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On the Regularity of Weak Solutions of the Boussinesq Equations in Besov Spaces

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Abstract The main issue addressed in this paper concerns an extension of a result by Zhang who proved, in the context of the homogeneous Besov space $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, that, if the solution of the Boussinesq equation (1) below (starting with an initial data in H^2) is such that $(\nabla u, \nabla \theta) \in L^2(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$, then the solution remains smooth forever after T. In this contribution, we prove the same result for weak solutions just by assuming the condition on the velocity u and not on the temperature θ.

Keywords Boussinesq equations · Besov space · Weak solution · Regularity criterion

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Dedicated to Enrique Zuazua on the occasion of his sixtieth birthday.

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1 Introduction

We are interested in the regularity of weak solutions of the Cauchy problem related to the Boussinesq equations in $\mathbb{R}^3$:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla\pi = \theta e_3, \\ \partial_t \theta + (u\cdot\nabla)\theta - \Delta\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^3$ and $t \geq 0$. Here, $u : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$ is the velocity field of the flow, $\pi = \pi(x,t) \in \mathbb{R}$ is a scalar function representing the pressure, $\theta : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$ represents the temperature of the fluid and $e_3 = (0,0,1)^T$. Note that $u_0(x)$ and $\theta_0(x)$ are given initial velocity and initial temperature with $\nabla\cdot u_0 = 0$ in the sense of distributions.

Owing to the physical importance and the mathematical challenges, the study of (1) which describes the dynamics of a viscous incompressible fluid with heat exchanges, has a long history and has attracted many contributions from physicists and mathematicians [17]. Although Boussinesq equations consist in a simplification of the original 3-D incompressible flow, they share a similar vortex stretching effect. For this reason they retains most of the mathematical and physical difficulties of the 3-D incompressible flow, and therefore, these equations have been studied and applied to various fields. Examples include for instance geophysical applications, where they serve as a model, see, e.g. [19]. There are several other results on existence and blowup criteria in different kinds of spaces which have been obtained, (see [1,3,8,7,9,27,28]).

The problem of the global-in-time well-posedness of (1) in a three-dimensional space is highly challenging, due to the fact that the system contains the incompressible 3D Navier–Stokes equations as a special case (obtained by setting $\theta = 0$), for which the issue of global well-posedness has not been proved until now. However, the question of the regularity of weak solutions is an outstanding open problem in mathematical fluid mechanics and many interesting results have been obtained (see e.g. [5,4,10–12,14,13,20,21,24,25,28,26]). We are interested in the classical problem of finding sufficient conditions for weak solutions of (1) such that they become regular.

Realizing the dominant role played by the velocity field in the regularity issue, Ishimura and Morimoto [10] were able to derive criteria in terms of the velocity field u alone. They showed that, if u satisfies

$$\nabla u \in L^1\left(0,T;L^\infty(\mathbb{R}^3)\right), \tag{2}$$

then the solution (u,θ) is regular on $[0,T]$. It is worthy to emphasize that there are no assumptions on the temperature θ. This assumption (2) was weakened in [6] with the L^∞-norm replaced by norms in Besov spaces $\dot{B}^0_{\infty,\infty}$. Quite recently, Zhang [23] showed that (u,θ) is a strong solution if

$$(\nabla u, \nabla\theta) \in L^2\left(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)\right), \tag{3}$$

where $\dot{B}^{-1}_{\infty,\infty}$ denotes the homogenous Besov space. A logarithmically improvement of Zhang's result, controlled by its H^3-norm, was given by Ye [22].

The main purpose of this work is to establish an improvement of Zhang's regularity criterion (3). Now, the refined regularity criterion in terms of the gradient of the velocity ∇u can be stated as follows:

Theorem 1 (Main result) *Assume that* $(u_0, \theta_0) \in L^2(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$. *Let* (u, θ) *be a weak solution to the Boussinesq equations on some interval* $(0, T)$ *with* $0 < T \leq \infty$. *If*

$$\nabla u \in L^2\left(0, T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)\right), \tag{4}$$

then the weak solution (u, θ) *is regular in* $(0, T]$, *that is* $(u, \theta) \in C^\infty(\mathbb{R}^3 \times (0, T])$.

Remark 1 This result is expected because of the fact that the (refinement of) Beale–Kato–Majda type criterion is well known in the class $\dot{B}^0_{\infty,\infty}$ for the 3D Boussinesq equations and one may replace the vorticity by ∇u since the Riesz transforms are continuous in $\dot{B}^0_{\infty,\infty}$. Then, the temperature plays a less dominant role than the velocity field does in the regularity theory of solutions to the Boussinesq equations. Furthermore, clearly Theorem 1 is an improvement of Zhang's regularity criterion (3).

By a weak solution, we mean that (u, θ, π) must satisfy (1) in the sense of distributions. In addition, we have the basic regularity for the weak solution

$$(u, \theta) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$$

for any $T > 0$. If a weak solution (u, θ) satisfies

$$(u, \theta) \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)),$$

then actually (u, θ) is a strong (classical) solution. It is worth to note that for strong solutions, we can gain more regularity properties.

Throughout this paper, C denotes a generic positive constant which may vary from one line to another.

2 Preliminaries

In this section we introduce the function spaces that will be used to state and prove the main result, and we collect and/or derive a number of auxiliary estimates that will be needed throughout the proof. Before introducing the homogeneous Besov and Triebel–Lizorkin spaces, we have to fix some notations. By $\mathcal{S}$ we denote the class of rapidly decreasing functions. The dual space of $\mathcal{S}$, i.e., the space of tempered distributions on $\mathbb{R}^3$ is denoted by $\mathcal{S}'$. For $u \in \mathcal{S}(\mathbb{R}^3)$, the Fourier transform of u is defined by

$$\mathcal{F}u(\omega) = \widehat{u}(\omega) = \int_{\mathbb{R}^3} u(x) e^{-ix\cdot\omega} dx, \quad \omega \in \mathbb{R}^3.$$

The homogeneous Littlewood–Paley decomposition relies upon a dyadic partition of unity. We can use for instance any $\varphi \in \mathcal{S}(\mathbb{R}^3)$, supported in $\mathcal{C} \triangleq \{\omega \in \mathbb{R}^3 : \frac{3}{4} \leq |\omega| \leq \frac{8}{3}\}$ such that

$$\sum_{l \in \mathbb{Z}} \varphi(2^{-l}\omega) = 1 \quad \text{if} \quad \omega \neq 0.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define dyadic blocks in this way:

$$\Delta_l u \triangleq \varphi(2^{-l}D)u = 2^{3l} \int_{\mathbb{R}^3} h(2^l y) u(x - y) dy \quad \text{for each } l \in \mathbb{Z},$$

and

$$S_l u \triangleq \sum_{k \leq l-1} \Delta_k u.$$

The formal decomposition

$$u = \sum_{l\in\mathbb{Z}} \Delta_l u$$

is called the homogeneous Littlewood–Paley decomposition.

Remark 2 The above dyadic decomposition has nice properties of quasi-orthogonality: with our choice of φ, we have,

$$\Delta_k \Delta_l u \equiv 0 \quad \text{if} \quad |k-l| \geq 2 \qquad \text{and} \qquad \Delta_k (S_{k-1} u \Delta_l u) \equiv 0 \quad \text{if} \quad |k-l| \geq 5.$$

With the introduction of Δ_l, let us recall the definition of homogeneous Besov and Triebel–Lizorkin spaces (see [23] for more details).

Definition 1 The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined by

$$\dot{B}^s_{p,q}(\mathbb{R}^3) = \left\{ u \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}(\mathbb{R}^3) : \ \|u\|_{\dot{B}^s_{p,q}} < \infty \right\}$$

for $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, where

$$\|u\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \|\Delta_j u\|^q_{L^p}\right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\Delta_j u\|_{L^p} & \text{if } q = \infty, \end{cases}$$

and $\mathcal{P}(\mathbb{R}^3)$ is the set of all scalar polynomials defined on $\mathbb{R}^3$. Similarly, the homogeneous Triebel–Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^3)$ is a quasi-normed space equipped with the family of seminorms $\|\cdot\|_{\dot{F}^s_{p,q}}$ which are defined by

$$\|u\|_{\dot{F}^s_{p,q}} = \begin{cases} \left\| \left(\sum_{j\in\mathbb{Z}} 2^{jsq} |\Delta_j u|^q\right)^{\frac{1}{q}} \right\|_{L^p} & \text{if } 1 \leq q < \infty, \\ \left\| \sup_{j\in\mathbb{Z}} 2^{js} |\Delta_j u| \right\|_{L^p} & \text{if } q = \infty. \end{cases}$$

Notice that there exists a universal constant C such that

$$C^{-1} \|u\|_{\dot{B}^s_{p,q}} \leq \|\nabla u\|_{\dot{B}^{s-1}_{p,q}} \leq C \|u\|_{\dot{B}^s_{p,q}}.$$

In particular,

$$u \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^3) \quad \Longleftrightarrow \quad \nabla u \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3).$$

From this observation we derive the following corollary to Theorem 1.

Corollary 1 *Suppose that (u,θ) is a weak solution of the Boussinesq equations on $(0,T)$. If*

$$u \in L^2\left(0,T; \dot{B}^0_{\infty,\infty}\left(\mathbb{R}^3\right)\right), \tag{5}$$

then the weak solution (u,θ) is regular in $(0,T]$.

Next, we introduce the following Bernstein lemma due to [6].

Lemma 1 (Bernstein) *For all $k \in \mathbb{N}$, $j \in \mathbb{Z}$, and $1 \leq p \leq q \leq \infty$, we have for all $f \in \mathcal{S}(\mathbb{R}^3)$:*

(i)

$$\sup_{|\alpha|=k} \left\|\nabla^\alpha \Delta_j f\right\|_{L^q} \leq C_1 2^{jk+3j(\frac{1}{p}-\frac{1}{q})} \|\Delta_j f\|_{L^p}.$$

(ii)
$$\|\Delta_j f\|_{L^p} \le C_2 2^{-jk} \sup_{|\alpha|=k} \left\|\nabla^\alpha \Delta_j f\right\|_{L^p},$$

where C_1, C_2 are positive constants independent of f and j.

The proof of the main result needs a logarithmic Sobolev inequality in terms of Besov space. It will play an important role in the proof of Theorem 1. The following is a well-known embedding result, (cf. [23, p. 244]):

$$L^\infty(\mathbb{R}^3) \hookrightarrow BMO(\mathbb{R}^3) = \dot{F}^0_{\infty,2} \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^3),$$

where $BMO(\mathbb{R}^3)$ stands for the Bounded Mean Oscillations space [23].

We state and prove the following lemma.

Lemma 2 *Suppose that $\nabla f \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ and $f \in H^s(\mathbb{R}^3)$ for all $s > \frac{3}{2}$. Then, there exists a constant $C > 0$ such that*

$$\|f\|_{L^\infty} \le C\left[1 + \|\nabla f\|_{\dot{B}^{-1}_{\infty,\infty}} \left(\ln^+ \|f\|_{H^s}\right)^{\frac{1}{2}}\right], \tag{6}$$

holds, where H^s denotes the standard Sobolev space and

$$\ln^+ x = \begin{cases} \ln x & \text{if } x > e, \\ 1 & \text{if } 0 < x \le e. \end{cases}$$

Proof The proof is an easy modification of the one in [15]. Owing the Littlewood–Paley decomposition, we can rewrite

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f = \sum_{j=-\infty}^{-N-1} \Delta_j f + \sum_{j=-N}^{N} \Delta_j f + \sum_{j=N+1}^{+\infty} \Delta_j f,$$

where N is a positive integer to be determined later. Bernstein's lemma and Young's inequality give rise to

$$\begin{aligned}
\|f\|_{L^\infty} &\le \sum_{j=-\infty}^{-N-1} \|\Delta_j f\|_{L^\infty} + \sum_{j=-N}^{N} \|\Delta_j f\|_{L^\infty} + \sum_{j=N+1}^{+\infty} \|\Delta_j f\|_{L^\infty} \\
&\le C \sum_{j<-N} 2^{\frac{3}{2}j} \|\Delta_j f\|_{L^2} + CN\|f\|_{\dot{B}^0_{\infty,\infty}} + C\sum_{j>N} 2^{(-s+\frac{3}{2})j} \|\Delta_j f\|_{L^2} 2^{js} \\
&\le C\left(2^{-\frac{3}{2}N}\|f\|_{L^2} + N\|\nabla f\|_{\dot{B}^{-1}_{\infty,\infty}} + \sum_{j>N} 2^{(-s+\frac{3}{2})j}\|f\|_{\dot{B}^s_{2,\infty}}\right) \\
&\le C\left(2^{-\frac{3}{2}N}\|f\|_{L^2} + N\|\nabla f\|_{\dot{B}^{-1}_{\infty,\infty}} + 2^{(-s+\frac{3}{2})N}\|f\|_{H^s}\right),
\end{aligned}$$

where we have used the fact that $s > \frac{3}{2}$ and the Besov embedding $H^s \hookrightarrow \dot{B}^s_{2,\infty}$.

Setting $\alpha = \min\left(s - \frac{3}{2}, \frac{3}{2}\right)$, we derive

$$\|f\|_{L^\infty} \le C\left(2^{-\alpha N}\|f\|_{H^s} + N\|\nabla f\|_{\dot{B}^{-1}_{\infty,\infty}}\right).$$

Now choose N such that $2^{-\alpha N}\|f\|_{H^s} \le 1$. Thus we get $N \ge \frac{\log\|f\|_{H^s}}{\alpha \log 2}$. □

Next, the following lemma is needed.

Lemma 3 *Let $g,h \in H^1(\mathbb{R}^3)$ and $f \in BMO(\mathbb{R}^3)$. Then we have*

$$\int_{\mathbb{R}^3} f \cdot \nabla(gh)dx \leq C\|f\|_{BMO}\left(\|\nabla g\|_{L^2}\|h\|_{L^2} + \|g\|_{L^2}\|\nabla h\|_{L^2}\right).$$

Proof The proof of the above lemma requires some paradifferential calculus. We have to recall here that paradifferential calculus enables to define a generalized product between distributions. It is continuous in many functional spaces where the usual product does not make sense (see the pioneering work of Bony [2]). The paraproduct between f and g is defined by

$$T_f g \triangleq \sum_{j\in\mathbb{Z}} S_{j-1} f \Delta_j g.$$

We thus have the following formal decomposition (modulo a polynomial):

$$fg = T_f g + T_g f + R(f,g)$$

with

$$R(f,g) = \sum_{|j-k|\leq 1} \Delta_j f \Delta_k g.$$

Coming back to the proof of Lemma 3, we split $\int_{\mathbb{R}^3} f \cdot \nabla(gh)dx$ into

$$\begin{aligned}\int_{\mathbb{R}^3} f \cdot \nabla(gh)dx &= \int_{\mathbb{R}^3} f \cdot \nabla(T_g h)dx + \int_{\mathbb{R}^3} f \cdot \nabla(g T_h)dx + \int_{\mathbb{R}^3} f \cdot \nabla R(g,h)dx \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Since we know that $BMO = \dot{F}^0_{\infty,2}$ (see [23, pp. 243–244]), the duality between $\dot{F}^0_{\infty,2}$ and $\dot{F}^0_{1,2}$ guarantees that

$$\begin{aligned}I_1 &= \int_{\mathbb{R}^3} f \cdot (T_{\nabla g} h)dx + \int_{\mathbb{R}^3} f \cdot (T_g \nabla h)dx \\ &\leq \|f\|_{BMO}\left(\left\|T_{\nabla g} h\right\|_{\dot{F}^0_{1,2}} + \left\|T_g \nabla h\right\|_{\dot{F}^0_{1,2}}\right) \\ &= \|f\|_{BMO}(I_{11} + I_{12}).\end{aligned}$$

In view of the boundedness of the Hardy–Littlewood maximal operator $\mathcal{M}$ in L^p spaces $(1 < p < \infty)$ (c.f. Stein [22, Chapter II, Theorem 1]), we can estimate the term I_{11} as follows:

$$\begin{aligned}I_{11} &\approx \left\|\left(\sum_{j\in\mathbb{Z}} \left|S_{j-1}(\nabla g)\right|^2 |\Delta_j h|^2\right)^{\frac{1}{2}}\right\|_{L^1} \leq C\left\|\mathcal{M}(\nabla g)\left(\sum_{j\in\mathbb{Z}} |\Delta_j h|^2\right)^{\frac{1}{2}}\right\|_{L^1} \\ &\leq C\|\mathcal{M}(\nabla g)\|_{L^2}\left\|\left(\sum_{j\in\mathbb{Z}} |\Delta_j h|^2\right)^{\frac{1}{2}}\right\|_{L^2} \leq C\|\nabla g\|_{L^2}\|h\|_{L^2}.\end{aligned}$$

Repeating the same arguments, we also have for I_{12}

$$\begin{aligned}I_{12} &\approx \left\|\left(\sum_{j\in\mathbb{Z}} \left|S_{j-1}(g)\right|^2 \left|\Delta_j(\nabla h)\right|^2\right)^{\frac{1}{2}}\right\|_{L^1} \leq C\left\|\mathcal{M}(g)\left(\sum_{j\in\mathbb{Z}} \left|\Delta_j(\nabla h)\right|^2\right)^{\frac{1}{2}}\right\|_{L^1} \\ &\leq C\|g\|_{L^2}\left\|\left(\sum_{j\in\mathbb{Z}} \left|\Delta_j(\nabla h)\right|^2\right)^{\frac{1}{2}}\right\|_{L^2} \leq C\|g\|_{L^2}\|\nabla h\|_{L^2}.\end{aligned}$$

Collecting these estimates, we obtain

$$I_1 \le C\|f\|_{BMO}\left(\|\nabla g\|_{L^2}\|h\|_{L^2} + \|g\|_{L^2}\|\nabla h\|_{L^2}\right).$$

As a result, estimating I_2 following the same arguments, we obtain

$$I_2 \le C\|f\|_{BMO}\left(\|\nabla g\|_{L^2}\|h\|_{L^2} + \|g\|_{L^2}\|\nabla h\|_{L^2}\right).$$

For the third term I_3, using the embedding relation $\dot{B}^0_{1,1}\dot{F}^0_{1,2}$ and in view of Bernstein's lemma, we can deduce that

$$\begin{aligned}
I_3 &\le \|f\|_{BMO}\|\nabla R(g,h)\|_{\dot{F}^0_{1,2}} \le C\|f\|_{BMO}\|R(g,h)\|_{\dot{B}^1_{1,1}} \\
&\le C\|f\|_{BMO}\sum_{j\in\mathbb{Z}} 2^j \left\|\Delta_j g\cdot\widetilde{\Delta}_j h\right\|_{L^1} \\
&\le C\|f\|_{BMO}\sum_{j\in\mathbb{Z}} 2^j \|\Delta_j g\|_{L^2}\left\|\widetilde{\Delta}_j h\right\|_{L^2} \\
&\le C\|f\|_{BMO}\|\nabla g\|_{L^2}\|h\|_{L^2}.
\end{aligned}$$

so that the proof of Lemma 3 is achieved. □

We often use the following well-known lemma.

Lemma 4 (Gagliardo–Nirenberg) *Let* $1 \le q, r < \infty$ *and* $m \le k$. *Suppose that* θ *and* j *satisfy* $m \le j \le k$, $0 \le \theta \le 1$ *and define* $p \in [1,+\infty]$ *by*

$$\frac{1}{p} = \frac{j}{3} + \theta\left(\frac{1}{r} - \frac{m}{3}\right) + (1-\theta)\left(\frac{1}{q} - \frac{k}{3}\right).$$

Then, the inequality

$$\left\|\nabla^j f\right\|_{L^p} \le C\|\nabla^m f\|_{L^q}^{1-\theta}\left\|\nabla^k f\right\|_{L^r}^{\theta} \qquad \text{for } f \in W^{m,q}(\mathbb{R}^3)\cap W^{k,r}(\mathbb{R}^3)$$

holds with some constant $C > 0$.

3 Proof of Theorem 1

Now we are ready to prove our main result of this section.

Proof First, note that a weak solution (u,θ) to (1) has at least one global weak solution

$$(u,\theta) \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)),$$

which satisfies the following energy inequality

$$\frac{1}{2}\left(\|u(\cdot,t)\|_{L^2}^2 + \|\theta(\cdot,t)\|_{L^2}^2\right) + \int_0^t \left(\|\nabla u(\cdot,\tau)\|_{L^2}^2 + \|\nabla\theta(\cdot,\tau)\|_{L^2}^2\right) d\tau \le \frac{1}{2}\left(\|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2\right)$$

for almost every $t \ge 0$.

In order to prove that $(u,\theta) \in C^\infty(\mathbb{R}^3\times(0,T])$, as it is well known, it suffices to show that the weak solution (u,θ) is also a strong solution on $(0,T]$, which means that:

$$(u,\theta) \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)).$$

Owing to (4), we know that for any small constant $\varepsilon > 0$, there exists $T_0 = T_0(\varepsilon) < T$ such that

$$\int_{T_0}^{T} \|\nabla u(\cdot,\tau)\|^2_{\dot{B}^{-1}_{\infty,\infty}} d\tau \leq \varepsilon.$$

To do so, we shall work on the local strong solution with the initial datum (u_0, θ_0) on its maximal existence time interval $(0, T_0)$. Then, we have only to show that

$$\sup_{0\leq t<T_0} \left(\|\nabla u(\cdot,t)\|^2_{L^2} + \|\nabla\theta(\cdot,t)\|^2_{L^2}\right) + \int_0^{T_0} \left(\|\nabla u(\cdot,\tau)\|^2_{L^2} + \|\nabla\theta(\cdot,\tau)\|^2_{L^2}\right) d\tau \leq C < \infty,$$

here and in what follows C denotes various positive constants which are independent from T_0.

Take the operator ∇ in equations $(1)_1$ and $(1)_2$, respectively, and the scalar product of them ∇u and $\nabla\theta$, respectively and add them together, to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|^2_{L^2} + \|\nabla\theta\|^2_{L^2}\right) + \|\Delta u\|^2_{L^2} + \|\Delta\theta\|^2_{L^2} \\
&= -\int_{\mathbb{R}^3} \theta e_3 \cdot \Delta u dx - \sum_{i=1}^{3}\int_{\mathbb{R}^3} (\partial_i u\cdot\nabla) u \partial_i u dx - \sum_{i=1}^{3}\int_{\mathbb{R}^3} (\partial_i u\cdot\nabla)\theta\partial_i\theta dx \\
&:= I_1 + I_2 + I_3. \qquad (7)
\end{aligned}$$

In the following, we estimate each term at the right-hand side of (7) separately below.

To bound I_1, we integrate by parts and apply Hölder's inequality to obtain

$$|I_1| \leq C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2} \leq C\left(\|\nabla u\|^2_{L^2} + \|\nabla\theta\|^2_{L^2}\right).$$

In order to deal with the terms I_2 and I_3, we need the following elegant Machihara–Ozawa inequality [16] (see also Meyer [18])

$$\|\nabla u\|^2_{L^4} \leq C\|u\|_{\dot{B}^0_{\infty,\infty}}\|\Delta u\|_{L^2}. \qquad (8)$$

We now bound I_2. By (8) and Young's inequality

$$\begin{aligned}
|I_2| &\leq C\|\nabla u\|_{L^2}\|\nabla u\|^2_{L^4} \\
&\leq C\|\nabla u\|_{L^2}\|u\|_{\dot{B}^0_{\infty,\infty}}\|\Delta u\|_{L^2} \\
&\leq C\|\nabla u\|_{L^2}\|u\|_{BMO}\|\Delta u\|_{L^2} \\
&\leq \frac{1}{2}\|\Delta u\|^2_{L^2} + C\|\nabla u\|^2_{L^2}\|u\|^2_{BMO}.
\end{aligned}$$

By integration by parts, we can rewrite and estimate I_3 as follows

$$\begin{aligned}
|I_3| &= \left|\sum_{i=1}^{3}\int_{\mathbb{R}^3} (\partial_i u\cdot\nabla)\theta\cdot\partial_i\theta dx\right| = \left|\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} \partial_i(\partial_i\theta_k\partial_k\theta_j)u_j dx\right| \\
&\leq C\|u\|_{BMO}\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^2} \\
&\leq \frac{1}{6}\|\Delta\theta\|^2_{L^2} + C\|u\|^2_{BMO}\|\nabla\theta\|^2_{L^2}.
\end{aligned}$$

Combining the estimates for I_1, I_2, and I_3, we find

$$\frac{d}{dt}\left(\|\nabla u\|^2_{L^2} + \|\nabla\theta\|^2_{L^2}\right) + \|\Delta u\|^2_{L^2} + \|\Delta\theta\|^2_{L^2} \leq C\left(1 + \|u\|^2_{BMO}\right)\left(\|\nabla u\|^2_{L^2} + \|\nabla\theta\|^2_{L^2}\right).$$

Using the Gronwall inequality on the time interval $[T_0, t]$, one has the following inequality

$$\begin{aligned}&\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla\theta(\cdot,t)\|_{L^2}^2+\int_{T_0}^t(\|\Delta u(\cdot,\tau)\|_{L^2}^2+\|\Delta\theta(\cdot,\tau)\|_{L^2}^2)d\tau\\&\le\left(\|\nabla u(\cdot,T_0)\|_{L^2}^2+\|\nabla\theta(\cdot,T_0)\|_{L^2}^2\right)\exp\left(C\int_{T_0}^t\|u(\cdot,\tau)\|_{BMO}^2d\tau\right).\end{aligned}$$

Let us denote for any $t\in[T_0,T)$,

$$F(t)\triangleq\max_{T_0\le\tau\le t}\left(\|u(\cdot,\tau)\|_{H^2}^2+\|\theta(\cdot,\tau)\|_{H^2}^2\right). \tag{9}$$

It should be noted that the function $F(t)$ is nondecreasing. Using (6), we obtain

$$\begin{aligned}&\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla\theta(\cdot,t)\|_{L^2}^2+\int_{T_0}^t(\|\Delta u(\cdot,\tau)\|_{L^2}^2+\|\Delta\theta(\cdot,\tau)\|_{L^2}^2)d\tau\\
&\le C(T_0)\exp\left(C\int_{T_0}^t\left(1+\|u(\cdot,\tau)\|_{\dot B^0_{\infty,\infty}}^2\log\left(\|u(\cdot,\tau)\|_{H^2}+\|\theta(\cdot,\tau)\|_{H^2}\right)\right)d\tau\right)\\
&\le C(T_0)\exp\left(C\int_{T_0}^t\|\nabla u(\cdot,\tau)\|_{\dot B^{-1}_{\infty,\infty}}^2\log\left(\|u(\cdot,\tau)\|_{H^2}^2+\|\theta(\cdot,\tau)\|_{H^2}^2\right)d\tau\right)\\
&\le C(T_0)\exp\left(C\int_{T_0}^t\|\nabla u(\cdot,\tau)\|_{\dot B^{-1}_{\infty,\infty}}^2d\tau\sup_{T_0\le\tau\le t}\log\left(\|u(\cdot,\tau)\|_{H^2}^2+\|\theta(\cdot,\tau)\|_{H^2}^2\right)\right)\\
&\le C(T_0)\exp\left(C\int_{T_0}^t\|\nabla u(\cdot,\tau)\|_{\dot B^{-1}_{\infty,\infty}}^2d\tau\log\sup_{T_0<\tau\le t}(\|u(\cdot,\tau)\|_{H^2}^2+\|\theta(\cdot,\tau)\|_{H^2}^2)\right)\\
&\le C(T_0)\exp(C\varepsilon\log F(t))\\
&\le C(T_0)[F(t)]^{C\varepsilon},\end{aligned}$$

where

$$C(T_0)=C\left(\|\nabla u(\cdot,T_0)\|_{L^2}^2+\|\nabla\theta(\cdot,T_0)\|_{L^2}^2\right).$$

Next, applying Δ to the equations $(1)_1$, $(1)_2$, taking the L^2 inner product of the obtained equations with $-\Delta u$ and $-\Delta\theta$, respectively, adding them up and using the incompressible conditions $\nabla\cdot u=0$, we arrive at

$$\begin{aligned}&\frac12\frac{d}{dt}\left(\|\Delta u\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2\right)+\|\nabla^3u\|_{L^2}^2+\|\nabla^3\theta\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}\Delta(\theta e_3)\cdot\Delta u dx-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla u)\cdot\Delta u dx-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla\theta)\cdot\Delta\theta dx\\
&\le\left|\int_{\mathbb{R}^3}\Delta(\theta e_3)\cdot\Delta u dx\right|+\left|\int_{\mathbb{R}^3}(\Delta u\cdot\nabla u)\cdot\Delta u dx\right|+2\sum_{i=1}^3\left|\int_{\mathbb{R}^3}(\partial_i u\cdot\nabla\partial_i u)\cdot\Delta u dx\right|\\
&\quad+\left|\int_{\mathbb{R}^3}(\Delta u\cdot\nabla\theta)\cdot\Delta\theta dx\right|+2\sum_{i=1}^3\left|\int_{\mathbb{R}^3}(\partial_i u\cdot\nabla\partial_i\theta)\cdot\Delta\theta dx\right|\\
&=\sum_{k=1}^5A_k.\end{aligned} \tag{10}$$

Now we will estimate the terms on the right-hand side of (10) one by one as follows. Let us begin with estimating the term A_1.

Using Lemma 4 with $p=q=r=j=2$, $k=3$ and $m=1$, A_1 can be bounded above as follows:

$$\begin{aligned}
A_1 &\le C\|\Delta u\|_{L^2}\|\Delta\theta\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2}^{\frac12}\|\nabla\theta\|_{L^2}^{\frac12}\|\nabla^3\theta\|_{L^2}^{\frac12}\\
&= \left(\|\nabla^3 u\|_{L^2}^2\right)^{\frac14}\left(\|\nabla^3\theta\|_{L^2}^2\right)^{\frac14}\left(C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}\right)^{\frac12}\\
&\le \frac{1}{16}\|\nabla^3 u\|_{L^2}^2+\frac{1}{16}\|\nabla^3\theta\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}\\
&\le \frac{1}{16}\|\nabla^3 u\|_{L^2}^2+\frac{1}{16}\|\nabla^3\theta\|_{L^2}^2+C\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right).
\end{aligned}$$

Let us now recall Gagliardo–Nirenberg's inequality

$$\|\Delta f\|_{L^4}\le C\|\nabla f\|_{L^2}^{\frac18}\|\nabla^3 f\|_{L^2}^{\frac78}.$$

Thus, we obtain

$$\begin{aligned}
A_2,A_3 &\le C\|\nabla u\|_{L^2}\|\Delta u\|_{L^4}^2\\
&\le C\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{\frac14}\|\nabla^3 u\|_{L^2}^{\frac74}\\
&= C\|\nabla u\|_{L^2}^{\frac54}\|\nabla^3 u\|_{L^2}^{\frac74}=\left(C\|\nabla u\|_{L^2}^{10}\right)^{\frac18}\left(\|\nabla^3 u\|_{L^2}^2\right)^{\frac78}\\
&\le \frac{1}{16}\|\nabla^3 u\|_{L^2}^2+C\|\nabla u\|_{L^2}^{10}.
\end{aligned}$$

Similarly to the estimate of A_1, the terms A_4 and A_5 can be bounded above as

$$\begin{aligned}
A_4,A_5 &\le C\|\nabla\theta\|_{L^2}\|\Delta\theta\|_{L^4}\|\Delta u\|_{L^4}\\
&\le C\|\nabla\theta\|_{L^2}\left(\|\Delta u\|_{L^4}^2+\|\Delta\theta\|_{L^4}^2\right)\\
&\le C\|\nabla\theta\|_{L^2}\|\nabla u\|_{L^2}^{\frac14}\|\nabla^3 u\|_{L^2}^{\frac74}+C\|\nabla\theta\|_{L^2}^{\frac54}\|\nabla^3\theta\|_{L^2}^{\frac74}\\
&\le \frac14\|\nabla^3 u\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^8\|\nabla u\|_{L^2}^2+\frac12\|\nabla^3\theta\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^{10}\\
&\le \frac{1}{16}\|\nabla^3 u\|_{L^2}^2+\frac14\|\nabla^3\theta\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^8\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right).
\end{aligned}$$

Summarizing all the estimates and absorbing the dissipative term, we can derive

$$\begin{aligned}
\frac{d}{dt}(\|\Delta u\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2) &\le C\|\nabla u\|_{L^2}^{10}+C\|\nabla\theta\|_{L^2}^8\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\\
&\le C\left(\|\nabla u\|_{L^2}^8+\|\nabla\theta\|_{L^2}^8\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\\
&\le C\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)^4\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\\
&\le C\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)^5\\
&\le C(T_0)[F(t)]^{5C\varepsilon}.
\end{aligned}$$

Integrating the above estimate over interval (T_0,t) and observing that $F(t)$ is a monotonically increasing function, we thus have

$$\|\Delta u(\cdot,t)\|_{L^2}^2+\|\Delta\theta(\cdot,t)\|_{L^2}^2\le\|\Delta u(\cdot,T_0)\|_{L^2}^2+\|\Delta\theta(\cdot,T_0)\|_{L^2}^2+C(T_0)\int_{T_0}^t[F(\tau)]^{5C\varepsilon}d\tau.$$

By using (9), it follows that

$$\begin{aligned} F(t) &\leq \|u(\cdot,T_0)\|_{H^2}^2 + \|\theta(\cdot,T_0)\|_{H^2}^2 + C\int_{T_0}^{t} [F(\tau)]^{5C\varepsilon} d\tau \\ &\leq \|u(\cdot,T_0)\|_{H^2}^2 + \|\theta(\cdot,T_0)\|_{H^2}^2 + C(T_0)(t-T_0)[F(t)]^{5C\varepsilon}. \end{aligned}$$

Choosing ε such that $5C\varepsilon < 1$, the above inequality yields for any $t \in [T_0, T)$

$$F(t) \leq C < \infty,$$

which implies that $(u,\theta) \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3))$. This completes the proof of Theorem 1. □

Remark 3 Comparing our result with [26], we have simplified the proof of Theorem 1.1 in [26], in fact we only need H^2 a priori estimates of solutions.

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