_1}$ and let X be the Cantor cube 2^{ω_1}. Recall that X has tightness $t(X) = \aleph_1$ and is therefore not sequential. For each $x \in X$, let $C_x = \{y \in X : |\{\alpha < \omega_1 : x(\alpha) \neq y(\alpha)\}| < \omega\}$. Observe for each $x \in X$ that C_x is a countably tight, dense subspace of X and that $\mathcal{C} = \{C_x : x \in X\}$ is a cover of X. Furthermore, our set- theoretic assumption implies $|\mathcal{C}| \leq 2^\omega$. As $t(X) = \aleph_1 \leq 2^\omega$, by Proposition 2.2 it follows that $wt(X) = \aleph_0$. (This can also be seen more directly as, under our assumption, $cl_{2^{\aleph_0}}(C_x) = cl_{2^{\aleph_1}}(C_x) = cl(C_x) = X$ for each $x \in X$). Thus X is a non-sequential compact group of tightness $\aleph_1$ such that, under $2^{\aleph_0} = 2^{\aleph_1}$, X is weakly countably tight. Note also that X is homogeneous.

As $\pi\chi(X) = \aleph_1$, this example also shows that Šapirovskiĭ's theorem $\pi\chi(X) \leq t(X)$ for any compactum X may fail by using the weak tightness, answering Question 4.8 in [9].

Juhász and van Mill [13] asked if there is a homogeneous compactum that is the countable union of countably tight subspaces (σ-CT) yet not countably tight. They

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observe that such a space can only exist in a model in which $2^{\aleph_0} = 2^{\aleph_1}$. We note that the above example is not σ-CT. (For if it were, by Lemma 2.4 in [13] X would have a point of countable π-character). However, under $2^{\aleph_0} = 2^{\aleph_1}$, X is nevertheless a $2^{\aleph_0}$-union of countably tight dense subspaces.

In light of our example, we ask the following.

Question 2.4 *Does there exist a compactum X with $wt(X) < t(X)$ in ZFC?*

Question 2.5 *Does there exists a compactum that is not countably tight and is the countable union of dense countably tight subspaces?*

3 S-free sequences and homogeneous compacta

The goal in this section is to give an improved bound for the cardinality of any homogeneous compactum. For a space X, the notion of a $\mathcal{C}$-saturated subset of a space X was introduced in [13]. Given a cover $\mathcal{C}$ of X, a subset $A \subseteq X$ is $\mathcal{C}$-*saturated* if $A \cap C$ is dense in A for every $C \in \mathcal{C}$. It is clear that the union of $\mathcal{C}$-saturated subsets is $\mathcal{C}$-saturated.

Let X be a space. If $wt(X) \leq \kappa$ then there is a cover $\mathcal{C}$ of X witnessing the properties in 2.1. By Proposition 3.2 in [9], for all $x \in X$ we can fix a $\mathcal{C}$-saturated set $S(x)$ such that $x \in S(x)$ and $|S(x)| \leq 2^\kappa$. For all $A \subseteq X$, set $S(A) = \bigcup\{S(x) : x \in A\}$ and note that $S(A)$ is $\mathcal{C}$-saturated. We will use the existence of the cover $\mathcal{C}$ and the family $\{S(A) : A \subseteq X\}$ implicitly in proofs where $wt(X) \leq \kappa$ without direct mention.

Recall that for a cardinal κ and a space X, a set $\{x_\alpha : \alpha < \kappa\} \subseteq X$ is a *free sequence* if $\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \alpha \leq \beta < \kappa\}} = \varnothing$ for all $\alpha < \kappa$. We define the notion of an S-free sequence below. Note that every S-free sequence is a free sequence and that an S-free sequence is defined by a cover witnessing $wt(X)$.

Definition 3.1 Let $wt(X) = \kappa$. A set $\{x_\alpha : \alpha < \lambda\}$ is an *S-free sequence* if $\overline{S(\{x_\beta : \beta < \alpha\})} \cap \overline{\{x_\beta : \alpha \leq \beta < \lambda\}} = \varnothing$ for all $\alpha < \lambda$.

For a compactum X, it is well known that $t(X) = F(X)$, where $F(X) = \sup\{|E| : E \text{ is a free sequence in } X\}$. The following proposition demonstrates that S-free sequences play a similar role for $wt(X)$ in κ^+-compact spaces. Recall that a space X is κ-compact if every subset of cardinality κ has a complete accumulation point.

Proposition 3.2 *If X is κ^+-compact where $\kappa = wt(X)$ then X does not contain an S-free sequence of length κ^+.*

Proof Assume the contrary and let $A = \{x_\alpha : \alpha < \kappa^+\}$ be an S-free sequence. Since X is κ^+-compact, the set A has a complete accumulation point x. Note that $x \in \overline{S(A)}$. Now, the assumption $wt(X) = \kappa$ and the fact that $\overline{S(A)} = \bigcup_{\alpha<\kappa^+} \overline{S(\{x_\beta : \beta < \alpha\})}$ ([9], Lemma 2.7) ensure the existence of an ordinal $\alpha < \kappa^+$ such that $x \in \overline{S(\{x_\beta : \beta < \alpha\})}$. But this implies that the set $X \setminus \overline{\{x_\beta : \alpha \leq \beta < \kappa^+\}}$ is a neighborhood of x meeting A in less than κ^+-many points, a contradiction. □

In Theorem 2.5 in [13], Juhász and van Mill showed that if a compact space X is a countable union of countable tight subspaces then there is a non-empty G_δ-set

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contained in the closure of a countable set. That is, a σ-CT compact space has a non-empty subseparable G_δ-set. The following theorem is a variation of this and, in the countable case, states that if there is a cover $\mathcal{C}$ of X such that $|\mathcal{C}| \leq \mathfrak{c}$, and for all $C \in \mathcal{C}$ (a) C is countably tight and (b) $X = cl_\mathfrak{c} C$, then there is a G_δ-set contained in the closure of a set of size $\mathfrak{c}$. The proof is a variation of the proof of 2.2.4 in [2] and uses S-free sequences. Recall that a closed set G of a compactum X is a G_κ-set if and only if $\chi(G, X) \leq \kappa$.

Theorem 3.3 *Let X be a compactum and let $\kappa = wt(X)$. Then there exists a non-empty closed set $G \subseteq X$ and a $\mathcal{C}$-saturated set $H \in [X]^{\leq 2^\kappa}$ such that $G \subseteq \overline{H}$ and $\chi(G, X) \leq \kappa$.*

Proof Define $\mathcal{F} = \{F \subseteq X : F \neq \emptyset, F$ is closed, and $\chi(F, X) \leq \kappa\}$. Suppose by way of contradiction that for all $F \in \mathcal{F}$ and all $\mathcal{C}$-saturated $H \in [X]^{\leq 2^\kappa}$ we have $F \backslash \overline{H} \neq \emptyset$. We construct a decreasing sequence $\mathcal{F}' = \{F_\alpha : \alpha < \kappa^+\} \subseteq \mathcal{F}$ and an S-free sequence $A = \{x_\alpha : \alpha < \kappa^+\}$.

For $\alpha < \kappa^+$, suppose $x_\beta \in X$ and F_β have been defined for all $\beta < \alpha < \kappa^+$, where $F_{\beta''} \subseteq F_{\beta'}$ for all $\beta' < \beta'' < \alpha$. Define $A_\alpha = \{x_\beta : \beta < \alpha\}$ and $G_\alpha = \bigcap\{F_\beta : \beta < \alpha\}$. As X is compact, $G_\alpha \neq \emptyset$ and $G_\alpha \in \mathcal{F}$. As $|A_\alpha| \leq \kappa$, we have $|S(A_\alpha)| \leq \kappa \cdot 2^\kappa = 2^\kappa$. Therefore, $G_\alpha \backslash \overline{S(A_\alpha)} \neq \emptyset$ as $S(A_\alpha)$ is $\mathcal{C}$-saturated.

Let $x_\alpha \in G_\alpha \backslash \overline{S(A_\alpha)}$. As $\chi(G_\alpha, X) \leq \kappa$, we have that G_α is a G_κ-set. There exists an open family $\mathcal{U}$ such that $|\mathcal{U}| \leq \kappa$ and $G_\alpha = \bigcap \mathcal{U}$. As X is regular, for all $U \in \mathcal{U}$ there exists a closed G_δ-set G_U such that $x_\alpha \in G_U \subseteq U \backslash \overline{S(A_\alpha)}$. Set $F_\alpha = \bigcap\{G_U : U \in \mathcal{U}\}$, and note $x_\alpha \in F_\alpha \subseteq G_\alpha \backslash \overline{S(A_\alpha)}$ and that F_α is a non-empty closed G_κ-set. As X is compactum, $\chi(F_\alpha, X) \leq \kappa$ and $F_\alpha \in \mathcal{F}$. The choice of x_α and F_α completes the construction of the sequences $\mathcal{F}'$ and A.

We show now that A is an S-free sequence. Again, let $\alpha < \kappa^+$. Note $\{x_\beta : \alpha \leq \beta < \kappa^+\} \subseteq F_\alpha$ as $\mathcal{F}'$ is a decreasing sequence. Thus $\overline{\{x_\beta : \alpha \leq \beta < \kappa^+\}} \subseteq F_\alpha$. Also, $\{x_\beta : \beta < \alpha\} = A_\alpha$ and so $\overline{S(\{x_\beta : \beta < \alpha\})} = \overline{S(A_\alpha)} \subseteq X \backslash F_\alpha$. Therefore, $\overline{S(\{x_\beta : \beta < \alpha\})} \cap \overline{\{x_\beta : \alpha \leq \beta < \kappa^+\}} = \emptyset$ and A is S-free. However, X is κ^+-compact and cannot have an S-free sequence of length κ^+ by Proposition 3.2. This is a contradiction and thus there exists G and H as required. □

The following improvement of a theorem of Pytkeev [14] was given in [9]. It is used in the proof of Theorem 3.6. For a cardinal κ, the space X_κ is the set X with the collection of G_κ-sets as a basis for its topology.

Theorem 3.4 ([9], Corollary 3.5) *If X is a compactum and κ is a cardinal, then $L(X_\kappa) \leq 2^{wt(X) \cdot \kappa}$.*

We will also need the following proposition. Its proof can be extracted from the proof of Lemma 3.4 in [9], but we give a self-contained proof here.

Proposition 3.5 *Let X be a regular space, $\kappa = wt(X)$, and $D \subseteq X$ be $\mathcal{C}$-saturated. Then $nw(\overline{D}) \leq |D|^\kappa$.*

Proof Let $\mathcal{N} = \{\overline{A} : A \in [D]^{\leq\kappa}\}$ and note $|\mathcal{N}| \leq |D|^\kappa$. We show $\mathcal{N}$ is a network for $\overline{D}$. Let $x \in V \cap \overline{D}$ where V is open in X. By regularity, there exists an open set U

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of X such that $x \in U \subseteq \overline{U} \subseteq V$. As $\mathcal{C}$ is a cover of X, there exists $C \in \mathcal{C}$ such that $x \in C$. As D is $\mathcal{C}$-saturated, we have that $cl_D(D \cap C) = D$. Thus, $D \subseteq \overline{D \cap C}$ and $\overline{D} \subseteq \overline{D \cap C}$.

If W is an arbitrary open set containing x, then $x \in U \cap W$ and as $x \in \overline{D} \subseteq \overline{D \cap C}$, we see that $U \cap W \cap D \cap C \neq \varnothing$. Thus $x \in \overline{U \cap D \cap C} \cap C = cl_C(U \cap D \cap C)$. As $t(C) \leq \kappa$, there exists $A \in [U \cap D \cap C]^{\leq\kappa}$ such that $x \in cl_C(A) \subseteq \overline{A}$. Since $\overline{A} \in \mathcal{N}$ and $\overline{A} \subseteq \overline{U} \cap \overline{D} \subseteq V \cap \overline{D}$, this shows $\mathcal{N}$ is a network for $\overline{D}$. □

Homogeneity is now applied in the next theorem. Using 3.4 and 3.5, we give an upper bound for the weight of a homogeneous compactum.

Theorem 3.6 *If X is a homogeneous compactum then $w(X) \leq 2^{wt(X)}$.*

Proof Let $\kappa = wt(X)$. By Theorem 3.3 there exists a non-empty closed set $G \subseteq X$ and a $\mathcal{C}$-saturated set $H \in [X]^{\leq 2^\kappa}$ such that $G \subseteq \overline{H}$ and $\chi(G, X) \leq \kappa$. Fix $p \in G$. As X is homogeneous, for all $x \in X$ there exists a homeomorphism $h_x : X \to X$ such that $h_x(p) = x$. Then $\mathcal{F} = \{h_x[G] : x \in X\}$ is a cover of X by compact sets of character at most κ and, for all $x \in X$, we have $h_x[G] \subseteq h_x[\overline{H}]$. By Theorem 3.4 there exists $A \subseteq X$ such that $|A| \leq 2^\kappa$ and

$$X = \bigcup_{x \in A} h_x[G] = \bigcup_{x \in A} h_x[\overline{H}].$$

As H is $\mathcal{C}$-saturated, by Proposition 3.5 it follows that $nw(\overline{H}) \leq |H|^\kappa \leq (2^\kappa)^\kappa = 2^\kappa$. Let $\mathcal{N}$ be a network for $\overline{H}$ such that $|\mathcal{N}| \leq 2^\kappa$. Let $\mathcal{M} = \{h_x[N] : N \in \mathcal{N}, x \in A\}$ and note $|\mathcal{M}| \leq 2^\kappa \cdot 2^\kappa = 2^\kappa$. We show $\mathcal{M}$ is a network for X. Let $y \in U$, where U is open in X. There exists $x \in A$ such that $y \in h_x[\overline{H}] \cap U$. Then $h_x^{\leftarrow}(y) \in \overline{H} \cap h_x^{-1}[U]$. As $\mathcal{N}$ is a network for $\overline{H}$, there exists $N \in \mathcal{N}$ such that $h_x^{-1}(y) \in N \subseteq \overline{H} \cap h_x^{\leftarrow}[U]$. Thus $y \in h_x[N] \subseteq h_x[\overline{H}] \cap U \subseteq U$. As $h_x[N] \in \mathcal{M}$ it follows that $\mathcal{M}$ is a network for X and $nw(X) \leq 2^\kappa$. Now recall that $nw(X) = w(X)$ for any compact space X. This completes the proof. □

Homogeneity is used in the proof of 3.6 above. It is used in different way in the next corollary. Simply apply the above and the straightforward fact that $|X| \leq d(X)^{\pi\chi(X)}$ for any homogeneous Hausdorff space. As $wt(X) \leq t(X)$ for any space X, and $\pi\chi(X) \leq t(X)$ for any compactum, Corollary 3.7 represents an improvement of de la Vega's Theorem [10].

Corollary 3.7 *If X is a homogeneous compactum then $|X| \leq 2^{wt(X)\pi\chi(X)}$.*

Below we isolate the case of 3.7 where all cardinal invariants involved are countable. It follows directly from Proposition 2.2 and the above.

Corollary 3.8 *Let X be a homogeneous compactum of countable π-character with a cover $\mathcal{C}$ such that $|\mathcal{C}| \leq \mathfrak{c}$ and for all $C \in \mathcal{C}$, C is countably tight and dense in X. Then $|X| \leq \mathfrak{c}$.*

We compare Corollary 3.8 with the following theorem of Juhász and van Mill. Observe they are variations of each other.

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Theorem 3.9 ([13], Theorem 4.1) *If a compactum X is the union of countably many dense countably tight subspaces and X^ω is homogeneous, then $|X| \leq \mathfrak{c}$.*

As Example 2.3 shows that $\pi\chi(X) \leq wt(X)$ may fail for homogeneous compacta, we ask the following question. It was asked in [9] for $wt(X) = \aleph_0$.

Question 3.10 *If X is a homogeneous compactum, is $|X| \leq 2^{wt(X)}$?*

We also recall the following still-open question of R. de la Vega.

Question 3.11 *If X is a homogeneous compactum, is $|X| \leq 2^{\pi\chi(X)}$?*

Theorem 3.9 was extended to the power homogeneous setting in [9, Theorem 4.6]. Recall a space X is *power homogeneous* if there exists a cardinal κ such that X^κ is homogeneous. Given 3.6 and 3.7 above, we ask the following.

Question 3.12 *If X is a power homogeneous compactum, is $w(X) \leq 2^{wt(X)}$?*

Question 3.13 *If X is a power homogeneous compactum, is $|X| \leq 2^{wt(X)\pi\chi(X)}$?*

4 Weak tightness and a second theorem of Balogh

Our aim in this section is to give a variation (Theorem 4.1 below) of Theorem 2.3 in Balogh [5] using the weak tightness $wt(X)$. It is also related to Lemma 3.2 in [13] and results in an alternative proof that $|X| \leq 2^{L(X)wt(X)\psi(X)}$ if X is Hausdorff, proved in [9].

Theorem 4.1 *Let X be a T_1 space. If $wt(X) \leq \kappa$, X is κ^+-compact, and $\psi(\overline{D}, X) \leq 2^\kappa$ for any $D \subseteq X$ satisfying $|D| \leq 2^\kappa$, then $d(X) \leq 2^\kappa$ and X has at most 2^κ-many G_κ-points.*

Proof For any $D \in [X]^{\leq\kappa}$ fix a family $\mathcal{U}_D$ of open sets such that $\bigcap \mathcal{U}_D = \overline{S(D)}$ and $|\mathcal{U}_D| \leq 2^\kappa$. We will define by transfinite induction a non-decreasing sequence of $\mathcal{C}$-saturated sets $\{H_\alpha : \alpha < \kappa^+\}$ satisfying:

(1_α) $|H_\alpha| \leq 2^\kappa$;
(2_α) if $X \setminus \bigcup \mathcal{V} \neq \varnothing$ for some $\mathcal{V} \in [\bigcup\{\mathcal{U}_D : D \in [H_\alpha]^{\leq\kappa}\}]^{\leq\kappa}$, then $H_{\alpha+1} \setminus \bigcup \mathcal{V} \neq \varnothing$.

Put $H_0 = S(x_0)$ for some $x_0 \in X$ and let $\phi : \mathcal{P}(X) \to X$ be a choice function extended by letting $\phi(\varnothing) = x_0$. Assume we have already defined the subsequence $\{H_\beta : \beta < \alpha\}$. If α is a limit ordinal, then put $H_\alpha = \bigcup\{H_\beta : \beta < \alpha\}$. If $\alpha = \gamma + 1$, then put $H_\alpha = H_\gamma \cup \bigcup\{S(\phi(X \setminus \bigcup \mathcal{V})) : \mathcal{V} \in \left[\bigcup\{\mathcal{U}_D : D \in [H_\gamma]^{\leq\kappa}\}\right]^{\leq\kappa}\}$. A counting argument ensures that H_α satisfies 1_α and 2_α.

Now, put $H = \bigcup\{H_\alpha : \alpha < \kappa^+\}$. It is clear that $|H| \leq 2^\kappa$. We show H is dense in X. Assume by contradiction that there is some $p \in X \setminus \overline{H}$. We will show that this assumption implies the existence of an S-free sequence $\{x_\alpha : \alpha < \kappa^+\} \subseteq H$ such that $S(\{x_\alpha : \alpha < \kappa^+\}) \subseteq H$. So, let $\alpha < \kappa^+$ and suppose we have already chosen points $\{x_\beta : \beta < \alpha\} \subseteq H$ and open sets $V_\beta \in \mathcal{U}_{\{x_\xi : \xi < \beta\}}$ such that $S(\{x_\beta : \beta < \alpha\}) \subseteq H$ and for every $\beta < \alpha$ we have $\overline{\bigcup\{S(x_\xi) : \xi < \beta\}} \subseteq V_\beta \subseteq X \setminus \{p\}$ and $\{x_\xi : \beta \leq \xi <$

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$\alpha\} \cap V_\beta = \varnothing$. If $D_\alpha = \{x_\beta : \beta < \alpha\}$, then what we are assuming implies $p \notin \overline{S(D_\alpha)}$. We may then pick $V_\alpha \in \mathcal{U}_{D_\alpha}$ such that $p \notin V_\alpha$. Furthermore, there exists an ordinal $\gamma(\alpha) < \kappa^+$ such that $D_\alpha \subseteq H_{\gamma(\alpha)}$ and so $\{V_\beta : \beta \leq \alpha\} \subseteq \bigcup\{\mathcal{U}_D : D \in [H_{\gamma(\alpha)}]^{\leq\kappa}\}$. Since $X \setminus \bigcup\{V_\beta : \beta \leq \alpha\} \neq \varnothing$, condition $2_{\gamma(\alpha)}$ guarantees that $H_{\gamma(\alpha)+1} \setminus \bigcup\{V_\beta : \beta \leq \alpha\} \neq \varnothing$. Let $x_\alpha = \phi\left(X \setminus \bigcup\{V_\beta : \beta \leq \alpha\}\right) \in H_{\gamma(\alpha)+1}$ and note $S(x_\alpha) \in H$. Therefore, the construction can be carried out for all $\alpha < \kappa^+$, producing an S-free sequence of length κ^+.

Now we show every G_κ point is contained in H. Assume by contradiction that there exists a G_κ point $p \in X \setminus H$ and fix a family of open sets $\{W_\alpha : \alpha < \kappa\}$ such that $\bigcap\{W_\alpha : \alpha < \kappa\} = \{p\}$. Then let $\mathcal{V} = \{U \in \bigcup\{\mathcal{U}_D : D \in [H]^{\leq\kappa}\} : p \notin U\}$. Observe that H cannot be covered by $\leq \kappa$ elements of $\mathcal{V}$ because such a family would be contained in $\bigcup\{\mathcal{U}_D : D \in [H_\alpha]^{\leq\kappa}\}$ for some $\alpha < \kappa^+$ and this in turn would contradict condition 2_α. So, there is an ordinal $\gamma < \kappa$ such that $H \setminus W_\gamma$ cannot be covered by $\leq \kappa$ elements of $\mathcal{V}$. Now, by mimicking the argument used in the paragraph above, we may again establish the existence of an S-free sequence $\{x_\alpha : \alpha < \kappa^+\} \subseteq H \setminus W_\gamma$. □

The following is well-known. We include its proof for completeness.

Proposition 4.2 *If F is a closed subspace of a T_1 space X, then $\psi(F, X) \leq (|F| \cdot \psi(X))^{L(X)}$.*

Proof Let $\lambda = |F| \cdot \psi(X)$ and $\kappa = L(X)$. For all $x \in F$ there exists an open family $\mathcal{U}_x$ such that $\{x\} = \bigcap \mathcal{U}_x$ and $|\mathcal{U}_x| \leq \lambda$. Let $\mathcal{U} = \bigcup\{\mathcal{U}_x : x \in F\}$ and $\mathcal{F} = [\mathcal{U}]^{\leq\kappa}$. Note $|\mathcal{U}| \leq \lambda \cdot \lambda = \lambda$ and $|\mathcal{F}| \leq \lambda^\kappa$.

Fix $y \in X \setminus F$. For all $x \in F$ there exists $U_x \in \mathcal{U}_x$ such that $y \notin U_x$. Then $\{U_x : x \in F\}$ is an open cover of F. As $L(X) \leq \kappa$ there exists $\mathcal{V}_y \in [\{\mathcal{U}_x : x \in F\}]^{\leq\kappa}$ such that $F \subseteq \bigcup \mathcal{V}_y$. Note $y \notin \bigcup \mathcal{V}_y$ and $\mathcal{V}_y \in \mathcal{F}$.

As $F \subseteq \bigcap\left\{\bigcup \mathcal{V}_y : y \in X \setminus F\right\}$ and $y \notin \bigcap\left\{\bigcup \mathcal{V}_y : y \in X \setminus F\right\}$ for all $y \in X \setminus F$, it follows that $F = \bigcap\left\{\bigcup \mathcal{V}_y : y \in X \setminus F\right\}$. Now, $\{\mathcal{V}_y : y \in X \setminus F\} \subseteq \mathcal{F}$ and therefore $|\{\bigcup \mathcal{V}_y : y \in X \setminus F\}| \leq |\mathcal{F}| \leq \lambda^\kappa$. This completes the proof. □

Corollary 4.3 ([9], Theorem 2.8) *If X is a Hausdorff space then $|X| \leq 2^{L(X)wt(X)\psi(X)}$.*

Proof Let $L(X)wt(X)\psi(X) = \kappa$. $L(X) \leq \kappa$ implies that X is κ^+-compact. By Theorem 2.5 in [9], $|D| \leq 2^\kappa$ implies $|\overline{D}| \leq |D|^{wt(X)\psi_c(X)} \leq |D|^{L(X)wt(X)\psi(X)} \leq 2^\kappa$. As $L(X)\psi(X) \leq \kappa$ and $|\overline{D}| \leq 2^\kappa$, by Proposition 4.2 it follows that $\psi(\overline{D}, X) \leq 2^\kappa$. Now use either conclusion of Theorem 4.1. □

5 Weak tightness and calibers

A cardinal κ is a *caliber* of a space X if every family of open sets of cardinality κ has a subfamily of cardinality κ with a non-empty intersection. At the end of a survey paper on the Souslin number [16], B.E. Šapirovskiĭ stated without proof the following:

Proposition 5.1 ([16], Theorem 5.23) *Let X be a Lindelöf, regular sequential space. If an uncountable cardinal $\lambda \leq \mathfrak{c}^+$ is a caliber of X then $|X| \leq \mathfrak{c}$.*

Later, A.V. Arhangel′skiĭ [3] published a detailed proof of this result in the special case $\lambda = \aleph_1$. His proof is based on the following:

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Proposition 5.2 ([3], Theorem 5.1) *Let X be a Lindelöf, regular space. If $t(X) = \aleph_0$ and $\aleph_1$ is a caliber of X, then $d(X) \leq \mathfrak{c}$.*

Our purpose here is to show that the previous result continues to hold by replacing tightness with weak tightness. To achieve it, we will modify the argument in the proof of Theorem 1 of [7]. Recall that a space X is said to be quasiregular if for any non-empty open set U there exists a non-empty open set V such that $\overline{V} \subseteq U$.

Theorem 5.3 *Let X be a quasiregular space and let $\kappa = L(X)wt(X)$. If a cardinal λ satisfying $\kappa < \lambda \leq (2^\kappa)^+$ is a caliber of X, then $d(X) \leq 2^\kappa$.*

Proof Assume by way of contradiction that $d(X) > 2^\kappa$. Fix a choice function $\eta : \mathcal{P}(X)\setminus\{\varnothing\} \to X$. We will define by induction an increasing family $\{A_\alpha : \alpha < \lambda\}$ of $\mathcal{C}$-saturated subsets of X of cardinality not exceeding 2^κ and a family $\{U_\alpha : \alpha < \lambda\}$ of non-empty open subsets of X in a such a way that for all $\alpha < \lambda$,

(1) $\overline{A_\alpha} \cap \overline{U_\alpha} = \varnothing$, and
(2) if $\mathcal{V} \subseteq \{U_\beta : \beta < \alpha\}$ satisfies $|\mathcal{V}| \leq \kappa$ and $\bigcap \mathcal{V} \neq \varnothing$, then $\eta(\bigcap \mathcal{V}) \in A_\alpha$.

To justify the above construction, let us assume to have already defined the $\mathcal{C}$-saturated sets $\{A_\beta : \beta < \alpha\}$ and the open sets $\{U_\beta : \beta < \alpha\}$. Since $\alpha < \lambda$ and $\lambda \leq (2^\kappa)^+$, we have $|\{U_\beta : \beta < \alpha\}| \leq |\alpha| \leq 2^\kappa$. Consequently, the set $B = \{\eta(\bigcap \mathcal{V}) : \mathcal{V} \subseteq \{U_\beta : \beta < \alpha\}, |\mathcal{V}| \leq \kappa$ and $\bigcap \mathcal{V} \neq \varnothing\}$ has cardinality not exceeding 2^κ. Let $A_\alpha = \bigcup\{S(x) : x \in B\} \cup \bigcup\{A_\beta : \beta < \alpha\}$ and note that $|A_\alpha| \leq 2^\kappa$. As we are assuming that $d(X) > 2^\kappa$, we may find a non-empty open set U_α such that $\overline{A_\alpha} \cap \overline{U_\alpha} = \varnothing$.

Since λ is a caliber of X, there exists a set $E \subseteq \lambda$ such that $|E| = \lambda$ and $\bigcap\{U_\alpha : \alpha \in E\} \neq \varnothing$. We may fix an increasing mapping $f : \lambda \to E$. Observe now that we are assuming $\kappa^+ \leq \lambda$. For any $\alpha < \kappa^+$ let $x_\alpha = \eta(\bigcap\{U_{f(\xi)} : \xi \leq \alpha\})$. If $\beta < \alpha < \kappa^+$, then $x_\beta \in A_{f(\beta)+1} \subseteq A_{f(\alpha)}$ and $x_\alpha \in U_{f(\alpha)}$ and so $x_\beta \neq x_\alpha$. Consequently, the set $Y = \{x_\alpha : \alpha < \kappa^+\}$ has cardinality κ^+ and therefore, by $L(X) \leq \kappa$, there exists a complete accumulation point p for Y.

As pointed out in Lemma 2.7 of [9], $\overline{\bigcup\{A_{f(\alpha)} : \alpha < \kappa^+\}} = \bigcup\{\overline{A_{f(\alpha)}} : \alpha < \kappa^+\}$. As $Y \subseteq \bigcup\{A_{f(\alpha)} : \alpha < \kappa^+\}$, there exists some $\gamma < \kappa^+$ such that $p \in \overline{A_{f(\gamma)}} \subseteq X\setminus\overline{U_{f(\gamma)}}$. Moreover, for each $\gamma \leq \beta < \kappa^+$ the set $U_{f(\gamma)}$ occurs in the definition of x_β and consequently $x_\beta \in U_{f(\gamma)}$. This means that the set $W = X\setminus\overline{U_{f(\gamma)}}$ is a neighborhood of p such that $|W \cap Y| \leq \kappa$, in contrast to the choice of p. This completes the proof. □

For a space X and $A \subseteq X$, the θ-*closure* of A is defined as $cl_\theta(A) = \{x \in X : \overline{U} \cap A \neq \varnothing$ whenever U is an open set containing $x\}$. A set $D \subseteq X$ is θ-*dense* if $X = cl_\theta(D)$ and the θ-*density* of X is is defined as $d_\theta(X) = \min\{|D| : D$ is θ-dense in $X\}$. Dropping the assumption of quasiregularity, the same proof above implies a similar result concerning $d_\theta(X)$.

Theorem 5.4 *Let X be a space and let $\kappa = L(X)wt(X)$. If a cardinal λ satisfying $\kappa < \lambda \leq (2^\kappa)^+$ is a caliber of X, then $d_\theta(X) \leq 2^\kappa$.*

For a space X, the *linear Lindelöf degree* $lL(X)$ [8] is the least infinite cardinal κ such that every increasing open cover of X has a subcover of size at most κ. X is

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linearly Lindelöf if $lL(X)$ is countable. It is clear that $lL(X) \leq L(X)$. It is well-known that if $lL(X) \leq \kappa$ for a cardinal κ then every set of cardinality κ^+ has a complete accumulation point. Examining the proofs of Theorems 5.3 and 5.4, we see that both still hold if $L(X)$ is substituted with $lL(X)$.

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