Miscellaneous Applications of Certain Minimax Theorems II

Biagio Ricceri

Acta Mathematica Vietnamica

ISSN 0251-4184
Volume 45
Number 2

Acta Math Vietnam (2020) 45:515-524
DOI 10.1007/s40306-020-00369-z

Acta Mathematica Vietnamica (2020) 45: 515–524
https://doi.org/10.1007/s40306-020-00369-z

Miscellaneous Applications of Certain Minimax Theorems II

Biagio Ricceri[1]

Dedicated to Professor Hoang Tuy with my greatest esteem

Received: 19 October 2018 / Revised: 18 June 2019 / Accepted: 25 June 2019 /
Published online: 6 June 2020

Abstract
In this paper, we present new applications of our general minimax theorems. In particular, one of them concerns the multiplicity of global minima for the integral functional of the Calculus of Variations.

Keywords Minimax theorem · Connectedness · Global minimum · Multiplicity · Integral functional · Neumann problem

Mathematics Subject Classification (2010) 49J35 · 49K35 · 49J45 · 49K27 · 35J92 · 90C47

1 Statements of the Main Results

This is the second ring of a chain of papers (started with [14]) which is devoted to consequences and applications of certain general minimax theorems that we have established in the past years [5–15].

The motivation for such papers is just to show the great flexibility and usefulness of those theorems.

The two main results that we want to prove in the present paper are Theorems 1.1 and 1.2 below.

A real-valued function f on a topological space is said to be inf-connected (resp. sup-connected) if $f^{-1}(]-\infty, r[)$ (resp. $f^{-1}(]r, +\infty[)$) is connected for all $r \in \mathbf{R}$.

Theorem 1.1 *Let X and Y be two real Banach spaces; $\Phi : X \to Y$ a surjective continuous linear operator; $\Psi : X \to Y$ a non-constant Lipschitzian operator with Lipschitz constant*

✉ Biagio Ricceri
ricceri@dmi.unict.it

[1] Department of Mathematics and Informatics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy

equal to L; $\varphi : Y \to \mathbf{R}$ a non-constant, continuous, concave and inf-connected functional; $[a, b]$ a closed sub-interval of $[-1, 1]$.

Then, for every continuous and concave function $\gamma : [a, b] \to \mathbf{R}$, one has

$$\max \left\{ \inf_{x \in X} \varphi \left(\Phi(x) + \frac{a}{\alpha_\Phi L} \Psi(x) \right) + \gamma(a), \inf_{x \in X} \varphi \left(\Phi(x) + \frac{b}{\alpha_\Phi L} \Psi(x) \right) + \gamma(b) \right\}$$
$$= \inf_{x \in X} \sup_{\lambda \in [a,b]} \left(\varphi \left(\Phi(x) + \frac{\lambda}{\alpha_\Phi L} \Psi(x) \right) + \gamma(\lambda) \right),$$

where

$$\alpha_\Phi = \sup_{\|y\|_Y \leq 1} \mathrm{dist}(0, \Phi^{-1}(y)).$$

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary and let $p > 1$. On the Sobolev space $W^{1,p}(\Omega)$, we consider the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

If $n \geq p$, we denote by $\mathcal{A}$ the class of all continuous functions $\psi : \mathbf{R} \to \mathbf{R}$ such that

$$\sup_{\xi \in \mathbf{R}} \frac{|\psi(\xi)|}{1 + |\xi|^q} < +\infty,$$

where $0 < q < \frac{pn}{n-p}$ if $p < n$ and $0 < q < +\infty$ if $p = n$. While, when $n < p$, $\mathcal{A}$ stands for the class of all continuous functions $\psi : \mathbf{R} \to \mathbf{R}$.

Recall that a function $\varphi : \Omega \times \mathbf{R}^m \to \mathbf{R}$ is said to be a normal integrand [16] if it is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$-measurable and $\varphi(x, \cdot)$ is lower semicontinuous for a.e. $x \in \Omega$. Here $\mathcal{L}(\Omega)$ and $\mathcal{B}(\mathbf{R}^m)$ denote the Lebesgue and the Borel σ-algebras of subsets of Ω and $\mathbf{R}^m$, respectively.

Recall that if φ is a normal integrand then, for each measurable function $u : \Omega \to \mathbf{R}^m$, the composite function $x \to \varphi(x, u(x))$ is measurable [16].

A real-valued function f on a convex set is said to be quasi-convex (resp. quasi-concave) if $f^{-1}(]-\infty, r[)$ (resp. $f^{-1}(]r, +\infty[)$) is convex for all $r \in \mathbf{R}$.

Theorem 1.2 *Let $\varphi : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ be a normal integrand such that $\varphi(x, \xi, \cdot)$ is convex for all $(x, \xi) \in \Omega \times \mathbf{R}$ and let $\psi \in \mathcal{A}$ be a strictly monotone function. Assume that*
(i) *There are $c, d > 0$ such that*

$$c|\eta|^p - d \leq \varphi(x, \xi, \eta)$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$ and

$$\lim_{|\xi| \to +\infty} \frac{\inf_{(x,\eta) \in \Omega \times \mathbf{R}^n} \varphi(x, \xi, \eta)}{|\psi(\xi)| + 1} = +\infty;$$

(ii) *For each $\xi \in \mathbf{R}$, the function $\varphi(\cdot, \xi, 0)$ lies in $L^1(\Omega)$ and the function $\int_\Omega \varphi(x, \cdot, 0) dx$ is not quasi-convex.*

Then, for every sequentially weakly closed set $V \subseteq W^{1,p}(\Omega)$, containing the constants, and for every convex set $Y \subseteq L^\infty(\Omega)$, dense in $L^\infty(\Omega)$, there exists $\alpha \in Y$ such that the restriction to V of the functional

$$u \to \int_\Omega \varphi(x, u(x), \nabla u(x)) dx + \int_\Omega \alpha(x) \psi(u(x)) dx$$

has at least two global minima. The same property holds also with $Y = C_0^\infty(\Omega)$.

2 Tools for Proving the Main Results

In this section, for reader's convenience, we collect the tools that we will use to prove Theorems 1.1 and 1.2. For the basic notions on multifunctions, we refer to [1]. The results without any reference are new.

Theorem 2.A [10, Theorem 5.7] *Let* X *be a topological space,* I *a compact real interval, and* $f : X \times I \to \mathbf{R}$ *a function which is lower semicontinuous in* X*, and quasi-concave and upper semicontinous in* I*. Moreover, assume that the set*

$$\{\lambda \in I \ : \ f(\cdot, \lambda) \text{ is inf-connected in } X\}$$

is dense in I.

Then, one has

$$\sup_I \inf_X f = \inf_X \sup_I f.$$

A real-valued function f on a topological space is said to be inf-compact if $f^{-1}(]-\infty, r])$ is compact for all $r \in \mathbf{R}$.

Theorem 2.B [15, Theorem 1.2] *Let* X *be a topological space,* E *a real vector space,* $Y \subseteq E$ *a non-empty convex set, and* $f : X \times Y \to \mathbf{R}$ *a function which is lower semicontinuous and inf-compact in* X*, and concave in* Y*. Moreover, assume that*

$$\sup_Y \inf_X f < \inf_X \sup_Y f.$$

Then, there exists $\hat{y} \in Y$ *such that the function* $f(\cdot, \hat{y})$ *has at least two global minima.*

Proposition 2.A [10, Proposition 5.6] *Let* X *and* Y *be two topological spaces,* $F : X \to 2^Y$ *a lower semicontinuous multifunction with non-empty values, and* $A \subseteq X$ *a connected set. Assume that the set*

$$\{x \in A \ : \ F(x) \text{ is connected}\}$$

is dense in A.

Then, the set $F(A)$ *is connected.*

Theorem 2.C [4, Théorème 2] *Let* X *and* Y *be two real Banach spaces,* $\Phi : X \to Y$ *a surjective continuous linear operator, and* $\Psi : X \to Y$ *a non-constant Lipschitzian operator with Lipschitz constant equal to* L*. Assume that* $\alpha_\Phi L < 1$.

Then, the multifunction $y \to (\Phi + \Psi)^{-1}(y)$ *is Lipschitzian in* Y *and its values are absolute retracts.*

We now are in a position to prove the following result from which we will draw Theorem 1.1:

Theorem 2.1 *Let* X *be a topological space;* E *a real topological vector space;* Y *a convex subset of* E*;* $\varphi : Y \to \mathbf{R}$ *a continuous, concave, and inf-connected functional;* I *a compact real interval; and* $f, g : X \to E$ *two continuous functions such that* $f(x) + \lambda g(x) \in Y$ *for all* $x \in X$*,* $\lambda \in I$*. Moreover, assume that there exists a set* $D \subseteq I$*, dense in* I*, with the* *following property: for each* $\lambda \in D$*, the function* $f + \lambda g$ *is onto* Y *and open with respect to the relative topology of* Y*, and there exists a set* $S_\lambda \subseteq Y$*, dense in* Y*, such that the set* $(f + \lambda g)^{-1}(y)$ *is connected for each* $y \in S_\lambda$*.*

Then, for every continuous and concave function $\gamma : I \to \mathbf{R}$*, one has*

$$\inf_{x\in X} \sup_{\lambda\in I}(\varphi(f(x) + \lambda g(x)) + \gamma(\lambda)) = \sup_{\lambda\in I} \inf_{x\in X}(\varphi(f(x) + \lambda g(x)) + \gamma(\lambda)).$$

Proof Consider the function $\psi : X \times I \to \mathbf{R}$ defined by

$$\psi(x, \lambda) = \varphi(f(x) + \lambda g(x)) + \gamma(\lambda)$$

for all $(x, \lambda) \in X \times I$. Clearly, for each $x \in X$, the function $\psi(x, \cdot)$ is concave and continuous in I. Now, fix $\lambda \in D$. Let $r \in \mathbf{R}$ be such that $\{x \in X : \psi(x, \lambda) < r\} \neq \emptyset$. Clearly, we have

$$\{x \in X \, : \, \psi(x, \lambda) < r\} = (f + \lambda g)^{-1}(\varphi^{-1}(] - \infty, r - \gamma(\lambda)[)).$$

Now, observe that $\varphi^{-1}(] - \infty, r - \gamma(\lambda)[)$ is open in Y and connected since φ is continuous and inf-connected. But, since $\lambda \in D$, the multifunction $y \to (f + \lambda g)^{-1}(y)$ is non-empty valued and lower semicontinuous in Y. Since $S_\lambda \cap \varphi^{-1}(] - \infty, r - \gamma(\lambda)[)$ is dense in $\varphi^{-1}(] - \infty, r - \gamma(\lambda)[)$, thanks to Proposition 2.A, we conclude that the set $(f + \lambda g)^{-1}(\varphi^{-1}(] - \infty, r - \gamma(\lambda)[))$ is connected. Clearly, $\psi(\cdot, \lambda)$ is continuous in X for all $\lambda \in I$. Now, the conclusion follows directly from Theorem 2.A. □

The following two results will be used jointly with Theorem 2.B to prove Theorem 1.2.

Proposition 2.1 *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded domain with smooth boundary, let* $p > 1$*, and let* $\varphi : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ *be a normal integrand such that, for some* $c, d > 0$*, one has*

$$c|\eta|^p - d \leq \varphi(x, \xi, \eta)$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$ *and*

$$\lim_{|\xi|\to+\infty} \inf_{(x,\eta)\in\Omega\times\mathbf{R}^n} \varphi(x, \xi, \eta) = +\infty.$$

Then, in $W^{1,p}(\Omega)$*, one has*

$$\lim_{\|u\|\to+\infty} \int_\Omega \varphi(x, u(x), \nabla u(x))dx = +\infty.$$

Proof Clearly, for each $u \in W^{1,p}(\Omega)$, the integral $\int_\Omega \varphi(x, u(x), \nabla u(x))dx$ exists and belongs to $] - \infty, +\infty]$. Fix a sequence $\{u_n\}$ in $W^{1,p}(\Omega)$ such that $\lim_{n\to\infty} \|u_n\| = +\infty$. We have to show, up to a sub-sequence, that

$$\lim_{n\to\infty} \int_\Omega \varphi(x, u_n(x), \nabla u_n(x))dx = +\infty.$$

If the sequence $\left\{\int_\Omega |\nabla u_n(x)|^p dx\right\}$ is unbounded, this clearly holds, due to the assumed growth of φ. So, assume that the sequence $\left\{\int_\Omega |\nabla u_n(x)|^p dx\right\}$ is bounded. Then, by the Poincaré-Wirtinger inequality, there exists a constant $\gamma > 0$ such that

$$\int_\Omega |u_n(x) - a_n|^p dx \leq \gamma$$

for all $n \in \mathbf{N}$, where

$$a_n = \frac{\int_\Omega u_n(x)dx}{m(\Omega)},$$

$m(\Omega)$ being the Lebesgue measure of Ω. Since

$$\lim_{n\to\infty}\int_\Omega |u_n(x)|^p dx = +\infty$$

we clearly have

$$\lim_{n\to\infty} |a_n| = +\infty.$$

Fix any $M > 0$. Then, there exists $\delta > 0$ such that

$$\varphi(x, \xi, \eta) \geq M$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$ with $|\xi| \geq \delta$. We now show that

$$\lim_{n\to\infty} m(A_n) = m(\Omega),$$

where

$$A_n = \{x \in \Omega : |u_n(x)| \geq \delta\}.$$

Arguing by contradiction, assume that

$$\liminf_{n\to\infty} m(A_n) < m(\Omega).$$

Fix ρ satisfying

$$\liminf_{n\to\infty} m(A_n) < \rho < m(\Omega).$$

Now, fix $\theta > \gamma$ and $n \in \mathbf{N}$ so that, at the same time, one has

$$|a_n| > \left(\frac{\theta}{m(\Omega) - \rho}\right)^{\frac{1}{p}} + \delta$$

as well as

$$m(A_n) < \rho.$$

Then, one has

$$\gamma \geq \int_\Omega |u_n(x) - a_n|^p dx \geq \int_{\Omega\setminus A_n} |u_n(x) - a_n|^p dx > (|a_n| - \delta)^p (m(\Omega) - \rho) > \theta,$$

an absurd. Now, for each $n \in \mathbf{N}$, we have

$$\int_\Omega \varphi(x, u_n(x), \nabla u_n(x))dx = \int_{A_n} \varphi(x, u_n(x), \nabla u_n(x))dx + \int_{\Omega\setminus A_n} \varphi(x, u_n(x), \nabla u_n(x))dx$$
$$\geq Mm(A_n) - m(\Omega \setminus A_n)d.$$

Therefore,

$$\liminf_{n\to\infty} \int_\Omega \varphi(x, u_n(x), \nabla u_n(x))dx \geq Mm(\Omega).$$

Since M is arbitrary, the sequence $\{\int_\Omega \varphi(x, u_n(x), \nabla u_n(x))dx\}$ diverges and the proof is complete. □

Proposition 2.2 *Let X and Y be two non-empty sets and $f : X \to \mathbf{R}$ and $g : X \times Y \to \mathbf{R}$ two given functions. Assume that there are two sets $A, B \subset X$ such that*
(a) $\sup_A f < \inf_B f$;
(b) $\sup_{y\in Y} \inf_{x\in A} g(x, y) \leq 0$;
(c) $\inf_{x\in B} \sup_{y\in Y} g(x, y) \geq 0$;
(d) $\inf_{x\in X\setminus B} \sup_{y\in Y} g(x, y) = +\infty$.

Then, one has

$$\sup_{y\in Y} \inf_{x\in X} (f(x) + g(x, y)) < \inf_{x\in X} \sup_{y\in Y} (f(x) + g(x, y)).$$

Proof Fix $y \in Y$ and $\epsilon \in]0, \inf_B f - \sup_A f[$ as well. Since $\inf_{x\in A} g(x, y) \leq 0$, there is $\tilde{x} \in A$ such that $g(\tilde{x}, y) < \epsilon$. Hence, we have

$$\inf_{x\in X}(f(x) + g(x, y)) \leq f(\tilde{x}) + g(\tilde{x}, y) < \sup_A f + \epsilon,$$

from which it follows that

$$\sup_{y\in Y}\inf_{x\in X}(f(x) + g(x, y)) \leq \sup_A f + \epsilon < \inf_B f. \tag{2.1}$$

On the other hand, in view of (c) and (d), we have

$$\inf_B f \leq \inf_{x\in B}(f(x) + \sup_{y\in Y} g(x, y)) = \inf_{x\in B}\sup_{y\in Y}(f(x) + g(x, y)) = \inf_{x\in X}\sup_{y\in Y}(f(x) + g(x, y)). \tag{2.2}$$

Now, the conclusion follows directly from (2.1) and (2.2). □

We also recall the following well-known fact:

Proposition 2.3 *Let $A \subseteq \mathbf{R}^n$ be any non-empty open set and let $v \in L^1(\Omega) \setminus \{0\}$.*

Then, one has

$$\sup_{\alpha\in C_0^\infty(A)}\int_A \alpha(x)v(x)dx = +\infty.$$

3 Proof and Corollary of Theorem 1.1

Let $\lambda \in]a, b[$. The Lipschitz constant of the operator $\frac{\lambda}{\alpha_\Phi L}\Psi$ is equal to $\frac{|\lambda|}{\alpha_\Phi}$ and so it is strictly less than $\frac{1}{\alpha_\Phi}$. Then, by Theorem 2.C, the multifunction $y \to (\Phi + \frac{\lambda}{\alpha_\Phi L}\Psi)^{-1}(y)$ is lower semicontinuous (since it is Lipschitzian) and its values are non-empty and connected (since they are absolute retracts). So, the operator $\Phi + \lambda\Psi$ is onto Y and open. Consequently, the operators Φ and Ψ satisfy the assumptions of Theorem 2.1, with $D =]a, b[$ and $S_\lambda = Y$. Hence, the conclusion is directly ensured by Theorem 2.1.

Let us notice explicitly the following corollary of Theorem 1.1.

Corollary 3.1 *Let X and Y be two real Banach spaces, with $\dim(Y) \geq 2$, $\Phi : X \to Y$ a surjective continuous linear operator, $\Psi : X \to Y$ a non-constant Lipschitzian operator with Lipschitz constant equal to L, and $[a, b]$ a closed sub-interval of $[-1, 1]$.*

Then, for each pair of continuous and convex functions $\theta : [0, +\infty[\to \mathbf{R}$, $\eta : [a, b] \to \mathbf{R}$, with θ strictly increasing, one has

$$\min\left\{\sup_{x\in X}\theta\left(\left\|\Phi(x) + \frac{a}{\alpha_\Phi L}\Psi(x)\right\|_Y\right) + \eta(a), \sup_{x\in X}\theta\left(\left\|\Phi(x) + \frac{b}{\alpha_\Phi L}\Psi(x)\right\|_Y\right) + \eta(b)\right\}$$
$$= \sup_{x\in X}\inf_{\lambda\in[a,b]}\left(\theta\left(\left\|\Phi(x) + \frac{\lambda}{\alpha_\Phi L}\Psi(x)\right\|_Y\right) + \eta(\lambda)\right).$$

Proof Since $\dim(Y) \geq 2$, the norm on Y is a convex and sup-connected functional and hence so is $\theta(\|\cdot\|_Y)$. Then, we can apply Theorem 2.1 taking $\varphi(\cdot) = -\theta(\|\cdot\|_Y)$ and $\gamma = -\eta$, and the conclusion follows. □

Remark 1 Notice that Corollary 3.1 does not hold, in general, if $Y = \mathbf{R}$. In this connection, it is enough to take $X = \mathbf{R}$, $\Phi(x) = x$, $\Psi(x) = |x|$, $[a, b] = [-1, 1]$, $\theta(t) = t$, $\eta = 0$. Hence, $L = \alpha_\Phi = 1$ and we have

$$\sup_{x\in\mathbf{R}} \inf_{|\lambda|\le 1} |x + \lambda|x|| = 0 < +\infty = \inf_{|\lambda|\le 1} \sup_{x\in\mathbf{R}} |x + \lambda|x||.$$

4 Proof of Theorem 1.2

First, notice that, in view of the Rellich-Kondrachov theorem, for each $u \in W^{1,p}(\Omega)$, we have $\psi \circ u \in L^1(\Omega)$ and for each $\alpha \in L^\infty(\Omega)$ the functional $u \to \int_\Omega \alpha(x)\psi(u(x))dx$ is sequentially weakly continuous. Moreover, by (i), the functional $u \to \int_\Omega \varphi(x, u(x), \nabla u(x)dx$ is sequentially weakly lower semicontinuous [2, Theorem 4.6.8]. Now, let V be a sequentially weakly closed subset of $W^{1,p}(\Omega)$ containing the constants and let Y be a dense subset of $L^\infty(\Omega)$. Put

$$X = \left\{u \in V : \int_\Omega \varphi(x, u(x), \nabla u(x))dx < +\infty\right\}.$$

By (ii), the constants belong to X. Fix $\alpha \in L^\infty(\Omega)$. By (i), there is $\delta > 0$ such that

$$\varphi(x, \xi, \eta) - 2\|\alpha\|_{L^\infty(\Omega)}|\psi(\xi)| \ge 0$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$ with $|\xi| > \delta$. So, we have

$$\frac{c}{2}|\eta|^p - d - \|\alpha\|_{L^\infty(\Omega)} \sup_{|\xi|\le\delta} |\psi(\xi)| \le \varphi(x, \xi, \eta) + \alpha(x)\psi(\xi)$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$ and, of course,

$$\lim_{|\xi|\to+\infty} \inf_{(x,\eta)\in\Omega\times\mathbf{R}^n} (\varphi(x, \xi, \eta) + \alpha(x)\psi(\xi)) = +\infty.$$

Consequently, in view of Proposition 2.1, we have, in $W^{1,p}(\Omega)$,

$$\lim_{\|u\|\to+\infty} \left(\int_\Omega \varphi(x, u(x), \nabla u(x))dx + \int_\Omega \alpha(x)\psi(u(x))dx\right) = +\infty.$$

This implies that, for each $r \in \mathbf{R}$, the set

$$\left\{u \in V : \int_\Omega \varphi(x, u(x), \nabla u(x))dx + \int_\Omega \alpha(x)\psi(u(x))dx \le r\right\}$$

is weakly compact by reflexivity and Eberlein-Smulyan's theorem. Of course, we also have

$$\left\{u \in V : \int_\Omega \varphi(x, u(x), \nabla u(x))dx + \int_\Omega \alpha(x)\psi(u(x))dx \le r\right\}$$
$$= \left\{u \in X : \int_\Omega \varphi(x, u(x), \nabla u(x))dx + \int_\Omega \alpha(x)\psi(u(x))dx \le r\right\}.$$

Since the function $\int_\Omega \varphi(x, \cdot, 0)dx$ is not quasi-convex, there are $\xi_1, \xi_2, \xi_3 \in \mathbf{R}$, with $\xi_1 < \xi_2 < \xi_3$, such that

$$\max\left\{\int_\Omega \varphi(x, \xi_1, 0)dx, \int_\Omega \varphi(x, \xi_3, 0)dx\right\} < \int_\Omega \varphi(x, \xi_2, 0)dx.$$

Now, observe that, if we put

$$A = \{\xi_1, \xi_3\}$$

and

$$B = \{\xi_2\},$$

and define $f : X \to \mathbf{R}$, $g : X \times Y \to \mathbf{R}$ by

$$f(u) = \int_\Omega \varphi(x, u(x), \nabla u(x))dx, \quad g(u, \alpha) = \int_\Omega \alpha(x)(\psi(u(x)) - \psi(\xi_2))dx$$

for all $u \in X$, $\alpha \in Y$, we clearly have

$$\sup_A f < \inf_B f$$

and

$$\inf_{u\in B} \sup_{\alpha\in Y} g(u, \alpha) = 0.$$

Since ψ is strictly monotone, the numbers $\psi(\xi_1) - \psi(\xi_2)$ and $\psi(\xi_3) - \psi(\xi_2)$ have opposite signs. This clearly implies that

$$\sup_{\alpha\in Y} \inf_{u\in A} g(u, \alpha) \leq 0.$$

Furthermore, if $u \in X \setminus \{\xi_2\}$, again by strict monotonicity, $\psi \circ u \neq \psi(\xi_2)$, and so, since Y is dense in $L^\infty(\Omega)$, we have

$$\sup_{\alpha\in Y} g(u, \alpha) = +\infty.$$

Therefore, the sets A and B and the functions f and g satisfy the assumptions of Proposition 2.2. Consequently, we have

$$\sup_{\alpha\in Y} \inf_{u\in X} \left(\int_\Omega \varphi(x, u(x), \nabla u(x))dx + \int_\Omega \alpha(x)\psi(u(x))dx \right)$$
$$< \inf_{u\in X} \sup_{\alpha\in Y} \left(\int_\Omega \varphi(x, u(x), \nabla u(x))dx + \int_\Omega \alpha(x)\psi(u(x))dx \right).$$

Now, the conclusion is a direct consequence of Theorem 2.B. When $Y = C_0^\infty(\Omega)$, the same proof as above holds in view of Proposition 2.3.

We conclude presenting an application of Theorem 1.2 to the Neumann problem.

We denote by $\tilde{\mathcal{A}}$ the class of all Carathéodory functions $\psi : \Omega \times \mathbf{R} \to \mathbf{R}$ such that

$$\sup_{(x,\xi)\in\Omega\times\mathbf{R}} \frac{|\psi(x, \xi)|}{1 + |\xi|^q} < +\infty,$$

where $0 < q < \frac{pn-n+p}{n-p}$ if $p < n$ and $0 < q < +\infty$ if $p = n$. While, when $n < p$, $\tilde{\mathcal{A}}$ stands for the class of all Carathéodory functions $\psi : \Omega \times \mathbf{R} \to \mathbf{R}$. Given $\psi \in \tilde{\mathcal{A}}$, consider the following Neumann problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \psi(x, u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \qquad (\mathrm{P}_\psi)$$

where ν is the outward unit normal to $\partial\Omega$. Let us recall that a weak solution of (P_ψ) is any $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx - \int_\Omega \psi(x, u(x))v(x)dx = 0$$

for all $v \in W^{1,p}(\Omega)$.

If $\psi \in \tilde{\mathcal{A}}$, we set $\Psi(x, \xi) = \int_0^\xi \psi(x, t)dt$. Clearly, $\Psi(x, \cdot)$ lies in $\mathcal{A}$.

Theorem 4.1 *Let $f, g : \mathbf{R} \to \mathbf{R}$ be two functions lying in $\tilde{\mathcal{A}}$ and satisfying the following conditions:*

(a_1) *The function g has a constant sign and* $\mathrm{int}(g^{-1}(0)) = \emptyset$*;*

(a_2) $\lim_{|\xi|\to+\infty} \frac{F(\xi)}{|G(\xi)|+1} = +\infty$*;*

(a_3) *The function $F - G$ is not quasi-convex.*

Then, for each $\beta \in L^\infty(\Omega)$, with $\inf_\Omega \beta > 0$, and for each convex set $Y \subseteq L^\infty(\Omega)$, dense in $L^\infty(\Omega)$, there exists $\alpha \in Y$ such that the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \alpha(x)g(u) - \beta(x)f(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

has at least three weak solutions.

Proof Fix $\beta \in L^\infty(\Omega)$, with $\inf_\Omega \beta > 0$, and a convex set $Y \subseteq L^\infty(\Omega)$, dense in $L^\infty(\Omega)$. We are going to apply Theorem 1.2, defining φ, ψ by

$$\varphi(x, \xi, \eta) = \frac{1}{p}|\eta|^p + \beta(x)(F(\xi) - G(\xi))$$

and

$$\psi(\xi) = -G(\xi)$$

for all $(x, \xi, \eta) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$. It is immediate to realize that, by (a_1)–(a_3), the above φ, ψ satisfy the assumptions of Theorem 1.2. Of course, the set $Y - \beta$ is convex and dense in $L^\infty(\Omega)$. Then, Theorem 1.2 ensures the existence of $\alpha \in Y$ such that the functional

$$u \to \int_\Omega \varphi(x, u(x), \nabla u(x))dx + \int_\Omega (\alpha(x) - \beta(x))\psi(u(x))dx$$
$$= \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx + \int_\Omega \beta(x)F(u(x))dx - \int_\Omega \alpha(x)G(u(x))dx$$

has at least two global minima in $W^{1,p}(\Omega)$. But, by classical results, such a functional is C^1 and satisfies the Palais-Smale condition and hence, by [3, Corollary 1], has at least three critical points which are weak solutions of problem (P). □

In conclusion, we want to remark a feature of Theorem 2.A, the main tool that we used to prove Theorem 1.1: the second variable of f runs over a real interval. An important contribution to this kind of result has been provided by H. Tuy in [17].

Acknowledgments The author has been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by the Università degli Studi di Catania, "Piano della Ricerca 2016/2018 Linea di intervento 2".

References

1. Denkowski, S.M., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Theory. Kluwer Academic Publishers (2003)
2. Denkowski, S.M., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Applications. Kluwer Academic Publishers (2003)
3. Pucci, P., Serrin, J.: A mountain pass theorem. J. Differential Equations **60**, 142–149 (1985)
4. Ricceri, B.: Structure, approximation et dépendance continue des solutions de certaines équations non linéaires. C. R. Acad. Sci. Paris, Série **I**(305), 45–47 (1987)

5. Ricceri, B.: Some topological mini-max theorems via an alternative principle for multifunctions. Arch. Math. (Basel) **60**, 367–377 (1993)
6. Ricceri, B.: On a topological minimax theorem and its applications. In: Ricceri, B., Simons, S. (eds.) Minimax Theory and Applications, pp. 191–216. Kluwer Academic Publishers (1998)
7. Ricceri, B.: A further improvement of a minimax theorem of Borenshtein and Shul'man. J. Nonlinear Convex Anal. **2**, 279–283 (2001)
8. Ricceri, B.: Minimax theorems for limits of parametrized functions having at most one local minimum lying in a certain set. Topol. Appl. **153**, 3308–3312 (2006)
9. Ricceri, B.: Recent advances in minimax theory and applications. In: Chinchuluun, A., Pardalos, P.M., Migdalas, A., Pitsoulis, L. (eds.) Pareto Optimality, Game Theory and Equilibria, pp. 23–52. Springer (2008)
10. Ricceri, B.: Nonlinear eigenvalue problems. In: Gao, D.Y., Motreanu, D. (eds.) Handbook of Nonconvex Analysis and Applications, pp. 543–595. International Press (2010)
11. Ricceri, B.: A strict minimax inequality criterion and some of its consequences. Positivity **16**, 455–470 (2012)
12. Ricceri, B.: Energy functionals of Kirchhoff-type problems having multiple global minima. Nonlinear Anal. **115**, 130–136 (2015)
13. Ricceri, B.: A minimax theorem in infinite-dimensional topological vector spaces. Linear Nonlinear Anal. **2**, 47–52 (2016)
14. Ricceri, B.: Miscellaneous applications of certain minimax theorems I. Proc. Dynam. Systems Appl. **7**, 198–202 (2016)
15. Ricceri, B.: On a minimax theorem: an improvement, a new proof and an overview of its applications. Minimax Theory Appl. **2**, 99–152 (2017)
16. Rockafellar, R.T.: Integral Functionals, Normal Integrands and Measurable Selections. Lecture Notes in Math, vol. 543, pp. 157–207. Springer, Berlin (1976)
17. Tuy, H.: A new topological minimax theorem with application. J. Global Optim. **50**, 371–378 (2011)