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An Improved Criterion for the Oscillation of Fourth-Order Differential Equations

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Abstract: The main purpose of this manuscript is to show asymptotic properties of a class of differential equations with variable coefficients $(r(v)(w'''(v))^\beta)' + \sum_{i=1}^j q_i(v)y^\kappa(g_i(v)) = 0$, where $v \geq v_0$ and $w(v) := y(v) + p(v)y(\sigma(v))$. By using integral averaging technique, we get conditions to ensure oscillation of solutions of this equation. The obtained results improve and generalize the earlier ones; finally an example is given to illustrate the criteria.

Keywords: fourth-order differential equations; neutral delay; oscillation

MSC: 34K11

1. Introduction

In this paper, we study the oscillatory properties of solutions of the following fourth-order neutral differential equation

$$(r(v)(w'''(v))^\beta)' + \sum_{i=1}^j q_i(v)y^\kappa(g_i(v)) = 0, v \geq v_0, \quad (1)$$

where $j \geq 1$ and

$$w(v) := y(v) + p(v)y(\sigma(v)).$$

From now on we make the following assumptions:

- β and κ are quotient of odd positive integers;
- $r \in C[v_0, \infty)$, $r(v) > 0$, $r'(v) \geq 0$ and under the condition

$$\int_{v_0}^{\infty} r^{-1/\beta}(s) ds = \infty; \quad (2)$$

- $p, q_i \in C[v_0, \infty)$, $q_i(v) > 0$, $0 \leq p(v) < p_0 < 1$, $i = 1, 2, \dots, j$;
- $\sigma, g_i \in C[v_0, \infty)$, $\sigma(v) \leq v$, $g_i(v) \leq v$, $\lim_{v \rightarrow \infty} \sigma(v) = \lim_{v \rightarrow \infty} g_i(v) = \infty$, $i = 1, 2, \dots, j$.

Here we present some preliminary definitions that will clarify the terms used throughout the paper.

Definition 1. The function $y \in C^3[v_y, \infty)$, $v \geq v_y \geq v_0$, is said to be a solution of (1), if $r(v)(w'''(v))^\beta \in C^1[v_y, \infty)$, and $y(v)$ satisfies (1) on $[v_y, \infty)$.

Definition 2. A solution of (1) is said to be oscillatory if it has arbitrarily large zeros on $[v_y, \infty)$. Otherwise, a solution that is not oscillatory is said to be nonoscillatory.

Definition 3. The equation (1) is said to be oscillatory if every solution of it is oscillatory.

Definition 4. A differential equation is said to be neutral if the highest-order derivative of the unknown function appears both with and without delay.

Definition 5. Let

$$D = \{(v, s) \in \mathbb{R}^2 : v \geq s \geq v_0\} \text{ and } D_0 = \{(v, s) \in \mathbb{R}^2 : v > s \geq v_0\}.$$

A kernel function $H_i \in C(D, \mathbb{R})$ is said to belong to the function class \mathfrak{S} if, for $i = 1, 2$

- (i) $H_i(v, s) = 0$ for $v \geq v_0$, $H_i(v, s) > 0$, $(v, s) \in D_0$;
- (ii) $\partial H_i / \partial s$ exists on D_0 and it is continuous and non-positive. Moreover, there exist three functions $\xi, \pi \in C^1([v_0, \infty), (0, \infty))$ and $h_i \in C(D_0, \mathbb{R})$ such that

$$\frac{\partial}{\partial s} H_1(v, s) + \frac{\xi'(s)}{\xi(s)} H_1(v, s) = h_1(v, s) H_1^{\beta/(\beta+1)}(v, s) \tag{3}$$

and

$$\frac{\partial}{\partial s} H_2(v, s) + \frac{\pi'(s)}{\pi(s)} H_2(v, s) = h_2(v, s) \sqrt{H_2(v, s)}. \tag{4}$$

The historical background of neutral differential equations is extremely varied. In fact, they find numerous applications in natural science but also in technology: in the study of distributed networks containing lossless transmission lines, in high-speed computers, in the theory of automatic control and in aeromechanical systems (see [1]). In last years, the asymptotic properties of solutions of differential equations has been the subject of intensive study (see [1–34]). The model of human balancing is considered in [21–23] where results on stability are presented.

Furthermore, many researchers investigate regularity and existence properties of solutions to partial differential equations. See for instance [11,24–27] and the references therein. Also, we mention the study of the exact solutions to partial differential equations performed with a Lie symmetry analysis. A recent result in this direction is represented by [9] where the authors study a modified Schrödinger equation.

Interesting applications of neutral differential equations can be found in the study of the effects of vibrating systems fixed to an elastic bar, for example the Euler equations of the fluid dynamics (see the recent paper [10]).

In [28], the author obtained the necessary and sufficient conditions under which a general fourth-order ordinary differential equation admits a unique Lagrangian. Nevertheless, there exist examples of fourth-order ordinary differential equations which do not have a second-order Lagrangian.

Many papers have been concerned to the solution of the inverse problem of calculus of variations, namely finding a Lagrangian of differential equations. Also, the use of the Jacobi last multiplier and its connection with Lie theory, in order to find the Lagrangian for ordinary differential equations, can be found in [29].

Now we state some preliminary and interesting results related to the contents of this paper. Zafer [33], Zhang and Yan [35] studied the equation

$$w^{(n)}(v) + q(v) y(g(v)) = 0 \tag{5}$$

where n is even and established some new sufficient conditions for oscillation.

Theorem 1 ([33]). *Let $n \geq 2$ such that*

$$\limsup_{\nu \rightarrow \infty} \int_{g(\nu)}^{\nu} \theta(s) ds > (n - 1) 2^{(n-1)(n-2)},$$

or

$$\liminf_{\nu \rightarrow \infty} \int_{g(\nu)}^{\nu} \theta(s) ds > \frac{(n - 1) 2^{(n-1)(n-2)}}{e}, \tag{6}$$

where $\theta(\nu) := g^{n-1}(\nu) (1 - p(g(\nu))) q(\nu)$, then every solution of (5) is oscillatory.

Theorem 2 ([35]). *Let $0 \leq p(\nu) < p_0 < 1$ and $n \geq 2$ such that*

$$\liminf_{\nu \rightarrow \infty} \int_{g(\nu)}^{\nu} \theta(s) ds > \frac{(n - 1)!}{e}, \tag{7}$$

or

$$\limsup_{\nu \rightarrow \infty} \int_{g(\nu)}^{\nu} \theta(s) ds > (n - 1)!,$$

where $\theta(\nu) := g^{n-1}(\nu) (1 - p(g(\nu))) q(\nu)$, then every solution of (5) is oscillatory.

Now, we consider the equation

$$\left(y(\nu) + \frac{1}{2}y\left(\frac{1}{2}\nu\right) \right)^{(iv)} + \frac{q_0}{\nu^4}y\left(\frac{9}{10}\nu\right) = 0, \nu \geq 1. \tag{8}$$

By applying condition (6), we find

$$\liminf_{\nu \rightarrow \infty} \int_{\frac{9}{10}\nu}^{\nu} \left(\frac{9}{10}\right)^3 \frac{q_0}{2} \frac{1}{s} ds = \frac{q_0}{2} \left(\frac{9}{10}\right)^3 \ln(10/9) > \frac{(3) 2^6}{e}.$$

By applying condition (7), we find

$$\liminf_{\nu \rightarrow \infty} \int_{\frac{9}{10}\nu}^{\nu} \left(\frac{9}{10}\right)^3 \frac{q_0}{2} \frac{1}{s} ds = \frac{q_0}{2} \left(\frac{9}{10}\right)^3 \ln(10/9) > \frac{6}{e}.$$

Thus, we get that (8) is oscillatory if

The condition	(6)	(7)
The criterion	$q_0 > 1839.2$	$q_0 > 59.5$

From above, we see that [35] enriched the results in [33].

Thus, the motivation in studying this paper is to extend the already interesting and pioneer results in [33,35]. By using integral averaging technique, new oscillatory criteria for (1) are established. Furthermore, in order to illustrate the criteria presented here, an example is given.

The following lemmas will be very useful:

Lemma 1 ([3], Lemma 2.2.3). *Let $y \in C^n([v_0, \infty), (0, \infty))$. Assume that $y^{(n)}(\nu)$ is of fixed sign and not identically zero on $[v_0, \infty)$ and that there exists a $\nu_1 \geq v_0$ such that $y^{(n-1)}(\nu) y^{(n)}(\nu) \leq 0$ for all $\nu \geq \nu_1$. If $\lim_{\nu \rightarrow \infty} y(\nu) \neq 0$, then for every $\mu \in (0, 1)$ there exists $\nu_\mu \geq \nu_1$ such that*

$$y(\nu) \geq \frac{\mu}{(n - 1)!} \nu^{n-1} |y^{(n-1)}(\nu)| \text{ for } \nu \geq \nu_\mu.$$

Lemma 2 ([16], Lemma 1.2). *If the function y satisfies $y^{(i)}(v) > 0, i = 0, 1, \dots, n$, and $y^{(n+1)}(v) < 0$, then*

$$\frac{y(v)}{v^n/n!} \geq \frac{y'(v)}{v^{n-1}/(n-1)!}.$$

Lemma 3 ([12], Lemma 1.1). *Let β be a ratio of two odd numbers, $V > 0$ and U are constants. Then*

$$Uy - Vy^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^\beta}.$$

Lemma 4 ([31], Lemma 1.2). *Assume that y is an eventually positive solution of (1). Then, there exist two possible cases:*

Case (N₁): $w'(v) > 0, w''(v) > 0, w'''(v) > 0, w^{(4)}(v) \leq 0, (r(v)(w'''(v))^\beta)' \leq 0,$

Case (N₂): $w'(v) > 0, w''(v) < 0, w'''(v) > 0, w^{(4)}(v) \leq 0, (r(v)(w'''(v))^\beta)' \leq 0,$
for $v \geq v_1$, where $v_1 \geq v_0$ is sufficiently large.

2. Oscillation Criteria

For convenience, we denote

$$S_1(v) = \zeta(v) \sum_{i=1}^j q_i(v) (1-p_0)^\kappa A_1^{\kappa-\beta} \left(\frac{g_i(v)}{v}\right)^{3\kappa},$$

$$\phi(v) = (1-p_0)^{\kappa/\beta} \pi(v) A_2^{\kappa/\beta-1}(v) \int_v^\infty \left(\frac{1}{r(u)} \int_u^\infty \sum_{i=1}^j q_i(s) \frac{g_i^\kappa(s)}{s^\kappa} ds\right)^{1/\beta} du$$

and

$$S_2(v) = \beta\mu_1 \frac{v^2}{2r^{1/\beta}(v)\zeta^{1/\beta}(v)}.$$

Lemma 5. *Let y is an eventually positive solution of (1). Then*

$$(r(v)(w'''(v))^\beta)' \leq -g_i(v)(w'''(g_i(v)))^\kappa, \tag{9}$$

where

$$g_i(v) = \sum_{i=1}^j q_i(v) (1-p_0)^\kappa \left(\frac{\mu}{6} g_i^3(v)\right)^\kappa.$$

Proof. Let y be an eventually positive solution of (1) on $[v_0, \infty)$. Since $\sigma(v) \leq v, w'(v) > 0$ and from the definition of w , we get

$$\begin{aligned} y(v) &= w(v) - p(v)y(\sigma(v)) \\ &\geq w(v) - p_0y(\sigma(v)) \\ &\geq w(v) - p_0w(\sigma(v)) \\ &\geq (1-p_0)w(v), \end{aligned}$$

which with (1) gives

$$(r(v)(w'''(v))^\beta)' + \sum_{i=1}^j q_i(v) (1-p_0)^\kappa w^\kappa(g_i(v)) \leq 0. \tag{10}$$

Using Lemma 1, we see that

$$w(v) \geq \frac{\mu}{6} v^3 w'''(v). \tag{11}$$

Combining (10) and (11), we find

$$\left(r(v) (w'''(v))^\beta \right)' + \sum_{i=1}^j q_i(v) (1 - p_0)^\kappa \left(\frac{\mu}{6} g_i^3(v) \right)^\kappa (w'''(g_i(v)))^\kappa \leq 0.$$

Thus, (9) holds. This completes the proof. \square

Lemma 6. Assume that y is an eventually positive solution of (1) and

$$\psi'(v) \leq \frac{\xi'(v)}{\xi(v)} \psi(v) - S_1(v) - \beta \mu_1 \frac{v^2}{2r^{1/\beta}(v) \xi^{1/\beta}(v)} \psi^{\frac{\beta+1}{\beta}}(v), \text{ if } w \text{ satisfies } (\mathbf{N}_1) \tag{12}$$

and

$$\omega'(v) \leq -\phi(v) + \frac{\pi'(v)}{\pi(v)} \omega(v) - \frac{1}{\pi(v)} \omega^2(v), \text{ if } w \text{ satisfies } (\mathbf{N}_2), \tag{13}$$

where

$$\psi(v) := \xi(v) \frac{r(v) (w'''(v))^\beta}{w^\beta(v)} \tag{14}$$

and

$$\omega(v) := \pi(v) \frac{w'(v)}{w(v)}, v \geq v_1. \tag{15}$$

Proof. Let y be an eventually positive solution of (1) on $[v_0, \infty)$. It follows from Lemma 4 that there exist two possible cases (\mathbf{N}_1) and (\mathbf{N}_2) .

Let Case (\mathbf{N}_1) holds. Using the definition of $\psi(v)$, we see that $\psi(v) > 0$ for $v \geq v_1$, and using (10), we obtain

$$\psi'(v) \leq \frac{\xi'(v)}{\xi(v)} \psi(v) - \xi(v) \sum_{i=1}^j q_i(v) (1 - p_0)^\kappa \frac{w^\kappa(g_i(v))}{w^\beta(v)} - \beta \xi(v) \frac{r(v) (w'''(v))^\beta}{w^{\beta+1}(v)} w'(v). \tag{16}$$

From Lemma 2, we have that $w(v) \geq \frac{v}{3} w'(v)$, and hence,

$$\frac{w(g_i(v))}{w(v)} \geq \frac{g_i^3(v)}{v^3}. \tag{17}$$

It follows from Lemma 1 that

$$w'(v) \geq \frac{\mu_1}{2} v^2 w'''(v), \tag{18}$$

for all $\mu_1 \in (0, 1)$ and every sufficiently large v . Thus, by (16)–(18), we get

$$\begin{aligned} \psi'(v) \leq & \frac{\xi'(v)}{\xi(v)} \psi(v) - \xi(v) \sum_{i=1}^j q_i(v) (1 - p_0)^\kappa w^{\kappa-\beta}(v) \left(\frac{g_i(v)}{v} \right)^{3\kappa} \\ & - \beta \mu_1 \frac{v^2}{2r^{1/\beta}(v) \xi^{1/\beta}(v)} \psi^{\frac{\beta+1}{\beta}}(v). \end{aligned}$$

Since $w'(v) > 0$, there exist a $v_2 \geq v_1$ and a constant $A_1 > 0$ such that

$$w(v) > A_1. \tag{19}$$

Thus, we obtain

$$\begin{aligned} \psi'(v) \leq & \frac{\xi'(v)}{\xi(v)} \psi(v) - \xi(v) \sum_{i=1}^j q_i(v) (1 - p_0)^\kappa A^{\kappa-\beta} \left(\frac{g_i(v)}{v} \right)^{3\kappa} \\ & - \beta \mu_1 \frac{v^2}{2r^{1/\beta}(v) \xi^{1/\beta}(v)} \psi^{\frac{\beta+1}{\beta}}(v), \end{aligned}$$

which yields

$$\psi'(v) \leq \frac{\xi'(v)}{\xi(v)} \psi(v) - S_1(v) - \beta \mu_1 \frac{v^2}{2r^{1/\beta}(v) \xi^{1/\beta}(v)} \psi^{\frac{\beta+1}{\beta}}(v).$$

Thus, (12) holds.

Assume that Case (N₂) holds. Integrating (10) from v to u , we obtain

$$r(u) (w'''(u))^\beta - r(v) (w'''(v))^\beta \leq - \int_v^u \sum_{i=1}^j q_i(s) (1 - p_0)^\kappa w^\kappa(g_i(s)) ds. \tag{20}$$

From Lemma 2, we get that $w(v) \geq vw'(v)$, and hence,

$$w(g_i(v)) \geq \frac{g_i(v)}{v} w(v). \tag{21}$$

For (20), letting $u \rightarrow \infty$ and using (21), we get

$$r(v) (w'''(v))^\beta \geq (1 - p_0)^\kappa w^\kappa(v) \int_v^\infty \sum_{i=1}^j q_i(s) \frac{g_i^\kappa(s)}{s^\kappa} ds.$$

Integrating this inequality again from v to ∞ , we get

$$w''(v) \leq - (1 - p_0)^{\kappa/\beta} w^{\kappa/\beta}(v) \int_v^\infty \left(\frac{1}{r(u)} \int_u^\infty \sum_{i=1}^j q_i(s) \frac{g_i^\kappa(s)}{s^\kappa} ds \right)^{1/\beta} du. \tag{22}$$

From the definition of $\omega(v)$, we see that $\omega(v) > 0$ for $v \geq v_1$, and using (19) and (22), we find

$$\begin{aligned} \omega'(v) &= \frac{\pi'(v)}{\pi(v)} \omega(v) + \pi(v) \frac{w''(v)}{w(v)} - \pi(v) \left(\frac{w'(v)}{w(v)} \right)^2 \\ &\leq \frac{\pi'(v)}{\pi(v)} \omega(v) - \frac{1}{\pi(v)} \omega^2(v) \\ &\quad - (1 - p_0)^{\kappa/\beta} \pi(v) w^{\kappa/\beta-1}(v) \int_v^\infty \left(\frac{1}{r(u)} \int_u^\infty \sum_{i=1}^j q_i(s) \frac{g_i^\kappa(s)}{s^\kappa} ds \right)^{1/\beta} du. \end{aligned}$$

Since $w'(v) > 0$, there exist a $v_2 \geq v_1$ and a constant $A_2 > 0$ such that

$$w(v) > A_2. \tag{23}$$

Thus, we obtain

$$\omega'(v) \leq -\phi(v) + \frac{\pi'(v)}{\pi(v)} \omega(v) - \frac{1}{\pi(v)} \omega^2(v).$$

Thus, (13) holds. This completes the proof. \square

Theorem 3. Let (28) holds. If there exist positive functions $\zeta, \pi \in C^1([v_0, \infty), \mathbb{R})$ such that

$$\limsup_{v \rightarrow \infty} \frac{1}{H_1(v, v_1)} \int_{v_1}^v \left(H_1(v, s) S_1(s) - \frac{h_1^{\beta+1}(v, s) H_1^\beta(v, s) 2^\beta r(s) \zeta(s)}{(\beta + 1)^{\beta+1} (\mu_1 s^2)^\beta} \right) ds = \infty \tag{24}$$

for all $\mu_2 \in (0, 1)$, and

$$\limsup_{v \rightarrow \infty} \frac{1}{H_2(v, v_1)} \int_{v_1}^v \left(H_2(v, s) \phi(s) - \frac{\pi(s) h_2^2(v, s)}{4} \right) ds = \infty, \tag{25}$$

then (1) is oscillatory.

Proof. Let y be a non-oscillatory solution of (1) on $[v_0, \infty)$. Without loss of generality, we can assume that y is eventually positive. It follows from Lemma 4 that there exist two possible cases (N_1) and (N_2) .

Assume that (N_1) holds. From Lemma 6, we get that (12) holds. Multiplying (12) by $H(v, s)$ and integrating the resulting inequality from v_1 to v , we find that

$$\begin{aligned} \int_{v_1}^v H_1(v, s) S_1(s) ds &\leq \psi(v_1) H_1(v, v_1) + \int_{v_1}^v \left(\frac{\partial}{\partial s} H_1(v, s) + \frac{\zeta'(s)}{\zeta(s)} H_1(v, s) \right) \psi(s) ds \\ &\quad - \int_{v_1}^v S_2(s) H_1(v, s) \psi^{\frac{\beta+1}{\beta}}(s) ds. \end{aligned}$$

From (3), we get

$$\begin{aligned} \int_{v_1}^v H_1(v, s) S_1(s) ds &\leq \psi(v_1) H_1(v, v_1) + \int_{v_1}^v h_1(v, s) H_1^{\beta/(\beta+1)}(v, s) \psi(s) ds \\ &\quad - \int_{v_1}^v S_2(s) H_1(v, s) \psi^{\frac{\beta+1}{\beta}}(s) ds. \end{aligned} \tag{26}$$

Using Lemma 3 with $V = S_2(s) H_1(v, s)$, $U = h_1(v, s) H_1^{\beta/(\beta+1)}(v, s)$ and $y = \psi(s)$, we get

$$\begin{aligned} &h_1(v, s) H_1^{\beta/(\beta+1)}(v, s) \psi(s) - S_2(s) H_1(v, s) \psi^{\frac{\beta+1}{\beta}}(s) \\ &\leq \frac{h_1^{\beta+1}(v, s) H_1^\beta(v, s) 2^\beta r(s) \zeta(s)}{(\beta + 1)^{\beta+1} (\mu_1 v^2)^\beta}, \end{aligned}$$

which with (26) gives

$$\frac{1}{H_1(v, v_1)} \int_{v_1}^v \left(H_1(v, s) S_1(s) - \frac{h_1^{\beta+1}(v, s) H_1^\beta(v, s) 2^\beta r(s) \zeta(s)}{(\beta + 1)^{\beta+1} (\mu_1 s^2)^\beta} \right) ds \leq \psi(v_1),$$

which contradicts (24).

Assume that (N_2) holds. From Lemma 6, we get that (13) holds. Multiplying (13) by $H_2(v, s)$ and integrating the resulting inequality from v_1 to v , we obtain

$$\begin{aligned} \int_{v_1}^v H_2(v, s) \phi(s) ds &\leq \omega(v_1) H_2(v, v_1) \\ &\quad + \int_{v_1}^v \left(\frac{\partial}{\partial s} H_2(v, s) + \frac{\pi'(s)}{\pi(s)} H_2(v, s) \right) \omega(s) ds \\ &\quad - \int_{v_1}^v \frac{1}{\pi(s)} H_2(v, s) \omega^2(s) ds. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{v_1}^v H_2(v, s) \phi(s) \, ds &\leq \omega(v_1) H_2(v, v_1) + \int_{v_1}^v h_2(v, s) \sqrt{H_2(v, s)} \omega(s) \, ds \\ &\quad - \int_{v_1}^v \frac{1}{\pi(s)} H_2(v, s) \omega^2(s) \, ds \\ &\leq \omega(v_1) H_2(v, v_1) + \int_{v_1}^v \frac{\pi(s) h_2^2(v, s)}{4} \, ds \end{aligned}$$

and so

$$\frac{1}{H_2(v, v_1)} \int_{v_1}^v \left(H_2(v, s) \phi(s) - \frac{\pi(s) h_2^2(v, s)}{4} \right) \, ds \leq \omega(v_1),$$

which contradicts (25). This completes the proof. \square

Corollary 1. Assume that (28) holds. If there exist positive functions $\zeta, \pi \in C^1([v_0, \infty), \mathbb{R})$ such that

$$\int_{v_0}^{\infty} \left(S_1(s) - \frac{2^\beta}{(\beta + 1)^{\beta+1}} \frac{r(s) (\zeta'(s))^{\beta+1}}{\mu_1^\beta s^{2\beta} \zeta^\beta(s)} \right) \, ds = \infty \tag{27}$$

and

$$\int_{v_0}^{\infty} \left(\phi(s) - \frac{(\pi'(s))^2}{4\pi(s)} \right) \, ds = \infty, \tag{28}$$

for some $\mu_1 \in (0, 1)$, then (1) is oscillatory.

Example 1. Consider the equation

$$(v(y + p_0 y(av))''')' + \frac{q_0}{v^4} y(\epsilon v) = 0, \quad v \geq 1, \tag{29}$$

where $p_0 \in [0, 1)$, $a, \epsilon \in (0, 1)$ and $q_0 > 0$. We note that $\beta = \kappa = 1$, $r(v) = v$, $p(v) = p_0$, $\sigma(v) = av$, $g_i(v) = \epsilon v$ and $q(v) = q_0/v^3$. Hence, if we set $\zeta(s) := v^3$ and $\pi(v) := v^2$, then we have

$$S_1(v) = \frac{q_0(1-p_0)\epsilon^3}{v}, \quad \phi(v) = \frac{q_0(1-p_0)\epsilon}{4v}.$$

Thus, (27) and (28) become

$$\int_{v_0}^{\infty} \left(S_1(s) - \frac{2^\beta}{(\beta + 1)^{\beta+1}} \frac{r(s) (\zeta'(s))^{\beta+1}}{\mu_1^\beta s^{2\beta} \zeta^\beta(s)} \right) \, ds = \int_{v_0}^{\infty} \left(\frac{q_0(1-p_0)\epsilon^3}{s} - \frac{2}{\mu_1 s} \right) \, ds$$

and

$$\int_{v_0}^{\infty} \left(\phi(s) - \frac{(\pi'(s))^2}{4\pi(s)} \right) \, ds = \int_{v_0}^{\infty} \left(\frac{q_0(1-p_0)\epsilon}{4s} - \frac{1}{4s} \right) \, ds.$$

Therefore, the conditions become

$$q_0 > \frac{2}{(1-p_0)\epsilon^3} \tag{30}$$

and

$$q_0 > \frac{1}{(1-p_0)\epsilon}.$$

Thus, by using Corollary 1, Equation (29) is oscillatory if (30) holds.

Remark 1. When taking $p_0 = \epsilon = 1/2$ and $a = 1/3$, then condition (30) will become $q_0 > 32$. Now, we compare our result with the earlier ones. By applying condition (6) in [33], we get

$$\liminf_{v \rightarrow \infty} \int_{\frac{1}{2}v}^v \frac{q_0}{2^4} \frac{1}{s} ds = \frac{q_0}{2^4} \ln 2 > \frac{(3)^{2^6}}{e},$$

and we find

$$q_0 > 1630.4$$

By applying condition (7) in [35], we get

$$\liminf_{v \rightarrow \infty} \int_{\frac{1}{2}v}^v \frac{q_0}{2^4} \frac{1}{s} ds = \frac{q_0}{2^4} \ln 2 > \frac{6}{e},$$

and we find

$$q_0 > 51$$

Therefore, our result improves the results contained in [33,35].

Remark 2. By applying condition (30) in Equation (8), we find

$$\begin{aligned} & \int_{v_0}^{\infty} \left(S_1(s) - \frac{2^\beta}{(\beta + 1)^{\beta+1}} \frac{r(s) (\zeta'(s))^{\beta+1}}{\mu_1^\beta s^{2\beta} \zeta^\beta(s)} \right) ds \\ &= \int_{v_0}^{\infty} \left(\left(\frac{9}{10} \right)^3 \frac{q_0}{2s} - \frac{9}{2\mu_1 s} \right) ds \\ &= \left(\left(\frac{9}{10} \right)^3 \frac{q_0}{2} - \frac{9}{2\mu_1} \right) \int_{v_0}^{\infty} \frac{1}{s} ds \end{aligned}$$

and we get

$$q_0 > 12.34.$$

Therefore, our result improves the results contained in [33,35].

Example 2. Consider the equation

$$\left(y(v) + \frac{1}{3}y\left(\frac{1}{2}v\right) \right)^{(iv)} + \frac{q_0}{v^4}y\left(\frac{1}{3}v\right) = 0, \quad v \geq 1, \tag{31}$$

where $q_0 > 0$. Let $\beta = \kappa = 1$, $r(v) = 1$, $p(v) = 1/3$, $\sigma(v) = v/2$, $g(v) = v/3$ and $q(v) = q_0/v^4$. Hence, if we set $\xi(s) := v^3$ and $\pi(v) := v$, then we have

$$S_1(v) = \frac{2q_0}{3^4v}, \quad \phi(v) = \frac{q_0}{3^3v}.$$

Thus, (27) and (28) become

$$\int_{v_0}^{\infty} 2 \left(S_1(s) - \frac{2^\beta}{(\beta + 1)^{\beta+1}} \frac{r(s) (\zeta'(s))^{\beta+1}}{\mu_1^\beta s^{2\beta} \zeta^\beta(s)} \right) ds = \int_{v_0}^{\infty} \left(\frac{2q_0}{3^4s} - \frac{3^2}{2\mu_1 s} \right) ds$$

and

$$\int_{v_0}^{\infty} \left(\phi(s) - \frac{(\pi'(s))^2}{4\pi(s)} \right) ds = \int_{v_0}^{\infty} \left(\frac{q_0}{3^3s} - \frac{1}{4s} \right) ds.$$

Therefore, the conditions become

$$q_0 > 182.25$$

and

$$q_0 > 6.75.$$

Thus, by using Corollary 1, Equation (31) is oscillatory if $q_0 > 182.25$.

3. Conclusions

This paper deals with a class of fourth-order neutral differential equations with variable coefficients. Using the famous Riccati's transformation, we establish a new asymptotic criterion that improves and complements the findings contained in [33,35]. Moreover, we get Philos type oscillation criteria to ensure oscillation of solutions of the Equation (1). Furthermore, in a future work we will get some oscillation criteria for (1) under the condition

$$\int_{v_0}^{\infty} \frac{1}{r^{1/\beta}(s)} ds < \infty.$$

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