A bi-preference interplay between transitivity and completeness: Reformulating and extending Schmeidler's theorem

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Abstract

Consumers' preferences and choices are traditionally described by appealing to two classical tenets of rationality: transitivity and completeness. In 1971, Schmeidler proved a striking result on the interplay between these properties: *On a connected topological space, a nontrivial bi-semicontinuous preorder is complete.* Here we reformulate and extend this well-known theorem. First, we show that the topology is not independent of the preorder, contrary to what the original statement suggests. In fact, Schmeidler's theorem can be restated as follows: *A nontrivial preorder with a connected order-section topology is complete.* Successively, we extend it to comonotonic bi-preferences: these are pairs of relations such that the first is a preorder, and the second consistently enlarges the first. In particular, a NaP-preference (necessary and possible preference, Giarlotta and Greco, 2013) is a comonotonic bi-preference with a complete second component. We prove two complementary results of the following kind: *Special comonotonic bi-preferences with a connected order-section topology are NaP-preferences.* Schmeidler's theorem is a particular case.

Key words: Rational behavior; transitivity; completeness; preorder; order-section topology; Schmeidler's theorem; bi-preference; necessary and possible preference.

1 Motivation and goal

Nearly invariantly, graduate textbooks on the foundation of microeconomics start the description of consumers' individual preferences and choice behavior by discussing the two (almost) undisputed tenets of economic rationality: transitivity and completeness. The classical textbook by Mas-Colell, Whinston, and Green (1995), as well as the recent treatise on the foundations of microeconomic theory by Kreps (2013), are no exception to this didactic rule; in fact, transitivity and completeness are extensively discussed starting from Chapter 1. These two features of rationality, which are defined for any binary relation S on a set X of alternatives, are the following:

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Transitivity: for all $x, y, z \in X$, if xSy and ySz, then xSz.

Completeness: for all distinct $x, y \in X$, either xSy or ySx (or both).[1]

The overwhelming influence of these two tenets of rationality[2] within economic theory calls for an attentive analysis of their mutual relationship. Schmeidler's theorem, proved almost fifty years ago, is one of the most well-known instances of this kind. Intuitively, this striking result states that, for a nontrivial preference on a connected topological space, the join of transitivity and weak continuity implies completeness. More formally:

Theorem (Schmeidler, 1971). *Let S be a nontrivial preorder on a connected topological space X. If S is closed-semicontinuous and $S^{>}$ is open-semicontinuous, then S is complete.*

(Recall that a *preorder* S is a reflexive transitive binary relation, the strict preference $S^{>}$ denotes the asymmetric part of S, and *nontrivial* means that $S^{>}$ is nonempty. The two *semicontinuity* properties assumed by Schmeidler consist of (1) the closeness of all upper and lower sections in S, and (2) the openness of all upper and lower sections in $S^{>}$.)[3]

The proof given by Schmeidler is neat and compact. However, two arguments of a different nature, one order-theoretic and one topological, are quite intertwined. Thus it is not clear to what extent the assumptions of connectedness (of the space) and double semicontinuity (of the preorder) are needed to yield completeness (of the preorder).

In this paper, we clarify how each hypothesis of Schmeidler's result contributes in deriving the thesis. It turns out that one can extract two preliminary facts from the original proof: the first is order-theoretic, and uses the two assumptions of semicontinuity to obtain that suitable sets are equal; the second is topological, and uses the connectedness of the space to conclude that some sets must be either empty or equal to the whole space. Then, starting from the nontriviality of the given preorder, these two facts deliver completeness.

In fact, one of the contributions of this paper is to fully understand the relationship between the topology and the preorder in the statement of Schmeidler's theorem. The original formulation suggests that the topology and the preorder are 'primitive', in the sense that they are given independently of each other. However, this is not true. Indeed, Schmeidler's theorem can be compactly reformulated in a way that the preorder is primitive, whereas the topology is induced by the preorder:

[1]Sometimes (see, e.g., Mas-Colell, Whinston, and Green, 1995), completeness is defined in a broader way, that is, for all (not necessarily distinct) pair of elements $x, y \in X$. In this extended sense, completeness implies reflexivity. We prefer the formulation of completeness given in the main body of the paper, since otherwise asymmetric linear orders would fail to be complete by definition.

[2]Mas-Colell, Whinston, and Green (1995) call a preference relation *rational* whenever both transitivity and completeness hold. To emphasize even more the relevance of these two properties, the notion of a rational preference is repeated twice, namely in the context of *Preference and Choice* (Chapter 1, page 6), and in the process of dealing with *Classical Demand Theory* (Chapter 3, page 42).

[3]For $x \in X$, the *upper section of x in S* is the set $\{y \in X : ySx\}$, and the *lower section of x in S* is the set $\{y \in X : xSy\}$. The *upper and lower sections of x in $S^{>}$* are defined similarly, with $S^{>}$ in place of S.

Theorem (Schmeidler, 1971, reformulated)**.** *A nontrivial preorder with a connected order-section topology is complete.*

(Here the *order-section topology* is a refinement of the order topology by means of the complements of the closed sections.) In other words, transitivity of a preference implies its completeness whenever a natural induced topology happens to be connected.

Although clarifying the nature of Schmeidler's result already provides us with a strong motivation for the present work, our aim is broader than singling out the main ingredients of his original proof. In fact, this paper provides an extension of this classical theorem to a more general setting, where two interconnected binary relations are involved. To that end, we develop a similar analysis in the realm of *bi-preferences*: these are pairs (R, S) of nested binary relations on the same set X such that R (the *Rigid* preference) is a preorder, and S (the *Soft* preference) is a transitively coherent enlargement of R. Here by '*transitively coherent enlargement*' we mean the following two facts: (i) S is a super-relation of R, i.e., the inclusion $R \subseteq S$ holds; and (ii) S is transitive with respect to R, i.e., the two inclusions $R \circ S \subseteq S$ and $S \circ R \subseteq S$ hold.[4]

The bi-preferences (R, S) we examine here are *comonotonic*, in the sense that $S^{>}$ is contained in $R^{>}$. These bi-preferences can be interpreted as follows in a decision making context. The rigid component R models the very core of an agent's mental judgement, hence it is assumed to be transitive, but it is typically quite far from being complete. The soft component S models a transitively coherent enrichment of R, which has the goal of 'smoothing' the agent's core judgement, due to the rising of reasonable doubts.[5]

In the special case that the soft component of a comonotonic bi-preference is also complete, we finally obtain a NaP*-preference (Necessary and Possible preference).* Originally introduced in multiple criteria analysis (Greco, Mousseau, and Słowiński, 2008), NaP-preferences have been extensively used in applications: see the surveys by Corrente et al. (2013), Corrente et al. (2016), and Greco, Figueira, and Ehrgott (2016). The economic rationale of NaP-preferences stems from an attempt to consistently combine *Knightian preferences* (Bewley, 1986) and *justifiable preferences* (Lehrer and Teper, 2011). NaP-preferences naturally arise in mathematical psychology, decision theory, operations research, multiple criteria decision analysis, and choice theory: see Giarlotta (2014, 2015) and Giarlotta and Watson (2017a, 2017b) for some theoretical aspects, as well as Alcantud, Biondo, and Giarlotta (2018) for a fuzzy modelization of political parties, and Alcantud and Giarlotta (2019) for group decisions based on hesitant fuzzy sets.[6]

[4]Here "$\circ$" denotes the *composition* of two binary relations, defined as follows: $x(R \circ S)y$ if there is $z \in X$ such that $xRzSy$. Since the transitivity of S can be equivalently defined by the inclusion $S \circ S \subseteq S$, the two inclusions $R \circ S \subseteq S$ and $S \circ R \subseteq S$ can be interpreted as a form of transitivity of S with respect to its sub-relation R.

[5]The requirement $S^{>} \subseteq R^{>}$ implies that, for instance, whenever $xR^{>}y$ holds, we may have either $xS^{>}y$ or $xS^{\sim}y$ (where $xS^{\sim}y$ means that x and y are S-indifferent, i.e., both xSy and ySx hold). In the described situation, $xS^{\sim}y$ says that the soft preference 'smoothens' the rigid judgement $xR^{>}y$ by adding a soft reverse preference ySx: this may typically happen because there are feasible scenarios in which the agent believes that x may fail to be strictly better than y. See Section 4.1.2 in Giarlotta (2019), p. 40–43, for an extensive discussion on the point.

[6]For an overview of motivations and applications of NaP-preferences, see Giarlotta (2019).

The vast range of applications of bi-preferences—and, in particular, NaP-preferences—suggests that the use of pairs of interconnected binary relations to model decision makers' attitude entails some advantages over the usual approach based on a single preference. Thus, from a theoretical point of view, it is natural to extend classical results in decision theory to a bi-preference setting. Here we exhibit a first instance of this kind by proving two generalizations of Schmeidler's theorem, in which a (enhanced form of) comonotonic bi-preference plays the role of a preorder, and a NaP-preference that of a total preorder.

The key property that shapes the structure of a comonotonic bi-preference (R, S) is *only* one of the two tenets of economic rationality, namely transitivity: in fact, R and $S^>$ are transitive,[7] and S is transitive with respect to R. However, no assumption is made about completeness, whose satisfaction has indeed become quite controversial in economic theory.[8] Schmeidler's theorem establishes a sharp relationship between the two tenets of rationality for a single preference in a connected topological space. The approach used in this paper is similar: we split the two hypotheses of semicontinuity over the two components of a comonotonic bi-preference, and, using transitivity and transitive coherence, we derive the completeness of the soft part in two different settings. More formally:

Theorem. *Let (R, S) be a nontrivial comonotonic bi-preference on a connected topological space. Suppose (R, S) is either (1) 'quasi-monotonic' or (2) 'everywhere nontrivial'. If R is closed-semicontinuous and $S^>$ is open-semicontinuous, then (R, S) is a NaP-preference.*

Using a compact (and more transparent) formulation, in which the only primitive element is a bi-preference (R, S), whereas the order-section topology is generated by the closed section of R and the open sections of S, our results can be rewritten as follows:

Theorem (reformulated)**.** *Let (R, S) be a nontrivial comonotonic bi-preference with a connected order-section topology. If (R, S) is either (1) 'quasi-monotonic' or (2) 'everywhere nontrivial', then it is a NaP-preference.*

As we shall clarify later on, the two additional properties (1) and (2) are somehow complementary to each other: in fact, (1) enhances the quasi-transitive structure of S, whereas (2) goes in the direction of its completeness by ensuring that $S^>$ has no isolated points. Schmeidler's theorem readily follows from our extensions by taking $R = S$.

Additional related literature

This paper aims at studying the interplay between the two classical tenets of rationality (transitivity and completeness) by discussing economically/psychologically sound generalizations of a well-known result by Schmeidler, which links them to each other in a topological setting. Before proceeding into a technical analysis, it is important to emphasize how the recent literature in mathematical psychology no longer considers transitivity

[7]The transitivity of $S^>$ is an easy consequence of the properties of a comonotonic bi-preference.

[8]Completeness is the most debated feature of rationality: see Aumann (1962) for the seminal work on incomplete preferences, after which a plethora of contributions in the same direction followed.

and completeness as undisputed tenets of rationality, due to several experimental counterexamples.[9]

Concerning transitivity, there is a long list of publications in which intransitive choice cannot be regarded *sic et simpliciter* as not rational. The most classical contribution of this kind dates back to Tversky (1969). More recent contributions detect different types of problems with the identification of rationality by the property of transitivity: see, among many others, Caravagnaro and Davis-Stober (2012), Regenwetter and Davis-Stober (2012), Regenwetter et al. (2011), and Tsai and Böckenholt (2006). Moreover, there is the problem with intransitive dice: this shows that intransitivity can be "very rational". For the paradox of three or more random variables, see Trybula (1961, 1963). Random utility aspects of this example are discussed in Suck (2002). Finally, a complete taxonomy of the 'transitive degree' of preference relations has recently been given by Giarlotta and Watson (2014, 2018).

Concerning completeness, the situation is even more controversial. A decision maker can be extremely rational and nevertheless have no preference for two alternatives. For example, two job candidates may have very different qualifications, which make them incomparable. In many experiments in psychology, completeness is obtained by forced choice. The above cited paper by Regenwetter and Davis-Stober (2012) shows that this can produce awkward choice behavior. Furthermore, in the theory of achievement tests, the incompleteness of difficult comparisons of items is common knowledge. Here, failure of completeness is by no means a lack of rationality in the subjects, but a feature of the complexity of the area for which the test is designed.

In view of the drawbacks and pitfalls of behavioral models that assume the full satisfaction of the axioms of transitivity and completeness, a new stream of research in mathematical psychology and theoretical economics has arisen. This alternative approach employs two binary relations—in place of one—to model decision makers' attitudes toward preference and choice: the two classical tenets of rationality are either relaxed or combined in different ways within the two relations. We have already mentioned some contributions of this kind; here we recall additional results in decision making under uncertainty and choice theory, as well as some work directly related to Schmeidler's theorem (which is extendable to bi-preferences).

Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) consider pairs of relations in an Anscombe-Aumann (1963) setting: in their approach, the rigid part is an incomplete preorder à la Bewley (1986), and the soft one is an uncertainty averse preference à la Gilboa and Schmeidler (1989). Additional contributions in the same stream of research are Cerreia-Vioglio (2016), Faro and Lefort (2019), and Kopylov (2009).

In the Anscombe-Aumann model described by Cerreia-Vioglio et al. (2018), two types of mutually consistent preferences are used. The first preference reflects the decision maker's judgments about well-being (her *mental preferences*), whereas the second represents the decision maker's choice behavior (her *behavioral preferences*). The resulting structure is, in fact, a NaP-preference.

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[9] We thank one of the referees for pointing out this aspect.

Nishimura and Ok (2018) study *preference structures*, which are bi-preferences (R, S) with the properties that the soft preference S is complete, and the strict soft preference $S^{>}$ extends the strict rigid preference $R^{>}$. These bi-preferences, which allow one to model a novel theory of choice, are complementary to NaP-preferences.

Within the theory of bounded rationality, several contributions employ more than one binary relation to explain choice behavior. See, among others, Apesteguía and Ballester (2013), Au and Kawai (2011), Kalai, Rubinstein, and Spiegler (2002), Manzini and Mariotti (2007), and Rubinstein and Salant (2006). It is of some interest to examine situations in which the 'rationales'—the binary relations that explain a choice behavior—display some structural connections, for instance forms of transitive coherence.[10]

Another important contribution related to Schmeidler's result is a recent paper by Khan and Uyanik (2019). The two authors examine in depth the relationship between various hypotheses of connectedness (of the underlying topological space) and some behavioral assumptions on single preference relations (transitivity, completeness, and continuity). In their analysis, the authors generalize not only Schmeidler's theorem, but also pioneering results of Eilenberg (1941), Sen (1969), and Sonnenschein (1965).[11] For a very recent bi-preference approach by the same authors, see Uyanik and Khan (2019).

Gerasimou (2013) characterizes the continuity of a preorder in terms of its semicontinuity, and, relying on Schmeidler's theorem, concludes that an approach based on a primitive weak preference leads to counter-intuitive behavioral predictions. Dubra (2011) shows that the analogue of Schmeidler's theorem holds for *lotteries* (probability distributions over a finite set of prizes). Karni (2011) shows that incompleteness is compatible with all continuity properties for some nonstandard notions of preference relations, and gives a revisited reading of Schmeidler's and Dubra's results. Finally, Schmeidler's theorem is explicitly used in proving the completeness of preorders derived from *Richter-Peleg multi-utility* representations:[12] see Alcantud, Bosi, and Zuanon (2016).

Organization of the paper

This paper is divided in three parts: (1) preliminaries and motivations; (2) an interlude; (3) results and proofs. Regarding (1), Section 2 collects basic notions on bi-preferences, and presents a wide range of examples. Regarding (2), Section 3 gives an alternative

[10] On the point, let us quote Kalai, Rubinstein, and Spiegler (2002, p. 2487) in their self-critique of the methodology RMR (*Rationalization by Multiple Rationales*) for the rationalizability of choice functions by means of (possibly several) linear orderings: "*We fully acknowledge the crudeness of this approach. The appeal of the RMR proposed for 'Luce and Raiffa's dinner' does not emanate only from its small number of orderings, but also from the simplicity of describing in which cases each of them is applied. ... More research is needed to define and investigate 'structured' forms of rationalization.*".

[11] An extension of their analysis to bi-preference structures appears possible and interesting.

[12] A *Richter-Peleg utility representation* of a preference R on X is a map $u \colon X \to \mathbb{R}$ such that, for all $x, y \in X$, xRy implies $u(x) \geqslant u(y)$, and $xR^{>}y$ implies $u(x) > u(y)$ (Richter, 1966; Peleg, 1970). A *multi-utility representation* of R is a family $\mathcal{U}$ of maps $u \colon X \to \mathbb{R}$ such that, for all $x, y \in X$, xRy if and only if $u(x) \geqslant u(y)$ for all $u \in \mathcal{U}$ (Ok, 2002; Evren and Ok, 2011; Evren, 2014). Finally, a *Richter-Peleg multi-utility representation* of R is a multi-utility representation of R whose elements are Richter-Peleg utility representations (Minguzzi, 2013, Sect. 5).

reading of Schmeidler's theorem, by providing several reformulations of it. Regarding (3), Sections 4 and 5 exhibit two different extensions of Schmeidler's theorem to comonotonic bi-preferences: since one of our goals is to sharpen Schmeidler's original argument, proofs are fully discussed in these sections. Conclusive remarks are in Section 6. The Appendix collects some technical facts related to our main results.

2 Preliminaries on bi-preferences

We first recall the basic notions on (single) preferences, and then introduce bi-preferences. Successively we analyze special types of bi-preferences, called uniform, which comprise comonotonic bi-preferences, NaP-preferences, and preference structures.

The purpose of this section is twofold: (i) to collect preliminary facts in an organized fashion; (ii) to show how bi-preferences naturally arise in several areas of economics and psychology. The reader who is more interested in results rather than motivations can quickly glance at it, and then go to Section 3. However, we do believe that the material included here allows one to gain valuable insight into the topic of bi-preferences, thus providing the needed motivation for our analysis.

Throughout the paper, X denotes a nonempty (typically infinite) set of alternatives.

2.1 Preferences

A *(weak) preference* on X is a reflexive binary relation on X, which usually satisfies some ordering properties.[13] We employ capital letters (R, S, E, F, etc.) to denote preferences. For any preference R and any pair of alternatives $x, y \in X$, we use xRy in place of $(x, y) \in R$, and interpret it from left to right: thus, xRy means "x is weakly preferred to y" or "x is at least as good as y". We slightly abuse notation, and write sequences of binary relationships in a compact way: for example, $xRySzRw$ stands for xRy, ySz, and zRw, where R, S are preferences on X, and $x, y, z, w \in X$.

Given a (weak) preference R on X, three binary relations are derived from it: the *strict preference* $R^{>}$, the *indifference* $R^{\sim}$, and the *incomparability* $R^{\perp}$. They are respectively defined by $xR^{>}y$ if xRy and $\neg(yRx)$, $xR^{\sim}y$ if xRy and yRx, and $xR^{\perp}y$ if $\neg(xRy)$ and $\neg(yRx)$, where $x, y \in X$ are arbitrary. Thus R is the union of $R^{>}$ and $R^{\sim}$, whereas R and $R^{\perp}$ are disjoint. The indifference $R^{\sim}$ is never empty, since it contains the *diagonal* $\Delta(X) = \{(x, x) : x \in X\}$ of X. On the contrary, $R^{>}$ may be empty; we call R *trivial* if $R^{>} = \varnothing$, and *nontrivial* otherwise.[14]

A (reflexive) preference R is *quasi-transitive* if $R^{>}$ is transitive, *Ferrers* if xRy and zRw imply xRw or zRy, and *semitransitive* if xRy and yRz imply xRw or wRz, where $x, y, z, w \in X$ are arbitrary. Further, R is *complete* (or *total*) if xRy or yRx holds for all

[13] In this paper, the terminology 'weak preference' will be used only when it is necessary to avoid confusion; otherwise, we shall simply speak of a 'preference'. Reflexivity is always assumed.

[14] We think that 'trivial' is a somehow misleading term, because the symmetric part of a weak preference typically represents meaningful judgements of similarity. However, in this paper we employ standard terminology, also because 'nontrivial' is the term originally used by Schmeidler in stating his result.

distinct $x, y \in X$. Then R is said to be a *preorder* if it is transitive, a *partial order* if it is transitive and antisymmetric, a *quasi-preorder* if it is quasi-transitive, an *interval order* (Fishburn, 1970a, 1985) if it is Ferrers, a *semiorder* (Luce, 1956, Pirlot and Vincke, 1997) if it is a semitransitive interval order, and a *linear order* if it is an antisymmetric total preorder.

Given two (not necessarily distinct or reflexive) relations R and S on X, the *composition* $R \circ S$ is defined by $x(R \circ S)y$ if there is $z \in X$ such that $xRzSy$. Notice that R is transitive if and only if $R \circ R \subseteq R$, and quasi-transitive if and only if $R^> \circ R^> \subseteq R^>$. A reflexive relation R is called *strongly quasi-transitive* if the two inclusions $R \circ R^> \subseteq R$ and $R^> \circ R \subseteq R$ hold, that is, $xRyR^>z$ or $xR^>yRz$ implies xRz for all $x, y, z \in X$; in this case, we shall also say that R is a *strong quasi-preorder*. Strong quasi-transitivity implies quasi-transitivity, but the converse is false; however, for a complete preference, quasi-transitivity and strong quasi-transitivity are equivalent.[15] Notice that an interval order—hence, a semiorder—is complete and (strongly) quasi-transitive.

Finally, we recall the notions of the upper and lower sections of an element (with respect to a given preference), which will play a main role in our extensions of Schmeidler's theorem. Let R be a (reflexive) preference on X. Given $x \in X$, define

$$\begin{aligned} x^{\downarrow,R} &:= \{w \in X : xRw\} && \textit{(weak lower section of } x\textit{)},\\ x^{\uparrow,R} &:= \{w \in X : wRx\} && \textit{(weak upper section of } x\textit{)},\\ x^{\downarrow,R^>} &:= \{w \in X : xR^>w\} && \textit{(strict lower section of } x\textit{)},\\ x^{\uparrow,R^>} &:= \{w \in X : wR^>x\} && \textit{(strict upper section of } x\textit{)}. \end{aligned}$$

The upper and lower sections are used to define forms of semicontinuity (see Definition 3.1 in Section 3). Further, the sections of a preference R yield a new relation derived from R, the *trace* R_{tr} of R, defined as follows for each $x, y \in X$:

$$xR_{\mathsf{tr}}y \quad \overset{\text{def}}{\iff} \quad x^{\downarrow,R} \supseteq y^{\downarrow,R} \text{ and } x^{\uparrow,R} \subseteq y^{\uparrow,R}.$$

In words, x is "trace-better" than y if the set of elements below x contains the set of elements below y, and the set of elements above x is contained in the set of elements above y. The trace of a preference is a preorder. The notion of trace dates back to the work of Luce (1956) on semiorders, and Fishburn (1970a) on interval orders.[16] For any R, the trace R_{tr} is the largest sub-relation of R such that the two inclusions $R \circ R_{\mathsf{tr}} \subseteq R$ and $R_{\mathsf{tr}} \circ R \subseteq R$ hold: see Bouyssou and Pirlot (2004) and Cerreia-Vioglio and Ok (2018).

[15]In fact, strong quasi-transitivity can be characterized as follows: *A preference R is strongly quasi-transitive if and only if R is quasi-transitive,* $(R \circ R^>) \cap R^\perp = \varnothing$, *and* $(R^> \circ R) \cap R^\perp = \varnothing$. The easy proof of this fact is left to the reader.

[16]On the topic, see also the paper by Bouyssou and Pirlot (2004). For a generalized notion of trace, called *sliced* (which is associated to universal types of semiorders, and is again a total preorder), see Giarlotta and Watson (2016). Very recently, the notion of trace has been axiomatically characterized, and suggestively renamed *transitive core*: see Nishimura (2017).

2.2 Bi-preferences

Bi-preferences were originally introduced by Giarlotta and Greco (2013) in their work on NaP-preferences, where they are named 'partial NaP-preferences'.[17] In this note we employ the more intuitive term 'bi-preferences' instead.

Definition 2.1 (Giarlotta and Greco, 2013)**.** A *bi-preference* on X is a pair (R, S) of binary relations on X such that the following properties hold:

- ◇ (Core Transitivity) R is a preorder,
- ◇ (Soft Extension) S includes R, and
- ◇ (Transitive Coherence) $R \circ S \subseteq S$ and $S \circ R \subseteq S$.

R is the *rigid preference*, S is the *soft preference*, and $G = S \backslash R$ is the *gap* of (R, S). A bi-preference (R, S) is *stable* if $R = S$. A *NaP-preference* (*necessary and possible preference*) on X is a bi-preference (R, S) on X such that the following additional property holds:

- ◇ (Mixed Completeness) for all $x, y \in X$, xRy or ySx.

In Giarlotta and Greco (2013), the authors are mostly concerned with NaP-preferences, rather than generic bi-preferences, because of their possible applications in several fields of research (decision theory, multiple criteria decision analysis, etc.). Under the Axiom of Choice (AC), these special bi-preference structures can be neatly characterized as follows:

Theorem 2.2 (Giarlotta and Greco, 2013)**.** *Under AC, a pair (R, S) of binary relations on X is a NaP-preference if and only if there is a family $\mathcal{F}$ of total preorders on X such that $R = \bigcap \mathcal{F}$ and $S = \bigcup \mathcal{F}$.*

It is worth noticing that—exactly in the spirit of the present paper—Theorem 2.2 provides a bi-preference extension of a well-known result by Donaldson and Weymark's (1998), which says that every preorder is the intersection of a family of total preorders.

Transitivity is the property that shapes the structure of an arbitrary bi-preference. In fact, Core Transitivity ensures that the rigid part R of an agent's preference structure is transitive, whereas Soft Extension and Transitive Coherence require that the soft preference S expands R in a transitively consistent way. However, Transitive Coherence does not guarantee the rationality of S by any means, since S may even fail to be quasi-transitive.

The rigid preference R represents the core of an agent's judgement, and models the preference rankings that 'must' happen. On the other hand, the soft preference S represents the admissibility/tolerability/enrichment of an agent's judgement, and models the preference rankings that 'may' happen. The gap $G = S \backslash R$ encodes a grey area of 'indecisiveness': the larger it is, the more indecisive the agent is. (Thus, an alternative term for 'stable' may be 'fully decisive'.) No completeness is assumed to hold in the general setting; however, several bi-preferences that arise in practice are complete. Uyanik and Khan (2019) effectively describe a bi-preference setting as follows:

[17] The reason for this terminology is that these pairs of relations satisfy all properties of NaP-preferences, with the possible exception of 'mixed completeness': see Section 2.4.

> [A bi-preference models] *a decision-maker who can be single-minded on some choices and double-minded regarding others.* [Such a decision-maker] *is not uniformly decisive or uniformly hesitant, never always assured and never always skeptical.*

Bi-preferences naturally appear in various applied areas. To wit, below we present a wide range of examples of bi-preferences. The first example collects some simple, yet significative, instances of bi-preferences.

Example 2.3 (Elementary bi-preferences)**.** Let R be a preference on X.

- (i) (R, R) is a (stable) bi-preference if and only if R is a preorder.
- (ii) $(\Delta(X), R)$ is a bi-preference. (Recall that $\Delta(X)$ is the diagonal of X.)
- (iii) If R is a preorder, then (R, X^2) is a bi-preference, in fact a NaP-preference.
- (iv) The pair $(\Delta(X), X^2)$ is the bi-preference on X with the largest gap.

Part (i) shows that bi-preferences generalize preorders. Parts (ii) and (iii) exhibit two limit instances of bi-preferences: in (ii) R is minimal, whereas in (iii) S is maximal. The trivial bi-preference (iv) is 'the least stable' pair in the family of all bi-preferences on X, since nothing must happen and everything may happen: in fact, the gap is maximum.[18]

A second example stems from utility theory.

Example 2.4 (Utility bi-preference)**.** Let $\mathcal{U} = \{u_i : i \in I\}$ be a family of real-valued functions $u_i \colon X \to \mathbb{R}$. Define two relations $R_{\mathcal{U}}$ and $S_{\mathcal{U}}$ on X as follows for each $x, y \in X$:

$$
\begin{aligned}
x \, R_{\mathcal{U}} \, y \quad &\overset{\text{def}}{\iff} \quad u_i(x) \geqslant u_i(y) \ \text{ for all } i \in I\,, \\
x \, S_{\mathcal{U}} \, y \quad &\overset{\text{def}}{\iff} \quad u_i(x) \geqslant u_i(y) \ \text{ for some } i \in I\,.
\end{aligned}
$$

Then the pair $(R_{\mathcal{U}}, S_{\mathcal{U}})$ is a bi-preference, called the *utility bi-preference* associated to $\mathcal{U}$. Utility bi-preferences are typical examples of preference structures that are derived from a family of functional evaluators by using universal and existential quantifiers. Notice that utility bi-preferences are, in fact, NaP-preferences.

Our third example of bi-preferences links a weak preference to its trace.

Example 2.5 (Tracing bi-preference)**.** Let S be a weak preference, and S_{tr} is its trace. Then (S_{tr}, S) is a bi-preference, called the *tracing bi-preference* of S. One can show that, among all bi-preferences having S as soft component, the tracing bi-preference (S_{tr}, S) is the one with the smallest gap.

Our forth example establishes a relationship between bi-preferences on one side, and the theory of individual choices and revealed preferences on the other one.

[18] See Section 5 in Giarlotta and Greco (2013) for the meaning of 'the least stable', which refers to both 'positive' information (encoded by the rigid preference) and 'negative' information (encoded by the complement of the soft preference).

Example 2.6 (Bi-preference revealed by a path independent choice)**.** Let $c\colon 2^X \to 2^X$ be a *choice correspondence* on X, that is, a map such that $c(A) \subseteq A$ for all $A \in 2^X$, and $c(A) = \varnothing$ if and only if $A = \varnothing$. (As usual, 2^X denotes the family of all subsets of X.) Define two relations R_c and S_c on X as follows for each $x, y \in X$:[19]

$$x \, R_c \, y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \text{for all } A \in 2^X, \text{ if } x \in A \text{ and } y \in c(A), \text{ then } x \in c(A);$$
$$x \, S_c \, y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \text{there is } A \in 2^X \text{ such that } x \in c(A) \text{ and } y \in A.$$

Assume that c satisfies the following property due to Plott (1973):

- ◇ (PI: Path Independence) $c(A \cup B) = c(c(A) \cup c(B))$ for all $A, B \in 2^X \backslash \{\varnothing\}$.

Then (R_c, S_c) is a bi-preference, called the *bi-preference revealed* by c.[20] Path independent choices are also important in combinatorial mathematics. In fact, as Koshevoy (1999) shows, path independent choices are "isomorphic" to abstract *convex geometries*, as defined by Edelman and Jamison (1985).

Our fifth example is a bi-preference that collects judgements of similarity.

Example 2.7 (NaP-indifference, Giarlotta and Watson, 2017b)**.** A *necessary and possible indifference* (for short, *NaP-indifference*) on X is a pair (E, F) of reflexive symmetric binary relations on X satisfying the following properties:

- ◇ (Symmetric Core Transitivity) E is an equivalence relation,
- ◇ (Symmetric Soft Extension) F is a symmetric extension of E, and
- ◇ (Transitive Coherence) $E \circ F \subseteq F$ and $F \circ E \subseteq F$.

NaP-indifferences are bi-preferences, and can be characterized in a way similar, *mutatis mutandis*, to NaP-preferences (cf. Theorem 2.2):

Theorem 2.8 (Giarlotta and Watson, 2017b)**.** *A pair (E, F) of binary relation on X is a NaP-indifference if and only if there is a family $\mathcal{E}$ of equivalence relations on X such that $E = \bigcap \mathcal{E}$ and $F = \bigcup \mathcal{E}$.*

---

[19] S_c the relation of *revealed preference*, connected to the work of Samuelson (1938), Houthakker (1950), Richter (1956), Arrow (1959), and Sen (1971), among many other contributions. Thus $xS_c\,y$ says that x is "(existentially) revealed to be preferred" to y if there exists a menu in which both x and y are available, and x is chosen. The relation R_c is the universal counterpart of the relation S_c: in fact, $xR_c\,y$ says that x is "(universally) revealed to be preferred" to y when, for all menus containing both x and y, if y is chosen then so is x.

[20] The proof of this fact is available upon request. As a matter of fact, (R_c, S_c) is a bi-preference under the weaker condition that c satisfies both the axiom of contraction consistency (α) (Chernoff, 1954) and the axiom of replacement consistency (ρ) (Cantone et al., 2016). Recall that (α) says that for any two menus $A, B \in 2^X$ and item $x \in X$, if $A \subseteq B$ and $x \in A \cap c(B)$, then $x \in c(A)$. On the other hand, (ρ) says that for any menu $A \in 2^X$ and items $x, y \in X$, if $y \in c(A)$ and $y \notin c(A \cup \{x\})$, then $x \in c(A \cup \{x\})$. It is not difficult to show that PI implies both (α) and (ρ), but the converse does not hold: thus, PI is strictly stronger that $(\alpha)\&(\rho)$. Finally, it is worth noticing that, for the special case of a choice c satisfying the Weak Axiom of Revealed Preference (WARP, see Samuelson (1938) and Arrow (1959)), the two relations of revealed preference R_c and S_c are equal, and coincide with a total preorder.

NaP-indifferences arise in the theory of choice, as ordered pairs of symmetric relations revealed by a choice. These relations of 'revealed similarity' link to each other those alternatives that are interchangeable according to the agent's choice behavior.[21]

Finally, we present a rather involved instance of a bi-preference, which arises in decisions under uncertainty. (Since the notions in the next example do not play a role in the sequel of the paper, the reader who does not wish to get distracted from the main themes may skip it, without altering the overall comprehension of the topic.)

Example 2.9 (Mental and behavioral bi-preference, Cerreia-Vioglio et al., 2018)**.** Within the stylized version of the Anscombe-Aumann (1963) setting analyzed by Fishburn (1970b), let A be a set of acts, i.e., simple measurable functions $f\colon W \to X$, with X convex set of outcomes, W set of states of the world endowed with an algebra Σ of events (and the set Δ of finitely additive probabilities on Σ is endowed with the event-wise convergence topology). For any relation R on A, the *strong-strict preference* $R^{\gg}$ is given by

$$f R^{\gg} g \overset{\text{def}}{\iff} \text{for all } h,k \in A \text{ there is } \epsilon > 0 : (1-\delta)f+\delta h \, R^{>} \, (1-\delta)g+\delta k \text{ for all } \delta \in [0,\epsilon].$$

The strong-strict preference $R^{\gg}$ is the *algebraic interior* of $R^{>}$. Intuitively, if R is a preorder that models mental preferences, then $R^{\gg}$ encodes a 'strong mental preference': in fact, $fR^{\gg}g$ holds if and only if "f plus a specification error is strictly better than g plus a specification error". Then, a bi-preference (R,S) on A is called:

- *algebraically monotonic*[22] if the inclusion $R^{\gg} \subseteq S^{>}$ holds;
- *mixed complete*[23] if, for all acts $f,g \in A$, $\neg(fRg)$ implies gSf.

Under standard expected utility assumptions, a mixed complete algebraically monotonic bi-preferences (R,S) on A is characterized by the existence of an affine utility function u on outcomes and a set C of probabilities on states, which jointly represent R and S by

$$\begin{aligned}
fRg &\iff \int u(f)dp \geqslant \int u(g)dp \quad \text{for all } p \in C,\\
fSg &\iff \int u(f)dp \geqslant \int u(g)dp \quad \text{for some } p \in C,\\
fR^{>}g &\iff \int u(f)dp \geqslant \int u(g)dp \quad \text{for all } p \in C, \text{ with at least one strict inequality,}\\
fS^{>}g &\iff \int u(f)dp > \int u(g)dp \quad \text{for all } p \in C \iff fR^{\gg}g.
\end{aligned}$$

Further, C is unique, and u is unique up to positive affine transformations.[24]

[21]Examples of NaP-indifferences in choice theory are those having the binary relation of *revealed indiscernibility* (Cantone, Giarlotta, and Watson, 2019a) as the rigid component, and some of the classical relations of revealed indifference as the soft component: see Section 3.2 in Giarlotta and Watson (2017b).

[22]Algebraic monotonicity is weaker than *monotonicity*, which is characterized by the satisfaction of the inclusion $R^{>} \subseteq S^{>}$: see Definition 2.10 below.

[23]This property is equivalent to the formulated in Definition 2.1.

[24]Notice that if S is finite, $X = \mathbb{R}$, $u = \mathrm{id}_{\mathbb{R}}$, and $C = \Delta$, then the mental preference R is the weak Pareto dominance $\geqslant$ on $\mathbb{R}^S$, the strict mental preference $R^{>}$ is the Pareto dominance $>$ on $\mathbb{R}^S$, and the strong mental preference $R^{\gg}$ is the strong Pareto dominance $\gg$ on $\mathbb{R}^S$.

2.3 Special bi-preferences

We single out special bi-preferences, and summarize their connections (cf. Figure 1).

Definition 2.10. A bi-preference (R, S) on X is:
- *complete* (resp. *nontrivial*) if S is complete (resp. nontrivial);
- *weakly monotonic* if $R^{>} \cap (S \circ S^{>}) \subseteq S^{>}$ and $R^{>} \cap (S^{>} \circ S) \subseteq S^{>}$;
- *weakly comonotonic* if, for each $x, y, z \in X$, we have:

$$\begin{aligned}\big(xS^{>}z \ \wedge \ yS^{>}z\big) \ \wedge \ xS^{\perp}y &\implies \big(xR^{>}z \ \wedge \ yR^{>}z\big)\\ \big(zS^{>}x \ \wedge \ zS^{>}y\big) \ \wedge \ xS^{\perp}y &\implies \big(zR^{>}x \ \wedge \ zR^{>}y\big);\end{aligned}$$

- *monotonic* if the inclusion $R^{>} \subseteq S^{>}$ holds;
- *comonotonic* if the inclusion $S^{>} \subseteq R^{>}$ holds;
- *strongly monotonic* if it is both monotonic and weakly comonotonic;
- *strongly comonotonic* if it is both comonotonic and weakly monotonic;
- *essentially rigid* if $S = X^2$;
- *monolithic* if $R^{>} = S^{>}$ (i.e., it is both monotonic and comonotonic);
- *stable* if $R = S$ (cf. Definition 2.1).

Let us quickly motivate the employed terminology. A bi-preference is complete (respectively, nontrivial) if the larger component is complete (respectively, nontrivial). The notions of monotonic and comonotonic bi-preference are dual to each other, since the inclusion $R^{>} \subseteq S^{>}$ reverses the inclusion $S^{>} \subseteq R^{>}$. Weak monotonicity is implied by monotonicity, since it states that a strict rigid preference $xR^{>}y$ implies a strict soft preference $xS^{>}y$ whenever X contains elements already witnessing that possibility, in the sense that either $xSzS^{>}y$ or $xS^{>}zSy$ holds for some $z \in X$. Similarly, weak comonotonicity is a weakening of comonotonicity. Strong comonotonicity and strong monotonicity are suitable enhancements of comonotonicity and monotonicity, respectively. Monolithic bi-preferences represent the limit case of mutual enhancement. Stable bi-preferences are in a one-to-one correspondence with preorders (cf. Example 2.3(i)). An essentially rigid bi-preference is shaped by its rigid part, since its soft part is everything.

Figure 1 summarizes the relationships existing among the bi-preferences introduced in Definition 2.10. The three shaded boxes are related to Schmeidler's theorem and its first generalization to bi-preferences. In fact, Schmeidler's theorem can be equivalently stated by saying that, under suitable topological assumptions, a nontrivial *stable* bi-preference is complete. Our generalization in Section 4 substantially weakens Schmeidler's hypotheses while deriving a stronger conclusion: in fact, it states that, under equivalent topological assumptions, a nontrivial *strongly comonotonic* bi-preference is complete *and transitive*.

2.4 Uniform bi-preferences

In passing from a preorder R to a bi-preference (R, S), the role of the soft component S is to extend R according to specific objectives. For instance, S may be used with the goal of 'sharpening' the judgement formulated by R, thus introducing new strict preferences. In this case we may have, e.g., $xS^{>}y$ and $xR^{\perp}y$. As an example, imagine that in the

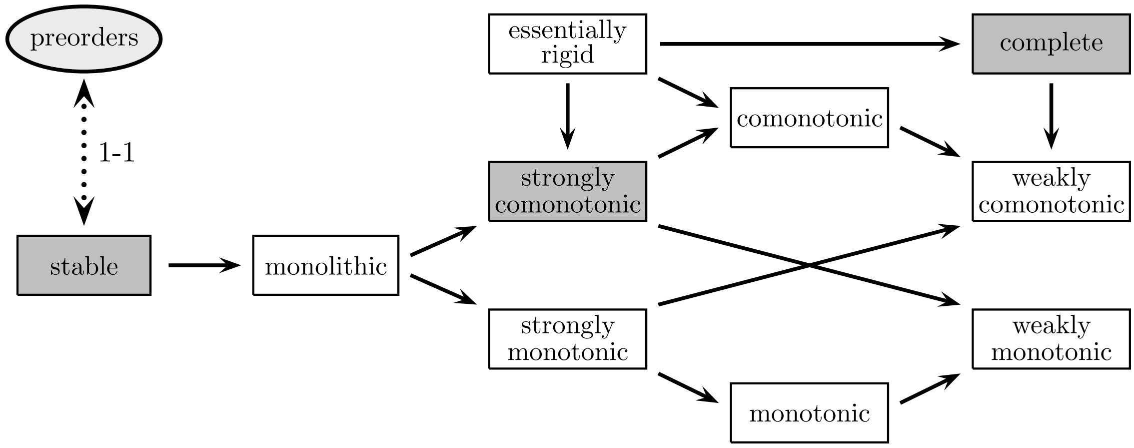

Figure 1: Implications between types of bi-preferences, represented by black arrows (which are transitive). The three shaded types of bi-preferences are related to Schmeidler's theorem and our first extension to strongly comonotonic bi-preferences.

process of selecting the set of best alternatives, an agent uses R to filter some of them according to a logic of dominance. Thus, an alternative x is pre-selected whenever there is no other alternative that R-dominates x, i.e., $yR^{>}x$ holds for no $y \in X \setminus \{x\}$. Then, in order to make a more accurate selection, the agent may asks an external source to rank the remaining non-dominated options, thus introducing new strict preferences via S.[25]

Quite in the opposite direction, it may happen that the soft component S is employed to extend R with the objective of 'smoothing' some strong judgements, having, e.g., $xS^{\sim}y$ despite being $xR^{>}y$. To justify this approach, imagine that an agent is required to sharply choose between two alternatives, say $xR^{>}y$. However, this forces her to neglect possible scenarios in which y is at least as good as x. The role of S is then to smoothen the rigid judgement modeled by R, thus transforming the rigid strict preference $xR^{>}y$ into a soft indifference $xS^{\sim}y$, which is obtained adding a soft preference ySx.

Of course, the soft relation S may be guided by a rationale of sharpening a judgement for some pairs of alternatives, and one of smoothing it for some others. The next definition describes the two limit cases, in which the logic of extension from the rigid preference to the soft preference is uniform for all pairs of alternatives.

Definition 2.11. A bi-preference (R, S) is *uniform* if the two strict preferences are nested: that is, (R, S) is either monotonic (i.e., $R^{>} \subseteq S^{>}$) or comonotonic (i.e., $S^{>} \subseteq R^{>}$).

Most of the bi-preferences already presented in Section 2.2 are indeed uniform.

[25]This procedure resembles the bounded rationality approach in choice theory called *rational shortlist method*, proposed by Manzini and Mariotti (2007), and further studied by Au and Kawai (2011), and Garcìa-Sanz and Alcantud (2015). However, in Manzini and Mariotti's model, the two preferences rationalizing a choice behavior usually fail to be nested into each other and structurally coherent.

Example 2.12. Any utility bi-preference (Example 2.4), the tracing bi-preference of any strong quasi-preorder (Example 2.5), and the bi-preference revealed by any path independent choice (Example 2.6) are all comonotonic. Further, any NaP-indifference (Example 2.7) is monolithic (both monotonic and comonotonic). The algebraically monotonic bi-preference defined in an Anscombe-Aumann setting (Example 2.9) is comonotonic.

If completeness is required along with uniformity, then we obtain either (i) preference structures (Nishimura and Ok, 2018), or (ii) necessary and possible preferences (Giarlotta and Greco, 2013). The next two examples describe these special bi-preferences.

Example 2.13 (Preference structures, Nishimura and Ok, 2018)**.** A *preference structure* is a complete monotonic bi-preference. Preference structures allow to model phenomena like rational choice, indecisiveness, imperfect ability of discrimination, regret, and advice taking, among others. Further, an alternative kind of choice theory, which uses the notion of *top cycles*, arises from preference structures.

Example 2.14 (NaP-preferences, Giarlotta and Greco, 2013)**.** Recall from Definition 2.1 that a NaP-preference (necessary and possible preference) on X is a bi-preference (R, S) on X such that Mixed Completeness holds. Notice that any utility bi-preference (Example 2.4) and the bi-preference revealed by any path independent choice (Example 2.6) are NaP-preferences. Further, the tracing bi-preference of any interval order or any semiorder (Example 2.5) is a NaP-preference. The bi-preference that describes decision makers' mental attitude and choice behavior in an Anscomb-Aumann framework (Example 2.9) is a NaP-preference, too.

The following additional characterization of NaP-preferences (the first is given by Theorem 2.2) will be useful in this paper:[26]

Lemma 2.15. *A bi-preference is a NaP-preference if and only if it is comonotonic and complete.*

Proof. Let (R, S) be a bi-preference on a nonempty set X.

For necessity, assume that (R, S) is a NaP-preference. Completeness of S is an immediate consequence of Soft Extension and Mixed Completeness. To prove that (R, S) is comonotonic, let $x, y \in X$ be such that $xS^{>}y$. Mixed Completeness readily yields xRy. Since yRx implies ySy, which is impossible, it follows that $xR^{>}y$, as required.

For sufficiency, suppose $x, y \in X$ satisfy $\neg(ySx)$. By the completeness of S, we get $xS^{>}y$, hence $xR^{>}y$ by comonotonicity. In particular, we have xRy, and so Mixed Completeness holds. Thus (R, S) is a NaP-preference. ■

We conclude this section mentioning a strong type of NaP-preference, in which both classical tenets of rationality hold for the soft component:

Definition 2.16. A NaP-preference (R, S) is *transitive* if S is transitive (that is, a total preorder). Equivalently, a transitive NaP-preference is an ordered pair of preorders (R, S) such that S is a completion[27] of R.

[26]For a third characterization of NaP-preferences, see Lemma 2.4 in Giarlotta and Watson (2017b).

[27]By "S is a *completion* of R" we mean that S is a complete super-relation of R.

In our first extension of Schmeidler's theorem (see Section 4) we shall obtain a bi-preference that is indeed a transitive NaP-preference.

3 Reformulating Schmeidler's theorem

This section is a brief interlude between preliminaries and new results. First, we introduce all notions of semicontinuity that are needed for a topological setting.[28] Successively, we state several equivalent reformulations of Schmeidler's theorem, which concern either a single binary relation or an ordered pair of binary relations.

The purpose of this interlude is to cast light on the overall significance of Schmeidler's theorem, also providing important insight into a bi-preference approach to the topic. In particular, we shall show that, although appearances say otherwise, Schmeidler's theorem is not about the relationship between a primitive preorder and an independently given primitive topology: in fact, the topology depends on the preorder.

Definition 3.1. Let (X, τ) be a topological space, S a preference on X, and $S^{>}$ its asymmetric part. We say that S is *closed-semicontinuous* if all weak lower sections $x^{\downarrow,S}$ and weak upper sections $x^{\uparrow,S}$ are closed subsets of (X, τ). Further, we say that $S^{>}$ is *open-semicontinuous* if all strict lower sections $x^{\downarrow,S^{>}}$ and strict upper sections $x^{\uparrow,S^{>}}$ are open subsets of (X, τ).

Let us recall again the main result under analysis:

Theorem (Schmeidler)**.** *Let S be a nontrivial preorder on a connected topological space X. If S is closed-semicontinuous and $S^{>}$ is open-semicontinuous, then S is complete.*

This theorem about the relationship between a preorder and a topology. The way it is stated suggests that the preorder and the topology are given independently of each other. To see why this conclusion is essentially false, we shall define a natural topology that is induced on any set endowed with a reflexive binary relation.

Definition 3.2. Let S be a weak preference on X. Define a topology $\tau_S^{\uparrow\downarrow}$ on X by taking as a subbase both the strict sections (upper or lower) and the complements of the weak sections (upper or lower). In other words, $\tau_S^{\uparrow\downarrow}$ is the topology obtained by declaring all strict sections open and all weak sections closed.[29] We call $\tau_S^{\uparrow\downarrow}$ the *order-section topology induced by S.*

If S is a total preorder, then the complement of a weak section is always a strict section. Thus the order-section topology induced by a total preorder S is, in fact, the usual *order topology* induced by S (that is, the topology having the open sections as subbase).

The order-section topology induced by a preference can be used to characterize semicontinuity (the proof of this fact is straightforward):

[28] Alternative settings are possible. See Wakker (1988) for an 'algebraic approach', which is shown to be more general than the topological one for the special case of additive representations.

[29] Thus, if $\mathcal{S}$ denotes the collection of all strict sections and all complements of weak sections, then the topology $\tau_S^{\uparrow\downarrow}$ is the set whose elements are arbitrary unions of finite intersections of elements in $\mathcal{S}$.

Lemma 3.3. *The following statements are equivalent for any preference S on a topological space (X, τ):*

(i) *S is closed-semicontinuous and $S^{>}$ is open-semicontinuous;*

(ii) *τ is a refinement of $\tau_S^{\uparrow\downarrow}$ (that is, $\tau \supseteq \tau_S^{\uparrow\downarrow}$).*

By Lemma 3.3, $\tau_S^{\uparrow\downarrow}$ is the coarsest topology on X such that S is closed-semicontinuous and $S^{>}$ is open-semicontinuous. Then we can restate Schmeidler's theorem as follows:

Theorem′ (Schmeidler, reformulation 1)**.** *Let S be a nontrivial preorder on a connected topological space (X, τ). If τ is a refinement of $\tau_S^{\uparrow\downarrow}$, then S is complete.*

Now we can see why the given (exogenous) topology τ plays no role. If τ is connected and $\tau_S^{\uparrow\downarrow}$ is included in τ, then $\tau_S^{\uparrow\downarrow}$ is connected. Thus we can reformulate the above result once again, so that the only topology that matters is the topology $\tau_S^{\uparrow\downarrow}$ induced by S.

Theorem″ (Schmeidler, reformulation 2)**.** *If S is a nontrivial preorder on X such that $\tau_S^{\uparrow\downarrow}$ is connected, then S is complete.*

Theorem′ is an immediate corollary of Theorem″, however Theorem″ does not mention any primitive topology τ at all. We can also reformulate the last fact in a way that the induced order-section topology is only in the conclusion:

Theorem‴ (Schmeidler, reformulation 3)**.** *If S is a nontrivial incomplete preorder on X, then $\tau_S^{\uparrow\downarrow}$ is not connected.*

In Schmeidler's theorem, a reflexive transitive relation S satisfies a double form of semicontinuity. In our extensions (see Sections 4 and 5), semicontinuity is split between the two components of a comonotonic bi-preference (R, S), with R being closed-semicontinuous and $S^{>}$ open-semicontinuous. The definition of an 'adequate' topology on X, induced by R and S, also yields several alternative reformulations in this extended setting.

Definition 3.4. Let R and S be weak preferences on X. Define a topology $\tau_{R,S}^{\uparrow\downarrow}$ on X by taking as a subbase the strict sections (upper or lower) of S and the complements of the weak sections (upper or lower) of R. In other words, $\tau_{R,S}^{\uparrow\downarrow}$ is the topology obtained by declaring the strict sections of S open and the weak sections of R closed. We call $\tau_{R,S}^{\uparrow\downarrow}$ the *order-section topology induced by* (R, S).

As it happens for $\tau_S^{\uparrow\downarrow}$, also $\tau_{R,S}^{\uparrow\downarrow}$ can be used to characterize semicontinuity:

Lemma 3.5. *The following statements are equivalent for any pair (R, S) of preferences on a topological space (X, τ):*

(i) *R is closed-semicontinuous and $S^{>}$ is open-semicontinuous;*

(ii) *τ is a refinement of $\tau_{R,S}^{\uparrow\downarrow}$ (that is, $\tau \supseteq \tau_{R,S}^{\uparrow\downarrow}$).*

By Lemma 3.5, $\tau_{R,S}^{\uparrow\downarrow}$ is the coarsest topology on X such that R is closed-semicontinuous and $S^{>}$ is open-semicontinuous. Then, a general statement (here called 'Desiderata') that extends Schmeidler's theorem to bi-preferences may be formulated as follows:

Desiderata. *Let (R, S) be a nontrivial bi-preference with property $\mathcal{P}$ on a connected topological space (X, τ). If R is closed-semicontinuous and $S^{>}$ is open-semicontinuous, then (R, S) is a* NaP*-preference (equivalently, S is complete).*

In this ideal result, $\mathcal{P}$ is a suitable property of the bi-preference (R, S). By Lemma 3.5, we can restate any such result as follows:

Desiderata′ (reformulation 1)**.** *Let (R, S) be a nontrivial bi-preference with property $\mathcal{P}$ on a connected topological space (X, τ). If τ is a refinement of $\tau^{\uparrow\downarrow}_{R,S}$, then (R, S) is a* NaP*-preference (equivalently, S is complete).*

And again, since the connectedness of τ plus the inclusion $\tau^{\uparrow\downarrow}_{R,S} \subseteq \tau$ implies that $\tau^{\uparrow\downarrow}_{R,S}$ is connected as well, the last statement is an immediate corollary of the following:

Desiderata″ (reformulation 2)**.** *If (R, S) is a nontrivial bi-preference with property $\mathcal{P}$ such that $\tau^{\uparrow\downarrow}_{R,S}$ is connected, then (R, S) is a* NaP*-preference (equivalently, S is complete).*

As for a single preference, the exogenously given topology τ has disappeared, and all that matters is whether the bi-preference (R, S) induces a connected order-section topology $\tau^{\uparrow\downarrow}_{R,S}$, which is an intrinsic feature of the preorder. Therefore, we can finally rephrase our ideal result as follows:

Desiderata‴ (reformulation 3)**.** *If (R, S) is a nontrivial bi-preference with property $\mathcal{P}$ such that S is incomplete, then $\tau^{\uparrow\downarrow}_{R,S}$ is not connected.*

The next two sections are the bulk of the paper. In fact, we shall discuss two extensions of Schmeidler's theorem, which are in the spirit of the above Desiderata. The properties $\mathcal{P}$ that we employ consist of two different enhancements of comonotonicity: (i) strong comonotonicity, and (ii) everywhere nontrivial comonotonicity.

We emphasize from the outset that the above enhancements of comonotonicity are independent of each other, in general. On the other hand, under the assumption that the order-section topology induced by a bi-preference is connected, property (i) is stronger than property (ii). Nevertheless, this does not affect the autonomous interest of our two main results by any means: in fact, under the stronger hypothesis (i), we shall obtain the stronger conclusion that the bi-preference is indeed a *transitive* NaP-preference.

4 First extension of Schmeidler's theorem

The first main result of this paper concerns strongly comonotonic bi-preferences, that is, comonotonic bi-preferences that are also weakly monotonic (see Definition 2.10). Our analysis is divided in three parts: (1) order-theoretic preliminaries, (2) topological preliminaries, and (3) main result.

To start, we mention the two key properties of a binary relation that are needed to extrapolate the basic ingredients from Schmeidler's original argument:

Definition 4.1. A preference S on X is called:

- *strictly covering* if $xS^{>}y$ implies $x^{\downarrow,S^{>}} \cup y^{\uparrow,S^{>}} = X$ (for all $x, y \in X$);
- *splitting* if $xS^{\perp}y$ implies $x^{\downarrow,S^{>}} \cap y^{\downarrow,S^{>}} = \varnothing$ and $x^{\uparrow,S^{>}} \cap y^{\uparrow,S^{>}} = \varnothing$ (for all $x, y \in X$).

In other words, S is strictly covering if the strict sections of any pair of elements witnessing the nontriviality of S suffice to cover X. Further, S is splitting if it fully separates the strict sections of incomparable elements.

Remark 4.2. The following facts, which describe the asymmetric part of strictly covering and splitting binary relations, are useful to clarify the structure of these special types of preferences. These results are essentially embodied in (or may be derived from) those of Sections 4 and 5; however, it worth to mention them from the outset, in order to help the reader getting some preliminary insight.[30]

(i) Let S be a strictly covering (reflexive) relation on X. Then $S^{>}$ is transitive, the incomparability relation of $S^{>}$ is transitive, and any two equivalence classes of the incomparability relation are comparable by $S^{>}$. In short, $S^{>}$ must be an *irreflexive weak order*, that is, the complement of a total preorder.[31]

(ii) Let S be a splitting (reflexive) relation on X. When X is finite, the alternatives of any maximal path in $S^{>}$ (where "maximal" means with respect to set-inclusion) are unrelated by $S^{>}$ to all alternatives lying outside the path. Consequently, X is a parallel, disjoint union of maximal paths, where "parallel" means that two alternatives lying in different maximal paths are incomparable for $S^{>}$.

4.1 Order-theoretic preliminaries

Here we are only concerned with properties of a strongly comonotonic bi-preference. No topological structure is assumed. In the next section, we shall derive some topological consequences under the hypothesis that the order-section topology is connected.

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[30] We thank the Editor-in-Charge, Jean-Paul Doignon, for pointing out all these very interesting facts. They suggest that the notions of strictly covering and splitting may be of independent (combinatorial) interest.

[31] In fact, strictly covering preferences can be characterized by using the notion of 'resolution of preferences', called 'composition of relations' by Bang-Jensen and Gutin (2001, p. 8). Recall that, given a binary relation R_X on X and a family $(R_x)_{x\in X}$ of binary relations on Y_x such that $Y_x \cap Y_{x'} = \varnothing$ for all distinct $x, x' \in X$, the *resolution of* R_X *into* $(R_x)_{x\in X}$ is the relation R_Z on $Z := \bigcup_{x\in X} Y_x$ defined by

$$zR_Zz' \iff \begin{cases} \text{either} & (\exists x \in X)\ (z, z' \in Y_x \ \wedge\ zR_xz')\,, \\ \text{or} & (\exists x, x' \in X)\ (x \neq x' \ \wedge\ z \in Y_x \ \wedge\ z' \in Y_{x'} \ \wedge\ xR_Xx') \end{cases}$$

where $z, z' \in Z$. Then, one can show that the following fact holds (proof is available upon request):

Lemma. *A preference is strictly covering if and only if it is the resolution of a linear order into trivial binary relations.*

The notion of preference resolution can be seen as a particular case of the notion of 'choice resolution', recently introduced by Cantone, Giarlotta, and Watson (2019b), and then adapted to 'convex geometries' (which are in one-to-one correspondence with path independent choice spaces) by Cantone et al. (2020).

Lemma 4.3. *The following are equivalent for a comonotonic bi-preference (R, S) on X:*

(i) *(R, S) is weakly monotonic;*

(ii) *for each $x, y \in X$, if $xS^{>}y$, then $x^{\downarrow,S^{>}} \cup y^{\uparrow,S^{>}} = x^{\downarrow,R} \cup y^{\uparrow,R}$;*

(iii) *for each $x, y \in X$, if $xR^{>}yS^{\sim}x$, then there is no $z \in X$ such that either $xS^{>}zS^{\sim}y$ and $zR^{\perp}y$, or $xS^{\sim}zS^{>}y$ and $xR^{\perp}z$ holds.*

Proof. Let (R, S) be a comonotonic bi-preference (R, S) on X.

(i) $\Rightarrow$ (ii). Assume that (R, S) is weakly monotonic. Let $x, y \in X$ be such that $xS^{>}y$. We shall prove that the equality $x^{\downarrow,S^{>}} \cup y^{\uparrow,S^{>}} = x^{\downarrow,R} \cup y^{\uparrow,R}$ holds. One inclusion readily follows from comonotonicity. For the reverse inclusion, we let $z \in x^{\downarrow,R} \cup y^{\uparrow,R}$, and show that $z \in x^{\downarrow,S^{>}} \cup y^{\uparrow,S^{>}}$. By duality, it suffices to prove the claim whenever $z \in x^{\downarrow,R}$, that is, if xRz holds. We deal separately with two possible cases: (a) zRx; (b) $xR^{>}z$.

In case (a), we get $zRxR^{>}y$ by hypothesis and comonotonicity, hence $zR^{>}y$ by the transitivity of R. Since $zSxS^{>}y$ holds, weak monotonicity yields $zS^{>}y$, that is, $z \in y^{\uparrow,S^{>}}$, as required. For case (b), if $xS^{>}z$, then we are immediately done. Next, assume that zSx. It follows that $zSxR^{>}y$ by hypothesis and comonotonicity, and so zSy by transitive coherence. Now if ySz were to hold, then we would get $xS^{>}ySz$ and $xR^{>}z$, hence $xS^{>}z$ by weak monotonicity, a contradiction. Therefore, $zS^{>}y$ holds, that is, $z \in y^{\uparrow,S^{>}}$, as claimed.

(ii) $\Rightarrow$ (i). Assume that (ii) holds. By symmetry, to show that (i) holds as well, it suffices to prove the inclusion $R^{>} \cap (S^{>} \circ S) \subseteq S^{>}$. Thus let $x, y, z \in X$ be such that $xS^{>}ySz$ and $xR^{>}z$. By extension, it follows that xSz. Toward a contradiction, assume that zSx holds. Since $z \in x^{\downarrow,R}$, the hypothesis yields $z \in y^{\uparrow,S^{>}}$, which is impossible.

(i) $\Rightarrow$ (iii). We prove the contrapositive. Assume that there are $x, y, z \in X$ such that $xR^{>}yS^{\sim}x$, $xS^{>}z$ and $zS^{\sim}yR^{\perp}z$. Thus, we have $xR^{>}y$ and $x(S^{>} \circ S)y$, but $\neg(xS^{>}y)$, that is, $R^{>} \cap (S^{>} \circ S) \nsubseteq S^{>}$. This shows that (R, S) fails to be weakly monotonic. Similarly, if $xR^{>}yS^{\sim}x$, $xS^{>}z$ and $zS^{\sim}yR^{\perp}z$ holds, then weak monotonicity fails again.

(iii) $\Rightarrow$ (i). Assume that (iii) holds. Let $x, y, z \in X$ be such that $xR^{>}y$ and $xS^{>}zSy$. By hypothesis, we cannot have ySx, hence we obtain $xS^{>}y$. Similarly, if $x, y, z \in X$ are such that $xR^{>}y$ and $xSzS^{>}y$, then the hypothesis yields $\neg(ySx)$, hence $xS^{>}y$ again. This shows that (R, S) is weakly monotonic, as claimed. ■

The following immediate consequence of Lemma 4.3 will be useful:

Lemma 4.4. *Let (R, S) be a strongly comonotonic bi-preference on X. For all $x, y \in X$,*

$$xS^{>}y \quad \Longrightarrow \quad x^{\downarrow,S^{>}} \cup y^{\uparrow,S^{>}} \;=\; x^{\downarrow,R} \cup y^{\uparrow,R}.$$

We still need an additional order-theoretic result, which we shall state in a more general way than needed, that is, for *weakly* comonotonic bi-preferences:

Lemma 4.5. *Let (R, S) be a weakly comonotonic bi-preference on X. For all $x, y \in X$,*

$$xS^{\perp}y \quad \Longrightarrow \quad \left(x^{\downarrow,S^{>}} \cap y^{\downarrow,S^{>}} \;=\; x^{\downarrow,R} \cap y^{\downarrow,R}\right) \,\wedge\, \left(x^{\uparrow,S^{>}} \cap y^{\uparrow,S^{>}} \;=\; x^{\uparrow,R} \cap y^{\uparrow,R}\right).$$

PROOF. Assume that $x, y \in X$ are such that $xS^{\perp}y$. We prove the first equality only, since the proof of the second is similar. Let $z \in x^{\downarrow,S^>} \cap y^{\downarrow,S^>}$, that is, $xS^>z$ and $yS^>z$. The hypothesis readily implies $xR^>z$ and $yR^>z$, and so $z \in x^{\downarrow,R} \cap y^{\downarrow,R}$. For the reverse inclusion, assume that $z \in x^{\downarrow,R} \cap y^{\downarrow,R}$, i.e., xRz and yRz. Since S extends R, we obtain xSz and ySz. If zSx were to hold, then we would get $yRzSx$, hence ySx by transitive coherence, a contradiction. It follows that $xS^>z$. A similar argument yields $yS^>z$. We conclude $z \in x^{\downarrow,S^>} \cap y^{\downarrow,S^>}$, as claimed. ■

4.2 Topological preliminaries

Here we present two topological consequences of Lemmas 4.4 and 4.5. Specifically, we show that whenever the order-section topology induced by a strongly comonotonic bi-preference is connected, the soft component is simultaneously strictly covering and splitting.

The first result of this section requires the full power of strong comonotonicity.

Lemma 4.6. *If (R, S) is a strongly comonotonic bi-preference such that $\tau^{\uparrow\downarrow}_{R,S}$ is connected, then S is strictly covering.*

PROOF. Suppose (R, S) is a strongly comonotonic bi-preference such that the order-section topology $\tau^{\uparrow\downarrow}_{R,S}$ is connected. By Lemma 3.5, R is closed-semicontinuous and $S^>$ is open-semicontinuous in $(X, \tau^{\uparrow\downarrow}_{R,S})$. To show that S is strictly covering, let $x, y \in X$ be such that $xS^>y$. Lemma 4.4 yields $x^{\downarrow,S^>} \cup y^{\uparrow,S^>} = x^{\downarrow,R} \cup y^{\uparrow,R}$. The closed-semicontinuity of R implies that the set $x^{\downarrow,R} \cup y^{\uparrow,R}$ is closed. Dually, the open-semicontinuity of $S^>$ implies that the set $x^{\downarrow,S^>} \cup y^{\uparrow,S^>}$ is open. It follows that the set $x^{\downarrow,S^>} \cup y^{\uparrow,S^>} = x^{\downarrow,R} \cup y^{\uparrow,R}$ is both closed and open; further, it is nonempty, since $x \in x^{\downarrow,R}$ by the reflexivity of R. Now the connectedness of $\tau^{\uparrow\downarrow}_{R,S}$ implies that $x^{\downarrow,S^>} \cup y^{\uparrow,S^>} = X$, that is, S is strictly covering. ■

Our second topological result holds under the milder assumption of weak comonotonicity.

Lemma 4.7. *If (R, S) is a weakly comonotonic bi-preference such that $\tau^{\uparrow\downarrow}_{R,S}$ is connected, then S is splitting.*

PROOF. Suppose (R, S) is a weakly comonotonic bi-preference such that the order-section topology $\tau^{\uparrow\downarrow}_{R,S}$ is connected. By Lemma 3.5, R is closed-semicontinuous and $S^>$ is open-semicontinuous in $(X, \tau^{\uparrow\downarrow}_{R,S})$. Let $x, y \in X$ be such that $xS^{\perp}y$. Lemma 4.5 yields the equalities $x^{\downarrow,S^>} \cap y^{\downarrow,S^>} = x^{\downarrow,R} \cap y^{\downarrow,R}$ and $x^{\uparrow,S^>} \cap y^{\uparrow,S^>} = x^{\uparrow,R} \cap y^{\uparrow,R}$. The semicontinuity hypothesis implies that the two sets $x^{\downarrow,S^>} \cap y^{\downarrow,S^>}$ and $x^{\uparrow,S^>} \cap y^{\uparrow,S^>}$ are clopen. Since x belongs to none of them, the connectedness of $\tau^{\uparrow\downarrow}_{R,S}$ implies that both intersections are empty. Thus S is splitting. ■

4.3 Main result and consequences

We are ready to prove the first extension of Schmeidler's theorem:[32]

Theorem 4.8. *If (R, S) is a nontrivial strongly comonotonic bi-preference such that $\tau^{\uparrow\downarrow}_{R,S}$ is connected, then S is complete and transitive (that is, a total preorder).*

Proof. Suppose (R, S) is a strongly comonotonic bi-preference on a connected topological space X. Assume further that R is closed-semicontinuous, and $S^{>}$ is nonempty and open-semicontinuous. Below we show that S is both complete and transitive. We shall argue by contradiction in both cases.

To prove completeness, suppose there are $a, b \in X$ such that $aS^{\perp}b$. By hypothesis, there are $p, q \in X$ such that $pS^{>}q$. Since S is strictly covering by Lemma 4.6, we obtain $a \in X = p^{\downarrow,S^{>}} \cup q^{\uparrow,S^{>}}$. It follows that at least one of the following two cases happens: (1) $pS^{>}a$; (2) $aS^{>}q$. By symmetry, it suffices to derive a contradiction in case (2). Assume that $aS^{>}q$, i.e., $q \in a^{\downarrow,S^{>}}$. Another application of Lemma 4.6 yields $b \in X = a^{\downarrow,S^{>}} \cup q^{\uparrow,S^{>}}$. Since $aS^{\perp}b$, we must have $bS^{>}q$, i.e., $q \in b^{\downarrow,S^{>}}$. It follows that $q \in a^{\downarrow,S^{>}} \cap b^{\downarrow,S^{>}}$. However, since S is splitting by Lemma 4.7, we also have $a^{\downarrow,S^{>}} \cap b^{\downarrow,S^{>}} = \varnothing$, which is impossible.

To prove transitivity, suppose $x, y, z \in X$ are such that $xSySz$ but $\neg(xSz)$. By the completeness of S, we get $zS^{>}x$. Since S is strictly covering by Lemma 4.6, we have $z^{\downarrow,S^{>}} \cup x^{\uparrow,S^{>}} = X$. Thus, either $zS^{>}y$ or $yS^{>}x$ holds. However, both cases contradict the hypothesis. This completes the proof. ■

In particular, we obtain Schmeidler's theorem:

Corollary 4.9. *A preorder R on a connected topological space such that R is closed-semicontinuous and $R^{>}$ is open-semicontinuous is either trivial or complete.*

Proof. Let (X, τ) be a connected topological space, and R a nontrivial preorder on X such that R is closed-semicontinuous and $R^{>}$ is open-semicontinuous. The pair (R, R) is a nontrivial and strongly comonotonic bi-preference on X such that R is closed-semicontinuous and $R^{>}$ is open-semicontinuous. By Lemma 3.5, τ contains the order-section topology $\tau^{\uparrow\downarrow}_{R,S}$. Thus, R is complete by Theorem 4.8. ■

Corollary 4.9 does not allow to deduce Theorem 4.8 by any means. In fact, if (R, S) is a bi-preference satisfying the hypothesis of Theorem 4.8, then we can apply Schmeidler's theorem to neither the pair (R, R) (because the hypothesis of the semicontinuity of $R^{>}$ is missing) nor the pair (S, S) (because S may possibly fail to be transitive or semicontinuous). The following result slightly generalizes the findings of Theorem 4.8:

Corollary 4.10. *Let (R, S) be a strongly comonotonic bi-preference on a topological space X. If R is closed-semicontinuous and $S^{>}$ is open-semicontinuous, then the restriction of S to each connected component of X is either trivial or complete.*

[32] This result was originally formulated with no deduction about the transitivity of S. During the process of revision of the paper, we became aware that, under the hypotheses of Theorem 4.8, S is not only complete but also transitive. We thank M. Ali Khan and Metin Uyanik for pointing out this fact. See also Uyanik and Khan (2019) for a different proof of a result similar to Theorem 4.8, in which the rigid component satisfies a mixed form of transitivity.

PROOF. Suppose K is a connected component of X. Let $R_K = R \cap K^2$ and $S_K = S \cap K^2$ be the restrictions of R and S to K. Then, (R_K, S_K) is a strongly comonotonic bi-preference on the connected topological space K, and R_K and $S_K^{>}$ are semicontinuous. If S_K is nontrivial, then S_K is complete by Theorem 4.8. ∎

A last consequence of Theorem 4.8 is the following:

Corollary 4.11. *A nontrivial strongly comonotonic bi-preference with a connected order-section topology is a transitive* NaP*-preference.*

PROOF. If (R, S) is a strongly comonotonic bi-preference on X such that $\tau^{\uparrow\downarrow}_{R,S}$ is connected, then S is transitive and complete by Theorem 4.8. Thus the claim readily follows from Lemma 2.15. ∎

We conclude this section by exhibiting an intransitive NaP-preference with a connected order-section topology.

Example 4.12 (Scott-Suppes representable semiorder, Scott and Suppes, 1958)**.** Let S be the binary relation on $\mathbb{R}$ defined as follows for each $x, y \in X$:

$$xSy \quad \overset{\text{def}}{\iff} \quad x + 1 \geqslant y.$$

Notice that $xS^{>}y$ if and only if $x > y + 1$, hence S is quasi-transitive; further, S is not transitive because of $S^{\sim}$. In fact, S is a *universal* Scott-Suppes representable semiorder,[33] in the sense that any other Scott-Suppes representable semiorder embeds into it. The trace S_{tr} of S is the usual linear order $\geqslant$ of the reals.[34] In particular, S_{tr} and $S^{>}$ are, respectively, closed and open in the product space $\mathbb{R} \times \mathbb{R}$. It follows that the tracing bi-preference (S_{tr}, S) is a nontrivial NaP-preference with the property that S_{tr} is closed in $\mathbb{R} \times \mathbb{R}$, and $S^{>}$ is open in $\mathbb{R} \times \mathbb{R}$. In particular, S_{tr} is closed-semicontinuous and $S^{>}$ is open-semicontinuous. By Lemma 3.5, $\tau^{\uparrow\downarrow}_{S_{\mathsf{tr}},S}$ is connected. Notice that Theorem 4.8 implies that (S_{tr}, S) is not weakly monotonic: indeed, $2.5\, S^{>}_{\mathsf{tr}}\, 2$ and $2.5\, S\, 3.3\, S^{>}\, 2$, but $\neg(2.5\, S^{>}\, 2)$.

5 Second extension of Schmeidler's theorem

Our second main result is about 'everywhere nontrivial' comonotonic bi-preferences. Everywhere nontriviality accounts for a mild richness of the strict soft component:

Definition 5.1. A weak preference S on X is *everywhere nontrivial* if for each $x \in X$ there is $y \in X$ such that either $xS^{>}y$ or $yS^{>}x$ holds. A bi-preference (R, S) is *everywhere nontrivial* if S is everywhere nontrivial.

[33] After Scott and Suppes (1958), a semiorder S on X is *Scott-Suppes representable* if there is a real-valued function $u\colon X \to \mathbb{R}$ such that, for each $x, y \in X$, xSy if and only if $u(x) + 1 \geqslant u(y)$.

[34] For a different kind of trace (*sliced*), see Example 4.4 in Giarlotta and Watson (2016): this trace is (isomorphic to) the lexicographic product $[0, 1) \times_{\text{lex}} \mathbb{Z}^*$, where $\mathbb{Z}^*$ is the reverse ordering of the integers.

The property described in Definition 5.1 is satisfied in most economic settings: for instance, it is trivially implied by the property of *local nonsatiation* in classical demand theory.[35] Notice that being everywhere nontrivial is also undemanding in terms of the transitive structure of the preference relation: in fact, such a relation S need not even be semitransitive, because a configuration of the type $xS^{>}yS^{>}z$, $\neg(xS^{>}t)$ and $\neg(tS^{>}z)$ is allowed, as long as there exists $w \in X$ such that either $tS^{>}w$ or $wS^{>}t$ holds.

We shall prove the following counterpart of Theorem 4.8, where 'everywhere nontrivial' replaces 'weakly monotonic' (but in this case the NaP-preference may fail to be transitive):

Theorem 5.2. *If (R, S) is an everywhere nontrivial comonotonic bi-preference such that $\tau^{\uparrow\downarrow}_{R,S}$ is connected, then S is complete (equivalently, (R, S) is a NaP-preference).*

For nontrivial bi-preferences, the two properties of (i) strong comonotonicity and (ii) everywhere nontrivial comonotonicity are, in general, independent of each other. For instance, the nontrivial bi-preference (R, S) on $X = \{x, y, z\}$, defined by $R = \{(x, z)\} \cup \Delta(X)$ and $S = X^2 \backslash \{(z, x)\}$, is a strongly comonotonic (in fact, monolithic) bi-preference, which fails to be everywhere nontrivial. However, under the assumption of a connected order-section topology, property (i) implies property (ii):

Lemma 5.3. *If (R, S) is a nontrivial strongly comonotonic bi-preference such that $\tau^{\uparrow\downarrow}_{R,S}$ is connected, then S is everywhere nontrivial.*

Proof. Suppose (R, S) is a nontrivial strongly comonotonic bi-preference such that the induced order-section topology $\tau^{\uparrow\downarrow}_{R,S}$ is connected. By the nontriviality of S, there are $a, b \in X$ such that $aS^{>}b$. Since S is strictly covering by Lemma 4.6, we have $X = a^{\downarrow,S^{>}} \cup b^{\uparrow,S^{>}}$. Thus, for any $x \in X$, we have either $aS^{>}x$ or $xS^{>}b$. This shows that S is everywhere nontrivial. ■

The remainder of this section is devoted to a formal proof of Theorem 5.2.

5.1 Order-theoretic preliminaries

Recall from Section 2.1 that a strong quasi-preorder is a *strongly quasi-transitive* preference S, i.e., the inclusions $S \circ S^{>} \subseteq S$ and $S^{>} \circ S \subseteq S$ hold. Strong quasi-transitivity implies quasi-transitivity, and the converse holds under completeness.[36]

The key result to derive Theorem 5.2 is the *Decomposition Lemma* (Lemma 5.10), which fully describes the structure of a strong quasi-preorder that is also splitting. To state this result, we need some new notions that concern a single binary relation.

Definition 5.4. Let S be a weak preference on X, and $A, B \subseteq X$. We say that A and B are *incomparable sets*, denoted by $AS^{\perp}B$, if $aS^{\perp}b$ for all $a \in A$ and $b \in B$. Similarly, we say that A and B are *indifferent sets*, denoted by $AS^{\sim}B$, if $aS^{\sim}b$ for all $a \in A$ and $b \in B$.

[35] A preference relation R on a metric space (X, d) is *locally nonsatiated* if, for every $x \in X$ and $\varepsilon > 0$, there is $y \in X$ such that $d(x, y) < \varepsilon$ and $yR^{>}x$: see Mas-Colell, Whinston, and Green (1995, page 42).

[36] See Footnote 15.

Remark 5.5. In graph-theoretic terms, the pair (X, S) can be looked at as a directed graph, where $e = (a, b)$ is an edge in (X, S) if and only if aSb. Then two sets A and B are incomparable (with respect to S) if there is no directed edge between an element of A and element of B. Notice that being incomparable sets says nothing about the internal structure of each of the two sets: in fact, there may well be directed edges inside A and B. A similar remark applies for the notion of indifferent sets.

The next definition establishes an equivalence relation on X on the basis of a sequence of 'mixed' strict preferences, in the sense that they can go in either directions.

Definition 5.6. Let S be a weak preference on X, and $a, b \in X$. A *mixed strict path* from a to b is a finite sequence $a = x_0\, T_1\, x_1\, T_2 \ldots T_n\, x_n = b$, with $n \in \mathbb{N}\backslash\{0\}$, $x_0, \ldots x_n \in X$, and $T_i \in \{S^{>}, S^{<}\}$ for all $i = 1, \ldots, n$ (where $xS^{<}y$ if and only if $yS^{>}x$).[37] Define a binary relation $\approx$ on X by letting, for each $a, b \in X$, $a \approx b$ if there is a mixed strict path from a to b. Then $\approx$ is an equivalence relation on X: we shall denote by $\mathrm{Mix}(a)$ the equivalence class of a.

Remark 5.7. In graph-theoretic terms, if (X, T) is the *undirected* graph obtained from the directed graph (X, S) by transforming each directed edge into an undirected edge, then the equivalence classes of (X, S) with respect to $\approx$ are exactly the *connected components* of (X, T), that is, the maximal connected subsets of the graph (X, T).

Then we have:

Lemma 5.8. *Assume S is a strongly quasi-transitive. For any $a, a', x \in X$, if $a \approx a'$ and $\neg(xS^{>}a'' \vee a''S^{>}x)$ for all $a'' \in \mathrm{Mix}(a)$, then $aS^{\sim}x$ if and only if $a'S^{\sim}x$.*

Proof. Let $a, a' \in X$ be such that $a \approx a'$. Further, let $x \in X$ be such that x is strictly S-related to no element of $\mathrm{Mix}(a)$ (the $\approx$-equivalence class of a). Since $a' \approx a$, there is a sequence $a = x_0, x_1, \ldots, x_n = a'$ of elements in X such that, for each $0 \leqslant i < n$, either $x_iS^{>}x_{i+1}$ or $x_{i+1}S^{>}x_i$ holds. Notice that the definition of $\mathrm{Mix}(a)$ yields that $x_i \in \mathrm{Mix}(a)$ for all $0 \leqslant i < n$.

To prove that $aS^{\sim}x$ if and only $a'S^{\sim}x$, it suffices to prove one direction only, since the proof of other direction is similar. Thus, assume that $x_0 = aS^{\sim}x$; we shall show that $a'S^{\sim}x$. If $x_0S^{>}x_1$ holds, then we get xSx_1 by the strong quasi-transitivity of S, and so $xS^{\sim}x_1$, since there is no strict preference between x and any element of $\mathrm{Mix}(a)$ by hypothesis. On the other hand, if $x_1S^{>}x_0$ holds, we get x_1Sx, and again $xS^{\sim}x_1$. Now iterate the argument to obtain $xS^{\sim}x_n = a'$. This completes the proof. ■

Strong quasi-preorders that are also splitting satisfy an additional property:

Lemma 5.9. *Assume S is strongly quasi-transitive and splitting. For all $x, y \in X$, if $xS^{\perp}y$ then* $\mathrm{Mix}(x)S^{\perp}\mathrm{Mix}(y)$.

[37] Examples of mixed strict paths from a to b (equivalently, from b to a) are (1) $aS^{>}x_1S^{<}b$, (2) $aS^{<}x_1S^{>}x_2S^{<}b$, (3) $aS^{>}x_1S^{>}x_2S^{>}x_3S^{>}b$, etc.

Proof. Assume S is strongly quasi-transitive and splitting.

Claim: *For each $A, B \subseteq X$, if $AS^{\perp}B$ and $a \in A$, then $\left(A \cup a^{\downarrow,S^{>}} \cup a^{\uparrow,S^{>}}\right)S^{\perp}B$.*

To prove the Claim, let $z \in a^{\downarrow,S^{>}} \cup a^{\uparrow,S^{>}}$ and $b \in B$. We show that $zS^{\perp}b$. Without loss of generality, suppose $z \in a^{\downarrow,S^{>}}$, i.e, $aS^{>}z$. If zSb, then aSb by the strong quasi-transitivity of S, which contradicts $aS^{\perp}b$. Further, we cannot have $bS^{>}z$, since otherwise z would contradict the fact that S is splitting. It follows that $zS^{\perp}b$. This proves the Claim.

Now we can iteratively apply the Claim to all $a \in \mathrm{Mix}(x)$ and $b \in \mathrm{Mix}(y)$, and get $\mathrm{Mix}(x)S^{\perp}\mathrm{Mix}(y)$. This completes the proof. ■

We can finally state the key result to prove Theorem 5.2.

Lemma 5.10 (Decomposition Lemma). *If S is strongly quasi-transitive and splitting, then X is the disjoint union of sets X_i such that each restriction $S \restriction X_i$ is complete, and distinct X_i's are either indifferent sets or incomparable sets.*

Proof. Let $\{X_i : i \in I\}$ be the family of all distinct equivalence classes with respect to $\approx$. Each equivalence class is complete, because incomparable elements are in different equivalence classes by Lemma 5.9. Notice also that the definition of mixed strict path implies that we can never have a relation of strict preference between elements of distinct equivalence classes. Finally, Lemma 5.8 rules out the possibility that there are distinct $i, j \in I$, $x, x' \in X_i$ and $y, y' \in X_j$ such that $xS^{\sim}y$ but $x'S^{\perp}y'$. It follows that, for each $i \neq j$, either $X_iS^{\sim}X_j$ or $X_iS^{\perp}X_j$ holds. This completes the proof. ■

5.2 Proof of main result

The next two results link properties of bi-preferences to properties of single preferences.

Lemma 5.11. *The soft component of a comonotonic bi-preference is strongly quasi-transitive.*

Proof. Let (R, S) be a comonotonic bi-preference on X. By symmetry, it suffices to show that $S \circ S^{>} \subseteq S$. Let $x, y \in X$ be such that $xSzS^{>}y$ for some $z \in X$. Now comonotonicity yields $xSzR^{>}y$, and transitive coherence entails xSy, as required. ■

Lemma 5.12. *The following statements are equivalent for a bi-preference (R, S) such that $\tau^{\uparrow\downarrow}_{R,S}$ is connected and S is strongly quasi-transitive:*

(i) *(R, S) is weakly comonotonic;*

(ii) *S is splitting.*

Proof. Assume that R and $S^{>}$ are semicontinuous, and S is strongly quasi-transitive. If (R, S) is weakly comonotonic, then S is splitting by Lemma 4.7: thus (i) implies (ii). Conversely, if S is splitting, the Decomposition Lemma ensures that both $(xS^{>}z \wedge yS^{>}z) \wedge xS^{\perp}y$ and $(xS^{>}z \wedge yS^{>}z) \wedge xS^{\perp}y$ never happen, hence (R, S) is (vacuously) weakly comonotonic. ■

We are now ready to prove Theorem 5.2.

Proof of Theorem 5.2. Let (R, S) be an everywhere nontrivial comonotonic bi-preference on X such that the order-section topology such that $\tau^{\uparrow\downarrow}_{R,S}$ is connected. Since S is strongly quasi-transitive by Lemma 5.11, and a comonotonic bi-preference is trivially weakly comonotonic, Lemma 5.12 yields that S is splitting. By the Decomposition Lemma, X can be partitioned into the disjoint union of sets X_i such that the restriction of S to each X_i is complete, and distinct X_i's are either pairwise indifferent or pairwise incomparable. Since S is everywhere nontrivial by hypothesis, it follows that each complete component X_i has size at least two. By the definition of X_i (see the proof of the Decomposition Lemma), we obtain

$$X_i = \bigcup \left\{ x^{\uparrow, S^>} : x \in X_i \right\} \cup \bigcup \left\{ x^{\downarrow, S^>} : x \in X_i \right\} \neq \varnothing$$

for all $i \in I$. Since $S^>$ is open-semicontinuous, each X_i is open. Now the connectedness of X implies that there is only one such X_i. It follows that S is complete or, equivalently by Lemma 2.15, (R, S) is a NaP-preference. ■

Remark 5.13. Theorem 5.2 holds in a more general form, with no requirement that the rigid preference R be transitive. In fact, if one goes through the proof of all preliminary results that are involved in the proof of Theorem 5.2 (Lemmas 5.11, 4.7, 3.5, 4.5, and 5.12) and the proof of Theorem 5.2 itself, it becomes apparent that the transitivity of R is never used. As a matter of fact, R need not satisfy any form of transitivity at all. Nevertheless, since here we are mostly interested in bi-preference structures for their amenability to applications, we prefer to state Theorem 5.2 in a particular case.[38] On the point, see also Uyanik and Khan (2019) for two additional extensions of Schmeidler's theorem to pairs of binary relations, which only require a weak form of transitivity of the rigid preference.

Example 4.12 differentiates the extent of the two main results of this paper. In fact, if S is the classical semiorder on $\mathbb{R}$ with discrimination threshold 1, and S_{tr} is its trace, then (S_{tr}, S) is a comonotonic bi-preferences such that S is everywhere nontrivial, $\tau^{\uparrow\downarrow}_{R,S}$ is connected, and yet S is not transitive. In fact, the most important difference between the two extensions of Schmeidler's theorem is that strong comonotonicity is not needed, whereas everywhere nontriviality is. In the Appendix, we exhibit a minimal failure of the property of being everywhere nontrivial, which in turn causes the failure of completeness.

We conclude the paper with an additional consequence of the Decomposition Lemma, which delivers the completeness of a *single* preference relation with a connected order-section topology, whenever its strict part satisfies suitable forms of transitivity and local completeness. Its proof is similar to that of Theorem 5.2.

Corollary 5.14. *A strongly quasi-transitive, splitting, and everywhere nontrivial preference with a connected order-section topology is always complete.*

[38]However, we believe that this second extension of Schmeidler's theorem is more far reaching than the first, especially in view of applications to weaker preference structures.

6 Conclusions

In this paper we have revisited and extended a celebrated theorem by Schmeidler on the completeness of a preorder. Specifically, we have shown that what appears to be a relationship between order and topology is, in fact, a topological property of a primitive preorder. Further, we have extended Schmeidler's theorem to bi-preferences, using an enhanced comonotonic bi-preference in place of a preorder in the hypothesis, and a (possibly transitive) NaP-preference in place of a complete preorder in the thesis. These extensions of Schmeidler's theorem contribute to better clarify the relationship between the two classical tenets of economic rationality.

Future research on the topic may try to derive an extension of Schmeidler's theorem to *monotonic* bi-preferences, instead of comonotonic ones. Furthermore, it appears possible to generalize the analysis between topological connectedness and behavioral assumptions on single preferences to bi-preferences: see Khan and Uyanik (2018).

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Appendix: 'Everywhere nontrivial' is needed

Here we show that the property of being everywhere nontrivial is needed in the statement of Theorem 5.2. In fact, we shall provide an instance of a comonotonic bi-preference with a connected order-section topology, which is everywhere nontrivial except at one point, and yet it fails to be complete. For the sake of brevity, most proofs of the results in this section are either sketched or left to the reader.

To start, we consider trivial extensions of bi-preferences by adding a new single point to the ground set.

Definition 6.1. Let (R, S) be a bi-preference on X. Select a point ∞ not in X, and set $X' := X \cup \{\infty\}$. The *one-point trivial extension* of (R, S) is the pair (R', S'), where R' and S' are the (unique) binary relations on X' such that $R' \cap X^2 = R$, $S' \cap X^2 = S$, $R' \subseteq S'$, and $x(S')^{\perp}\infty$ for all $x \in X$.

In words, R' and S' are the binary relations which are obtained from R and S by (1) adding a new point to the ground set X, and (2) declaring this new point incomparable to all points of X, both for R' and S'. The following fact is obvious:

Lemma 6.2. *One-point trivial extensions of bi-preferences preserve the following properties: (i) being a bi-preference, (ii) being comonotonic, (iii) having a rigid preference that is a partial order, and (iv) having a nontrivial soft component. However, one-point trivial extensions are always incomplete.*

The following topological fact will be needed.

Lemma 6.3. *Let X be a connected locally compact non-compact Hausdorff topological space, and $X' = X \cup \{\infty\}$ its one-point (Alexandroff) compactification. Further, let (R, S) a bi-preference on X such that R is closed-semicontinuous and $S^{>}$ is open-semicontinuous, and let (R', S') be its one-point trivial extension to X'. If all closed sections $x^{\uparrow,R}$ and $x^{\downarrow,R}$ are compact, then R' and $(S')^{>}$ are, respectively, closed-semicontinuous and open-semicontinuous in X'.*

Next, we construct a bi-preference on a product space starting from two partial orders.

Definition 6.4. Let $(A, \geqslant_A)$ and $(B, \geqslant_B)$ be partially ordered sets (posets). Define two binary relations R and S on $A \times B$ as follows. For all $(a, b), (a', b') \in A \times B$, set

$$(a,b)R^{>}(a',b') \iff (a >_A a' \wedge b >_B b') \vee (a = a' \wedge b >_B b') \vee (a >_A a' \wedge b = b'),$$
$$(a,b)R^{\sim}(a',b') \iff a = a' \wedge b = b',$$

and $(a, b)R^{\perp}(a', b')$ otherwise. Further, set

$$(a,b)S^{>}(a',b') \iff (a >_A a' \wedge b >_B b'),$$

and $(a, b)S^{\sim}(a', b')$ otherwise. We call (R, S) the *product bi-preference* induced by the posets $(A, \geqslant_A)$ and $(B, \geqslant_B)$.

The terminology employed in Definition 6.4 is justified by the following fact:

Lemma 6.5. *Let $(A, \geqslant_A)$ and $(B, \geqslant_B)$ be posets on connected locally compact topological spaces. Suppose $\geqslant_A$, $>_A$, $\geqslant_B$, $>_B$ are semicontinuous, and $\geqslant_A, \geqslant_B$ are nontrivial. The product bi-preference (R, S) induced by $(A, \geqslant_A)$ and $(B, \geqslant_B)$ has the following properties:*

(i) *(R, S) is a nontrivial* NaP*-preference on $A \times B$;*

(ii) *R is a partial order on $A \times B$;*

(iii) *R is closed-semicontinuous and $S^{>}$ is open-semicontinuous;*

(iv) *$A \times B$ is a connected and locally compact topological space.*

Proof. Most properties are obvious. Here we only check a part of (i), proving that Transitive Coherence holds. By symmetry, it suffices to show that $(a, b)R(a', b')S(a'', b'')$ implies $(a, b)S(a'', b'')$ for all $(a, b), (a', b'), (a'', b'') \in A \times B$. Suppose $(a, b)R(a', b')$ and $(a', b')S(a'', b'')$. By the completeness of S, proving that $(a, b)S(a'', b'')$ holds is equivalent to proving that $(a'', b'')S^{>}(a, b)$ fails. If $(a, b)R^{\sim}(a', b')$, then the claim holds trivially. Thus, let $(a, b)R^{>}(a', b')$. Three cases are possible: (1) $a >_A a'$ and $b >_B b'$; (2) $a = a'$ and $b >_B b'$; (3) $a >_A a'$ and $b = b'$. It is immediate to show that $(a'', b'')S^{>}(a, b)$ (i.e., $a'' >_A a$ and $b'' >_B b$) generates a contradiction in each of the above cases. ■

As a simple application of Lemma 6.5, we get:

Lemma 6.6. *There is a comonotonic bi-preference (R, S) on a connected locally compact non-compact Hausdorff topological space such that S is everywhere nontrivial, R is closed-semicontinuous, $S^{>}$ is open-semicontinuous, and all sections $x^{\uparrow,R}, x^{\downarrow,R}$ are compact.*

Proof. Let $(A, \geqslant_A) = (B, \geqslant_B) = (\mathbb{R}, \geqslant)$ in Lemma 6.5, and take the restriction of the product bi-preference (R, S) to the unbounded strip $Y = \{(a, b) \in \mathbb{R}^2 : |a + b| \leqslant 1\}$. ∎

Finally, we obtain what we were after:

Proposition 6.7. *There is a comonotonic bi-preference (R, S) with a connected order-section topology such that S is incomplete and everywhere nontrivial except at one point.*

Proof. Apply Lemmas 3.5, 6.2, 6.3, and 6.6. ∎

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