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# Pseudoradial vs strongly pseudoradial

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Dedicated to our friend Filippo Cammaroto

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ABSTRACT

We consider the notion of strongly pseudoradial space. Among other things, we examine its relation with various similar notions, including the weak Whyburn property. Our investigation will suggest several interesting questions.



## 1. Introduction

In [16] J. Brazas and the second author have recently introduced and studied strongly pseudoradial spaces, which form a quite natural subclass of pseudoradial spaces. Roughly speaking, the difference between them is to replace in the latter transfinite converging sequences with compact ordinals. As a countable convergent sequence is always a topological copy of $\omega + 1$, we immediately see that every sequential space is strongly pseudoradial. In [16] the major emphasis was made in the categorical structure, while here we will focus on

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the topological properties. For instance, it turns out that strongly pseudoradial spaces have a nice behavior in products, see Theorems 4.4 and 4.10. We will also compare this notion with other similar ones, including the weak Whyburn property.

Several questions, not only related to strongly pseudoradial spaces, are formulated.

## 2. Preliminaries and basic definitions

All spaces here are assumed $T_2$. For notations and undefined concepts we refer to [22] and [19].

A transfinite sequence $\langle x_\alpha : \alpha < \kappa \rangle$ converges to a point $x$ provided that for any neighborhood $U$ of $x$ there exists some $\beta < \kappa$ such that $\{x_\alpha : \beta < \alpha < \kappa\} \subseteq U$.

Recall that a topological space $X$ is *pseudoradial* if for every non-closed subset $A \subseteq X$ there is a point $x \in \overline{A} \setminus A$ and a *(transfinite) sequence* $S \subseteq A$ converging to $x$.

The systematic investigation of the topological properties of these spaces was initiated by Arhangel'skiĭ more than 40 years ago. Since then, several subclasses of pseudoradial spaces have been considered by many authors (see e.g. [14]). Some of them will be considered here and mentioned below.

Very recently the further notion of *strongly pseudoradial space* appeared in the literature [16]. However, this notion seems to have been considered already in 1970 by S. Mrówka under the name of $\mathfrak{R}_w$ [25].

**Definition 2.1.** A topological space $X$ is called *strongly pseudoradial* if for any non-closed subset $A \subseteq X$ there is a limit ordinal $\gamma$ and a continuous map $f : \gamma + 1 = \gamma \cup \{\infty_\gamma\} \to X$ such that $f(\gamma) \subseteq A$ and $f(\infty_\gamma) \notin A$ (here we write $\gamma + 1 = \gamma \cup \{\infty_\gamma\}$ to avoid confusion). The ordinal $\gamma + 1$ is considered here as a topological space with the order topology.

In [16] the authors pointed out that, without any loss of generality, in the above definition the ordinal $\gamma$ can be assumed to be a regular cardinal and the function $f$ injective.

Observe that in the usual definition of pseudoradial spaces, we actually consider on the ordinal $\kappa + 1$ the so called *directed topology*, where points $\beta < \kappa$ are isolated while neighborhoods of $\infty_\kappa$ are the sets $]\beta, \kappa] = ]\beta, \infty_\kappa]$, $\beta < \kappa$. Obviously, in the uncountable case, this topology is strictly finer than the order topology.

By replacing in the definition of pseudoradial space *sequence* with *thin sequence* or *essential sequence*, we obtain the notion of *almost radial space* [2] and *essentially pseudoradial space* [13].

A sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$ converging to $x$ is *thin* if for any $\beta < \kappa$ we have that $x \notin \overline{\{x_\alpha : \alpha < \beta\}}$.

A sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$ converging to $x$ is *essential* [13] if it is injective and $\overline{\{x_\alpha : \alpha < \kappa\}} = \{x_\alpha : \alpha < \kappa\} \cup \{x\}$.

Of course, a countable convergent sequence is thin and, if injective, essential.

The radial character of a pseudoradial space $X$, denoted by $\chi_R(X)$, is the smallest cardinal $\kappa$ such that the definition of pseudoradiality for $X$ works by taking only transfinite sequences of length not exceeding $\kappa$.

Clearly, the pseudoradial spaces of countable radial character are precisely the sequential spaces.

Finally, the classes of spaces that are now known as Whyburn and weakly Whyburn were independently discovered several times under different names. Gordon T. Whyburn in 1955 introduced spaces *accessible by closed sets* that he called *spaces having property H* [40]. Later on, in the class of $T_1$ spaces, Whyburn considered the *accessibility spaces* or spaces *approximately accessible by closed sets* [41]; for regular spaces, accessibility spaces coincide with spaces having property H. Siwiec and Mancuso in their paper [36] cite accessibility spaces and remember their characterization in terms of maps. In 1982 Okromeshko [30] gave a characterization of such spaces. In 1987 Dimov, Isler and Tironi [17] introduced the classes of *gF-spaces* and *gs-spaces* (to be identified with the Whyburn and weakly Whyburn spaces respectively) and briefly described some of their properties. In 1993 Pultr and Tozzi [32] introduced spaces having the property of *approximation by points* (AP) and shortly after Simon [35] studied also spaces having the property of *weak approximation by points* (WAP). Then, Bella [5] pointed out the nice relationship between WAP and pseudoradial spaces.

AP are the same as gF-spaces and are called now Whyburn spaces; WAP are the same as the gs-spaces and are called now weakly Whyburn spaces. Names for these classes of spaces continued to fluctuate until A.V. Arhangel'skiĭ noticed and communicated to the authors of [33] that the concept of AP space was first introduced by Whyburn.

**Definition 2.2.** A topological space $X$ is said to be Whyburn if for every non-closed subset $A$ of $X$ and every $x \in \overline{A} \setminus A$ there is a subset $B \subseteq A$ such that $\{x\} = \overline{B} \setminus A$.

A topological space $X$ is said to be weakly Whyburn if for every non-closed subset $A$ of $X$ there is $x \in \overline{A} \setminus A$ and a subset $B \subseteq A$ such that $\{x\} = \overline{B} \setminus A$.

Whyburn spaces are a generalization of Fréchet–Urysohn spaces and weakly Whyburn spaces are a generalization of sequential spaces. Clearly every Whyburn space is weakly Whyburn. A space $X$ is hereditarily weakly Whyburn if any subset $Y \subseteq X$ is weakly Whyburn. Any Whyburn space is hereditarily weakly Whyburn, but it is not true that any hereditarily weakly Whyburn is Whyburn. A simple example is the space $\omega_1 + 1$ with the order topology (see [39,29]).

**Remark 2.3.** Taking into account the definition of radial space, one may be tempted to consider the analogous notion of strongly radial space, by requiring that for any set $A \subseteq X$ and any point $x \in \overline{A} \setminus A$ there is a limit ordinal $\gamma$ and a continuous function $f : \gamma + 1 \to X$ satisfying $f(\gamma) \subseteq A$ and $f(\infty_\gamma) = x$. However, it turns out that such a definition is meaningless. Arguing as in Theorem 3.1, we see that a strongly radial space should be Whyburn. But every compact Whyburn space is actually Fréchet [15, Theorem 1.1]. This suffices to conclude that a space is strongly radial if and only if it is Fréchet.

## 3. Strongly pseudoradial spaces: general facts and relations with other notions

As already mentioned, a strongly pseudoradial space is basically a pseudoradial space in which the closure of a set is determined by using homeomorphic copies of compact ordinals. This gives several additional properties to such spaces.

Recall that a space $X$ is a $k$-space provided that a set $A \subseteq X$ is closed whenever $A \cap K$ is closed for each compact set $K$.

**Theorem 3.1.** *A strongly pseudoradial space is essentially pseudoradial, almost radial, weakly Whyburn and a $k$-space.*

**Proof.** Let $X$ be a space and $A \subseteq X$ be a non-closed set. If $X$ is strongly pseudoradial, then there is a regular cardinal $\kappa$ and a continuous injective map $f : \kappa \cup \{\infty_\kappa\} \to X$ such that $f(\kappa) \subseteq A$ and $f(\infty_\kappa) \in \overline{A} \setminus A$. As we are assuming that $X$ is $T_2$, we see that the set $S = f(\kappa \cup \{\infty_\kappa\})$ is a closed subset of $X$ homeomorphic to $\kappa + 1 = \kappa \cup \{\infty_\kappa\}$ with the order topology. This immediately shows that $S$ is both a thin and an essential sequence in $X$. Furthermore, the set $S$ also witnesses that $X$ is weakly Whyburn and a $k$-space. □

The one-point Lindelöfication of an uncountable discrete space $D$ is almost radial, essentially pseudoradial, Whyburn, but not strongly pseudoradial. Indeed, let $X = D \cup \{p\}$ be such space. It suffices to observe that every compact set in $X$ is finite and therefore Theorem 3.1 would imply that $X$ is discrete.

A nicer example, which is also a topological group, is the space $C_p(\kappa)$, where $\kappa$ is a regular $\omega$-inaccessible cardinal with the order topology. Recall that $\omega$-inaccessible means that $\lambda^\omega < \kappa$ for every $\lambda < \kappa$. $C_p(\kappa)$ is not sequential, but it is almost radial [18] and weakly Whyburn [12]. However, $C_p(\kappa)$ cannot be strongly pseudoradial because by Theorem 3.1 it should be a $k$-space and Pytkeev [34] has shown that if $C_p(X)$ is a $k$-space, then it is sequential. An argument used in [6] shows that under certain circumstances the above $C_p(\kappa)$ can be even essentially pseudoradial.

For compact spaces the situation appears much more delicate.

**Question 3.2.** Find a compact essentially pseudoradial and almost radial space which is not strongly pseudoradial.

An important subclass of almost radial spaces is in the next definition.

A space $X$ is *semiradial* if every non-$\kappa$-closed subset $A$ contains a sequence $S$ converging to a point outside $A$ and satisfying $|S| \leq \kappa$. As mentioned below, these spaces play an important role when dealing with the product operation (see Theorem 4.2).

In view of Theorem 3.1, one may wonder whether strongly pseudoradial spaces are also semiradial.

It turns out that this is false even within the class of compact spaces.

Recall that a tower is a well-ordered, by reverse almost inclusion, family of infinite subsets of $\omega$ with no infinite pseudo intersection. In other words, the family $\{B_\alpha : \alpha \in \gamma\}$ of infinite subsets of $\omega$ is a tower provided that:

(i) if $\alpha < \beta$, then $|B_\alpha \setminus B_\beta| = \omega$ and $|B_\beta \setminus B_\alpha| < \omega$;

(ii) if $A$ is an infinite subset of $\omega$, then there exists some $\alpha$ such that $|A \setminus B_\alpha| = \omega$. The smallest cardinality of a tower is denoted by $\mathfrak{t}$.

**Example 3.3.** A compact strongly pseudoradial space which is not semiradial.

*Construction:* Let $\{A_\alpha : \alpha \in \mathfrak{t}\}$ be a family of subsets of $\omega$ such that $\{\omega \setminus A_\alpha : \alpha \in \mathfrak{t}\}$ is a tower. Furthermore put $A_\mathfrak{t} = \omega$. Define a topology on the set $X = \omega \cup \{x_\alpha : \alpha \leq \mathfrak{t}\}$ by declaring each point of $\omega$ isolated and taking as a local base at $x_\alpha$ the sets $\{x_\gamma : \beta < \gamma \leq \alpha\} \cup (A_\alpha \setminus (A_\beta \cup F))$, where $F$ is a finite subset of $\omega$ and $\beta < \alpha$. It is easy to see that the space $X$ is compact and Hausdorff.

We check first that $X$ is not semiradial. This happens because the set $X \setminus \{x_\mathfrak{t}\}$ is not $\omega$-closed, but no countable sequence in it converges to $x_\mathfrak{t}$. Indeed, let $S \subset X \setminus \{x_\mathfrak{t}\}$ be a countable set. If $S \subset X \setminus \omega$, then it cannot converge to $x_\mathfrak{t}$ because $X \setminus \omega$ is a topological copy of $\mathfrak{t}+1$ and $\mathfrak{t}$ has uncountable cofinality. If $S \subset \omega$, then by property (ii) there exists some $\alpha \in \mathfrak{t}$ such that $|S \cap A_\alpha| = \omega$ and we see that the set $S \cap A_\alpha$ is a sequence converging to $x_\alpha$. So, again $S$ cannot converge to $x_\mathfrak{t}$.

Now, let us prove that $X$ is strongly pseudoradial. Take a non-closed set $A$ and let $\gamma$ be the smallest ordinal such that $x_\gamma \in \overline{A} \setminus A$.

Case 1: $x_\gamma \in \overline{A \setminus \omega}$. Since $X \setminus \omega$ is homeomorphic to the set $\mathfrak{t}+1$ equipped with the order topology, we may fix a minimal cofinal set $C \subseteq A \cap \{x_\alpha : \alpha < \gamma\}$. The order type of $C$ is a cardinal $\kappa$ and it is easy to see that the isomorphism $f : \kappa \to C$ induces a continuous injection $\overline{f} : \kappa \cup \{\infty_\kappa\} \to C \cup \{x_\gamma\}$ and obviously $f(\kappa) = C$ converges to $x_\gamma$. Thus, in this case we have found the desired "strong sequence" converging to $x_\gamma$.

Case 2: $x_\gamma \notin \overline{A \setminus \omega}$. Then there exists some $\beta < \gamma$ such that $x_\alpha \notin A$ for all $\beta < \alpha < \gamma$. By the minimality of $\gamma$, we see that the set $\{x_\gamma\} \cup (A \cap (A_\gamma \setminus A_\beta))$ is closed in $X$ and hence compact. Moreover, this set is countable and has $x_\gamma$ as the only non-isolated point. Therefore, $A \cap (A_\gamma \setminus A_\beta)$ is a sequence converging to $x_\gamma$. This suffices, since ordinary sequences are "strong sequences". Thus $X$ is strongly pseudoradial. □

As we observed in the introduction, all sequential spaces are strongly pseudoradial, but this is the only case when we can easily determine if a space has this property. The passage from $\omega + 1$ to the successor of an uncountable cardinal appears much more difficult. However, a remarkable consequence of the proper forcing axiom (PFA) describes a possibility to do it for $\kappa = \omega_1$ (see also [27], Theorem 5.14).

**Theorem 3.4.** *[3] PFA implies that in every countably compact $T_3$ space of character at most $\omega_1$ the closure of a subset $A$ can be obtained by first adding all limits of convergent sequences and then adding to the resulting set $\hat{A}$ all points $x$ for which there is a copy $W$ of $\omega_1$ in $\hat{A}$ such that $W \cup \{x\}$ is homeomorphic to $\omega_1 + 1$.*

**Corollary 3.5.** *If PFA holds, then every countably compact $T_3$ space of character at most $\omega_1$ is strongly pseudoradial.*

In view of Theorem 3.4, it may be possible to obtain the answer to the following special case of Question 3.2.

**Question 3.6.** PFA. Let $X$ be a compact essentially radial and almost radial space of radial character $\omega_1$. Is $X$ strongly pseudoradial?

## 4. Products

A central and old problem is whether the class of compact pseudoradial spaces is finitely productive. This problem is still open in ZFC. In this direction, the best known facts are:

**Theorem 4.1** *(Juhász–Szentmiklossy). [24, $\mathfrak{c} \leq \omega_2$] The product of countably many compact pseudoradial spaces is pseudoradial.*

**Theorem 4.2** *(Bella–Gerlits). [11] The product of a compact pseudoradial space and a compact semiradial space is pseudoradial.*

**Theorem 4.3** *(Bella–Dow–Tironi). [10, $\mathfrak{c} \leq \mathfrak{p}^+$] The product of two compact pseudoradial spaces is pseudoradial provided that one of them has radial character not exceeding $\omega_1$.*

The notion of strongly pseudoradiality enables us to add one more non-trivial item to the previous list, as well as other results concerning the product operation.

**Theorem 4.4.** *If $X$ is a compact pseudoradial space and $Y$ a strongly pseudoradial space, then $X \times Y$ is pseudoradial.*

**Proof.** Denote by $\pi_Y : X \times Y \to Y$ the projection. Let $A$ be a non-closed subset of $X \times Y$ and fix $\langle x, y\rangle \in \overline{A} \setminus A$. If $\langle x, y\rangle \in \overline{A \cap (X \times \{y\})}$, then, since $X \times \{y\} \equiv X$ is pseudoradial, there exists a transfinite sequence in $A \cap (X \times \{y\}) \subseteq A$ which converges to some point $\langle x', y\rangle \notin A$. If $\langle x, y\rangle \notin \overline{A \cap (X \times \{y\})}$, then there exists a closed neighborhood $V$ of $x$ such that $(V \times \{y\}) \cap A = \emptyset$. Replacing $A$ with $A \cap (V \times Y)$, we can assume that $y \notin \pi_Y[A]$. Since $y \in \overline{\pi_Y[A]}$, it follows that $\pi_Y[A]$ is not closed. Now, being $Y$ strongly pseudoradial, there exists a cardinal $\kappa$ and a continuous function $f : \kappa + 1 = \kappa \cup \{\infty_\kappa\} \to Y$ such that $f(\kappa) \subseteq \pi_Y[A]$ and $f(\infty_\kappa) \notin \pi_Y[A]$. Let $Z = f(\kappa + 1)$ and $\phi : X \times (\kappa + 1) \to X \times Z$ be the function defined by $\phi(x, \alpha) = \langle x, f(\alpha)\rangle$ for every $\alpha \leq \kappa + 1$. By Theorem 4.2 the space $X \times (\kappa + 1)$ is pseudoradial. Furthermore, the set $B = A \cap (X \times Z)$ is not closed and, being $\phi$ closed and surjective, even $\phi^{-1}(B)$ is not closed. Therefore, there exists a transfinite sequence $S \subseteq \phi^{-1}(B)$ converging to some point $p \notin \phi^{-1}(B)$. Now, the set $\phi(S) \subseteq B \subseteq A$ is a sequence converging to $\phi(p) \notin B$. But $B = A \cap (X \times Z)$ and consequently $\phi(p) \notin A$. This completes the proof. □

We wish to emphasize that Theorem 4.4 requires compactness in the first factor only. This cannot be done in Theorem 4.2: The product of the unit interval $[0, 1]$ (a very nice compact pseudoradial space) with the one-point Lindelöfication of a discrete space of size $\aleph_1$ (which is obviously semiradial) is not pseudoradial. Linked to the previous theorem is the following:

**Question 4.5.** Is the product of a compact almost radial space with a strongly pseudoradial space almost radial?

Indeed, we cannot mimic the proof of Theorem 4.4. To do it, we need to have a positive answer to the following:

**Question 4.6.** Is it true that a continuous image of a compact almost radial space is still almost radial?

Notice that, as already observed in [17], an open (hence quotient) image of an almost radial space may fail to be almost radial.

A non-trivial result in [16] is:

**Lemma 4.7.** *[16, Lemma 6.5] The product of two compact ordinals is strongly pseudoradial.*

Here we will show that a much more general result holds (Theorem 4.10 below). Let $X$ be a space. A set $A \subseteq X$ is *SPR-closed* if for every infinite cardinal $\kappa$ and every continuous function $f : \kappa+1 \to X$, $f(\kappa) \subseteq A$ implies $f(\kappa+1) \subseteq A$. Of course, a space is strongly pseudoradial whenever every SPR-closed set is closed.

**Lemma 4.8.** *Suppose $X$ is strongly pseudoradial, suppose $f : X \to Y$ is continuous, and $A \subseteq Y$ is SPR-closed. Then $f^{-1}(A)$ is closed in $X$.*

**Proof.** Let $B = f^{-1}(A)$. Suppose $\kappa$ is an infinite cardinal, $g : \kappa+1 \to X$ a continuous function and $g(\kappa) \subseteq B$. Since $A$ is SPR-closed and $f \circ g(\kappa) \subseteq A$, we have $f \circ g(\kappa+1) \subseteq A$. This means $g(\kappa+1) \subseteq B$ and so $B$ is SPR-closed. Since $X$ is strongly pseudoradial, $B$ is actually closed. □

**Lemma 4.9.** *If $X$ is a compact ordinal and $Y$ a strongly pseudoradial space, then $X \times Y$ is strongly pseudoradial.*

**Proof.** Suppose $Z \subseteq X \times Y$ is SPR-closed. Suppose $\langle x, y\rangle \notin Z$. Since $X$ is $T_3$, there is an open $U \ni x$ so that $Z \cap (\overline{U} \times \{y\}) = \emptyset$. Let $B = \{y \in Y : (\overline{U} \times y) \cap Z \neq \emptyset\}$. Suppose $\kappa$ is an infinite cardinal and $f : \kappa+1 \to Y$ is a continuous function satisfying $f(\kappa) \subseteq B$. By Lemma 4.7 $id \times f : \overline{U} \times (\kappa+1) \to X \times Y$ is a continuous function defined on a strongly pseudoradial space. Thus by Lemma 4.8 $Z' = (id \times f)^{-1}(Z)$ is closed. Let $C \subset \kappa+1$ denote the image of $Z'$ under the (closed) projection $\overline{U} \times (\kappa+1) \to \kappa+1$. As $C$ is closed, we must have $C = \kappa+1$. Therefore, $f(\kappa+1) \subseteq B$ and so $B$ is SPR-closed. Since $Y$ is strongly pseudoradial, $B$ is actually closed in $Y$. Note $\langle x, y\rangle \in U \times (Y \setminus B)$, and $U \times (Y \setminus B)$ is open and disjoint from $Z$, and hence $Z$ is closed in $X \times Y$. □

**Theorem 4.10.** *If $X$ is compact strongly pseudoradial and $Y$ is strongly pseudoradial, then $X \times Y$ is strongly pseudoradial.*

**Proof.** It is enough to argue as in the proof of Theorem 4.4, by using Lemma 4.9 and the fact that strong pseudoradiality is preserved by quotient images. □

**Corollary 4.11.** *The class of compact strongly pseudoradial spaces is finitely productive.*

Another partial positive result can be obtained by weakening the compactness hypothesis in Theorem 4.10, as follows:

**Theorem 4.12.** *If $X$ is a countably compact $T_3$ strongly pseudoradial space and $Y$ a sequential space, then $X \times Y$ is strongly pseudoradial.*

**Proof.** Denote by $\pi_Y : X \times Y \to Y$ the projection. Let $A$ be a non-closed subset of $X \times Y$ and fix $\langle x, y \rangle \in \overline{A} \setminus A$. Arguing as at the beginning of the proof of Theorem 4.4, we can assume that $y \notin \pi_Y[A]$. Since $y \in \overline{\pi_Y[A]}$, it follows that $\pi_Y[A]$ is not closed. Now, being $Y$ sequential, there exists a countable set $C \subseteq A$ such that $\pi_Y[C]$ is a sequence converging to some point $y' \notin \pi_Y[A]$. Since $X$ is clearly sequentially compact, even $X \times (\pi_Y[C] \cup \{y'\})$ is sequentially compact. Therefore, $C$ contains a convergent subsequence $S$. The limit point of $S$ is outside $A$ and we are done. $\square$

In Theorems 4.10 and 4.12 the compactness assumption in the first factor cannot be omitted: Example 3.3.29 in [22] provides two $T_3$ sequential spaces whose product is not a $k$-space, therefore by Theorem 3.1 such a product cannot be strongly pseudoradial.

## 5. Miscellanea

First of all, we mention some more examples.

**Example 5.1.** There are examples of spaces that are zero-dimensional, separable, pseudoradial, weakly Whyburn, but not almost radial (and hence not strongly pseudoradial). Such a space was constructed by Juhász and Weiss [23]. (See also [37] for a Hausdorff example.)

**Proof.** The space is $Z = X \cup \{\infty\}$ where $X$ is a locally compact, locally countable, countably tight, zero-dimensional Hausdorff space, whose closed sets have cardinality either $\leq \aleph_0$ or the continuum. Also $Z$ has countable tightness. A neighborhood base of the point $\infty$ consist of all sets of the form $\{\infty\} \cup (X \setminus A)$, where $A$ is a closed countable set in $X$. No sequence can converge to the point $\infty$. In fact its range $B = \{x_n : n \in \omega\}$ is a closed discrete set and $\{\infty\} \cup (X \setminus B)$ is a neighborhood of $\infty$ disjoint with $B$. However the space is weakly Whyburn. Given any non-closed set $A$, either there is $x \in (\overline{A} \setminus A) \cap X$, and a countable sequence with range in $A$ converging to $x$, or the only point in $\overline{A} \setminus A$ is $\infty$ and so $B = A$ shows the space is weakly Whyburn. A well ordering of $A$ in type $\mathfrak{c}$, also shows that the space is essentially pseudoradial (see [13]). As $Z$ is a space of countable tightness that is not sequential, it is not almost radial. So it is not even strongly pseudoradial. $\square$

Similar considerations (see [13]) can be made on the one-point compactifications of Ostaszewski's space, constructed using $\diamondsuit$ [31]. The advantage here is to have a compact example with all the properties of the previous one.

We finish with some more remarks and questions on weakly Whyburn spaces. Let us recall the following:

**Proposition 5.2.** *[5] Any compact weakly Whyburn space is pseudoradial.*

The above result cannot be reversed. Indeed, Dow [20] managed to construct, under $\diamondsuit$, a compact pseudoradial space which is not weakly Whyburn.

However, it is still unclear the relation between almost radial and weakly Whyburn spaces. An easy inequality, true for every almost radial space $X$ is $|X| \leq d(X)^{t(X)}$ [6]. Here $d(X)$ and $t(X)$ denote density and tightness of $X$. More recently, Alas et al. [1] were able to prove that the same inequality holds for weakly Whyburn spaces. Incidentally, the latter result can be used to obtain another compact pseudoradial not weakly Whyburn space (this time consistent with MA $+\neg$ CH), as a by-product of a recent result of Dow [21]: there is a model with a compact separable space of countable tightness and cardinality bigger than $\mathfrak{c} = \omega_2$. By the result in [1], this space is not weakly Whyburn. In addition, in this space each infinite compact subspace has a point of first countability and so it is sequentially compact, hence pseudoradial because $\mathfrak{c} = \omega_2$ and a Theorem in [24]. The example also shows that the inequality $|X| \leq d(X)^{t(X)}$ may fail for a compact pseudoradial space!

Certainly, Dow's example is not almost radial and one may then wonder whether there is a compact almost radial space which is not weakly Whyburn. Perhaps, such a space does not exist, but the interesting point here is that we do not know the answer even to the following very natural question. Surprisingly, it was not considered so far.

**Question 5.3.** Is there an almost radial not weakly Whyburn space?

Both examples of Dow are compact spaces. With no compactness, one may expect to obtain such a counterexample quite easily in ZFC, but nothing exists at moment in the literature.

**Question 5.4.** Is there in ZFC a pseudoradial not weakly Whyburn space?

Still in connection with Proposition 5.2, it is worth mentioning that compactness is essential there. Spadaro has recently constructed a $T_3$ countably compact weakly Whyburn space which is not pseudoradial [38].

Notice that a semiradial space is always weakly Whyburn [5]. In [5] it was also shown that the product of a compact weakly Whyburn space and a compact semiradial space is weakly Whyburn and consequently the product of finitely many compact semiradial spaces is weakly Whyburn. Furthermore, in [7] it is established that the product of countably many compact semiradial spaces is almost radial. We strongly believe the next question should have a positive answer.

**Question 5.5.** Is the product of countably many compact semiradial spaces weakly Whyburn?

Using the second result of [5] mentioned above and mimicking the proof of Theorem 4.4, we obtain the following:

**Theorem 5.6.** *The product of a compact weakly Whyburn space with a strongly pseudoradial space is weakly Whyburn.*

The behavior of the weak Whyburn property in product is more intriguing than the pseudoradial case. For instance, we do not have a result analogous to Theorem 4.1.

**Question 5.7.** Let $X$ and $Y$ be weakly Whyburn (compact) spaces and assume that $X \times Y$ is pseudoradial. Is it true that $X \times Y$ is weakly Whyburn?

The Tychonoff cube $2^{\omega_1}$ has two nice characterizations: $2^{\omega_1}$ is pseudoradial if and only if $\mathfrak{s} > \omega_1$ [26, Theorem 3] and $2^{\omega_1}$ is semiradial if and only if $\mathfrak{p} > \omega_1$ [8, Theorem 1]. In addition, by Corollary 3.5, if PFA holds, then $2^{\omega_1}$ is even strongly pseudoradial.

Under $\mathfrak{p} > \omega_1$, $2^{\omega_1}$ is indeed a compact separable semiradial not sequential space. Another example of this kind is constructed in [27] under the assumption $\mathfrak{d} = \omega_1$. What remains open is the following:

**Question 5.8** *(ZFC).* Is there a compact separable semiradial not sequential space?

A more general (and perhaps easier) version of this question asks for a compact semiradial non-R-monolithic space. In an attempt to find a positive solution, Nyikos in [28] constructed a space of this kind by means of a nice application of PCF theory. Unfortunately, this example requires the axiom $\square_{\aleph_\omega}$, so the ZFC case remains still open.

A pseudoradial space $X$ is R-monolithic provided that $\chi_R(\overline{A}) \leq |A|$ holds for every $A \subseteq X$. Notice that, a separable R-monolithic space is sequential. This explains why R-monolithicity serves to generalize Question 5.8. The reader may consult [4] and [9] for some further results related to these spaces.

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