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Renormings concerning the lineability of the norm-attaining functionals

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ABSTRACT

We prove that if a Banach space admits a biorthogonal system whose dual part is norming, then the set of norm-attaining functionals is lineable. As a consequence, if a Banach space admits a biorthogonal system whose dual part is bounded and its weak-star closed absolutely convex hull is a generator system, then the Banach space can be equivalently renormed so that the set of norm-attaining functionals is lineable. Finally, we prove that every infinite dimensional separable Banach space whose dual unit ball is weak-star separable has a linearly independent, countable, weak-star dense subset in its dual unit ball. As a consequence, we show the existence of linearly independent norming sets which are not the dual part of a biorthogonal system.

1. Introduction

Filling subspaces of ℓ_∞ have been studied in [9] where it is shown that, whereas not every subspace of ℓ_∞ verifies that the set of its norm-attaining functionals is lineable, filling subspaces do.

Following the notion of a "big set" in the measure theory sense (the complementary of a measure zero set) and in the Baire theory sense (a comeager set), Gurariy coined in 1991 (see [11]) a new version of this notion in the linear sense: *lineability* and *spaceability*. However, this did not appear in the literature until the early 2000's in [3,12]. For the last decade there has been an intensive trend to search for large algebraic and linear structures of special objects. We would like to mention the nice survey paper [6] related to this topic and the very recent monograph [2]. Let us introduce what we are meaning: A subset M of a Banach space X is said to be *lineable* (*spaceable*) if $M \cup \{0\}$ contains an infinite dimensional (closed) vector subspace. By λ-lineable (λ-spaceable) we mean that $M \cup \{0\}$ contains a (closed) vector subspace of dimension λ.

Throughout this paper, we will deal with a special friend: $\mathsf{NA}(X)$, the set of norm-attaining functionals on a Banach space X. By a classical Bishop–Phelps's theorem it is known that $\mathsf{NA}(X)$ is always "topologically

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generic", that is, dense in X^*, therefore it seems natural to raise the following question (originally posed by Godefroy in [10]).

Problem 1.1. (See Godefroy, [10].) Given an infinite dimensional Banach space X, is $\mathsf{NA}(X)$ always lineable?

Very recently, Rmoutil in [14] observed that the example of Read [13] of a Banach space with no proximinal subspaces of codimension 2 is also an example of a Banach space whose set of norm-attaining functionals does not contain subspaces of dimension 2. In [1] it has been shown that the above question has a positive answer for some classical Banach spaces like the $\mathcal{C}(K)$ and the $\mathsf{L}_1(\mu)$ spaces. Concerning Question 1.1 in terms of spaceability, the main effort has been done by Bandyopadhyay and Godefroy in [5], where it was shown that Asplund Banach spaces with the Dunford–Pettis property cannot be equivalently renormed to make the norm-attaining functionals spaceable. In particular, if K is an infinite Hausdorff scattered compact topological space, then $\mathsf{NA}(\mathcal{C}(K))$ is lineable but not spaceable.

As far as we know, the main result obtained until now concerning the isomorphic lineability of $\mathsf{NA}(X)$ was obtained in [8], where it is shown that every Banach space admitting an infinite dimensional separable quotient can be equivalently renormed so that the set of its norm-attaining functionals is lineable.

All the Banach spaces throughout this manuscript will be considered infinite dimensional.

2. Filling subspaces of $\ell_\infty(\Lambda)$

We refer the reader to [9, Definition 2.4] for the original definition of filing subspace of ℓ_∞. Here we will generalize it for $\ell_\infty(\Lambda)$. From now on and unless explicitly stated, Λ will stand for an infinite set.

Given a subset V of $\ell_\infty(\Lambda)$, we define the supporting set of V as

$$\operatorname{supp}(V) := \bigcup \{\operatorname{supp}(v) : v \in V\},$$

where as expected $\operatorname{supp}(v) := \{\lambda \in \Lambda : v(\lambda) \neq 0\}$. Observe that if $\operatorname{supp}(V)$ is finite and V is a subspace, then V is finite dimensional.

An infinite dimensional closed subspace V of $\ell_\infty(\Lambda)$ is said to be filling provided that for every infinite subset A of $\operatorname{supp}(V)$ there exists $x \in \mathsf{S}_V$ with $\operatorname{supp}(x) \subseteq A$ and x attains its sup norm.

It is not hard to see that every infinite dimensional closed subspace of $\ell_\infty(\Lambda)$ containing $c_{00}(\Lambda)$ is filling.

Also recall that for every $\lambda \in \Lambda$, the evaluation functional δ_λ on $\ell_\infty(\Lambda)$ is defined by $\delta_\lambda(x) = x(\lambda)$. It is not hard to see that $\delta_\lambda \in \mathsf{S}_{\ell_\infty(\Lambda)^*}$, and if $\lambda_1, \dots, \lambda_p \in \Lambda$ are all different, then

$$\|\alpha_1 \delta_{\lambda_1} + \cdots + \alpha_p \delta_{\lambda_p}\| = |\alpha_1| + \cdots + |\alpha_p|.$$

Theorem 2.1. *Every filling subspace V of $\ell_\infty(\Lambda)$ verifies that the set of its norm-attaining functionals is* $\operatorname{card}(\operatorname{supp}(V))$*-lineable.*

Proof. Since $\operatorname{supp}(V)$ is infinite, we can decompose it as $\operatorname{supp}(V) = \dot{\bigcup}_{\lambda \in \operatorname{supp}(V)} A_\lambda$ with $\operatorname{card}(A_\lambda) = \aleph_0$ for all $\lambda \in \operatorname{supp}(V)$. By hypothesis, for every $\lambda \in \operatorname{supp}(V)$, there exists $x_\lambda \in \mathsf{S}_V$ such that $\operatorname{supp}(x_\lambda) \subseteq A_\lambda$ and there exists $\gamma_\lambda \in A_\lambda$ with $|x_\lambda(\gamma_\lambda)| = 1$. We will show now that $\operatorname{span}\{\delta_{\gamma_\lambda} : \lambda \in \operatorname{supp}(V)\} \subseteq \mathsf{NA}(V)$. Indeed, let $\lambda_1, \dots, \lambda_p \in \operatorname{supp}(V)$ all different and $\alpha_1, \dots, \alpha_p \in \mathbb{K}$. By keeping in mind that the A_λ's are all disjoint, note that

$$\left\|\operatorname{sgn}(\alpha_1) x_{\lambda_1} + \cdots + \operatorname{sgn}(\alpha_p) x_{\lambda_p}\right\|_\infty = 1$$

and

$$\begin{aligned}&\left(\alpha_1\delta_{\gamma_{\lambda_1}}+\cdots+\alpha_p\delta_{\gamma_{\lambda_p}}\right)\left(\operatorname{sgn}(\alpha_1)x_{\lambda_1}+\cdots+\operatorname{sgn}(\alpha_p)x_{\lambda_p}\right)\\&=|\alpha_1|+\cdots+|\alpha_p|\\&=\left\|\alpha_1\delta_{\gamma_{\lambda_1}}+\cdots+\alpha_p\delta_{\gamma_{\lambda_p}}\right\|. \qquad \square\end{aligned}$$

Corollary 2.2. *Every infinite dimensional closed subspace of* $\ell_\infty(\Lambda)$ *containing* $c_{00}(\Lambda)$ *verifies that the set of its norm-attaining functionals is* $\operatorname{card}(\Lambda)$*-lineable.*

3. Main results

This section is devoted to find isometric and isomorphic conditions to accomplish the lineability of the norm-attaining functionals.

3.1. Biorthogonal systems

A biorthogonal system in a vector space X is a family of the form $(x_i, x_i^*)_{i\in I} \subseteq X \times X^*$ such that $x_i^*(x_j) = \delta_{ij}$. Notice that we do not require any norm-condition. Biorthogonal systems allow the construction of filling subspaces associated to a Banach space.

Theorem 3.1. *Let X be a Banach space and consider a biorthogonal system $(x_i, x_i^*)_{i\in I} \subseteq X \times X^*$ such that $\{x_i^* : i \in I\}$ is bounded. If $\Lambda := \{x_i^* : i \in I\}$, then the subspace $V := \left\{(x_i^*(x))_{i\in I} \in \ell_\infty(\Lambda) : x \in X\right\}$ contains $c_{00}(\Lambda)$ and thus is filling.*

Proof. We will show that the canonical basis $\left(e_{x_i^*}\right)_{i\in I}$ of $\ell_\infty(\Lambda)$ is contained in V. Indeed, fix an arbitrary $i_0 \in I$. Then $e_{x_{i_0}^*} = \left(x_i^*\left(x_{i_0}\right)\right)_{i\in I} \in V$. $\square$

Recall that, given a Banach space X, a subset Λ of X^* is said to be separating provided that the pre-annihilator of Λ is null, that is,

$$\Lambda^\top := \Lambda^\perp \cap X = \bigcap_{x^*\in\Lambda} \ker(x^*) = \{0\}.$$

The dual part $\{x_i^* : i \in I\}$ of a biorthogonal system $(x_i, x_i^*)_{i\in I}$ does not necessarily need to be separating unless, for instance, $X = \operatorname{span}\{x_i : i \in I\}$.

On the other hand, a subset $\Lambda := \{x_i^* : i \in I\} \subseteq X^*$ is the dual part of a biorthogonal system if and only if $\bigcap_{i\in I\setminus\{i_0\}} \ker(x_i^*) \nsubseteq \ker(x_{i_0}^*)$ for all $i_0 \in I$. In particular, $\Lambda \setminus \{x_{i_0}^*\}$ is not separating for all $i_0 \in I$. Conversely, if Λ is separating and $\Lambda \setminus \{x_{i_0}^*\}$ is not separating for all $i_0 \in I$, then Λ is the dual part of a biorthogonal system.

Theorem 3.2. *If V is a filling subspace of $\ell_\infty(\Lambda)$, then there exists a biorthogonal system $(v_i, v_i^*)_{i\in I} \subseteq V \times V^*$ such that $\{v_i^* : i \in I\}$ is norming.*

Proof. We use the same decomposition as in the proof of Theorem 2.1 to write $\operatorname{supp}(V) = \dot{\bigcup}_{\lambda\in\operatorname{supp}(V)} A_\lambda$ with $\operatorname{card}(A_\lambda) = \aleph_0$ for all $\lambda \in \operatorname{supp}(V)$. By hypothesis, for every $\lambda \in \operatorname{supp}(V)$, there exists $x_\lambda \in \mathsf{S}_V$ such that $\operatorname{supp}(x_\lambda) \subseteq A_\lambda$ and there exists $\gamma_\lambda \in A_\lambda$ with $x_\lambda(\gamma_\lambda) = 1$. The biorthogonal system that we will consider is $(x_\lambda, \delta_{\gamma_\lambda})_{\lambda\in\operatorname{supp}(V)}$. Finally, note that $\{\delta_\lambda : \lambda \in \Lambda\}$ is norming on $\ell_\infty(\Lambda)$, so $\{\delta_{\gamma_\lambda} : \lambda \in \operatorname{supp}(V)\}$ is norming on V. $\square$

3.2. Norming sets

We recall the reader that a subset Λ of the dual X^* of a Banach space X is norming, that is, $\|x\| = \sup\{|y^*(x)| : y^* \in \Lambda\}$ for all $x \in X$, if and only if the linear operator

$$\begin{array}{ccc} X & \to & \ell_\infty(\Lambda) \\ x & \mapsto & \begin{array}{ccc} \Lambda & \to & \mathbb{K} \\ y^* & \mapsto & y^*(x) \end{array} \end{array}$$

is an isometry over its range. Norming sets are separating and always contained in the dual unit ball.

We also recall the reader about the polar set Λ^0 of a subset Λ of X, that is,

$$\Lambda^0 := \{x^* \in X^* : |x^*(x)| \leq 1 \text{ for all } x \in \Lambda\}.$$

If $\Lambda \subseteq X^*$, then $\Lambda_0 := \Lambda^0 \cap X$.

Theorem 3.3. *Let X be a Banach space. Let Λ be a subset of X^*. The following conditions are equivalent:*

(1) Λ *is norming.*
(2) $\overline{\text{co}}^{w^*}(\Lambda) = \mathsf{B}_{X^*}$.
(3) $\Lambda_0 = \mathsf{B}_X$.
(4) $\overline{\text{aco}}^{w^*}(\Lambda) = \mathsf{B}_{X^*}$.

Proof.

(1) ⇒ (2) Let $x^* \in \mathsf{B}_{X^*} \setminus \overline{\text{co}}^{w^*}(\Lambda)$. By the Hahn–Banach Theorem there exists $x \in \mathsf{S}_X$ such that

$$x(x^*) > \sup x\left(\overline{\text{co}}^{w^*}(\Lambda)\right) = 1,$$

which is impossible.

(2) ⇒ (3) Observe that

$$\mathsf{B}_X = (\mathsf{B}_{X^*})_0 = \left(\overline{\text{co}}^{w^*}(\Lambda)\right)_0 = \Lambda_0.$$

(3) ⇒ (4) Note that

$$\mathsf{B}_{X^*} = (\mathsf{B}_X)^0 = (\Lambda_0)^0 = \overline{\text{aco}}^{w^*}(\Lambda).$$

(4) ⇒ (1) Finally

$$\begin{aligned} \|x\| &= \sup\{x^*(x) : x^* \in \mathsf{B}_{X^*}\} \\ &= \sup\left\{x^*(x) : x^* \in \overline{\text{aco}}^{w^*}(D)\right\} \\ &= \sup\{x^*(x) : x^* \in D\} \end{aligned}$$

for all $x \in X$. □

Theorem 3.4. *Let X be a Banach space. Let Λ be a subset of X^*. Then following conditions are equivalent:*

(1) *Λ is bounded.*
(2) *Λ_0 is a neighborhood of* 0.
(3) $\operatorname{span}(\Lambda_0) = X$.

Proof.

(1) ⇒ (2) If Λ is bounded, then $\Lambda \subseteq \alpha \mathsf{B}_{X^*}$ for some $\alpha > 0$, so $\alpha \mathsf{B}_X = (\alpha \mathsf{B}_{X^*})_0 \subseteq \Lambda_0$.
(2) ⇒ (3) It is clear.
(3) ⇒ (2) Observe that Λ_0 is absolutely convex and a generator system, therefore we are entitled to apply [7, Lemma 2.4] to conclude that Λ_0 is absorbing, which makes it a barrel of X. Since X is complete, we deduce that Λ_0 is a neighborhood of 0.
(2) ⇒ (1) If Λ_0 is a neighborhood of 0, then we can find $\beta > 0$ so that $\beta \mathsf{B}_X \subseteq \Lambda_0$, which means that

$$\Lambda \subseteq \overline{\operatorname{aco}}^{w^*}(\Lambda) = (\Lambda_0)^0 \subseteq (\beta \mathsf{B}_X)^0 = \beta \mathsf{B}_{X^*}. \qquad \square$$

Theorem 3.5. *Let X be a Banach space. Let Λ be a subset of X^*. Then following conditions are equivalent:*

(1) *Λ_0 is bounded.*
(2) *$\overline{\operatorname{aco}}^{w^*}(\Lambda)$ is a neighborhood of* 0.
(3) $\operatorname{span}\left(\overline{\operatorname{aco}}^{w^*}(\Lambda)\right) = X^*$.

Proof.

(1) ⇒ (2) If Λ_0 is bounded, then $\Lambda_0 \subseteq \alpha \mathsf{B}_X$ for some $\alpha > 0$, so

$$\alpha \mathsf{B}_{X^*} = (\alpha \mathsf{B}_X)^0 \subseteq (\Lambda_0)^0 = \overline{\operatorname{aco}}^{w^*}(\Lambda).$$

(2) ⇒ (3) It is clear.
(3) ⇒ (2) Observe that $\overline{\operatorname{aco}}^{w^*}(\Lambda)$ is absolutely convex and a generator system, therefore we are entitled to apply [7, Lemma 2.4] to conclude that $\overline{\operatorname{aco}}^{w^*}(\Lambda)$ is absorbing, which makes it a barrel of X^*. Since X^* is complete, we deduce that $\overline{\operatorname{aco}}^{w^*}(\Lambda)$ is a neighborhood of 0.
(2) ⇒ (1) If $\overline{\operatorname{aco}}^{w^*}(\Lambda)$ is a neighborhood of 0, then we can find $\beta > 0$ so that $\beta \mathsf{B}_{X^*} \subseteq \overline{\operatorname{aco}}^{w^*}(\Lambda)$, which means that

$$\Lambda_0 = \left(\overline{\operatorname{aco}}^{w^*}(\Lambda)\right)_0 \subseteq (\beta \mathsf{B}_{X^*})_0 = \beta \mathsf{B}_X. \qquad \square$$

Corollary 3.6. *Let X be a Banach space. Let Λ be a subset of X^*. The following conditions are equivalent:*

(1) *Λ and Λ_0 are both bounded.*
(2) *There exists an equivalent norm on X that makes Λ is norming.*

Now we are ready to state and prove the main result in this subsection.

Theorem 3.7. *Let X be a Banach space.*

(1) *There exists a biorthogonal system $(x_i, x_i^*)_{i\in I}$ such that $\{x_i^* : i \in I\}$ is norming if and only if X is linearly isometric to a filling subspace of $\ell_\infty(\Lambda)$. In this situation, $\mathsf{NA}(X)$ is $\operatorname{card}(\Lambda)$-lineable.*
(2) *There exists a biorthogonal system $(x_i, x_i^*)_{i\in I}$ such that $\{x_i^* : i \in I\}$ is bounded and $\operatorname{span}\left(\overline{\operatorname{aco}}^{w^*}\{x_i^* : i \in I\}\right) = X^*$ if and only if X is isomorphic to a filling subspace of $\ell_\infty(\Lambda)$. In this situation, X can be equivalently renormed to make $\mathsf{NA}(X)$ be $\operatorname{card}(\Lambda)$-lineable.*

Proof.

(1) Assume first that there exists a biorthogonal system $(x_i, x_i^*)_{i\in I}$ such that $\Lambda := \{x_i^* : i \in I\}$ is norming. Then X is linearly isometric to $\{(x_i^*(x))_{i\in I} : x \in X\}$, which is a filling subspace of $\ell_\infty(\Lambda)$ in virtue of Theorem 3.1. Theorem 2.1 assures that $\mathsf{NA}(X)$ is card(Λ)-lineable. Conversely, in virtue of Theorem 3.2 we deduce that there exists a biorthogonal system $(x_i, x_i^*)_{i\in I}$ such that $\{x_i^* : i \in I\}$ is norming.

(2) Assume first that there exists a biorthogonal system $(x_i, x_i^*)_{i\in I}$ such that $\Lambda := \{x_i^* : i \in I\}$ is bounded and span $\left(\overline{\text{aco}}^{w^*}(\Lambda)\right) = X^*$. By applying Theorem 3.5 we have that Λ_0 is bounded. Now, it suffices to call on Corollary 3.6 to conclude that there exists an equivalent norm on X for which Λ is norming. The first item of this theorem applied to this new equivalent norm allows us to deduce that X is isomorphic to a filling subspace of $\ell_\infty(\Lambda)$ and thus X can be equivalently renormed to make $\mathsf{NA}(X)$ be card(Λ)-lineable. Conversely, if X is isomorphic to a filling subspace of $\ell_\infty(\Lambda)$, then in accordance with Theorem 3.2 X can be equivalently renormed to have a biorthogonal system $(x_i, x_i^*)_{i\in I}$ such that $\{x_i^* : i \in I\}$ is norming. Then in the original norm of X, $\{x_i^* : i \in I\}$ is bounded and span $\left(\overline{\text{aco}}^{w^*}\{x_i^* : i \in I\}\right) = X^*$ (recall Corollary 3.6 and Theorem 3.5). □

We would like to make the reader beware that Rmoutil example of a Banach space whose set of norm-attaining functionals is not even 2-lineable is an equivalent renorming of c_0, thus we are talking about a Banach space with a Schauder basis, which will not be monotone in virtue of [1, Theorem 3.1]. As a consequence, the existence of a biorthogonal system is not enough to assure the lineability of the norm-attaining functionals.

3.3. w^*-Separable sets

Notice that if B_{X^*} is w^*-separable, then X^* is w^*-separable. Indeed, let Λ be a countable w^*-dense set in B_{X^*}, then

$$\overline{\bigcup_{n\in\mathbb{N}} n\Lambda}^{w^*} \subseteq \bigcup_{n\in\mathbb{N}} n\mathsf{B}_{X^*} = X^*$$

and $\bigcup_{n\in\mathbb{N}} n\Lambda$ is countable. However, the converse to the previous assertion does not hold in virtue of [4].

Theorem 3.8. *Let X be an infinite dimensional Hausdorff locally convex topological vector space. If D is a separable, first countable, bounded and absolutely convex subset of X, then there exists $\Lambda \subseteq D$ countable, dense in D, and linearly independent.*

Proof. Let $(V_n)_{n\in\mathbb{N}}$ be a nested basis of open neighborhoods of 0 in D in such a way that $V_n = U_n \cap D$, where U_n is an open absolutely convex and absorbing neighborhood of 0 in X, for every $n \in \mathbb{N}$. Observe that by hypothesis, we have that $\left(\frac{1}{n}V_n\right)_{n\in\mathbb{N}}$ is a basis of open neighborhoods of 0 in D. Let $(d_n)_{n\in\mathbb{N}}$ be a dense sequence in D. We will follow an inductive process:

- For $n = 1$, we take $z_1 = d_1$.
- For $n = 2$, since $\frac{1}{2}d_2 + \frac{1}{2}V_2 \nsubseteq \mathbb{K}z_1$, there exists $z_2 \in \left(\frac{1}{2}d_2 + \frac{1}{2}V_k\right) \setminus \mathbb{K}z_1$.
- For $n = k$, since $\left(1 - \frac{1}{k}\right)d_k + \frac{1}{k}V_k \nsubseteq \text{span}\{z_1, \ldots, z_{k-1}\}$, there exists $z_k \in \left(\left(1 - \frac{1}{k}\right)d_k + \frac{1}{k}V_k\right) \setminus \text{span}\{z_1, \ldots, z_{k-1}\}$.

Following this process we obtain a sequence $(z_n)_{n>1}$ verifying the following conditions:

(1) $z_n \in \left(1 - \frac{1}{n}\right)d_n + \frac{1}{n}V_n \subseteq D$ due to the convexity of D.

(2) $(z_n)_{n\in\mathbb{N}}$ is dense in D. Indeed, write $z_n = \left(1 - \frac{1}{n}\right) d_n + \frac{1}{n} v_n$ with $v_n \in V_n$. Fix an arbitrary $m \in \mathbb{N}$. We can find $n_m > m$ such that $d_{n_m} \in V_m$. Then $z_{n_m} \in V_m$ by the convexity of V_m and the fact that $V_{n_m} \subseteq V_m$.

(3) $z_n \notin \operatorname{span}\{z_1, \ldots, z_{n-1}\}$.

Now it is easy to understand that $\Lambda := \{z_n : n \in \mathbb{N}\}$ verifies the desired properties. □

As a direct consequence of Theorem 3.8 we obtain the following result.

Corollary 3.9. *If X is an infinite dimensional separable Banach space such that B_{X^*} is w^*-separable, then there exists $\Lambda \subseteq \mathsf{B}_{X^*}$ countable, w^*-dense in B_{X^*}, and linearly independent.*

The final example in this section shows the existence of countable norming linearly independent sets which are not the dual part of a biorthogonal system.

Example 3.10. Let X be c_0 endowed with the norm given in [14], which makes $\mathsf{NA}(X)$ not even 2-lineable. According to Corollary 3.9, there exists $\Lambda \subseteq \mathsf{B}_{X^*}$ countable, w^*-dense in B_{X^*}, and linearly independent. We clearly have that Λ is norming by Theorem 3.3, and thus separating. However, Theorem 3.7 allows us to conclude that Λ cannot be the dual part of a biorthogonal system.

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