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Non-expandable non-overlapping sets of pictures ☆

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ABSTRACT

The non-overlapping sets of pictures are sets such that no two pictures in the set (properly) overlap. They are the generalization to two dimensions of the cross-bifix-free sets of strings. Non-overlapping sets of pictures are non-expandable when no other picture can be added without violating the property. We propose a general construction method for non-expandable non-overlapping (NENO) sets based on some structural properties of NENO sets. As an application, we show a first example of a family of NENO sets.

1. Introduction

The digital technology that pervades every aspect of our lives is bringing communications more and more towards pictorial (two-dimensional) environments. The generalization to two dimensions of the formal study of all structures and special patterns of the strings is then gaining a growing interest in the scientific community. The two-dimensional strings are called *pictures* and they are represented by two-dimensional (rectangular) arrays over a finite alphabet Σ. The set of all pictures over Σ is usually denoted by Σ^{**}. Extending results from the formal (string) language theory to two dimensions is a very challenging task. The two-dimensional structure in fact imposes some intrinsic difficulties even in the basic concepts. For example, we can define two concatenation operations (horizontal and vertical concatenations) between two pictures, but they are only partial operations and do not induce a monoid structure to the set Σ^{**}. Moreover, for example, the definition of "prefix" of a string can be extended to a picture by considering its rectangular portion in the top-left corner; nevertheless, if one deletes a prefix from a picture, the remaining part is not a picture anymore.

Several results from string language theory have been usefully extended to pictures. Many researchers have investigated how the notion of recognizability can be transferred to two dimensions to accept picture languages (see for example [1–9]).

A relevant notion on strings is the one of "border". Given a string s, a *bifix* or a *border* of s is a substring x that is both prefix and suffix of s. A string s is *bifix-free* or *unbordered* if it has no other bifixes, besides the empty string and s itself. Bifix-free strings are strictly related to the theory of codes [10] and are involved in the data structures for pattern matching algorithms [11,12]. From a more applicative point of view, bifix-free strings are suitable as synchronization patterns in

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digital communications and similar communication protocols [13]. In this framework, the *cross-bifix-free codes* have been introduced in [14] by reviving a notion introduced in early sixties. Cross-bifix-free codes are strictly connected with circular codes. A set of strings X is a cross-bifix-free code when no prefix of any string is the suffix of any other string in X; it is *non-expandable* if no other element can be added to X without falsifying the property of the set. Several efforts have been made in the last years in order to construct families of non-expandable cross-bifix-free codes. A first family has been exhibited in [15]. Subsequently, other families based on different approaches have been shown in [16,17], with the aim of finding codes of bigger cardinality.

The notion of border extends very naturally from strings to pictures since it is not related to any scanning direction. Informally, we can say that a picture p is *bordered* if a copy p' of p can be overlapped on p by putting a corner of p' somewhere on some position in p. Observe that, differently from the string case, depending on the position of the corner of p', several different types of picture overlaps are possible. We stay in the general situation when the overlaps can be made on any position in p and therefore the borders can be of any size. This leads to a quite different scenario with respect to the string case. In fact, we have two pairs of symmetric cases; either p can be overlapped by putting its top-left corner in a copy p' (i.e. the bottom-right corner of p' is inside p) or we can put the bottom-left corner of p somewhere in some position of its copy p' (i.e. the top-right corner of p' is inside p). Unbordered pictures, in this general setting, were first investigated in [18], where an algorithm for the construction of the set of all unbordered pictures of a fixed size is proposed.

Moving from pictures to sets of pictures, we consider the notion of *non-overlapping set of pictures*. In a non-overlapping set of pictures, each picture is unbordered and moreover no picture can be overlapped on another one of the same set. This notion gives therefore the generalization of cross-bifix-free code of strings and provides a family of picture codes (see [19–24] for some recent results on picture codes). In particular, non-overlapping sets of pictures are two-dimensional comma-free codes (recently studied in [25]) with a stronger property.

An example of a non-overlapping set of pictures was recently given in [26]; its cardinality is calculated using generalized Fibonacci sequences. Note that non-overlapping sets of pictures of a fixed size (m,n) could be also found by considering the set of all possible pictures of that size and choosing some of them by checking that there are no overlaps. The challenge is to find, for each size (m,n), large non-overlapping sets of pictures which cannot be expanded and, at the same time, to provide a description for any size of the pictures. In the same paper [26], it is left as main open problem whether it is possible to construct a *non-expandable non-overlapping* set of pictures (we will call it a NENO set, for short). A solution to this problem is a generalization to two dimensions of the above cited results on non-expandable cross-bifix-free sets of strings.

This paper shows an example of a family of NENO sets. It is obtained following a construction method based on some properties of NENO sets. The method can be applied to obtain further examples. More precisely, first, we show some simple necessary conditions which are satisfied by any non-overlapping set of pictures. Subsequently, we identify some conditions which ensure that a set of pictures is non-overlapping non-expandable. The focus is first on the properties that the frames of the pictures may have, and then on the internal part of the pictures. As an application, we exhibit the family of NENO sets $Y(m,n)$, for any $m,n \geq 4$. It is the first example in the literature.

2. Preliminaries

In this section we recall all the definitions on strings and pictures needed in the rest of the paper. For more details see [10] and [5].

2.1. Basic notations on strings

A string is a sequence of zero or more symbols from an alphabet Σ. Let $s = s_1 s_2 \ldots s_n$ be a string of length n over Σ. The length of s is denoted $|s|$. A string w over Σ of length h, $h \leq n$, is a substring (or factor) of s if $s = uwv$ for $u, v \in \Sigma^*$. Moreover we say that w occurs at position j of s if $w = s_j \ldots s_{j+h-1}$. A string x of length $m < n$ is a *prefix* of s if x is a substring that occurs in s at position 1; it is a *suffix* of s if it is a substring that occurs in s at position $n - m + 1$. A string x that is both prefix and suffix of s is called a *border* or a *bifix* of s. The empty string and s itself are *trivial* borders of s. A string s is *unbordered* or *bifix-free* if it has no borders other than the trivial ones. Unbordered strings have received a lot of attention since they occur in many applications like message synchronization or string matching. In [13] P.T. Nielsen proposed a procedure to generate all bifix-free strings of a given length.

Finally, two strings s and s' *overlap* if there exists a string x that is a suffix of s and a prefix of s', or vice versa; the string x is their overlap and $|x|$ is the length of the overlap. We will equivalently say that s overlaps s'.

2.2. Basic notations on pictures

A *picture* over a finite alphabet Σ is a two-dimensional rectangular array of elements of Σ. Given a picture p, $|p|_{row}$ and $|p|_{col}$ denote the number of rows and columns, respectively while $size(p) = (|p|_{row}, |p|_{col})$ denotes the picture *size*. The pictures of size $(m,0)$ or $(0,n)$ for all $m,n \geq 0$, called *empty* pictures, will be never considered in this paper. The set of all pictures over Σ of fixed size (m,n) is denoted by $\Sigma^{m,n}$, while the set of all pictures over Σ is denoted by Σ^{**}.

Let p be a picture of size (m, n). The set of coordinates $dom(p) = \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ is referred to as the *domain* of a picture p. We let $p(i, j)$ denote the symbol in p at coordinates (i, j). We assume the top-left corner of the picture to be at position $(1, 1)$. Moreover, to easily detect border positions of pictures, we use initials of words "top", "bottom", "left" and "right"; then, for example, the *tl-corner* of p refers to position $(1, 1)$ while the *br-corner* refers to position (m, n). Furthermore, we denote by $r_F(p), r_L(p) \in \Sigma^n$ the first and the last row of p, respectively and by $c_F(p), c_L(p) \in \Sigma^m$ the first and the last column of p, respectively. Then, the *frame* of p is $frame(p) = (r_F(p), r_L(p), c_F(p), c_L(p))$.

For the sequel, it is convenient to extend the notation for the frame of a picture to languages. Let $X \subseteq \Sigma^{m,n}$. Let us denote by $R_F(X) \subseteq \Sigma^n$ the set $R_F(X) = \{r_F(p) \mid p \in X\}$ of the first rows of all pictures in X. In a similar way, $R_L(X)$, $C_F(X)$, and $C_L(X)$ will denote the sets of the last rows, of the first columns, and of the last columns of all pictures in X, respectively. The *frame* of X is the quadruple $frame(X) = (R_F(X), R_L(X), C_F(X), C_L(X))$.

A *subdomain* of $dom(p)$ is a set d of the form $\{i, i+1, \ldots, i'\} \times \{j, j+1, \ldots, j'\}$, where $1 \le i \le i' \le m$, $1 \le j \le j' \le n$, also specified by the pair $[(i, j), (i', j')]$. The portion of p corresponding to positions in subdomain $[(i, j), (i', j')]$ is denoted by $p[(i, j), (i', j')]$. Then, a non-empty picture x is *subpicture of* p if $x = p[(i, j), (i', j')]$, for some $1 \le i \le i' \le m$, $1 \le j \le j' \le n$; we say that x *occurs* at position (i, j) (its tl-corner).

Observe that the notion of subpicture generalizes very naturally to two dimensions the notion of substring. On the other hand, the notions of prefix and suffix of a string implicitly assume the left-to-right reading direction. In two dimensions, there are four corners and four scanning-directions from a corner toward the opposite one. Hence, we introduce the definition of four different "prefixes" of a picture, each one referring to one corner.

Definition 1. Given pictures $p \in \Sigma^{m,n}$, $x \in \Sigma^{h,k}$, with $1 \le h \le m$, $1 \le k \le n$,
x is a *tl-prefix* of p if x is a subpicture of p occurring at position $(1, 1)$,
x is a *tr-prefix* of p if x is a subpicture of p occurring at position $(1, n-k+1)$,
x is a *bl-prefix* of p if x is a subpicture of p occurring at position $(m-h+1, 1)$,
x is a *br-prefix* of p if x is a subpicture of p occurring at position $(m-h+1, n-k+1)$.

Several operations can be defined on pictures (cf. [5]). Two concatenation products are usually considered, the column and the row concatenation. The reverse operation on strings can be generalized to pictures and gives rise to two different mirror operations (called *row-* and *col-mirror*) obtained by reflecting with respect to a vertical and a horizontal axis, respectively. Another operation that has no counterpart in one dimension is the *rotation*. The rotation of a picture p of size (m, n), is the clockwise rotation of p by 90°, denoted by p^{90°. Note that p^{90° has size (n, m). All the operations defined on pictures can be extended in the usual way to sets of pictures.

We conclude by remarking that any string $s = y_1 y_2 \cdots y_n$ can be identified either with a single-row or with a single-column picture, i.e. a picture of size $(1, n)$ or $(n, 1)$, whereas any picture in $\Sigma^{m,n}$ can be viewed as a string of length n on the alphabet of the columns Σ^m, and as a string of length m on the alphabet of the rows Σ^n.

3. Non-overlapping sets of pictures

In this section we set up all the necessary definitions and notations on non-overlapping pictures. The notion is strictly related to the one of "unbordered picture" already introduced and studied in [18]. Here, we state directly the definition of non-overlapping pictures obtaining the notion of unbordered picture as a particular case.

Recall that two strings overlap when the prefix of one of them is the suffix of the other one. This notion can be extended very naturally to two dimensions by taking into account that now four different corners exist. Informally, we say that two pictures p and q overlap when we can find the same rectangular portion at a corner of p and at the opposite corner of q. Observe that there are two different kinds of overlaps depending on the pair of opposite corners involved.

Definition 2. Let $p \in \Sigma^{m,n}$ and $q \in \Sigma^{m',n'}$.

The pictures p and q *tl-overlap* if there exists a picture $x \in \Sigma^{h,k}$, with $1 \le h \le \min\{m, m'\}$ and $1 \le k \le \min\{n, n'\}$, which is a tl-prefix of p and a br-prefix of q, or vice versa.

The pictures p and q *bl-overlap* if there exists a picture $x \in \Sigma^{h,k}$, with $1 \le h \le \min\{m, m'\}$ and $1 \le k \le \min\{n, n'\}$, which is a bl-prefix of p and a tr-prefix of q, or vice versa.

The pictures p and q *overlap* if they tl-overlap or they bl-overlap.

The picture x is called an *overlap* of p and q, and its size (h, k) is the *size of the overlap*.

For the sequel, it is useful to identify some special cases of picture overlaps and we list them in the definition below. Examples are given in Fig. 1 where the first pair of pictures tl-overlap, the second pair h-slide overlap, the third one v-slide overlap, and the last pair shows two pictures that frame-overlap (and also bl-overlap).

Definition 3. Let $p \in \Sigma^{m,n}$ and $q \in \Sigma^{m',n'}$, then

- p and q *properly overlap* if they have an overlap $x \in \Sigma^{h,k}$ with $x \ne p$ and $x \ne q$

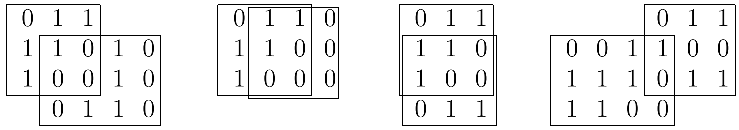

Fig. 1. From left to right: a pair of pictures that tl-overlap, h-slide overlap, v-slide overlap, frame-overlap (and also bl-overlap).

- p and q *h-slide overlap* if they have an overlap $x \in \Sigma^{h,k}$ with $h = m = m'$
- p and q *v-slide overlap* if they have an overlap $x \in \Sigma^{h,k}$ with $k = n = n'$
- p and q *frame-overlap* if they have an overlap $x \in \Sigma^{h,k}$ with $h = 1$ or $k = 1$

The case when p overlaps with itself leads to the notions of border of a picture, self-overlapping and unbordered pictures. As for the overlaps, there are two different kinds of borders (*tl-borders* and *bl-border*) depending on the pair of opposite corners that hold the border. A tl-border is called a diagonal border in [27].

The notion of "non-overlapping" is naturally extended to sets of pictures in order to generalize the notion of cross-bifix-free sets of strings, introduced in [14]. Notice that, in analogy to the case of cross-bifix-free sets of strings, we will consider sets of pictures of fixed size.

Definition 4. A set of pictures $X \subseteq \Sigma^{m,n}$ is *non-overlapping* if for any $p, q \in X$, p and q do not properly overlap.

Moreover, a set $X \subseteq \Sigma^{m,n}$ is *non-expandable non-overlapping*, *NENO* for short, if X is non-overlapping and for any $p \in \Sigma^{m,n} \setminus X$, there exists $q \in X$ such that p and q overlap.

In the string case, for any length n, the set containing only the string 10^n is non-overlapping (cross-bifix-free) and non-expandable. The real challenge is to find large sets of strings satisfying the property, as discussed in the Introduction. In the two-dimensional case, it seems that there is not a NENO set containing only one picture, for each size (m, n). This because there are a lot of different kinds of overlaps to be taken into account. However, we can show some particular examples, if we fix the size. For example, a set with only the following picture $\boxed{\begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{matrix}}$ is a NENO set of size $(2, 4)$. This can be easily verified in an exhaustive way, because there are only four unbordered pictures of size $(2, 4)$ (see [18]).

Very recently, in [26], a set of non-overlapping matrices has been presented by exploiting some techniques from the string case. The authors leave open the problem of finding non-expandable non-overlapping sets. And this appears to be a difficult problem. No examples of NENO sets defined for any size (m, n) are found in the literature.

Note that, in general, a NENO set X of size (m, n) could be produced by an exhaustive procedure that first generates all unbordered pictures in $\Sigma^{m,n}$, then systematically chooses $p \in \Sigma^{m,n}$, and finally adds p to X when it verifies that p does not overlap any picture already in X. This procedure is not only computationally heavy, but it also produces a set whose pictures can be given only by enumeration; the pictures are not classified by their structural properties nor by a uniform description.

In the sequel, we propose some conditions for NENO sets that will be used in Section 5 to show a family of NENO sets $Y(m, n)$, for any size (m, n). Observe that further examples can be simply obtained applying to a NENO set the operations of rotation, col- and row-mirror operations, and permutation or renaming of symbols in Σ.

Let us now consider some properties which are necessarily satisfied by any non-overlapping set of pictures. First, observe that any picture in a non-overlapping set is necessarily unbordered. In order to show some properties on the frames of the pictures in a non-overlapping set, let us introduce the following definition. Note that in a cross-non-overlapping pair (S_1, S_2), it is not required that S_1 and S_2 are cross-bifix-free sets.

Definition 5. Let S_1, $S_2 \subseteq \Sigma^n$ and $S_1 \cap S_2 = \emptyset$.

The pair (S_1, S_2) is *cross-non-overlapping* if for any $s_1 \in S_1$, $s_2 \in S_2$, s_1 and s_2 do not overlap.

Proposition 6. *Let $X \subseteq \Sigma^{m,n}$. If X is non-overlapping then the two pairs $(R_F(X), R_L(X))$ and $(C_F(X), C_L(X))$ are cross-non-overlapping.*

Proof. Suppose to the contrary that there exist $s_1 \in R_F(X)$, $s_2 \in R_L(X)$, and s_1 and s_2 overlap. Then any pictures $p_1, p_2 \in X$ with $r_F(p_1) = s_1$ and $r_L(p_2) = s_2$ are such that p_1 and p_2 overlap, against X non-overlapping. An analogous reasoning holds for the sets of columns. $\square$

Some more properties can be stated in the case of the binary alphabet. If X is a non-overlapping set over a binary alphabet then the four corners of any picture in X carry the same quadruple of symbols and not any quadruple is allowed. We state this simple necessary condition in the lemma below.

Let $corners(p) = (p(1,1), p(1,n), p(m,1), p(m,n))$, for any picture $p \in \Sigma^{m,n}$.

Lemma 7. *Let $\Sigma = \{0, 1\}$ and $X \subseteq \Sigma^{m,n}$. If X is a non-overlapping set then only four cases are possible*

a) $corners(p) = (0, 0, 1, 1)$, for any picture $p \in X$
b) $corners(p) = (1, 1, 0, 0)$, for any picture $p \in X$
c) $corners(p) = (1, 0, 1, 0)$, for any picture $p \in X$
d) $corners(p) = (0, 1, 0, 1)$, for any picture $p \in X$.

Proof. Given an unbordered picture p, $corners(p)$ must be of the form a), b), c) or d) otherwise p would have a border of size $(1, 1)$. Since any picture in a non-overlapping set is necessarily unbordered, we have that, for any $p \in X$, $corners(p)$ must be of the form a), b), c) or d). The proof is completed by noting that, for any $p, q \in X$, it must be $corners(p) = corners(q)$; if the sets $corners(p)$ and $corners(q)$ were of two different forms among a), b), c) or d), then p and q would have an overlap of size $(1, 1)$ against the hypothesis that X is non-overlapping. □

4. Properties and conditions for a NENO set

In this section we present some properties on non-expandable non-overlapping sets. The focus will be firstly on the frames of the pictures in a NENO set and their properties related to the non-expandability of the set (see Proposition 12). Subsequently, we will present some further conditions on the internal part of the pictures in a NENO set (see Proposition 14). The obtained conditions, altogether, will be exploited to present an example of a NENO set in the next section.

Let us introduce the following definition on string sets, with the aim of investigating the frames of the pictures in NENO sets.

Definition 8. Let S_1, $S_2 \subseteq \Sigma^n$ and $S_1 \cap S_2 = \emptyset$. The pair (S_1, S_2) is *full* if for any $s \notin S_1 \cup S_2$, there exist $s_1 \in S_1$, $s_2 \in S_2$ such that s and s_1 overlap, and s and s_2 overlap.

Example 9. Let $\Sigma = \{0, 1\}$ and let $S_1, S_2 \subseteq \Sigma^n$ be the languages $S_1 = \{1^n\}$ and $S_2 = \{0w0 \mid w \in \{0, 1\}^{n-2}\}$. The pair (S_1, S_2) is cross-non-overlapping, since the strings in S_1 do not contain occurrences of symbol 0.

Let us show that it is also full. Let s be any string $s \notin S_1 \cup S_2$. Since $s \notin S_2$, three cases are possible, $s = 0x1$, $s = 1y0$, or $s = 1z1$. If $s = 0x1$ or $s = 1y0$ then s overlaps the string in S_1 and any string in S_2 with an overlap of length 1. If $s = 1z1$ then z contains at least one occurrence of 0, because $s \notin S_1$. Then $s = 1^k0r1$, for some $k \geq 1$ and $r \in \Sigma^*$. Therefore, s and 1^n overlap, and also s and $0r10^k$ overlap, with $0r10^k \in S_2$.

Example 10. Let $\Sigma = \{0, 1\}$ and let $S_3, S_4 \subseteq \Sigma^m$ be the languages $S_3 = \{110^{m-2}\}$ and $S_4 = \{1w0 \mid |w| = m - 2, w \neq 0^{m-2}, 1w0$ with no suffix in $110^+\}$. The pair (S_3, S_4) is cross-non-overlapping. Indeed, consider $s' = 110^{m-2} \in S_3$ and $s'' = 1w0 \in S_4$ and suppose that there exists u such that $s' = xu$ and $s'' = uy$ or $s' = ux$ and $s'' = yu$. In the first case, it must be $|u| = m - 1$ and, hence, $w = 0^{m-2}$ against the definition of S_4. In the second case, it must be $|u| \geq 3$ and, hence, w would have a suffix in 110^+ against the definition of S_4.

Let us show that the pair (S_3, S_4) is full. Let s be any string $s \notin S_3 \cup S_4$. If $s \in 0\Sigma^*$ or $s \in \Sigma^*1$ then s overlaps the string in S_3 and any string in S_4 with an overlap of length 1. Consider the case that $s = 1w0$. Since $s \notin S_4$, $w = 0^{m-2}$ or w has a suffix in 110^+. In the first case $s = 10^{m-1}$. Then s and the string in S_3 overlap with an overlap of length $m - 1$; and s overlaps any string in S_4 that has 010 as a suffix, with an overlap of length 2.

In the second case $s = 1x110^k$ with $k \geq 1$, and $x \in \Sigma^*$. Then s and the string in S_3 overlap with an overlap of length $k + 2$, and s and any string $10^ky0 \in S_4$, for some y, overlap with an overlap of length $k + 1$.

Recall that the frame of a language X is the quadruple of the sets of its first and last rows and columns, $frame(X) = (R_F(X), R_L(X), C_F(X), C_L(X))$.

Note that not any quadruple of string languages can be the frame of a set of pictures, since their strings need to match in the lengths and in the corner positions. Given four string languages $S_1, S_2 \subseteq \Sigma^n$ and $S_3, S_4 \subseteq \Sigma^m$, we say that the quadruple (S_1, S_2, S_3, S_4) is *frame-compatible* if there exists a picture language $X \subseteq \Sigma^{m,n}$ such that $frame(X) = (S_1, S_2, S_3, S_4)$. In the case of non-overlapping sets of pictures, the constrains are even stronger. As previously remarked (see Lemma 7), if X is a non-overlapping set on $\Sigma = \{0, 1\}$, then all the pictures in X have the same set of corners and, for this set, only four cases are possible, following which symbols appear in the corners of pictures in X. Hence, for example, a quadruple (S_1, S_2, S_3, S_4) can be the frame of a non-overlapping set X such that, for all $p \in X$, $corners(p) = (0, 0, 1, 1)$, if $S_1 \subseteq 0\Sigma^*0$, $S_2 \subseteq 1\Sigma^*1$, and $S_3, S_4 \subseteq 0\Sigma^*1$.

We give the following definition.

Definition 11. Let $X \subseteq \Sigma^{m,n}$. X is *frame-complete* if for any $p, q \in X$, p and q do not frame-overlap, and if for any picture $p \in \Sigma^{m,n} \setminus X$ there exists a picture $q \in X$ such that p and q frame-overlap.

Note that being frame-complete is a *sufficient condition* for a set X to be non-expandable with respect to the overlapping. Any picture not in the language overlaps the frame of some picture in the language. However, no warranty is given at this stage on the non-overlapping property of the pictures inside the set X. Frame-complete sets of pictures can be obtained due to the following proposition. Recall that (in view of Proposition 6) the pairs of languages that form the frame of a non-overlapping set of pictures are necessarily cross-non-overlapping.

Proposition 12. *Let $S_1, S_2 \subseteq \Sigma^n$, $S_3, S_4 \subseteq \Sigma^m$ and (S_1, S_2, S_3, S_4) be a quadruple of frame-compatible string languages.*

If the pairs (S_1, S_2) and (S_3, S_4) are cross non-overlapping and full then the set X of all the pictures p with $frame(p) \in S_1 \times S_2 \times S_3 \times S_4$ is frame-complete.

Proof. The frame-compatibility of (S_1, S_2, S_3, S_4) guarantees that X is not empty.

Moreover, since the pairs (S_1, S_2) and (S_3, S_4) are cross-non-overlapping, then, for any $p, q \in X$, we are sure that p and q do not frame-overlap.

Now let p be any picture $p \in \Sigma^{m,n} \setminus X$. Let us show that there exists a picture $q \in X$ such that p and q frame-overlap. The definition of X implies that $frame(p) \notin S_1 \times S_2 \times S_3 \times S_4$. Hence, either $r_F(p) \notin S_1$, or $r_L(p) \notin S_2$, or $c_F(p) \notin S_3$, or $c_L(p) \notin S_4$. Suppose without loss of generality that $r_F(p) \notin S_1$.

If also $r_F(p) \notin S_2$ then there exists $s \in S_2$ such that $r_F(p)$ and s overlap (because (S_1, S_2) is full). Consider any picture q in X with $r_L(q) = s$. Then, p and q frame-overlap.

If $r_F(p) \in S_2$ then consider any picture q in X with $r_L(q) = r_F(p)$. Then, p and q frame-overlap (and also v-slide overlap). □

Example 13. Let $\Sigma = \{0, 1\}$. Referring to the sets S_1, S_2, S_3, and S_4 in Examples 9 and 10, let $X(m, n) \subseteq \Sigma^{m,n}$, with $m, n \geq 4$, be the set of all the pictures with $R_F(X(m, n)) = S_1$, $R_L(X(m, n)) = S_2$, $C_F(X(m, n)) = S_3$, and $C_L(X(m, n)) = S_4$. The sets $R_F(X(m, n))$, $R_L(X(m, n))$, $C_F(X(m, n))$, $C_L(X(m, n))$ satisfy the conditions of Proposition 12 and therefore $X(m, n)$ is frame-complete.

The next proposition states some sufficient conditions for a subset of a frame-complete set to be NENO. The interest is now devoted also to the internal part of the pictures.

Proposition 14. *Let $X \subseteq \Sigma^{m,n}$ be a frame-complete set. If a subset Y of X is such that*

a) $frame(Y) = frame(X)$
b) Y is non-overlapping
c) for any $p \in X \setminus Y$ there exists $q \in Y$ such that p and q overlap

then Y is a NENO set.

Proof. The set Y is non-overlapping by condition b). Let us show that Y is non-expandable. Let p be any picture $p \in \Sigma^{m,n} \setminus Y$, and let us show that there exists $q \in Y$ such that p and q overlap. If $p \in X$ then condition c) implies the goal. Suppose $p \notin X$. Then there exists $x \in X$ such that p and x frame-overlap, because X is frame-complete. Condition a) guarantees that there exists $q \in Y$ such that $frame(q) = frame(x)$ and hence p and q frame-overlap, too. □

5. Construction of a family of NENO sets

In this section we exhibit a family of non-expandable non-overlapping sets over the binary alphabet $\Sigma = \{0, 1\}$. This is accomplished in two main steps.

The goal of the first step is to meet the non-expandability property, and it is accomplished by a proper definition of the frames of the pictures. It consists in the construction, for all $m, n \geq 4$, of a frame-complete set $X(m, n)$ of pictures. We choose a quadruple of frame-compatible string languages, (S_1, S_2, S_3, S_4), with $S_1, S_2 \subseteq \Sigma^n$ and $S_3, S_4 \subseteq \Sigma^m$, such that the pairs (S_1, S_2) and (S_3, S_4) are cross non-overlapping and full. Subsequently, we define the set $X(m, n)$ of all the pictures p with $frame(p) \in S_1 \times S_2 \times S_3 \times S_4$. The set $X(m, n)$ is frame-complete by Proposition 12.

In the subsequent step, the non-overlapping property is attained by imposing some conditions on the internal part of the pictures. To be more precise, the second step consists in selecting a subset $Y(m, n)$ of $X(m, n)$. If $Y(m, n)$ is chosen satisfying conditions a), b), and c) in Proposition 14, then Proposition 14 ensures that $Y(m, n)$ is a NENO set. Observe that the second step of the construction needs to balance accurately two opposite operations. On one side, one has to remove from $X(m, n)$ those pictures that overlap other pictures in the set. On the other hand, it is necessary not to remove “too many” pictures, in order to achieve the non-expandability property. The following family $X(m, n)$, for $m, n \geq 4$, reaches this goal.

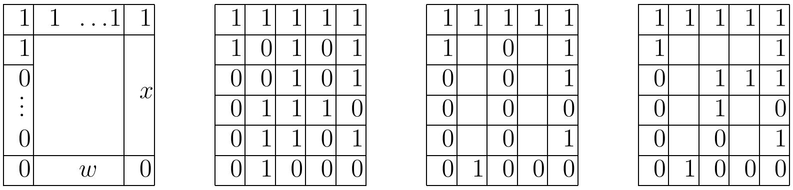

Fig. 2. From left to right: a generic picture in $X(m,n)$, a picture in $Y(m,n)$ and pictures in $X(m,n) \setminus Y(m,n)$ that violate condition 1) and 2), respectively.

Definition 15. Let $\Sigma = \{0, 1\}$ and $m, n \geq 4$.

Let $S_1, S_2 \subseteq \Sigma^n$ and $S_3, S_4 \subseteq \Sigma^m$ be the languages $S_1 = \{1^n\}$, $S_2 = \{0w0 \mid w \in \{0,1\}^{n-2}\}$, $S_3 = \{110^{m-2}\}$, and $S_4 = \{1w0 \mid |w| = m-2, w \neq 0^{m-2}, 1w0$ with no suffix in $110^+\}$.

Let $X(m,n) \subseteq \Sigma^{m,n}$ be the set of all the pictures with $R_F(X(m,n)) = S_1$, $R_L(X(m,n)) = S_2$, $C_F(X(m,n)) = S_3$, and $C_L(X(m,n)) = S_4$.

The set $Y(m,n) \subseteq X(m,n)$ is the set of all pictures $p \in X(m,n)$ such that

1) there exists no $2 \leq j \leq n$ such that $p(2, j) = p(3, j) = \cdots = p(m-1, j) = 0$
2) there exists no (i, j), with $(i, j) \neq (1, 1)$ such that $p(i, j) = p(i, j+1) = \cdots = p(i, n) = 1$ and $p(i, j)p(i+1, j) \cdots p(m, j) \in 110^*$.

Fig. 2 shows a generic picture in $X(m,n)$, a picture in $Y(m,n)$, and two pictures which are in $X(m,n)$, but are not in $Y(m,n)$ because they violate condition 1) and 2), respectively.

Note that $X(m,n) \subseteq \Sigma^{m,n}$ in the definition is the frame-complete set introduced in Example 13. The set $Y(m,n)$ has been extracted from $X(m,n)$ in such a way that it satisfies the sufficient conditions in Proposition 14. More precisely, conditions 1) and 2) in the definition of $Y(m,n)$ are designed in order to avoid the overlaps between two pictures in $Y(m,n)$. In particular, condition 1) prevents bl-overlaps. The bl-corner of a picture p in $X(m,n)$ cannot be placed inside another picture p' of $X(m,n)$, unless when it is placed on the $(m-1)$-th row (in all the other cases the 0's in the first column of p would meet the 1's in the first row of p'). Condition 1) prevents this possibility in the pictures in $Y(m,n)$. In a similar way, condition 2) forbids tl-overlaps. On the contrary, the pictures in $X(m,n)$ which do not satisfy conditions 1) and 2), i.e. the pictures in $X(m,n) \setminus Y(m,n)$, will necessarily overlap some pictures in $Y(m,n)$.

The following proposition shows, in a more accurate way, that $Y(m,n)$ satisfies the conditions a), b), and c) of Proposition 14, and then $Y(m,n)$ is a NENO set.

Proposition 16. *The language $Y(m,n)$ in Definition 15 is a NENO set, for any $m, n \geq 4$.*

Proof. Let $X(m,n)$ and $Y(m,n)$ be the languages defined in Definition 15. The set $X(m,n)$ is frame-complete, as shown in Example 13. Let us show that $Y(m,n)$ satisfies the conditions a), b), and c), of Proposition 14.

a) $frame(Y(m,n)) = frame(X(m,n))$.
Since $Y(m,n) \subseteq X(m,n)$, we have $frame(Y(m,n)) \subseteq frame(X(m,n))$. Let us show the inverse inclusion, namely, that for any $p \in X(m,n)$ there exists $q \in Y(m,n)$ with $frame(q) = frame(p)$. It is trivial if $p \in Y(m,n)$. Suppose $p \in X(m,n) \setminus Y(m,n)$. Then, p does not satisfy condition 1) or condition 2) (or both) in Definition 15. It is possible to obtain a picture $q \in Y(m,n)$ and $frame(q) = frame(p)$, by exchanging some symbols in p. More precisely, for any $1 \leq j \leq n$ such that $p(2, j) = p(3, j) = \cdots = p(m-1, j) = 0$, one can replace with 1 one occurrence of 0 among $p(3, j), \cdots, p(m-1, j)$. This replacement reduces the violations of condition 1) and does not introduce any violation to condition 2).
Suppose now that there exists (i, j), with $(i, j) \neq (1, 1)$ such that $p(i, j) = p(i, j+1) = \cdots = p(i, n) = 1$ and $p(i, j)p(i+1, j) \cdots p(m, j) \in 110^*$. Recalling that $m, n \geq 4$, there always exists one position among (i, j), $(i, j+1), \cdots, (i, n)$ and $(i+1, j), \cdots, (m, j)$ that is not on the frame of p. Exchanging the symbol in such position removes the violation and does not introduce any violation to condition 1).
b) *$Y(m,n)$ is non-overlapping.*
Let $p, q \in Y(m,n)$. The pictures p and q cannot frame-overlap, because the pairs $(R_F(X(m,n)), R_L(X(m,n)))$, $(C_F(X(m,n)), C_L(X(m,n)))$ are cross-non-overlapping (as shown in Examples 9 and 10).
The pictures p and q cannot h-slide overlap, because otherwise there would exist a position $(1, j)$ with $(i, j) \neq (1, 1)$ violating condition 2).
The pictures p and q cannot v-slide overlap because the first column of any picture in $Y(m,n)$ is unbordered.
Moreover, p and q cannot tl-overlap with an overlap of size (r, c) with $1 < r \leq m-1$ and $1 < c \leq n-1$. Suppose to the contrary that there exists x of size (r, c) with $1 < r \leq m-1$ and $1 < c \leq n-1$, such that x is a tl-prefix of p and a br-prefix of q.
Finally, p and q cannot bl-overlap with an overlap of size (r, c) with $1 < r \leq m-1$ and $1 < c \leq n-1$. Suppose to the contrary that there exists x of size (r, c) with $1 < r \leq m-1$ and $1 < c \leq n-1$, such that x is a bl-prefix of p and a tr-prefix of q.

Since $r_F(q) = 1^n$, r cannot be strictly less than $m-1$. On the other hand, if $r = m-1$ then the picture q violates condition 1) in the column $j = n-c+1$.

c) *For any $p \in X(m,n) \setminus Y(m,n)$ there exists $q \in Y(m,n)$ such that p and q overlap.*

If $p \in X(m,n) \setminus Y(m,n)$ then $frame(p) \in S_1 \times S_2 \times S_3 \times S_4$, but p does not satisfy condition 1) or condition 2) (or both) in Definition 15.

If p does not satisfy condition 1) then let j be the greatest index $2 \leq j \leq n$ such that $p(2,j) = p(3,j) = \cdots = p(m-1,j) = 0$. Then there exists $q \in Y(m,n)$ with the bl-prefix of q of size $(m-1, n-j+1)$ equal to the tr-prefix of p of size $(m-1, n-j+1)$. Note that since $j \neq 1$ the last column of q can be constructed so that no violation of condition 2) appears in q, by inserting some occurrences of 0's where necessary. Furthermore, such 0's are not necessary in each position $q(2,n), q(3,n), \cdots, q(m-1,n)$ since the last column of p cannot be in 1^+0^+.

If p satisfies condition 1), but not condition 2), then let (i,j) be the lowest among the rightmost positions such that $p(i,j) = p(i,j+1) = \cdots = p(i,n) = 1$ and $p(i,j)p(i+1,j)\cdots p(m,j) \in 110^*$. Then there exists $q \in Y(m,n)$ with the tl-prefix of q of size $(m-i+1, n-j+1)$ equal to the br-prefix of p of size $(m-i+1, n-j+1)$. $\square$

We conclude the section with some remarks. Following the results presented in Section 4, for any $m, n \geq 4$, we have extracted from the frame-complete set $X(m,n)$ a NENO set $Y(m,n)$, by imposing two conditions on the internal part of its pictures. Such conditions are designed so that the pictures in $X(m,n)$ that satisfy them cannot overlap one another, whereas as soon as a picture in $X(m,n)$ does not satisfy a condition, it necessarily overlaps a picture in $Y(m,n)$.

Let us emphasize that not any choice of the conditions has this nice property. There are conceivable conditions avoiding any overlap among pictures in $X(m,n)$ when satisfied, which do not necessarily imply an overlap among pictures in $Y(m,n)$ when they are not satisfied. As an example, consider the language $M(m,n) \subseteq X(m,n)$, defined by replacing in the definition of $Y(m,n)$, condition 1) with the following condition.

1)bis if there exists (i,j), with $(i,j) \neq (1,n)$ such that $p(i,1) = p(i,2) = \cdots = p(i,j) = 1$ then $p(i,j)p(i+1,j)\cdots p(m,j)$ has a suffix in 110^*.

Condition 1)bis seems to avoid bl-overlaps in a similar way as condition 1) does. Nevertheless, one can show that $M(m,n)$ is non-overlapping, but it is not non-expandable. A counter-example, for $m = n = 5$, is the picture $p = \begin{array}{|ccccc|} \hline 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ \hline \end{array}$. Picture p belongs to $X(m,n) \setminus M(m,n)$, but there is no picture $q \in X$ such that p and q overlap.

6. Cardinality of non-overlapping sets of pictures

In the previous sections we have discussed a construction method of NENO sets and presented a family of NENO sets. We consider now the question of how large this family is and how large a NENO set can be. The question has been extensively investigated in the string case. Let $C(n,q)$ denote the maximum size of a cross-bifix-free code of strings of length n over an alphabet of cardinality q. In [17], it is shown that $C(n,q) \leq \frac{q^n}{2n-1}$. The non-expandable cross-bifix-free sets of strings introduced in [15–17] have been compared with this bound.

Let us apply the upper bound on strings, in order to obtain a simple upper bound on the size of a non-overlapping set of pictures. Let $C(m,n,q)$ denote the maximum size of a non-overlapping set of pictures of size (m,n) over an alphabet of cardinality q.

Proposition 17. *Let m, n, q be integers and $N = \max\{m,n\}$. Then*

$$C(m,n,q) \leq \frac{q^{mn}}{2N-1}.$$

Proof. Let $X \subseteq \Sigma^{m,n}$ and $|\Sigma| = q$. Note that X can be viewed as a set $col(X)$ of strings of length n on the alphabet of the columns Σ^m, and $|\Sigma^m| = q^m$. The key observation is that if X is non-overlapping then $col(X)$ is cross-bifix-free. Actually, an overlapping of two strings in $col(X)$ would give a h-slide-overlap of two pictures in X. Therefore, applying the upper bound given in [17] on the cardinality of the cross-bifix-free sets of strings, $|X| \leq \frac{(q^m)^n}{2n-1}$. In an analogous manner, X can be viewed as a set $row(X)$ of strings of length m on the alphabet of the rows Σ^n, and $|\Sigma^n| = q^n$. Hence, $|X| \leq \frac{(q^n)^m}{2m-1}$. Finally, $|X| \leq \min\{\frac{q^{mn}}{2n-1}, \frac{q^{mn}}{2m-1}\} = \frac{q^{mn}}{2N-1}$. $\square$

Let us now evaluate the cardinality of the NENO sets $Y(m,n)$ introduced in Definition 15. We are going to present a lower bound on the cardinality of $Y(m,n)$.

Let $\Sigma = \{0,1\}$. Define the set $Z(m,n)$ of all the pictures $z \in \Sigma^{m,n}$ with $frame(z) \in frame(Y(m,n))$ and such that

1) there exists no $2 \le j \le n$ such that $p(2, j) = p(3, j) = \cdots = p(m-1, j) = 0$
2)bis there exists no (i, j) such that $p(i, j)p(i+1, j)\cdots p(m, j) \in 110^*$, with $(i, j) \neq (1, 1)$.

Observe that $Z(m,n) \subseteq Y(m,n)$, since condition 2)bis implies condition 2) in the definition of $Y(m,n)$. Therefore, $|Y(m,n)| \ge |Z(m,n)|$. In the next proposition we will evaluate $|Z(m,n)|$, obtaining a lower bound on $|Y(m,n)|$.

Proposition 18. *Let $Y(m,n)$ be the NENO set in Definition 15. Then*

$$|Y(m,n)| \ge \left(2^{m-2}-1\right)^{n-2} \cdot\ 2^{m-3}$$

Proof. Let $\Sigma = \{0, 1\}$ and $Z(m,n) \subseteq \Sigma^{m,n}$ be the set defined above. We prove that $|Z(m,n)| = \left(2^{m-2}-1\right)^{n-2} \cdot\ 2^{m-3}$. This shows the statement since $Z(m,n) \subseteq Y(m,n)$.

Define the sets $I(m), L(m) \subseteq \Sigma^m$ as follows.

$I(m) = \{1ya \mid a \in \Sigma,\ y \neq 0^{m-2} \text{ and } 1ya \text{ with no suffix in } 110^*\}$

$L(m) = \{1x0 \mid x \neq 0^{m-2} \text{ and } 1x0 \text{ with no suffix in } 110^+\}$.

Consider any picture $z \in Z(m,n)$. The first column of z is 110^{m-2}; the internal columns of z, that is the columns with index $2 \le j \le n-1$, are all strings in $I(m)$, while the last column of z is a string in $L(m)$.

Let us show that $|I(m)| = 2^{m-2} - 1$. There are $\sum_{i=0}^{m-3} 2^i = 2^{m-2} - 1$ strings of the form $1ya$ and length m with a suffix in 110^+, while there are 2 strings of this form and length with $y = 0^{m-2}$ are 2. Therefore, $|I(m)| = 2^{m-1} - \left(2^{m-2}-1\right) - 2 = 2^{m-2} - 1$.

Let us show that $|L(m)| = 2^{m-3}$. There are $\sum_{i=0}^{m-4} 2^i = 2^{m-3} - 1$ strings of the form $1x0$ and length m with a suffix in 110^*, while there is just one string of this form and length with $x = 0^{m-2}$. Therefore, $|L(m)| = 2^{m-2} - \left(2^{m-3}-1\right) - 1 = 2^{m-3}$.

Any picture $z \in Z(m,n)$ is just the catenation of the column 110^{m-2} with $n-2$ columns in $I(m)$ and then a column in $L(m)$, with no other constraints. Hence $|Z(m,n)| = |I(m)|^{n-2} \cdot |L(m)| = \left(2^{m-2}-1\right)^{n-2} \cdot\ 2^{m-3}$. $\square$

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