Arch. Math. Logic (2016) 55:767–784
DOI 10.1007/s00153-016-0493-8

Mathematical Logic

Menger remainders of topological groups

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Received: 27 March 2015 / Accepted: 18 May 2016 / Published online: 13 June 2016

Abstract In this paper we discuss what kind of constrains combinatorial covering properties of Menger, Scheepers, and Hurewicz impose on remainders of topological groups. For instance, we show that such a remainder is Hurewicz if and only it is σ-compact. Also, the existence of a Scheepers non-σ-compact remainder of a topological group follows from CH and yields a P-point, and hence is independent of ZFC. We also make an attempt to prove a dichotomy for the Menger property of remainders of topological groups in the style of Arhangel'skii.

Keywords Remainder · Topological group · Menger space · Hurewicz space · Scheepers space · Ultrafilter · Forcing

The research of Angelo Bella that led to the present paper was partially supported by a grant of the group GNSAGA of INdAM. Seçil Tokgöz would like to thank Hacettepe University BAP project 014 G 602 002 for its support. Lyubomyr Zdomskyy would like to thank the Austrian Academy of Sciences (APART Program) as well as the Austrian Science Fund FWF (Grant I 1209-N25) for generous support for this research.

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Mathematics Subject Classification Primary 03E75 · 54D40 · 54D20; Secondary 03E35 · 54D30 · 54D80

1 Introduction

All topological spaces are assumed to be completely regular. All undefined topological notions can be found in [13]. For a space X and its compactification bX the complement $bX\setminus X$ is called a *remainder* of X. The interplay between the properties of spaces and their remainders has been studied since more than 50 years and resulted in a number of duality results describing properties of X in terms of those of their remainders. A typical example of such a duality is the celebrated result of Henriksen and Isbell stating that a topological space X is Lindelöf if and only if all (equivalently any) of its remainders is of *countable type*, that is, any compact subspace can be enlarged to another compact subspace with countable outer base.

In the last years, remainders in compactifications of topological groups have been a popular topic. This is basically due to the fact that topological groups are much more sensitive to the properties of their remainders than topological spaces in general. A major role in this study was played by Arhangel′skii, who initiated a systematic study of this topic. Among many other things, he obtained two elegant results, which are dichotomies for non-locally compact topological groups.

Theorem 1.1 ([5]) *Let G be a topological group. If bG is a compactification of G, then $bG\setminus G$ is either Lindelöf or pseudocompact.*

Theorem 1.2 ([4]) *Let G be a topological group. If bG is a compactification of G, then $bG\setminus G$ is either σ-compact or Baire.*

We recall that a topological space X is *Baire* if the intersection of countably many open dense subsets is dense.

In this paper we will focus our attention on topological properties which are strictly in between σ-compact and Lindelöf. Recall from [22] that a space X is *Menger* (or has the *Menger property*) if for any sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X one may pick finite sets $\mathcal{V}_n \subset \mathcal{U}_n$ in such a way that $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a cover of X. A family $\{W_n : n \in \omega\}$ of subsets of X is called an ω-cover (resp. γ-cover) of X, if for every $F \in [X]^{<\omega}$ the set $\{n \in \omega : F \subset W_n\}$ is infinite (resp. co-finite). The properties of *Scheepers* and *Hurewicz* are defined in the same way as the Menger property, the only difference being that we additionally demand that $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a ω-cover (resp. γ-cover) of X. It is immediate that

$$\sigma\text{-compact} \Rightarrow \text{Hurewicz} \Rightarrow \text{Scheepers} \Rightarrow \text{Menger} \Rightarrow \text{Lindelöf}.$$

The properties mentioned above have recently received great attention, mainly because of their combinatorial nature and game-theoretic characterizations. One of the most striking results about the Menger property is due to Aurichi who proved [6] that any Menger space is a D-space.

Our initial idea was to find counterparts of the properties of Menger, Scheepers, and Hurewicz in the style of Theorems 1.1 and 1.2. It turned out that the counterpart of the Hurewicz property is already given by Theorem 1.1 because of the following result, see Sect. 2 for its proof.

Theorem 1.3 *Let G be a topological group. If $\beta G\setminus G$ is Hurewicz, then it is σ-compact.*

Let us note that there are ZFC examples of Hurewicz sets of reals which are not σ-compact (see [17, Theorem 5.1] or [25, Theorem 2.12]), and thus Theorem 1.3 is specific for remainders of topological groups.

As it follows from the theorems below, which are the main results of this paper, for the properties of Scheepers and Menger the situation depends on the ambient set-theoretic universe. Each subspace of $\mathcal{P}(\omega)$ (e.g., an ultrafilter) is considered with the subspace topology. Let us recall from [17, Theorem 3.9] that if all finite powers of a topological space X are Menger then X is Scheepers. The converse of this statement fails consistently: under CH there exists a Hurewicz subspace of $\mathcal{P}(\omega)$ whose square is not Menger, see [17, Theorem 3.7].

Theorem 1.4 *There exists a Scheepers ultrafilter iff there exists a topological group G such that $\beta G\setminus G$ is Scheepers and not σ-compact iff there exists a topological group G such that all finite powers of $\beta G\setminus G$ are Menger and not σ-compact.*

Corollary 1.5 *The existence of a topological group G such that $\beta G\setminus G$ is Scheepers (resp. has all finite powers Menger) and not σ-compact is independent from ZFC. More precisely, such a group exists under $\mathfrak{d}=\mathfrak{c}$, and its existence yields P-points.*

Theorem 1.4 and Corollary 1.5 are proved in Sect. 3. Let us note that there exists a ZFC example of a dense Baire subspace X of $[\omega]^\omega$ all of whose finite powers are Menger (and thus also Scheepers), and hence it is indeed essential in Theorem 1.4 and Corollary 1.5 that we consider remainders of topological groups. In fact, such a subspace X can be chosen to be a filter, see [11, Claim 5.5] and the proof of [20, Theorem 1].

It is worth mentioning here that Scheepers ultrafilters have been studied intensively under different names for decades: In [10] Canjar proved that $\mathfrak{d}=\mathfrak{c}$ implies the existence of an ultrafilter whose Mathias forcing does not add dominating reals. By Chodounsky et al. [11, Theorem 1.1] the latter property for filters on ω is equivalent to the Menger one, and by Chodounsky et al. [11, Claim 5.5] all finite powers of a Menger filter are Menger. Combining this with [17, Theorem 3.9] we conclude that for filters on ω, the Menger property is equivalent to the Scheepers one, and hence Scheepers (equivalently Menger [in all finite powers]) ultrafilters are exactly those studied in [10]. Another descriptions of such ultrafilters may be found in [9,15], where they were characterized as ultrafilters with certain topological and combinatorial properties stronger than being a P-point, respectively. In particular, there are no Scheepers ultrafilters in models of ZFC without P-points.

Regarding the Menger property, we have the following partial result established in Sect. 4. Note that the assumption on the remainder we make in it is formally weaker than that made in Theorem 1.4, see the last equivalent statement there.

Theorem 1.6 *It is consistent that for any topological group G and compactification bG, if $(bG\setminus G)^2$ is Menger, then it is σ-compact.*

Let us note that the properties of Menger, Scheepers, Hurewicz, having Menger square, etc., are preserved by perfect maps in both directions. This implies that if one of the remainders of a space X has one of these covering properties, then all others also have it, see the beginning of Sect. 3 for more details and corresponding definitions. Thus Theorems 1.3, 1.4, and 1.6 admit several equivalent reformulations, given by the freedom to consider either all or some (specific) compactifications.

In light of Theorems 1.4 and 1.6 it is natural to ask the following questions.

Question 1.7 Is there a ZFC example of a topological group with a Menger non-σ-compact remainder?

Question 1.8 Is it consistent that there exists a topological group G such that $\beta G\setminus G$ is Menger and not Scheepers? Does CH imply the existence of such a group?

Since we do not have an analogous statement to Theorem 1.4 for the Menger property (in Theorem 1.6 we make a somewhat unpleasant assumption that the square of the remainder is Menger), it may still be the case that for the Menger property there exists a dichotomy similar to Theorems 1.1 and 1.2. In Sect. 5 we analyze some properties which might be counterparts of the Menger one for remainders of topological groups.

2 Hurewicz remainders

According to the definition on [3, p. 235], a topological group G is *feathered* if it contains a non-empty compact subspace with countable outer base. Recall that a family $\mathcal{U}$ of open subsets of a topological space X is an *outer base* for a subset A of X if $A \subset U$ for all $U \in \mathcal{U}$, and for every open $O \supset A$ there exists $U \in \mathcal{U}$ such that $U \subset O$. By Arhangel'skii and Tkachenko [3, Lemma 4.3.10] every feathered group has a compact subgroup with countable outer base. A topological group G is *Raikov-complete* if it is complete in the uniformity generated by sets $\{(x, y) \in G^2 : xy^{-1}, x^{-1}y \in U\}$, where U is a neighbourhood of the neutral element of G, see [3, § 3.6] and references therein.

The following fact is probably well-known.

Lemma 2.1 *For a feathered group G the following conditions are equivalent:*

(1) *G is Čech-complete;*
(2) *Each closed subgroup G_0 of G admitting a dense σ-compact subspace is Čech-complete;*
(3) *There exists a compact subgroup H of G with countable outer base such that $\overline{\langle QH\rangle}$ is Čech-complete for every countable $Q \subset G$, where for $X \subset G$ we denote by $\langle X\rangle$ the smallest subgroup of G containing X.*

Proof The implication (1) $\to$ (2) is straightforward, and (2) $\to$ (3) is a direct consequence of [3, Prop. 4.3.11]. The proof of (3) $\to$ (1) will be obtained by a tiny modification of that of [3, Theorem 4.3.15]. In particular, all details missing below can be found in the proof of the above-mentioned theorem.

Let $\langle V_n : n \in \omega\rangle$ be a decreasing sequence of open symmetric neighbourhoods of H such that $V_{n+1}^2 \subset V_n$ for all n and $H = \bigcap_{n\in\omega} V_n$. By Arhangel'skii and Tkachenko [3, Lemma 3.3.10] there exists a continuous prenorm[1] on G which satisfies

$$\left\{x \in G : N(x) < 1/2^n\right\} \subset V_n \subset \left\{x \in G : N(x) < 2/2^n\right\}$$

for all $n \in \omega$. Thus $H = \{x \in G : N(x) = 0\}$. Let ρ be a pseudometric on G defined by $\rho(x, y) = N(xy^{-1}) + N(x^{-1}y)$ for all $x, y \in G$. Then ρ is continuous and $\rho(x, y) = 0$ iff $xy^{-1}, x^{-1}y \in H$. Consider the equivalence relation $\sim$ on G defined by $x \sim y$ iff $\rho(x, y) = 0$ and denote by X the quotient space $G/{\sim}$. Let $\pi : G \to X$ be the quotient map and set $\rho^*(\pi(x), \pi(y)) = \rho(x, y)$ for any $x, y \in G$. It follows that π is perfect , ρ^* is well-defined, and is a metric generating the quotient topology on X.

The proof of [3, Theorem 4.3.15] is done as follows: Assuming that G is Raikov-complete, it is shown that ρ^* is complete. The same argument, applied to any subgroup G' of G which contains H (and hence is closed under $\sim$) yields that if G' is Raikov-complete, then $\rho^* \restriction \pi[G']$ is complete. Our item (3) implies that $\overline{\langle QH\rangle}$ is Raikov-complete for every countable $Q \subset G$ because Čech-complete groups are Raikov-complete, see, e.g., [3, Theorem 4.3.7]. Therefore by (3) we have that $\rho^* \restriction \pi[\overline{\langle QH\rangle}]$ is complete for any countable $Q \subset G$. Since π is perfect, the latter gives that $\rho^* \restriction Y$ is complete for any separable closed $Y \subset X$, and hence ρ^* is complete. therefore G is Čech-complete being a perfect preimage of a complete metric space, see [13, Theorems 3.9.10 and 4.3.26]. □

We are in a position now to present the

Proof of Theorem 1.3 Let G be a topological group such that $\beta G\setminus G$ is Hurewicz. Then $\beta G\setminus G$ is Lindelöf, hence G is of countable type [14], and therefore it is feathered. By Lemma 2.1 it is enough to show that each closed subgroup G_0 of G admitting a dense σ-compact subspace is Čech-complete. Let G_0 be as above, F be a dense σ-compact subspace of G_0, and $X = \overline{G_0}\setminus G_0$, where the closure is taken in βG. Since X is closed in $\beta G\setminus G$, it is Hurewicz. Applying [8, Theorem 27][2] to the Hurewicz space X and Čech-complete space $\overline{G_0}\setminus F$ containing it, we conclude that there exists a σ-compact space F' such that $X \subset F' \subset \overline{G_0}\setminus F$, which implies that $\overline{G_0}\setminus F'$ is a dense (because it contains F) Čech-complete subspace of G_0. Thus G_0 is Čech-complete by Arhangel'skii [5, Theorem 1.2], which completes our proof. □

It is well-known [27, Lemma 22] that if player *II* has a winning strategy in the Menger game on a space X (see Sect. 4 for its definition) then X is Hurewicz. Therefore Theorem 1.3 generalizes [7, Corollary 3.5].

[1] Following [3, § 3.3] we call a function $N : G \to \mathbb{R}_+$ a *prenorm*, if $N(e) = 0$, $N(x^{-1}) = N(x)$, and $N(xy) \leq N(x) + N(y)$ for all $x, y \in G$.

[2] We refer here to the online version of the paper available from the web-pages of the authors, which is an extended version of the published one.

3 Scheepers remainders

In the proof of Theorem 1.4, which is the main goal of this section, we shall need set-valued maps, see [19] for more information on them. By a *set-valued map* Φ from a set X into a set Y we understand a map from X into $\mathcal{P}(Y)$ and write $\Phi : X \Rightarrow Y$ (here $\mathcal{P}(Y)$ denotes the set of all subsets of Y). For a subset A of X we set $\Phi(A) = \bigcup_{x\in A} \Phi(x) \subset Y$. A set-valued map Φ from a topological space X to a topological space Y is said to be

- *compact-valued*, if $\Phi(x)$ is compact for every $x \in X$;
- *upper semicontinuous*, if for every open subset V of Y the set $\Phi^{-1}(V) = \{x \in X : \Phi(x) \subset V\}$ is open in X.

To abuse terminology, we shall call compact-valued upper semicontinuous maps *cvusc* maps. It is known [28, Lemma 1] that all combinatorial covering properties considered in this paper are preserved by cvusc maps. Also, if $f : X \to Y$ is a perfect map, then $f^{-1} : Y \Rightarrow X$ assigning to $y \in Y$ the subset $f^{-1}(y)$ of X, is a cvusc maps. Therefore the properties of Menger, Scheepers, Hurewicz, having Menger square, etc., are preserved by perfect maps in both directions. That is, if f is perfect and $Z \subset X$ (resp. $Z \subset Y$) has one of these properties, then so does $f(Z)$ (resp. $f^{-1}(Z)$). In particular, this implies that if one of the remainders of a space X has one of these covering properties, then all others also have it. In addition, all these properties are preserved by product with ω equipped with the discrete topology.[3]

We shall also need some additional notation. For $X \subset \mathcal{P}(\omega)$ we shall denote by $\sim X$ the set $\{\omega\backslash x : x \in X\}$. Note that $\sim X$ is homeomorphic to X because $x \mapsto \omega\backslash x$ is a homeomorphism from $\mathcal{P}(\omega)$ to itself. For subsets a, b of ω (resp. $a, b \in \omega^\omega$) $a \subset^* b$ (resp. $a \leq^* b$) means $|a\backslash b| < \omega$ (resp. $|\{n : a(n) > b(n)\}| < \omega$). A collection $\mathcal{F}$ of infinite subsets of ω is called a *semifilter* if for any $a \in \mathcal{F}$ and $a \subset^* b$ we have $b \in \mathcal{F}$. For a semifilter $\mathcal{F}$ we set $\mathcal{F}^+ = \{x \subset \omega : \forall a \in \mathcal{F}(a \cap x \neq \emptyset)\}$. Note that $\mathcal{F}^+ = \mathcal{P}(\omega)\backslash\sim\mathcal{F}$. $\mathfrak{Fr}$ denotes the minimal with respect to inclusion semifilter which consists of all co-finite sets. For the other notions used in the proof of the following statement we refer the reader to [3].

Lemma 3.1 *Suppose that G is a topological group, K is a compact subgroup of G with countable outer base in G, and QK is dense in G for some countable $Q \subset G$. Let* P *be a property of topological spaces preserved by images under cvusc maps and product with ω equipped with the discrete topology.*

If $\beta G\backslash G$ has P *and is not σ-compact, then there exists a semifilter $\mathcal{F}$ such that $\mathcal{F}^+ \subset \mathcal{F}$ and $\mathcal{F}$ has* P*. If, moreover, $(\beta G\backslash G)^2$ is Menger, then there exists a semifilter $\mathcal{F}$ such that $\mathcal{F} = \mathcal{F}^+$ and $\mathcal{F}^2$ is Menger.*

Proof Observe that G is not locally compact because otherwise its remainders would be compact. Since G is feathered, there exists a Čech-complete group $\tilde{G}$ containing G as a dense subgroup, see [3, Theorem 4.3.16]. Let $\beta\tilde{G}$ be the Stone-Čech compactification of $\tilde{G}$. It follows that $\beta\tilde{G}\backslash G$ is not σ-compact, and hence $G \neq \tilde{G}$. Fix

[3] For the Scheepers property this fact is slightly non-trivial and follows from [26, Proposition 4.7].

$g \in \tilde{G} \setminus G$ and note that gG is dense in $\beta\tilde{G}$ and $gG \cap G = \emptyset$. Therefore both $\beta\tilde{G} \setminus G$ and $\beta\tilde{G} \setminus gG$ have property P being remainders of spaces homeomorphic to G, and $\beta\tilde{G} = (\beta\tilde{G} \setminus G) \bigcup (\beta\tilde{G} \setminus gG)$.

Note that K has a countable outer base also in $\tilde{G}$, and hence the quotient space $X := \tilde{G}/K = \{zK : z \in \tilde{G}\}$ is metrizable. It is also separable by our assumption on G, so there exists a metrizable compactification bX of X. In addition, the quotient map $\pi_K : \tilde{G} \to X$, $\pi_K(z) = zK$, is perfect by Arhangel'skii and Tkachenko [3, Theorem 1.5.7], and hence by Engelking [13, Theorem 3.7.16] it can be extended to a (perfect) map $\pi : \beta\tilde{G} \to bX$ such that $\pi(\beta\tilde{G} \setminus \tilde{G}) = bX \setminus X$. $\pi \restriction \tilde{G} = \pi_K$, hence $G = \pi^{-1}(\pi(G))$, $gG = \pi^{-1}(\pi(gG))$, and consequently $A := \pi(\beta\tilde{G} \setminus gG)$ and $B := \pi(\beta\tilde{G} \setminus G)$ are both co-dense subsets of bX with property P covering bX.

Note that bX has no isolated points because both $G = \pi^{-1}(\pi(G))$ and $\beta\tilde{G} \setminus G = \pi^{-1}(\pi(\beta\tilde{G} \setminus G))$ are nowhere locally compact. Since bX is a metrizable compact, there exists a continuous surjective map $f : \mathcal{P}(\omega) \to X$. Applying [13, 3.1.C(a)] we can find a closed subspace T of $\mathcal{P}(\omega)$ such that $f \restriction T \to bX$ is surjective and irreducible, i.e., $f[T'] \neq bX$ for any closed $T' \subsetneq T$. T has no isolated points: if $t \in T$ were isolated then the irreducibility of $f \restriction T$ would give that $t = (f \restriction T)^{-1}[f(t)]$, which infers that $f(t)$ is isolated in bX and this leads to a contradiction. Therefore T is homeomorphic to $\mathcal{P}(\omega)$, and hence there exists a continuous surjective irreducible $h : \mathcal{P}(\omega) \to bX$. Since h is irreducible, both $C := h^{-1}(A)$ and $D := h^{-1}(B)$ are co-dense. Since h is perfect, they also have P: both of them are cvusc images of $\beta\tilde{G} \setminus G$. Note also that $\mathcal{P}(\omega) = C \bigcup D$.

The Cantor set $\mathcal{P}(\omega)$ has the following fundamental property (see [2] and references therein) which can be directly proved by Cantor's celebrated back-and-forth argument: for any countable dense subsets I_0, I_1, J_0, J_1 of $\mathcal{P}(\omega)$ such that $I_0 \cap I_1 = \emptyset$ and $J_0 \cap J_1 = \emptyset$ there exists a homeomorphism $i : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ with the property $i(I_0) = J_0$ and $i(I_1) = J_1$. Therefore there is no loss of generality to assume that $[\omega]^{<\omega} \subset \mathcal{P}(\omega) \setminus C$ and $\mathfrak{Fr} \subset \mathcal{P}(\omega) \setminus D$. Set

$$\mathcal{F}_0 = \left\{x \subset \omega : \exists c \in C, u \in [\omega]^{<\omega}, v \subset \omega \; (x = (c \setminus u) \cup v)\right\},$$
$$\mathcal{I}_0 = \left\{x \subset \omega : \exists d \in D, u \in [\omega]^{<\omega}, v \subset \omega \; (x = (d \cup u) \setminus v)\right\},$$

and note that both $\mathcal{F}_0$ and $\sim\mathcal{I}_0$ are semifilters. It follows that both $\mathcal{F}_0$ and $\mathcal{I}_0$ are countable unions of continuous images of $C \times \mathcal{P}(\omega)$ and $D \times \mathcal{P}(\omega)$, respectively, and consequently they are cvusc images of $(\beta\tilde{G} \setminus G) \times \omega$, where ω is considered with the discrete topology. Since the property P is preserved by product with ω, we conclude that both $\mathcal{F}_0$ and $\mathcal{I}_0$ have it.

Set $\mathcal{F} = \mathcal{F}_0 \cup \sim\mathcal{I}_0$ and note that it has property P for the same reason as $\mathcal{F}_0, \mathcal{I}_0$ do. Since $C \subset \mathcal{F}_0 \subset \mathcal{F}$ and $D \subset \mathcal{I}_0 = \sim(\sim\mathcal{I}_0) \subset \sim\mathcal{F}$, we have that $\mathcal{P}(\omega) = \mathcal{F} \bigcup \sim\mathcal{F}$. Therefore $\mathcal{F}^+ \subset \mathcal{F}$ because $\mathcal{F}^+ = \mathcal{P}(\omega) \setminus \sim\mathcal{F}$.

To prove the "moreover" part assume that $\mathcal{F}^2$ is Menger and consider the following map $\phi : (\mathcal{F} \cap \sim\mathcal{F}) \to [\omega]^\omega$:

$$\phi(a) = (a \cup \{n+1 : n \in a\}) \setminus a.$$

Note that $\mathcal{F} \cap \sim\mathcal{F}$ is homeomorphic to the closed subset $\{(x, x) : x \in \mathcal{P}(\omega)\} \cap (\mathcal{F} \times \sim\mathcal{F})$ of the Menger space $\mathcal{F} \times \sim\mathcal{F}$ and thus is Menger itself. Therefore $\phi(\mathcal{F} \cap \sim\mathcal{F})$ is not dominating.

Every strictly increasing sequence $\bar{k} = (k_n)_{n\in\omega}$ of integers such that $k_0 = 0$ generates a monotone surjection $\psi_{\bar{k}} : \omega \to \omega$ by letting $\psi_{\bar{k}}^{-1}(n) = [k_n, k_{n+1})$. We claim that there exists $\bar{k}$ as above such that $\psi_{\bar{k}}(\mathcal{F})^+ = \psi_{\bar{k}}(\mathcal{F})$. Suppose to the contrary that for every $\bar{k}$ there exists $a_{\bar{k}} \in \mathcal{F}$ such that $\psi_{\bar{k}}(a_{\bar{k}}) \in \psi_{\bar{k}}(\mathcal{F}) \setminus \psi_{\bar{k}}(\mathcal{F})^+$, i.e., $\omega \setminus \psi_{\bar{k}}(a_{\bar{k}}) \in \psi_{\bar{k}}(\mathcal{F})$. Then both $b_{\bar{k}} := \psi_{\bar{k}}^{-1}(\psi_{\bar{k}}(a_{\bar{k}}))$ and $\omega \setminus b_{\bar{k}}$ are in $\mathcal{F}$, and therefore $b_{\bar{k}} \in \mathcal{F} \cap \sim\mathcal{F}$. Note, however, that $\phi(b_{\bar{k}}) \subset \{k_n : n \in \omega\}$, which means that $\bar{k} \leq^* \phi(b_{\bar{k}})$. Since $\bar{k}$ was chosen arbitrarily we get that $\phi(\mathcal{F} \cap \sim\mathcal{F})$ is dominating, which is impossible. This contradiction implies that $\psi(\mathcal{F})^+ = \psi(\mathcal{F})$ for some monotone surjection $\psi : \omega \to \omega$, and then $\psi(\mathcal{F})$ is the semifilter with Menger square we were looking for. □

Recall that a family $\mathcal{X} \subset \mathcal{P}(\omega)$ is *centered* if $\bigcap \mathcal{X}'$ is infinite for every $\mathcal{X}' \in [\mathcal{X}]^{<\omega}$. We are in a position now to present the

Proof of Theorem 1.4 If $\mathcal{F}$ is a Scheepers ultrafilter, then $\sim\mathcal{F}$ is a subgroup of $(\mathcal{P}(\omega), \Delta)$ and $\mathcal{F} \cup \sim\mathcal{F} = \mathcal{P}(\omega)$. Thus $\mathcal{F}$ is a Scheepers non-σ-compact (no ultrafilter can be Borel) remainder of the group $(\sim\mathcal{F}, \Delta)$. Moreover, all finite powers of $\mathcal{F}$ are Menger (and hence also Scheepers) by Chodounsky et al. [11, Claim 5.5].

Let us now prove the "if" part, i.e., assume that $(\beta G \setminus G)$ is Scheepers and not σ-compact. In the same way as at the beginning of the proof of Theorem 1.3 we conclude that G is feathered. By Lemma 2.1 we may assume without loss of generality that G satisfies the premises of Lemma 3.1. Applying this lemma for P being the Scheepers property, we conclude that there exists a Scheepers semifilter $\mathcal{F}$ such that $\mathcal{F}^+ \subset \mathcal{F}$. For every $n \in \omega$ let us denote by O_n the open subset $\{x \subset \omega : n \in x\}$ of $\mathcal{P}(\omega)$ and note that each $x \in \mathcal{F}$ belongs to infinitely many members of $\mathcal{U}_0 = \{O_n : n \in \omega\}$. Applying [21, Theorem 21] (namely the implication (1) → (2) there) we conclude that there exists an increasing number sequence $(n_k)_{k\in\omega}$ such that $n_0 = 0$ and

$$\left\{\left\{\bigcup O_n : n \in [n_k, n_{k+1}\right\} : k \in \omega\right\}$$

is an ω-cover of $\mathcal{F}$. The latter means that for any family $\{x_0, \ldots, x_l\} \subset \mathcal{F}$ there exist infinitely many $k \in \omega$ such that $x_i \cap [n_k, n_{k+1}) \neq \emptyset$ for all $i \leq l$.

Let us define $\phi : \omega \to \omega$ by letting $\phi^{-1}(k) = [n_k, n_{k+1})$ for all k and set

$$\mathcal{S} = \left\{s \subset \omega : \phi^{-1}(s) \in \mathcal{F}\right\} = \{\phi(x) : x \in \mathcal{F}\}.$$

Then $\mathcal{S}$ is a Scheepers semifilter being a continuous image of $\mathcal{F}$. □

Claim 3.2 *$\mathcal{S}$ is centered.*

Proof Given any $s_0, \ldots, s_l \in \mathcal{S}$, set $x_i = \phi^{-1}(s_i)$ and take $k \in \omega$ such that $x_i \cap [n_k, n_{k+1}) = x_i \cap \phi^{-1}(k) \neq \emptyset$ for all $i \leq l$. There are infinitely many such k's, and each of them is an element of s_i for all i because $s_i = \phi(x_i)$. □

Claim 3.3 $\mathcal{S}^+ \subset \mathcal{S}$.

Proof Take any $x \in \mathcal{S}^+$ and set $y = \phi^{-1}(x)$. Then $y \in \mathcal{F}^+$: given $u \in \mathcal{F}$, note that $\phi(u) \in \mathcal{S}$, and hence $|\phi(u) \cap x| = \omega$, which implies that $|u \cap y| = \omega$ and thus y meets all elements of $\mathcal{F}$. Since $\mathcal{F}^+ \subset \mathcal{F}$, we have that $y \in \mathcal{F}$, and consequently $x = \phi(y) \in \mathcal{S}$.

Let us fix now $s_0, \dots, s_i \in \mathcal{S}$ and take arbitrary $s \in \mathcal{S}$. Claim 3.2 implies that $|s \cap \bigcap_{i \leq l} s_i| = \omega$, hence $\bigcap_{i \leq l} s_i \in \mathcal{S}^+$, and therefore $\bigcap_{i \leq l} s_i \in \mathcal{S}$ by Claim 3.3. Thus $\mathcal{S}$ is a filter, and consequently it is an ultrafilter by Claim 3.3. This completes our proof. □

We call a semifilter $\mathcal{F}$ a *P-semifilter* if for every sequence $(F_n)_{n\in\omega} \in \mathcal{F}^\omega$ there exists a sequence $(A_n)_{n\in\omega}$ such that $A_n \in [F_n]^{<\omega}$ and $\bigcup_{n\in\omega} A_n \in \mathcal{F}$. Note that if $\mathcal{F}$ is a filter then we get a standard definition of a P-filter. P-filters which are ultrafilters are nothing else but P-points.

Recall that for every $n \in \omega$ we denote by O_n the clopen subset $\{x \subset \omega : n \in x\}$ of $\mathcal{P}(\omega)$. The following fact is straightforward.

Observation 3.4 *Let $A \subset \omega$ and $\mathcal{F}$ be a semifilter. Then $\{O_n : n \in A\}$ covers $\mathcal{F}^+$ iff $A \in \mathcal{F}$. Consequently, if $\mathcal{F}^+$ is Menger, then $\mathcal{F}$ is a P-semifilter.*

Proof of Corollary 1.5 It is known [10] that under $\mathfrak{d} = \mathfrak{c}$ there exists an ultrafilter $\mathcal{F}$ on ω such that the Mathias forcing $\mathbb{M}(\mathcal{F})$ does not add dominating reals, see [10] for corresponding definitions. Applying [11, Theorem 1] we conclude that $\mathcal{F}$ is Menger when considered with the topology inherited from $\mathcal{P}(\omega)$. By Chodounsky et al. [11, Claim 5.5] we have that all finite powers of $\mathcal{F}$ are Menger, and hence $\mathcal{F}$ is Scheepers by Just et al. [17, Theorem 3.9].

Now suppose that $\mathcal{F}$ is a Menger ultrafilter. By the maximality of $\mathcal{F}$ we have $\mathcal{F} = \mathcal{F}^+$. Now it suffices to apply Observation 3.4. □

4 Menger remainders

This section is devoted to the proof of Theorem 1.6 which is divided into a sequence of lemmata. In the proof of the next lemma we shall need the following game of length ω on a topological space X: In the nth move player *I* chooses an open cover $\mathcal{U}_n$ of X, and player *II* responds by choosing a finite $\mathcal{V}_n \subset \mathcal{U}_n$. Player *II* wins the game if $\bigcup_{n\in\omega} \bigcup \mathcal{V}_n = X$. Otherwise, player *I* wins. We shall call this game *the Menger game* on X. It is well-known that X is Menger if and only if player *I* has no winning strategy in the Menger game on X, see [16] or [22, Theorem 13].

Formally, a strategy for player *I* is a map $\S : \tau^{<\omega} \to \mathcal{O}(X)$, where τ is the topology of X and $\mathcal{O}(X)$ is the family of all open covers of X. The strategy $\S$ is winning if $\bigcup_{n\in\omega} U_n \neq X$ for any sequence $(U_n)_{n\in\omega} \in \tau^\omega$ such that U_n is a union of a finite subset of $\S(U_0, \dots, U_{n-1})$ for all $n \in \omega$.

Lemma 4.1 *Suppose that $\mathcal{F}$ is a Menger semifilter. Then for every sequence $\langle B_i : i \in \omega\rangle \in (\mathcal{F}^+)^\omega$ and increasing $h \in \omega^\omega$ there exists increasing $\delta \in \omega^\omega$ such that*

$$\bigcup_{i\in\omega} B_i \cap [h(2\delta(i)), h(2\delta(i+1))) \in \mathcal{F}^+.$$

Proof For every $n \in \omega$ let us denote by O_n the subset $\{x \subset \omega : n \in x\}$ of $\mathcal{P}(\omega)$ and note that O_n is clopen. It is easy to see that for $B \subset \omega$ the collection $\mathcal{U}_B := \{O_n : n \in B\}$ is an open cover of $\mathcal{F}$ if and only if $B \in \mathcal{F}^+$. Set $\delta(0) = 0$ and consider the following strategy for player *I* in the Menger game on $\mathcal{F}$: In the 0th move he chooses $\mathcal{U}_{B_0 \setminus h(0)} = \mathcal{U}_{B_0 \setminus h(\delta(0))}$. Suppose that for some $i \in \omega$ we have already defined $\delta(i)$. Then player *I* chooses $\mathcal{U}_{B_i \setminus h(2\delta(i))}$. If player *II* responds by choosing $\mathcal{V}_i \in [\mathcal{U}_{B_i \setminus h(2\delta(i))}]^{<\omega}$, then we define $\delta(i+1)$ to be so that $\mathcal{V}_i \subset \{O_n : n \in [h(2\delta(i)), h(2\delta(i+1))) \cap B_i\}$, and the next move of player *I* is $\mathcal{U}_{B_{i+1} \setminus h(2\delta(i+1))}$.

The strategy for player *I* we described above is not winning, so there exists a run in the Menger game in which he uses this strategy and looses. Let δ be the function defined in the course of this run. It follows that $\bigcup\{\mathcal{V}_i : i \in \omega\} \supset \mathcal{F}$, where the $\mathcal{V}_i$s are the moves of player *II*, and hence

$$\bigcup_{i \in \omega} B_i \cap [h(2\delta(i)), h(2\delta(i+1))) \in \mathcal{F}^+$$

because $\mathcal{V}_i \subset \{O_n : n \in [h(2\delta(i)), h(2\delta(i+1))) \cap B_i\}$. □

For a semifilter $\mathcal{F}$ we denote by $\mathbb{P}_{\mathcal{F}}$ the poset consisting of all partial maps p from $\omega \times \omega$ to 2 such that for every $n \in \omega$ the domain of $p_n : k \mapsto p(n,k)$ is an element of $\sim\mathcal{F}$. If, moreover, we assume that and $\mathrm{dom}(p_n) \subset \mathrm{dom}(p_{n+1})$ for all n, the corresponding poset will be denoted by $\mathbb{P}^*_{\mathcal{F}}$. A condition q is stronger than p (in this case we write $q \leq p$) if $p \subset q$. For filters $\mathcal{F}$ the poset $\mathbb{P}^*_{\mathcal{F}}$ is obviously dense in $\mathbb{P}_{\mathcal{F}}$, and the latter is proper and ω^ω-bounding if $\mathcal{F}$ is a non-meager P-filter [23, Fact VI.4.3, Lemma VI.4.4]. In light of Observation 3.4, the following lemma may be thought of as a topological counterpart of [23, Fact VI.4.3, Lemma VI.4.4].

Lemma 4.2 *If $\mathcal{F}^+$ is a Menger semifilter, then both $\mathbb{P}_{\mathcal{F}}$ and $\mathbb{P}^*_{\mathcal{F}}$ are proper and ω^ω-bounding.*

Proof We shall present the proof for $\mathbb{P}_{\mathcal{F}}$. The one for the poset $\mathbb{P}^*_{\mathcal{F}}$ is completely analogous.

To prove the properness let us fix a countable elementary submodel $M \ni \mathbb{P}_{\mathcal{F}}$ of $H(\theta)$ for θ big enough, a condition $p \in \mathbb{P}_{\mathcal{F}} \cap M$, and list all open dense subsets of $\mathbb{P}_{\mathcal{F}}$ which are elements of M as $\{D_i : i \in \omega\}$. Let us denote by τ the collection of all open subsets of $\mathcal{P}(\omega)$. For every $s \in [\omega]^{<\omega}$ we shall denote by O_s the set $\{x \subset \omega : x \cap s \neq \emptyset\}$. O_s is clearly a clopen subset of $\mathcal{P}(\omega)$.

In what follows we shall define a strategy $\S : \tau^{<\omega} \to \mathcal{O}(\mathcal{F}^+)$ of player *I* in the Menger game on $\mathcal{F}^+$ as well as a map $\S_0 : \tau^{<\omega} \cap M \to \mathbb{P}_{\mathcal{F}} \cap M$. Set $p^0 = p$, $\S_0(\emptyset) = p^0$, and

$$\S(\emptyset) = \left\{O_s : \exists l \in \omega \left[s = \left(\omega \setminus \mathrm{dom}(p^0_0)\right) \cap l\right]\right\}.$$

Now suppose that for some $n \in \omega$ and all sequences $(U_k)_{k<n}$ of open subsets of $\mathcal{P}(\omega)$ we have defined $p^n = \S_0((U_k)_{k<n})$ and $\S((U_k)_{k<n})$, and fix such a sequence $(U_k)_{k\leq n}$ of length n. If U_n is not of the form $\bigcap_{i \leq n} O_{s^n_i}$, where $s^n_i = (\omega \setminus \mathrm{dom}(p^n_i)) \cap l$ for some

$l \in \omega$, then $\S_0((U_k)_{k\le n})$ and $\S((U_k)_{k\le n})$ are irrelevant. Otherwise write $\prod_{i\le n} 2^{\{i\}\times s_i^n}$ in the form $\{(t_i^{n,j})_{i\le n} : j \le N\}$, set $p^{n,-1} = p^n$, and by induction on $j \le N$ define a decreasing sequence $(p^{n,j})_{j\le N}$ of conditions in $\mathbb{P}_{\mathcal{F}} \cap M$ with the following properties:

(i) $\mathrm{dom}(p_i^{n,j}) \cap s_i^n = \emptyset$ for all $j \le N$ and $i \le n$;
(ii) $p^{n,j} \cup \bigcup_{i\le n} t_i^{n,j} \in D_n$ for all $j \le N$.

Then we let $p^{n+1} = p^{n,N}$, $\S_0((U_k)_{k\le n}) = p^{n+1}$ and

$$\S((U_k)_{k\le n}) = \left\{ \bigcap_{i\le n+1} O_{s_i^{n+1}} : \exists l \in \omega \forall i \le n+1 \left[s_i^{n+1} = (\omega \backslash \mathrm{dom}(p_i^{n+1})) \cap l \right] \right\}.$$

Since $\mathcal{F}^+$ is Menger, $\S$ cannot be a winning strategy for player I, and hence there exists a sequence $(U_n)_{n\in\omega}$ of open subsets of $\mathcal{P}(\omega)$ with the following properties:

(iii) $p^n := \S_0((U_i)_{i<n}) \in \mathbb{P}_{\mathcal{F}} \cap M$ for all $n \in \omega$;
(iv) For every $n \in \omega$ there exists $l_n \in \omega$ such that $U_n = \bigcap_{i\le n} O_{s_i^n}$, where $s_i^n = (\omega \backslash \mathrm{dom}(p_i^n)) \cap l_n$;
(v) $l_n \le l_{n+1}$ for all $n \in \omega$;
(vi) $\mathrm{dom}(p_i^{n+1}) \cap s_i^n = \emptyset$; and
(vii) $\mathcal{F}^+ \subset \bigcup_{n\in\omega} U_n$.

Items (iv) and (vii) imply that for every $i \in \omega$ we have $\mathcal{F}^+ \subset \bigcup_{n\ge i} O_{s_i^n}$, therefore for every $x \in \mathcal{F}^+$ there exists $n \ge i$ such that $x \cap s_i^n \ne \emptyset$, which is equivalent to $y_i := \bigcup_{n\ge i} s_i^n \in (\mathcal{F}^+)^+ = \mathcal{F}$. By the definition of s_i^n in (iv) together with items (v) and (vi) we have that $s_i^n \subset s_i^{n+1}$ for all $n \in \omega$ and $i \le n$, and hence $y_i \cap \bigcup_{n\ge i} \mathrm{dom}(p_i^n) = \emptyset$. Since $i \in \omega$ was chosen arbitrarily, we conclude that $q := \bigcup_{n\in\omega} p_n \in \mathbb{P}_{\mathcal{F}}$.

We claim that q is $(M, \mathbb{P}_{\mathcal{F}})$-generic. Indeed, pick $q' \le q$, $n \in \omega$, and $r \le q'$ such that $\mathrm{dom}(r_i) \supset s_i^n$ for all $i \le n$. Then there exists $j \le N$ such that $r_i \restriction (\{i\}\times s_i^n) = t_i^{n,j}$ for all $i \le n$, and consequently

$$r \le q \cup \bigcup_{i\le n} t_i^{n,j} \le p^{n,j} \cup \bigcup_{i\le n} t_i^{n,j} \in D_n$$

by (ii). This implies that r is compatible with an element of $D_n \cap M$ and thus completes our proof of the properness.

Note that for every n we have found a finite subset A_n (namely $\{p^{n,j} : j \le N\}$) of $D_n \cap M$ such that any extension of q is compatible with some element of A_n. If $\dot{f} \in M$ is a $\mathbb{P}_{\mathcal{F}}$-name for a real, then the open dense subset of $\mathbb{P}_{\mathcal{F}}$ consisting of those conditions which determine $\dot{f}(k)$ equals D_{n_k} for some $n_k \in \omega$. It follows from the above that q forces that $\dot{f}(k)$ cannot exceed $\max\{l : \exists u \in A_{n_k} (u \Vdash \dot{f}(k) = \check{l})\}$, and therefore $\mathbb{P}_{\mathcal{F}}$ is ω^ω-bounding. □

For a relation R on ω and $x, y \in \omega^\omega$ we denote by $[x \, R \, y]$ the set $\{n : x(n) \, R \, y(n)\}$.

Lemma 4.3 *Suppose that $\mathcal{F} = \mathcal{F}^+$ is a semifilter with Menger square. Let x be $\mathbb{P}^*_{\mathcal{F}}$-generic, $\mathbb{Q} \in V[x]$ be an ω^ω-bounding poset, and H be a $\mathbb{Q}$-generic over $V[x]$. Then in $V[x * H]$ there is no semifilter $\mathcal{G} = \mathcal{G}^+$ containing $\mathcal{F}$ such that $\mathcal{G}^2$ is Menger.*

Proof Throughout the proof we shall identify x with $\cup x : \omega \times \omega \to 2$. Suppose to the contrary that such a $\mathcal{G}$ exists. Set $x_j(n) = x(j, n)$. In $V[x * H]$, the following 2 cases are possible.

(a) For every $m \in \omega$ there exists $k > m$ such that $\bigcup_{j \in [m,k)} [x_j = x_m] \in \mathcal{G}$. Then we can inductively construct an increasing sequence $\langle m_k : k \in \omega \rangle$ such that

$$\bigcup_{j \in [m_k, m_{k+1})} [x_j = x_{m_k}] \in \mathcal{G} \text{ for all } k. \tag{1}$$

Since $\mathbb{P}^*_{\mathcal{F}} * \mathbb{Q}$ is ω^ω-bounding, we may additionally assume that this sequence is in V.

(b) There exists m such that $\bigcup_{j \in [m,k)} [x_j = x_m] \in {\sim}\mathcal{G}$ for all $k > m$. This means that $\bigcap_{j \in [m,k)} [x_j \neq x_m] \in \mathcal{G}$ for all $k > m$. Then

$$[x_i = x_{i+1}] \supset [x_i \neq x_m] \cap [x_{i+1} \neq x_m] \supset \bigcap_{j \in [m, i+2)} [x_j \neq x_m] \in \mathcal{G}$$

for all $i > m$. Thus the sequence $m_k = m + 1 + 2k$ satisfies (1), and hence there always exists a sequence $\langle m_k : k \in \omega \rangle \in V$ satisfying (1).

Set $A_k = \bigcup_{j \in [m_k, m_{k+1})} [x_j = x_{m_k}] \in \mathcal{G}$ and $\mathcal{U}_k = \{U^k_n : n \in \omega\}$, where

$$U^k_n = \{\langle X, Y \rangle \in \mathcal{P}(\omega)^2 : \forall i \leq k \left((X \cap A_i \cap [k, n) \neq \emptyset) \wedge (Y \cap A_i \cap [k, n) \neq \emptyset)\right)\}.$$

Since $A_k \in \mathcal{G} = \mathcal{G}^+$ for all k, $\mathcal{U}_k$ is easily seen to be an open cover of $\mathcal{G}^2$. The Menger property of $\mathcal{G}^2$ yields a strictly increasing $f \in \omega^\omega \cap V[x * H]$ such that $\{U^k_{f(k)} : k \in \omega\}$ covers $\mathcal{G}^2$. Since $\mathbb{P}^*_{\mathcal{F}} * \mathbb{Q}$ is ω^ω-bounding, we could additionally assume that $f \in V$. Set $h(0) = f(0) + 1$ and $h(l + 1) = f(h(l)) + 1$ for all l. Note that $U^k_n = (W^k_n)^2$, where

$$W^k_n = \{X \in \mathcal{P}(\omega) : \forall i \leq k \; (X \cap A_i \cap [k, n) \neq \emptyset)\}.$$

Therefore there exists $\epsilon \in 2$ such that

$$O_\epsilon := \bigcup \left\{ W^k_{f(k)} : k \in \bigcup_{l \in \omega} [h(2l + \epsilon), h(2l + \epsilon + 1)) \right\} \supset \mathcal{G} :$$

If there were $X_\epsilon \in \mathcal{G} \setminus O_\epsilon$ for all $\epsilon \in 2$, then $\langle X_0, X_1 \rangle$ could not be an element of $U^k_{f(k)}$ for any k thus contradicting the choice of f. Without loss of generality $\epsilon = 0$ is as above.

Claim 4.4 *Let $\delta \in \omega^\omega$ be strictly increasing. Then*

$$A_\delta := \bigcup_{i\in\omega} A_i \cap [h(2\delta(i)), h(2\delta(i+1))) \in \mathcal{G}.$$

Proof Given any $X \in \mathcal{G}$, find $l \in \omega$ and $k \in [h(2l), h(2l+1))$ such that $X \in W^k_{f(k)}$. Let $i \in \omega$ be such that $l \in [\delta(i), \delta(i+1))$. Note that $i \le l \le k$, hence $X \in W^k_{f(k)}$ implies $X \cap A_i \cap [k, f(k)) \neq \emptyset$. It follows that

$$[k, f(k)) \subset [h(2l), f(h(2l+1))) \subset [h(2l), h(2l+2)) \subset [h(2\delta(i)), h(2\delta(i+1))),$$

consequently $X \cap A_i \cap [h(2\delta(i)), h(2\delta(i+1))) \neq \emptyset$, which implies $\bigcup_{i\in\omega} A_i \cap [h(2\delta(i)), h(2\delta(i+1))) \in \mathcal{G}^+$. □

Let us fix any $p \in \mathbb{P}^*_\mathcal{F}$ and set $B_i = \omega\setminus\mathrm{supp}(p_{m_{i+1}}) \in \mathcal{F}^+$. By Lemma 4.1 used in V there exists an increasing δ such that $B := \bigcup_{i\in\omega} B_i \cap [h(2\delta(i)), h(2\delta(i+1))) \in \mathcal{F}^+ = \mathcal{F}$. For every $m \in \omega$ find i such that $m \in [m_i, m_{i+1})$ and set

$$q_m = p_m \cup \big(B_i \cap [h(2\delta(i)), h(2\delta(i+1))) \times \{0\}\big)$$

if $m = m_i$ and

$$q_m = p_m \cup \big(B_i \cap [h(2\delta(i)), h(2\delta(i+1))) \times \{1\}\big)$$

otherwise. This q obviously forces (i.e., any condition in $\mathbb{P}^*_\mathcal{F} * \mathbb{Q}$ whose first coordinate is q forces) that $B_i \cap \dot{A}_i \cap [h(2\delta(i)), h(2\delta(i+1))) = \emptyset$ for all i, and hence it also forces $B \cap \dot{A}_\delta = \emptyset$. Thus the set of those $q \in \mathbb{P}^*_\mathcal{F}$ which force $B \cap \dot{A}_\delta = \emptyset$ is dense, which means that $B \cap A_\delta = \emptyset$ (here $A_\delta = \dot{A}_\delta^{G*H}$). However, $B \in \mathcal{F} \subset \mathcal{G}$ by the choice of δ and $A_\delta \in \mathcal{G}$ by Claim 4.4, and therefore $B \cap A_\delta = \emptyset$ contradicts $\mathcal{G} = \mathcal{G}^+$. This contradiction completes our proof. □

Proof of Theorem 1.6 Suppose that there exists a topological group G and a compactification bG, such that $(bG\setminus G)^2$ is Menger but not σ-compact. Then in the same way as at the beginning of the proof of Theorem 1.3 we conclude that G is feathered. By Lemma 2.1 we may assume without loss of generality that G satisfies the premises of Lemma 3.1. Applying this lemma for P being the property of having the Menger square we get a semifilter $\mathcal{F} = \mathcal{F}^+$ such that $\mathcal{F}^2$ is Menger. Thus the theorem will be proved as soon as we construct a model of ZFC in which there are no semifilters $\mathcal{F} = \mathcal{F}^+$ with Menger square.

To this end let us assume that GCH holds in V and consider a function $B : \omega_2 \to H(\omega_2)$, the family of all sets whose transitive closure has size $< \omega_2$, such that for each $x \in H(\omega_2)$ the family $\{\alpha : B(\alpha) = x\}$ is ω_1-stationary. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \beta < \alpha \le \omega_2\rangle$ be the following iteration with at most countable supports: If $B(\alpha)$ is a $\mathbb{P}_\alpha$-name for $\mathbb{P}^*_{\dot{\mathcal{F}}}$ for some semifilter $\dot{\mathcal{F}}$ such that $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathcal{F}} = \dot{\mathcal{F}}^+$ and $\dot{\mathcal{F}}^2$ is Menger", then $\dot{\mathbb{Q}}_\alpha = \mathbb{P}^*_{\dot{\mathcal{F}}}$. Otherwise we let $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\alpha$-name for the trivial forcing. Then $\mathbb{P}_{\omega_2}$ is ω^ω-bounding forcing notion with ω_2-c.c. being a countable support iteration of length ω_2 of proper ω^ω-bounding posets of size ω_1 over a model of CH.

Let G be a $\mathbb{P}_{\omega_2}$-generic over V and suppose that $\mathcal{F} \in V[G]$ is a semifilter such that $\mathcal{F} = \mathcal{F}^+$ and $\mathcal{F}^2$ is Menger. Then the set $\{\alpha : \mathcal{F}_\alpha := (\mathcal{F} \cap V[G \cap \mathbb{P}_\alpha]) \in V[G \cap \mathbb{P}_\alpha]$, $\mathcal{F}_\alpha = \mathcal{F}^+_\alpha$ and $\mathcal{F}^2_\alpha$ is Menger in $V[G \cap \mathbb{P}_\alpha]\}$ contains an ω_1-club subset of ω_2, and hence for one of these α we have that $\dot{\mathbb{Q}}_\alpha = \mathbb{P}^*_{\dot{\mathcal{F}}_\alpha}$, where $\dot{\mathcal{F}}_\alpha$ is a $\mathbb{P}_\alpha$-name such that $\dot{\mathcal{F}}_\alpha^{G \cap \mathbb{P}_\alpha} = \mathcal{F} \cap V[G \cap \mathbb{P}_\alpha]$. Now, a direct application of Lemmata 4.3 and 4.2 implies that $\mathcal{F}_\alpha \subset \mathcal{F}$ cannot be enlarged to any semifilter $\mathcal{U} \in V[G]$ such that $\mathcal{U}^2$ is Menger and $\mathcal{U}^+ = \mathcal{U}$, which contradicts our choice of $\mathcal{F}$. $\square$

Let us note that in the proof of Theorem 1.6 above we have also proven the following

Theorem 4.5 *It is consistent with ZFC that there are no semifilters $\mathcal{F}$ such that $\mathcal{F} = \mathcal{F}^+$ and $\mathcal{F}^2$ is Menger.*

5 On a possible dichotomy for the Menger property

Our first attempt to find a counterpart of the Menger property is based on its game characterization we have exploited in Sect. 4. As the Menger game produces a strengthening of the Lindelöf property, we should consider a game which produces a strengthening of the Baire property.

There is an obvious candidate for this purpose: the Banach–Mazur game, see for instance [18] for more information. This game BM(X) is played on the space X in ω-many innings between two players α and β as follows. β makes the first move by choosing a non-empty open set U_0 and α responds by taking a non-empty open set $V_0 \subseteq U_0$. In general, at the n-th inning β chooses a non-empty open set $U_n \subseteq V_{n-1}$ and α responds by taking a non-empty open set $V_n \subseteq U_n$. The rule is that α wins if and only if $\bigcap\{V_n : n < \omega\} \neq \emptyset$. The relationship of the Banach-Mazur game with Baire spaces is given by the following [18, Theorem 8.11].

Theorem 5.1 *A space X is Baire if and only if player β does not have a winning strategy in BM(X).*

Consequently, if α has a winning strategy, then the space is Baire.

Definition A space X is *weakly α-favorable* if player α has a winning strategy in the Banach-Mazur game. X is said to be *α-favorable* if player α has a winning tactic, i. e. a winning strategy depending only on the last move of β.

Every pseudocompact space is α-favorable: player α has an easy winning tactic by choosing for any U_n a non-empty open set V_n such that $\overline{V_n} \subseteq U_n$. Of course, every weakly α-favorable space is Baire. Moreover, the following observation shows that being weakly α-favorable often contradicts the Menger property.

Observation 5.2 *No nowhere locally compact weakly α-favorable subset X of the real line is Menger.*

Proof Since X is nowhere locally compact, we may assume that $X \subset \mathbb{R}\backslash\mathbb{Q}$, and the latter we shall identify with ω^ω. By Kechris et al. [18, Theorem 8.17(1)] $X \supset Y$ for some dense G_δ subset Y of ω^ω. By the Baire category theorem Y cannot be contained in a σ-compact subspace of ω^ω, and hence it contains a copy Z of ω^ω which is closed in ω^ω according to [18, Corollary 21.23]. Therefore Z is a closed in X copy of ω^ω, which implies that X is not Menger as the Menger property is inherited by closed subspaces. □

Therefore, weak α-favorability seems to be a good candidate to be the counterpart of the Menger property. However, this is not the case by Theorem 5.6 below. Let us recall that a set $S \subset \mathbb{R}$ is a Bernstein set provided that both S and $\mathbb{R}\backslash S$ meet every closed uncountable subset of $\mathbb{R}$. The next two lemmas seem to be known, but we were not able to find them in the literature. That is why we present their proofs.

Lemma 5.3 *There is a subgroup G of the real line $\mathbb{R}$ containing the rationals which is a Bernstein set.*

Proof Let $\{C_\alpha : \alpha < \mathfrak{c}\}$ be the collection of all closed uncountable subsets of $\mathbb{R}$. Here, we will consider $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Choose a point $x_0 \in C_0$ and denote by G_0 the vector subspace of $\mathbb{R}$ generated by $\{1, x_0\}$. Obviously, we have $|G_0| = \omega$. Then, pick a point $y_0 \in C_0\backslash G_0$. We proceed by transfinite induction, by assuming to have already constructed a non decreasing family of vector subspaces $\{G_\beta : \beta < \alpha\}$ of $\mathbb{R}$ satisfying $|G_\beta| \le |\beta| + \omega$ for each β and points $x_\beta, y_\beta \in C_\beta$ in such a way that $x_\beta \in G_\beta$ and $\{y_\beta : \beta < \alpha\} \cap \bigcup\{G_\beta : \beta < \alpha\} = \emptyset$. The set $H_\alpha = \bigcup\{G_\beta : \beta < \alpha\}$ has cardinality not exceeding $|\alpha| + \omega$ and therefore even the vector subspace K_α generated by the set $H_\alpha \cup \{y_\beta : \beta < \alpha\}$ has cardinality less than $\mathfrak{c}$. So we may pick a point $x_\alpha \in C_\alpha\backslash K_\alpha$. Then, let G_α be the vector subspace generated by $H_\alpha \cup \{x_\alpha\}$ and finally pick a point $y_\alpha \in C_\alpha\backslash G_\alpha$. It is clear that $|G_\alpha| \le |\alpha| + \omega$. To complete the induction, we need to show $y_\beta \notin G_\alpha$ for each $\beta < \alpha$. Indeed, if we had $y_\beta \in G_\alpha$ for some β, then $y_\beta = z + qx_\alpha$, where $z \in H_\alpha$ and $q \in \mathbb{Q}\backslash\{0\}$. But, this would imply $x_\alpha = q^{-1}z - q^{-1}z \in K_\alpha$, in contrast with the way x_α was chosen.

Now, we let $G = \bigcup\{G_\alpha : \alpha < \mathfrak{c}\}$. It is clear that G is a $\mathbb{Q}$-vector subspace, and hence a subgroup, of $\mathbb{R}$ which is also a Bernstein set. □

Lemma 5.4 *A Bernstein set $X \subset \omega^\omega$ does not have the Menger property.*

Proof For any $f \in \omega^\omega$ there exists some $g \in X$ such that $f(n) < g(n)$ for each $n \in \omega$. This comes from the fact that X must meet the Cantor set $\prod_{n<\omega}\{f(n)+1, f(n)+2\}$. To finish, recall that a dominating subset of ω^ω is never Menger. Indeed, for any $n < \omega$ let $\pi_n : \omega^\omega \to \omega$ be the projection onto the n-th factor and put $\mathcal{U}_n = \{\pi_n^{-1}(k) \cap X : k \in \omega\}$. Each $\mathcal{U}_n$ is an open cover of X. For any choice of a finite set $\mathcal{V}_n \subseteq \mathcal{U}_n$, we may define a function $g : \omega \to \omega$ by letting $g(n) = \max \pi_n(\bigcup \mathcal{V}_n)$, if $\mathcal{V}_n \neq \emptyset$, and $g(n) = 0$ otherwise. Since X is dominating, there is some $f \in X$ such that $g(n) < f(n)$ for each n. Clearly, $f \notin \bigcup\{\bigcup \mathcal{V}_n : n < \omega\}$ and so X is not Menger. □

Lemma 5.5 *A Bernstein set $X \subseteq \mathbb{R}$ is not weakly α-favorable.*

Proof By Kechris [18, Theorem 8.17(1)] any weakly α-favorable subspace of $\mathbb{R}$ is comeager, while no Bernstein set can be comeager because any comeager subspace of $\mathbb{R}$ contains homeomorphic copies of the Cantor set. □

These three lemmas imply:

Theorem 5.6 *There exists a topological group G and its compactification bG such that the remainder $bG\setminus G$ is neither Menger nor weakly α-favorable.*

Proof Recall that the set of irrationals $\mathbb{R}\setminus\mathbb{Q}$ is homeomorphic to ω^ω. Let G be such as in Lemma 5.3. By Lemma 5.4 $\mathbb{R}\setminus G \subseteq \omega^\omega$ is not Menger, and by Lemma 5.5 $\mathbb{R}\setminus G$ is not weakly α-favorable. Now, it suffices to take as bG the compactification of $\mathbb{R}$ obtained by adding two end-points. □

Theorem 5.6 implies that the counterpart of the Menger property should be in between of weakly α-favorable and Baire.

6 Miscellanea

A very important example of a topological group is $C_p(X)$, the subspace of $\mathbb{R}^X$ with the Tychonoff product topology consisting of all continuous functions. We expect that the remainder of $C_p(X)$ cannot distinguish between being Menger and σ-compact, but we cannot prove this.

Question 6.1 Is it true that a remainder of $C_p(X)$ is Menger if and only if it is σ-compact?

Below we present some results giving a partial solution of Question 6.1.

Proposition 6.2 *Let Z be a compactification of $C_p(X)$. If $Z\setminus C_p(X)$ is Menger, then $C_p(X)$ is first countable and hereditarily Baire.*

Proof Since Menger spaces are Lindelöf, by Henriksen–Isbell's theorem [14], $C_p(X)$ is of countable type, and therefore it contains a compact subgroup with countable outer base according to [3, Lemma 4.3.10]. It is easy to see that there is no compact subgroup of $C_p(X)$ except for $\{0\}$: for any $f \in C_p(X)\setminus\{0\}$, the set $\{nf : n \in \omega\}$ is not contained in any compact $K \subset C_p(X)$ because $\{nf(x) : n \in \omega\}$ is unbounded in $\mathbb{R}$ if $f(x) \neq 0$. Therefore $C_p(X)$ is first-countable, and hence X is countable.

Since $Z\setminus C_p(X)$ is Menger, it follows that $C_p(X)$ contains no closed copy of $\mathbb{Q}$. Now, a theorem of Debs [12] implies that $C_p(X)$ is hereditarily Baire. □

The following fact together with Proposition 6.2 gives the positive answer to Question 6.1 for spaces containing non-trivial convergent sequences.

Observation 6.3 *If X contains a non-trivial convergent sequence then $C_p(X)$ is not Baire.*

Proof There is nice characterization of the Bairness for spaces of the form $C_p(X)$ due to Tkachuk. However, we shall present here a direct elementary proof. Suppose that $(x_n)_{n\in\omega}$ is an injective sequence converging to x. Set

$$F_n = \left\{ f \in C_p(X) : \forall m \geq n\ (|f(x) - f(x_m)| \leq 1) \right\}.$$

It is easy to check that each F_n is closed nowhere dense in $C_p(X)$ and $C_p(X) = \bigcup_{n\in\omega} F_n$. □

By a theorem of Lutzer (see Problem 265 in [24]), $C_p(X)$ is Čech-complete if and only if X is countable and discrete. So to answer Question 6.1 in the affirmative we need to show that $C_p(X)$ has a Menger remainder only if X is countable and discrete.

Note that in the proof of the "if" part of Theorem 1.4 the compactification was a topological group itself, namely $(\mathcal{P}(\omega), \Delta)$. We do not know whether complements to Menger subspaces in other Polish groups (e.g., $\mathbb{R}$) may consistently be subgroups. The next proposition imposes some restrictions.

Proposition 6.4 *Let G be an analytic topological group and M be a non-empty Menger subspace of G. If $G\backslash M$ is a subgroup of G, then G is σ-compact and M contains a topological copy of $\mathcal{P}(\omega)$.*

Proof Suppose that $H = G\backslash M$ is a subgroup of G and fix $g \in M$. Then $H \subset g^{-1} * M$, where $*$ is the underlying operation on G. Therefore $G = M \cup g^{-1} * M$ is Menger, and hence it is σ-compact, see [1].

Now suppose that M contains no topological copy of $\mathcal{P}(\omega)$ and let $X \subset G$ be homeomorphic to $\mathcal{P}(\omega)$. If $X \subset H$ then $g * X \subset g^* H \subset M$ which is impossible by our assumption above. Thus $X \cap M \neq \emptyset$. Since M contains no copy of $\mathcal{P}(\omega)$, $X\backslash M$ is dense in X, and hence there exists a countable dense subset Q of X disjoint from M. Then $X \cap M = (X\backslash Q) \cap M$ is a closed subset of M. Note that $(X\backslash Q)$ is a copy of ω^ω and $M \cap (X\backslash Q)$ is a Bernstein set in $(X\backslash Q)$. To finish, it suffices to apply Lemma 5.4. □

The following statement shows that the classical Cantor–Bendixon inductive procedure does not have any variant allowing to separate a "nowhere perfect" core of a Menger space from its "σ-compact part".

Proposition 6.5 *There exists a Baire dense nowhere locally compact subgroup $\mathcal{I}$ of $\mathcal{P}(\omega)$ with the Menger property such that for every σ-compact subspace $\mathcal{S}$ of $\mathcal{I}$ there exists $\mathcal{K} \subset \mathcal{I}$ homeomorphic to $\mathcal{P}(\omega)$ such that $\mathcal{K} \cap \mathcal{S} = \emptyset$.*

Proof It is well-known that there exists a non-meager Menger filter $\mathcal{F}$ on ω, see, e.g., the proof of Theorem 1 in [20]. Let $\mathcal{I}$ be the dual ideal of $\mathcal{F}$. Then $\mathcal{I}$ is Menger, nowhere locally compact, and non-meager being homeomorphic to $\mathcal{F}$. Also, $\mathcal{I}$ is a subgroup of $\mathcal{P}(\omega)$, and hence it is Baire because each non-meager topological group is so. Note that $\mathcal{I}$ contains copies of $\mathcal{P}(\omega)$: for every infinite $I \in \mathcal{I}$ the set $\mathcal{P}(I) \subset \mathcal{I}$ is such a copy. Let us fix $\mathcal{X} \subset \mathcal{I}$ homeomorphic to $\mathcal{P}(\omega)$ and a σ-compact $\mathcal{S} \subset \mathcal{I}$. Then there exists $I \in \mathcal{I}\backslash(\mathcal{S} + \mathcal{X})$ because $\mathcal{S} + \mathcal{X}$ is σ-compact and $\mathcal{I}$ is not. It follows that $\mathcal{K} := \{I\} - \mathcal{X}$ is a copy of $\mathcal{P}(\omega)$ disjoint from $\mathcal{S}$. □

Acknowledgments The authors wish to thank Masami Sakai and Boaz Tsaban for many useful comments. We also thank the anonymous referee for a careful reading and many suggestions which improved the paper.

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