On the Schrödinger-Poisson system with indefinite potential and 3-sublinear nonlinearity

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Abstract

We consider a class of stationary Schrödinger-Poisson systems with a general nonlinearity $f(u)$ and coercive sign-changing potential V so that the Schrödinger operator $-\Delta+V$ is indefinite. Previous results in this framework required f to be strictly 3-superlinear, thus missing the paramount case of the Gross-Pitaevskii-Poisson system, where $f(t)=|t|^2t$; in this paper we fill this gap, obtaining non-trivial solutions when f is not necessarily 3-superlinear.

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1. Introduction

The dynamic of a Bose-Einstein condensate can be described (see [1, 2]) by the Gross-Pitaevskii equation

$$i\,\partial_t\psi = -\Delta\psi + V\,\psi + g\,|\psi|^2\psi$$

where $\psi : \mathbb{R}^3\times[0,+\infty[\to\mathbb{C}$ is the wave function of the condensate, $V=V(x)$ is the potential, $|\psi|^2$ is the particle-density, whose integral gives the total (large) number of particles N and g is related to the scattering length of the mutual short-range atomic interaction (resulting in positive g for repulsive interaction and negative for attractive ones). The Gross-Pitaevskii equation is a particular case of the nonlinear Schrödinger equation

$$i\,\partial_t\psi = -\Delta\psi + V\,\psi + g\,|\psi|^{p-1}\psi$$

for $p>1$. If the particles are electrically charged, long-range electrostatic interaction can be effectively modelled by a potential term (see [3] for a formal justification), so that $V=V_{\rm ext}+\phi$, where $V_{\rm ext}$ is the external potential and ϕ is the electrostatic potential determined by the Poisson equation with charge density $k|\psi|^2$ (typically, $k>0$ giving repulsive interactions). This gives rise to the Schrödinger-Poisson system

$$\begin{cases} i\,\partial_t\psi = -\Delta\psi + (V_{\rm ext}+\phi)\psi + g\,|\psi|^{p-1}\psi \\ -\Delta\phi = k\,|\psi|^2 \end{cases} \tag{1.1}$$

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which has been object of extensive studies in the last decades. Assuming vanishing boundary conditions at infinity, the total energy

$$E := \int_{\mathbb{R}^3} \frac{1}{2}|\nabla\psi|^2 + \frac{V_{\text{ext}}}{2}|\psi|^2 + \frac{k}{4}|\nabla\phi|^2 + \frac{g}{p+1}|\psi|^{p+1}\,dx$$

is conserved along the motion, as well as the total mass $N = |\psi|_2^2$, where $|\cdot|_q$ stands for the L^q-norm over $\mathbb{R}^3$. A particularly interesting case of the previous system is when long- and short-range mutual strengths compete, for example when $k > 0$ (repulsive electrostatic interaction) and $g < 0$ (short-range binding). This is the case for a Bose-Einstein condensate of charged ions with attractive interatomic interaction, trapped in a potential well.

Standing waves for (1.1) are obtained through the ansatz $\psi(x,t) = e^{-i\omega t}u(x)$ with $u : \mathbb{R}^3 \to \mathbb{R}$, resulting in

$$\begin{cases} -\Delta u + V\,u + \phi\,u - f(u) = 0 \\ -\Delta\phi = u^2 \end{cases} \tag{1.2}$$

where we set $f(t) = |t|^{p-1}t$, $V = V_{\text{ext}} - \omega$ and $k = 1$ for simplicity of notation. Conservation of total energy E and mass N gives the relation $\omega\,N = E$, so that ω represents the energy per particle of the standing wave. A natural question, to which we will give a positive answer in the present paper, is *wether standing waves of arbitrarily large energy per particle can occur*. Notice that for large values of ω, the potential $V = V_{\text{ext}} - \omega$ is sign-changing and the linearisation of the first equation turns out to be an indefinite Schrödinger operator.

Formally, the functional $\mathcal{E} : H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ given by

$$\mathcal{E}(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V\,u^2\right)\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}\phi\,u^2\,\mathrm{d}x - \int_{\mathbb{R}^3}F(u)\,\mathrm{d}x$$

where

$$F(t) = \int_0^t f(s)\,\mathrm{d}s,$$

is such that critical points (u,ϕ) of $\mathcal{E}$ are solutions of (1.2). However, since $\mathcal{E}$ is strongly indefinite and thus difficult to deal with, Benci *et al.* [4, 5] proposed the following reduction procedure. For $u \in H^1(\mathbb{R}^3)$ let $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ be the unique solution of $-\Delta\phi = u^2$ in (1.2). Then, u is a critical point of the functional

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V\,u^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u\,u^2\,\mathrm{d}x - \int_{\mathbb{R}^3}F(u)\,\mathrm{d}x, \tag{1.3}$$

if and only if (u,ϕ_u) solves (1.2); see [4, 5] or [6, pp. 4929–4932] for more details.

Based on this reduction method, ground states and, more generally, positive solutions to system (1.2) have been obtained in a wide variety of assumptions, both on f and on V. With no attempt to give a complete account of the literature, we mention [7, 8, 9] for $V \equiv \text{const} > 0$ and [10, 11, 12] for V radial. More general asymptotically constant potentials are considered in [13, 14, 15, 16]; [17, 18, 19] treat periodic ones and [6, 20] deal with (weakly) coercive potentials. Sign-changing solutions are found in [21, 22] in the case $V \equiv \text{const} > 0$, in [23] when V is asymptotically constant. Motivated by [24], nodal solutions to (1.2) with coercive potential are constructed in [25, 26].

We emphasize that in the aforementioned papers the Schrödinger operator $-\Delta + V$ is always assumed to be positive definite (as when $\inf_{\mathbb{R}^3} V > 0$), so that $u \equiv 0$ is a local minimizer of J, leading to a mountain pass geometry if f is 3-superlinear (actually, superquadratic will often suffices, but require more intricate arguments). However, if we seek for standing waves with large ω, then $V = V_{\text{ext}} - \omega$ will be negative somewhere, disrupting the mountain pass geometry. For stationary NLS equations

$$-\Delta u + V\,u = f(u) \tag{1.4}$$

with indefinite Schrödinger operator $-\Delta + V$, one usually applies the linking theorem to get solutions, see *e.g.* [27, 28]. For system (1.2), however, it seems hard to verify the linking geometry due to the nonnegative and nonlocal term involving ϕ_u; see [29, p. 47] for further discussion on this issue. This is probably one of the reasons why there are very few existence results for (1.2) if the Schrödinger operator $-\Delta + V$ is indefinite. We are only aware of the works [30, 31] which actually infer linking from a perturbative argument, (thus obtaining solutions under a smallness assumption on the nonlocal term) and [29, 32], where critical points for J are obtained via the *local* linking theory [33, 34].

In the previous papers, however, f is always assumed to be 3-superlinear, i.e.

$$\lim_{|t|\to+\infty} \frac{f(x,t)t}{t^4} = +\infty \qquad \text{locally uniformly} \tag{1.5}$$

holds and, as far as we know, currently there is no existence result for system (1.2) with indefinite potential without (1.5). The relevance of this latter framework is clear from the previous discussion on Bose-Einstein condensates, since the reaction $f(t) = |t|^2 t$ corresponding to the Gross-Pitaevskii equation is exactly 3-linear.

Our first and by far easier result treats subquadratic nonlinearities. By $\sigma(-\Delta + V)$ we mean the spectrum of $-\Delta + V$, understood as the natural self-adjoint operator corresponding to the bilinear form given by (2.2) below.

Theorem 1.1. *Suppose that V is coercive in the following sense*

(V_0) *$V \in C(\mathbb{R}^3)$ is bounded from below and $|\{V \le k\}| < \infty$ for all $k \in \mathbb{R}$,*

and that $\inf \sigma(-\Delta + V) \le 0$. *If there exist $C, v > 0$, $p, q \in \,]1, 2[$ such that*

$$|f(t)| \le C(|t|^p + |t|^q) \tag{1.6}$$

and $F(t) = \int_0^t f(\tau)\,\mathrm{d}\tau \ge c\,|t|^{p+1}$ for all $t \in \mathbb{R}$, then there are at least two nontrivial solutions to (1.2).

Under the stated assumptions, the functional J given in (1.3) is coercive, hence Palais-Smale sequences are automatically bounded and precompact by (V_0). Since J has a local linking at 0, Theorem 1.1 directly follows from [35, Theorem 2.2].

Our next and main result deals with the superquadratic case, including $f(u) = |u|^{p-1}u$ with $p \in \,]2, 5[$. In addition to (V_0), we will need the following assumptions

(V_1) $V \in C^1(\mathbb{R}^3)$ and there exists $R > 0$ such that $2V(x) + \nabla V(x) \cdot x \ge 0$ for $|x| \ge R$.

(V_2) There exists $\kappa > 0$ and $m \in \mathbb{R}$ such that $|\nabla V(x) \cdot x| \le \kappa\,(V(x) + m) =: \kappa\,\tilde{V}(x)$ for all $x \in \mathbb{R}^3$.

(f_1) $f \in C(\mathbb{R})$ and $|f(t)| \le C(|t| + |t|^p)$ for some $p \in (1, 5)$, $C > 0$.

(f_2) There exists $\mu > 3$ such that $f(t)t \ge \mu F(t) > 0$ for all $t \in \mathbb{R}\setminus\{0\}$.

Clearly, any coercive, radially increasing potential with polynomial growth satisfies our assumptions, an explicit example being $V(x) = |x|^2 - \omega$ (which, for ω large enough, gives rise to an indefinite Schrödinger operator). For a more detailed discussion on (V_1) and (V_2), we refer to the beginning of Section 3. Under assumption (V_0), the Schrödinger operator operator $-\Delta + V$ is essentially self-adjoint with discrete spectrum and we will let X_+, X_- and X_0 denote its positive, negative and null eigenspaces, respectively.

Theorem 1.2. *Assume (V_0)–(V_2), (f_1)–(f_2) hold. If either*

1. $\dim X_- > 0$, $\dim X_0 = 0$
2. $\dim X_0 > 0$ *and* $F(t) = \int_0^t f(\tau)\,\mathrm{d}\tau \geq c\,|t|^\nu$ *for some* $\nu < 4$,

then problem (1.2) *has at least a nontrivial solution.*

Let us discuss some features of Theorem 1.2 in the model case $f(u) = |u|^{p-1}\,u$. The main difficulty in studying (1.2) in the range $p \in\,]2, 3]$ is that it is not known whether Palais-Smale (or even Cerami) sequences are bounded or not. This issue disappears when $p > 3$, thus allowing an easier application of variational methods (this is also why most of the previous papers on the subject assume (1.5)). To overcome this difficulty there are typically two approaches:

- Seek for a minimum on a suitable manifold $\mathcal{M}$. Coercivity is still an issue for $p \in\,]2, 3]$ and the standard Nehari manifold won't help, so one usually works on the Pohozaev manifold (or variants of it);

- Employ Struwe's monotonicity trick, i.e., define monotonic perturbations of J and find a solution for almost every perturbation. Boundedness of the resulting sequence is proved via the Pohozaev identity.

Both techniques can be successful when the Schrödinger operator $-\Delta + V$ is positive definite, but run into serious issues when it is not, for reasons which we will briefly outline.

For the NLS (1.4) with indefinite potential, the Nehari manifold $\mathcal{N}$ can be modified as in [36] to produce ground states. This is possible thanks to a linearity feature of the Nehari manifold, as $\mathcal{N}$ is the set of critical points of J along *lines* through 0. On the contrary, the natural curves defining the Pohozaev manifold (which is the one apparently needed to get coercivity) are highly nonlinear and may have nothing to do with the orthogonal decomposition of the space dictated by the linear operator $-\Delta + V$.

On the other hand, Struwe's monotonicity trick is usually successful when a uniform mountain pass geometry or more general linking geometry holds for the family of monotonic perturbations of J, see [37, 38] respectively. This cannot hold for the indefinite Schrödinger-Poisson systems we are considering because, as pointed out in [29, p. 47], our functional J only has a *local* linking at the origin. Currently, it seems unclear how to implement the monotonicity trick in a local linking geometry, due to the lack of an explicit minimax description of the critical values in this setting.

To get around these difficulties, inspired by [39], we add a dummy variable and consider an augmented functional $\tilde{J} : \mathbb{R} \times X \to \mathbb{R}$, see (3.4). It turns out that $\tilde{J}$ solves all the issues J had: it satisfies the (PS) condition (Theorem 3.6), and if $(\bar{s}, \bar{u})$ is critical point of $\tilde{J}$, then $\bar{u}$ is critical point of J (Lemma 3.5). Moreover, $\tilde{J}$ preserves the local geometry of J at zero, namely it has a local linking with a precise relationship with the one of J and, through a highly nonlinear version of a by-now classical argument, its homology at infinity is trivial. Eventually, we will apply Morse theory to get a critical point of $\tilde{J}$ and thus of J.

The paper is organized as follows. In Section 2 we recall the functional analytic tools we'll need and prove Theorem 1.1. In Section 3, we deal with the superquadratic case and present the proof of Theorem 1.2. To shorten the notation, all integrals will be on the whole $\mathbb{R}^3$, unless otherwise specified.

2. The coercive case

Let us discuss some first consequences of (V_0), which we'll assume from now on. From the lower boundedness we can henceforth fix $m > 2$ such that

$$\tilde{V}(x) := V(x) + m > \frac{m}{2} > 1, \qquad \text{for all } x \in \mathbb{R}^3. \tag{2.1}$$

By [40] we see that the Hilbert space

$$X := \left\{ u \in H^1(\mathbb{R}^3) : \int \tilde V\, u^2 \,\mathrm{d}x < +\infty \right\}, \qquad (u,v)_X = \int \left[\nabla u \cdot \nabla v + \tilde V\, u\, v \right] \mathrm{d}x$$

compactly embeds in $L^2(\mathbb{R}^3)$. Notice that since X also embeds into $L^6(\mathbb{R}^3)$, by interpolation, $X \hookrightarrow L^r(\mathbb{R}^3)$ compactly for all $r \in [2,6[$. From the compactness of $X \hookrightarrow L^2(\mathbb{R}^3)$, we deduce that the bilinear form

$$Q(u,v) = \frac{1}{2} \int (\nabla u \cdot \nabla v + V\, u\, v)\, \mathrm{d}x, \qquad u, v \in X, \tag{2.2}$$

is essentially selfadjoint (by Kato's criterion), semibounded from below on $X \subseteq L^2(\mathbb{R}^3)$ and the spectrum of the corresponding Schrödinger operator $\sigma(-\Delta + V)$ is discrete (with finite multiplicity) and bounded from below. In the following, we will denote by X_+, X_- and X_0 respectively the positive, negative and null eigenspaces of the Schrödinger operator, and by $u \mapsto u_\pm$ and $u \mapsto u_0$ the corresponding orthogonal projections. Accordingly, there exists $\lambda_\pm > 0$ such that

$$\pm\, Q(u,u) \geq \lambda_\pm \|u_\pm\|^2 \qquad \text{for } u \in X_\pm \oplus X_0, \text{ respectively.} \tag{2.3}$$

As already pointed out, solving (1.2) is equivalent to finding critical points of the C^1 functional $J : X \to \mathbb{R}$,

$$J(u) = \frac{1}{2} \int \left[|\nabla u|^2 + V\, u^2 \right] \mathrm{d}x + \frac{1}{4} \int \phi_u\, u^2 \,\mathrm{d}x - \int F(u)\,\mathrm{d}x,$$

where ϕ_u is the unique solution of $-\Delta\phi = u^2$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Recall that

$$0 \leq \int \phi_u\, u^2 \,\mathrm{d}x \leq C\|u\|^4, \tag{2.4}$$

see *e.g.* [6]. We will also need the following estimate, whose proof is similar to [7, Eqn (19)], therefore is omitted.

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^3)$ we have*

$$\int |u|^3 \,\mathrm{d}x \leq \frac{1}{2} \int \left[|\nabla u|^2 + \phi_u\, u^2 \right] \mathrm{d}x. \tag{2.5}$$

Given a Hilbert space X, we say that a functional $J \in C^1(X)$ has a local linking at 0 if $X = X^- \oplus X^+$ for some closed proper subspaces $X^\pm$ and for some $\rho > 0$ there holds

$$\begin{cases} J > 0 & \text{in } B_\rho \cap (X^+ \setminus \{0\}), \\ J \leq 0 & \text{in } B_\rho \cap X^-, \end{cases}$$

where B_ρ denotes the open ball in X of radius ρ and centered at zero. This implies that $u = 0$ is a trivial critical point of J. The following three critical point theorem can be found in [35, Theorem 2.2], which is a special case of [41, Theorem 2.1].

Theorem 2.2. *Let $J \in C^1(X)$ satisfy the (PS)-condition, have a local linking at 0 with $\dim X^- < \infty$, and be bounded from below. Then J has at least two nontrivial critical points.*

Now we can start our investigation for the functional J.

Proposition 2.3. *Suppose that (V_0) holds and that there exist $C \geq 0$, $p, q \in [1,2[$ such that*

$$|F(t)| \leq C(|t|^{p+1} + |t|^{q+1}) \tag{2.6}$$

for all $t \in \mathbb{R}$, then J is coercive on X.

Proof. Let us choose $m > 0$ as in (2.1) and set $\Lambda : X \to \mathbb{R}$,

$$\Lambda(u) = -\frac{m}{4}\int u^2\,\mathrm{d}x + \frac{1}{2}\int |u|^3\,\mathrm{d}x + \frac{1}{4}\int V\,u^2\,\mathrm{d}x - \int |F(u)|\,\mathrm{d}x.$$

For $u \in X$, using (2.5) we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\int \left[|\nabla u|^2 + V\,u^2\right]\,\mathrm{d}x + \frac{1}{4}\int \phi_u\,u^2\,\mathrm{d}x - \int F(u)\,\mathrm{d}x \\
&= \frac{1}{4}\int \left[|\nabla u|^2 + V\,u^2\right]\,\mathrm{d}x + \frac{1}{4}\int \left[|\nabla u|^2 + \phi_u\,u^2\right]\,\mathrm{d}x + \frac{1}{4}\int V\,u^2\,\mathrm{d}x - \int F(u)\,\mathrm{d}x \\
&\geq \frac{1}{4}\,\|u\|^2 - \frac{m}{4}\int u^2\,\mathrm{d}x + \frac{1}{2}\int |u|^3\,\mathrm{d}x + \frac{1}{4}\int V\,u^2\,\mathrm{d}x - \int |F(u)|\,\mathrm{d}x \\
&= \frac{1}{4}\,\|u\|^2 + \Lambda(u).
\end{aligned}$$

Therefore, it suffices to show that the functional Λ is bounded from below.

For any $M > m$, since $V(x) \geq -m$ for all $x \in \mathbb{R}$, we have

$$\int V\,u^2\,\mathrm{d}x = \int_{\{V>M\}} V\,u^2\,\mathrm{d}x + \int_{\{V\leq M\}} V\,u^2\,\mathrm{d}x \geq M\int_{\{V>M\}} u^2\,\mathrm{d}x - m\int_{\{V\leq M\}} u^2\,\mathrm{d}x,$$

so that

$$\Lambda(u) \geq \frac{M}{4}\int_{\{V>M\}} u^2\,\mathrm{d}x - \frac{m}{4}\int_{\{V\leq M\}} u^2\,\mathrm{d}x - \frac{m}{4}\int u^2\,\mathrm{d}x + \frac{1}{2}\int |u|^3\,\mathrm{d}x - \int |F(u)|\,\mathrm{d}x. \tag{2.7}$$

Accordingly, we split all the remaining integrals on the two sets $\{V > M\}$ and $\{V \leq M\}$, proving boundedness of the corresponding quantities separately.

On $\{V \leq M\}$, which has finite measure by assumption (V_0), Hölder inequality gives

$$\int_{\{V\leq M\}} u^2\,\mathrm{d}x \leq C_2\left(\int_{\{V\leq M\}} |u|^3\,\mathrm{d}x\right)^{\frac{2}{3}}.$$

Similarly, using (2.6) as well,

$$\int_{\{V\leq M\}} |F(u)|\,\mathrm{d}x \leq C_p\left(\int_{\{V\leq M\}} |u|^3\,\mathrm{d}x\right)^{\frac{p+1}{3}} + C_q\left(\int_{\{V\leq M\}} |u|^3\,\mathrm{d}x\right)^{\frac{q+1}{3}}$$

for some constants C_r depending on M, V and $r \in [1, 2[$. Hence

$$\begin{aligned}
\Lambda_M^-(u) &:= -\frac{m}{2}\int_{\{V\leq M\}} u^2\,\mathrm{d}x + \frac{1}{2}\int_{\{V\leq M\}} |u|^3\,\mathrm{d}x - \int_{\{V\leq M\}} |F(u)|\,\mathrm{d}x \\
&\geq \frac{1}{2}\int_{\{V\leq M\}} |u|^3\,\mathrm{d}x - \frac{m\,C_2}{2}\left(\int_{\{V\leq M\}} |u|^3\,\mathrm{d}x\right)^{\frac{2}{3}} \\
&\qquad - C_p\left(\int_{\{V\leq M\}} |u|^3\,\mathrm{d}x\right)^{\frac{p+1}{3}} - C_q\left(\int_{\{V\leq M\}} |u|^3\,\mathrm{d}x\right)^{\frac{q+1}{3}}
\end{aligned}$$

and since $q, p \in [1, 2[$, for any choice of M, m the right hand side is clearly bounded from below.

Consider now

$$
\begin{aligned}
\Lambda_M^+(u) &:= \frac{M-m}{4}\int_{\{V>M\}} u^2\,\mathrm{d}x + \frac{1}{2}\int_{\{V>M\}} |u|^3\,\mathrm{d}x - \int_{\{V>M\}} |F(u)|\,\mathrm{d}x \\
&\geq \frac{M-m}{4}\int_{\{V>M\}} u^2\,\mathrm{d}x + \frac{1}{2}\int_{\{V>M\}} |u|^3\,\mathrm{d}x \\
&\qquad - C\int_{\{V>M\}} |u|^{p+1}\,\mathrm{d}x - C\int_{\{V>M\}} |u|^{q+1}\,\mathrm{d}x.
\end{aligned} \tag{2.8}
$$

For $r \in \{p+1, q+1\}$, by the interpolation inequality we have

$$
\int_{\{V>M\}} |u|^r\,\mathrm{d}x \leq \left(\int_{\{V>M\}} u^2\,\mathrm{d}x\right)^{\frac{r\theta_r}{2}} \left(\int_{\{V>M\}} |u|^3\,\mathrm{d}x\right)^{\frac{r(1-\theta_r)}{3}} \tag{2.9}
$$

for $\theta_r \in\ [0,1[$ satisfying

$$
\frac{r\,\theta_r}{2} + \frac{r\,(1-\theta_r)}{3} = 1.
$$

We can suppose that $u \neq 0$ on $\{V > M\}$ (otherwise $\Lambda_M^+(u) = 0$) and set

$$
R_u = \left(\int_{\{V>M\}} u^2\,\mathrm{d}x\right)^{-1} \int_{\{V>M\}} |u|^3\,\mathrm{d}x.
$$

Then applying (2.9) for $r = q+1, p+1$ to (2.8) we have

$$
\begin{aligned}
\Lambda_M^+(u) \geq{} & \frac{M-m}{4}\int_{\{V>M\}} u^2\,\mathrm{d}x + \frac{1}{2}\int_{\{V>M\}} |u|^3\,\mathrm{d}x \\
& - C\left(\int_{\{V>M\}} u^2\,\mathrm{d}x\right)^{\frac{(p+1)\theta_{p+1}}{2}} \left(\int_{\{V>M\}} |u|^3\,\mathrm{d}x\right)^{\frac{(p+1)(1-\theta_{p+1})}{3}} \\
& - C\left(\int_{\{V>M\}} u^2\,\mathrm{d}x\right)^{\frac{(q+1)\theta_{q+1}}{2}} \left(\int_{\{V>M\}} |u|^3\,\mathrm{d}x\right)^{\frac{(q+1)(1-\theta_{q+1})}{3}} \\
={} & \left(\frac{M-m}{4} + \frac{R_u}{2} - C\,R_u^{\frac{(p+1)(1-\theta_{p+1})}{3}} - C\,R_u^{\frac{(q+1)(1-\theta_{q+1})}{3}}\right)\int_{\{V>M\}} u^2\,\mathrm{d}x.
\end{aligned}
$$

Because $\frac{r(1-\theta_r)}{3} < 1$ for $r = p+1, q+1$, there exists $M > m$ such that

$$
\frac{M-m}{4} + \frac{R}{2} - C\,R^{\frac{(p+1)(1-\theta_{p+1})}{3}} - C\,R^{\frac{(q+1)(1-\theta_{q+1})}{3}} > 0, \qquad \text{for all } R > 0. \tag{2.10}
$$

With this choice of M at the very beginning, the above argument shows that $\Lambda_M^+(u) \geq 0$. Since $\Lambda_M^-(u) + \Lambda_M^+(u)$ is exactly the right hand side of (2.7), we deduce that Λ is bounded from below and the proof is concluded. $\square$

Remark 2.4.

- Regarding the potential, coercivity holds under slightly weaker assumptions, namely that the measure of $\{V \leq M\}$ is finite for a suitable large M prescibed by the validity of (2.10). However, without assuming the full (V_0) in the previous proposition, compactness starts becoming the main issue to prove existence of a solution.

- The case $1 \leq p < q = 2$ can also be treated but is border-line: consider in (1.2) the nonlinearity $f(t) = \lambda|t|t$ for $\lambda > 0$. The previous proof still works for $\lambda \leq \lambda_0$ being λ_0 a small positive number that can be explicitly computed, but fails for $\lambda > \lambda_0$. The arguments in [7, Theorem 4.1] show that for $\lambda > \lambda_0$ there are actually no solutions.

Now we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. Notice that condition (2.6) trivially follows from (1.6). The arguments of [6, p.4933] and the coercivity of J imply that the (PS) condition holds. Let X_-, X_+ and X_0 be the negative, positive and zero eigenspaces of the bilinear form Q defined in (2.2), with u_-, u_+ and u_0 being the respective orthogonal projections of u. We claim that J has a local linking at 0 with respect to the decomposition $(X_- \oplus X_0) \oplus X_+$. By the compactness of $X \hookrightarrow L^2(\mathbb{R}^3)$, both X_- and X_0 are finite dimensional and $(X_- \oplus X_0)$ has positive dimension because $\inf \sigma(-\Delta + V) \leq 0$. By the embedding $X \hookrightarrow L^r(\mathbb{R}^3)$ for $r \in \{p+1, q+1\}$, there holds

$$\left| \int F(u)\,\mathrm{d}x \right| \leq C\left(\|u\|^{p+1} + \|u\|^{q+1}\right),$$

so that

$$J(u) = Q(u,u) + G(u)$$

where, recalling that $p, q > 1$ and (2.4)

$$G(u) := \int \left[\frac{1}{4}\phi_u\, u^2 - F(u)\right] \mathrm{d}x = o(\|u\|^2), \qquad \text{as } \|u\| \to 0.$$

This immediately forces $J > 0$ on $B_R \cap X_+ \setminus \{0\}$ for suitably small $R > 0$. For u in the finite dimensional space $X_- \oplus X_0$ (where all norms are equivalent), it holds

$$\int F(u)\,\mathrm{d}x \geq c \int |u|^{p+1}\,\mathrm{d}x \geq \tilde{c}\|u\|^{p+1},$$

for some $\tilde{c} > 0$. Therefore, by (2.4) and (2.3) we deduce

$$J(u) \leq -\lambda_- \|u_-\|^2 + C\,\|u\|^4 - \tilde{c}\,\|u\|^{p+1} \qquad \text{for } u \in X_- \oplus X_0,$$

Since $p + 1 < 4$, this implies that $J < 0$ in $B_R \cap (X_- \oplus X_0) \setminus \{0\}$ for an even smaller $R > 0$. The conclusion now follows from Theorem 2.2. □

Remark 2.5. The condition $F(t) \geq c|t|^{p+1}$ is only used to deal with the case $\dim X_0 > 0$. If $\dim X_0 = 0$ it is not needed and the multiplicity result above holds under the sôle assumption (1.6) with $p, q \in]1, 2[$.

3. The superquadratic case

We first briefly discuss the assumptions used in this section to prove Theorem 1.2, namely $(V_0) - (V_2)$, (f_1) and (f_2), as well as some of their consequences.

- Assumption (V_1) can be seen as a lack oscillation condition at infinity. For coercive radial potentials $V(x) = v(|x|)$ it can be rephrased requiring that $r \mapsto v(r)\,r^2$ is non-decreasing. An example of coercive potential failing to satisfy (V_1) is $V(x) = |x|^2 + |x| \sin |x|^2$.

- Condition (V_2) rules out exponentially growing potentials for which the implication $u \in X \Rightarrow u(\lambda\,\cdot) \in X$ may fail for $\lambda \neq 1$. For example, if $V(x) = e^{|x|}$ and $u = e^{-|x|}/(1+|x|^4)$, then certainly $u \in X$ but $u(\lambda\,\cdot)$ fails to be in X for any $\lambda \in]0, 1[$. A quantitative version of this is given in Lemma 3.1.

- Notice that condition (f_1) implies that $|F(t)| \le C\,(|t|^2 + |t|^{p+1})$, $p < 5$, so that J is well defined on X. We avoid the critical case $p = 5$, since the functional should satisfy $(PS)_c$ for c less than some positive number c_0. Since our functional only has a local linking structure, and currently all critical point theorems related to local linking require $(PS)_c$ for all $c \in \mathbb{R}$, at present the critical case cannot be treated.

- Hypothesis (f_2) is a 3-superlinear condition of Ambrosetti-Rabinowitz type. By standard arguments, it implies that $t \mapsto F(t)/|t|^{\mu-1}t$ is non-decreasing and therefore,

$$F(t) \le C\,|t|^{\mu} \quad \text{for } |t| \le 1, \qquad F(t) \ge C\,|t|^{\mu} \quad \text{for } |t| \ge 1. \tag{3.1}$$

Lemma 3.1. *Assume (V_2) holds and let $\tilde{V} := V + m$. Then for any $t > 0$, $x \in \mathbb{R}^3$*

$$\tilde{V}(tx) \le \max\{t^{\kappa}, t^{-\kappa}\}\,\tilde{V}(x). \tag{3.2}$$

Proof. For $t \ge 1$ we have

$$\begin{aligned} \log \frac{\tilde{V}(tx)}{\tilde{V}(x)} &= \log \tilde{V}(tx) - \log \tilde{V}(x) = \int_1^t \frac{\mathrm{d}}{\mathrm{d}s} \log \tilde{V}(sx)\,\mathrm{d}s \\ &= \int_1^t \frac{\nabla \tilde{V}(sx)\cdot(sx)}{\tilde{V}(sx)}\,\frac{1}{s}\,\mathrm{d}s \le \int_1^t \frac{\kappa}{s}\,\mathrm{d}s = \log t^{\kappa}. \end{aligned}$$

Therefore $\tilde{V}(tx) \le t^{\kappa}\,\tilde{V}(x)$. The argument for the case $0 < t < 1$ is similar. □

For any $t > 0$ and $u \in X$ define

$$u_t(x) = t^2\,u(tx), \tag{3.3}$$

and define on the Hilbert space $\mathbb{R}\times X$ (with natural norm $\|(s,u)\|^2 = s^2 + \|u\|^2$) the augmented functional

$$\tilde{J}(s,u) := \frac{s^2}{2} + J(u_{e^s}). \tag{3.4}$$

Remark 3.2. Obviously, for $s, t > 0$, from (3.3) we have $(u_t)_s = u_{ts} = (u_s)_t$.

Proposition 3.3. *Assume (V_2) and (f_1). Then the functional $\tilde{J}$ is well defined on $\mathbb{R}\times X$, of class C^1 and*

$$\tilde{J}(s,u) = \frac{s^2}{2} + \frac{e^{3s}}{2}\int \left[|\nabla u|^2 + \frac{1}{2}\phi_u\,u^2\right]\mathrm{d}x + \frac{e^s}{2}\int V(xe^{-s})\,u^2\,\mathrm{d}x - e^{-3s}\int F(e^{2s}u)\,\mathrm{d}x, \tag{3.5}$$

$$\langle \partial_u \tilde{J}(s,u), \varphi\rangle = e^{3s}\int \left[\nabla u\,\nabla\varphi + \phi_u\,u\,\varphi\right]\mathrm{d}x + e^s\int V(xe^{-s})\,u\,\varphi\,\mathrm{d}x - e^{-s}\int f(e^{2s}u)\,\varphi\,\mathrm{d}x, \tag{3.6}$$

$$\begin{aligned} \partial_s \tilde{J}(s,u) = s + \frac{3}{2}e^{3s}\int \left[|\nabla u|^2 + \frac{1}{2}\phi_u\,u^2\right]\mathrm{d}x + \frac{e^s}{2}\int \left[V(xe^{-s}) - \nabla V(xe^{-s})\cdot xe^{-s}\right]u^2\,\mathrm{d}x \\ - e^{-3s}\int \left[2f(e^{2s}u)\,e^{2s}u - 3F(e^{2s}u)\right]\mathrm{d}x. \end{aligned} \tag{3.7}$$

Proof. By changing variables, it suffices to prove the statement for $J(t,u) = J(u_t)$ on $\mathbb{R}_+\times X$. A simple scaling argument shows that $\phi_{u_t}(x) = t^2\phi_u(tx)$, so that the following change of variables is justified by $\phi_u \in L^6(\mathbb{R}^3)$ and $u \in L^{12/5}(\mathbb{R}^3)$

$$\int \phi_{u_t}\,u_t^2\,\mathrm{d}x = t^3\int \phi_u\,u^2\,\mathrm{d}x.$$

Similarly,

$$\int |\nabla u_t|^2 \,\mathrm{d}x = t^3 \int |\nabla u|^2 \,\mathrm{d}x \le t^3 \|u\|^2,$$
$$\left| \int F(u_t)\,\mathrm{d}x \right| \le Ct \int u^2 \,\mathrm{d}x + C|t|^{2p-3} \int |u|^p \,\mathrm{d}x.$$

For the potential term, thanks to the continuity of V the change of variable $x = y/t$ is justified on any fixed ball B_R and the previous Lemma ensures

$$\int_{B_R} \tilde{V}\, u_t^2 \,\mathrm{d}x = t \int_{B_{tR}} \tilde{V}(y/t) u^2(y)\, dy \le \max\{t^\kappa, t^{-\kappa}\} \int_{B_{tR}} \tilde{V}\, u^2 \, dy \le c_t \|u\|^2,$$

so that letting $R \to +\infty$ proves that $\tilde{J}$ is well-defined. Formula (3.5) directly follows from changing variables.

Formula (3.6) can be computed in a standard way, while (3.7) is obtained by deriving under the integral sign in (3.5). Observe that

$$\left|2f(e^{2s}u)\, e^{2s}u - 3F(e^{2s}u)\right| \le C(e^{4s}|u|^2 + e^{2ps}|u|^6) \tag{3.8}$$

by the growth condition (f_1) and

$$\begin{aligned} \left|V(xe^{-s}) - \nabla V(xe^{-s}) \cdot xe^{-s}\right| &\le \left|V(xe^{-s})\right| + \left|\nabla V(xe^{-s}) \cdot xe^{-s}\right| \\ &\le m + (1+\kappa)\, \tilde{V}(xe^{-s}) \le m + (1+\kappa)\, e^{\kappa|s|}\, \tilde{V}(x) \end{aligned} \tag{3.9}$$

due to $|V| \le \tilde{V} + m$, (V_2) and Lemma 3.1. Moreover both

$$s \mapsto \int \left|2f(e^{2s}u)\, e^{2s}u - 3F(e^{2s}u)\right| \,\mathrm{d}x, \qquad s \mapsto \int \left|V(xe^{-s}) - \nabla V(xe^{-s}) \cdot xe^{-s}\right| u^2 \,\mathrm{d}x$$

are continuous by dominated convergence and standard arguments yields the differentiation formula (3.7). Finally, the estimates (3.8), (3.9) ensure the continuity of the corresponding Nemitskii operators appearing in (3.6) and (3.7), so that $\tilde{J}$ is of class C^1. □

Proposition 3.4 (Pohozaev identity)**.** *Assume (V_2) and (f_1) and let u be a critical point of J on X. Then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} J(u_t) = 0.$$

Proof. Let $u_{(t)}(x) = u(tx)$. The same argument used in the proof of Proposition 3.3 shows that under assumption (V_2) the curve $t \mapsto u_{(t)}$ is continuous in X at $t = 1$, and the functions $\varphi(t) := J(u_t)$ and $\psi(t) := J(u_{(t)})$ are differentiable at $t = 1$. By the mean value theorem

$$\begin{aligned} \varphi(t) - \psi(t) = J(u_t) - J(u_{(t)}) &= \langle DJ(\xi_t), u_t - u_{(t)} \rangle \\ &= \langle DJ(\xi_t), \left(t^2 - 1\right) u_{(t)} \rangle \\ &= (t^2 - 1)\langle DJ(\xi_t), u_{(t)} \rangle \end{aligned} \tag{3.10}$$

for some ξ_t lying on the segment from u_t to $u_{(t)}$. Therefore $\xi_t \to u$ in X as $t \to 1$, because both u_t and $u_{(t)}$ possess this property. Consequently, $DJ(\xi_t) \to DJ(u) = 0$ and

$$\left|\langle DJ(\xi_t), u_{(t)} \rangle\right| \le \|DJ(\xi_t)\|\, \|u_{(t)}\| \to 0 \qquad \text{for } t \to 1$$

because $u_{(t)}$ is continuous. It follows from (3.10) and $\varphi(1) = \psi(1)$ that $\varphi'(1) = \psi'(1)$, that is,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} J(u_t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} J(u_{(t)}).$$

By the usual form of the Pohozaev identity[3] the last term vanishes, thus proving the theorem. $\square$

In the following, we denote by $\tilde{D}\tilde{J}$ the total differential of $\tilde{J}$ with respect to both variables s and u.

Lemma 3.5. *If* (V_2) *and* (f_1) *hold, then*

$$\tilde{D}\tilde{J}(\bar{s},\bar{u}) = 0 \quad \Leftrightarrow \quad \bar{s} = 0 \ \text{ and } \ DJ(\bar{u}) = 0.$$

Proof. $(\Leftarrow)$ From (3.6), it follows that $DJ(\bar{u}) = 0$ implies $\partial_u \tilde{J}(0,\bar{u}) = 0$. Therefore, it suffices to prove that $\partial_s \tilde{J}(0,\bar{u}) = 0$. From $DJ(\bar{u}) = 0$, Proposition 3.4 gives

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} J(\bar{u}_t) = 0. \tag{3.11}$$

The map $t \mapsto J(\bar{u}_t)$ is C^1 by Proposition 3.3 and

$$J(\bar{u}_t) = \tilde{J}(\log t,\bar{u}) - \frac{\log^2 t}{2},$$

so that (3.11) reads

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \left(\tilde{J}(\log t,\bar{u}) - \frac{\log^2 t}{2}\right) = \left.\left(\partial_s \tilde{J}(\log t,\bar{u})\frac{1}{t} - \frac{\log t}{t}\right)\right|_{t=1} = \partial_s \tilde{J}(0,\bar{u}).$$

$(\Rightarrow)$ From $\tilde{D}\tilde{J}(\bar{s},\bar{u}) = 0$, being $\tilde{D} = (\partial_s, \partial_u)$, we immediately infer $0 = \partial_u \tilde{J}(\bar{s},\bar{u}) = \partial_u J(\bar{u}_{e^{\bar{s}}})$ and we only have to prove that $\bar{s} = 0$. Proposition 3.4 applied to $v := \bar{u}_{e^{\bar{s}}}$ gives

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} J(v_t) = 0.$$

The function $t \mapsto J(v_t)$ is C^1 by Proposition 3.3 and

$$J(v_t) = J(\bar{u}_{te^{\bar{s}}}) = \tilde{J}(\bar{s} + \log t,\bar{u}) - \frac{(\bar{s} + \log t)^2}{2},$$

where the first equality is due to Remark 3.2. By the Chain Rule and $\partial_s \tilde{J}(\bar{s},\bar{u}) = 0$ we have

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} J(v_t) = \left.\left(\partial_s \tilde{J}(\bar{s} + \log t,\bar{u})\frac{1}{t} - \frac{\bar{s} + \log t}{t}\right)\right|_{t=1} = \partial_s \tilde{J}(\bar{s},\bar{u}) - \bar{s} = -\bar{s}. \quad \square$$

Theorem 3.6. *Suppose that* (V_0)*–*(V_2) *and* (f_1)*–*(f_2) *hold. Then* $\tilde{J}$ *satisfies the* (PS) *condition.*

Proof. Let $\{(s_n, u_n)\}$ be a (PS)-sequence for $\tilde{J}$ in $\mathbb{R} \times X$. Then

$$|\tilde{J}(s_n, u_n)| + |\partial_s \tilde{J}(s_n, u_n)| = O(1).$$

[3] This follows from the standard technique (see [42, 43]) of multiplying the strong form of the equation by $\nabla u \cdot x$ (notice that $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^3)$ by elliptic regularity), integrate by parts in B_R and use the finiteness of the energy to get rid of the corresponding boundary terms for a suitable sequence of radii $R_n \to +\infty$. The nonlocal term involving ϕ_u can be treated through [44, Proof of Theorem 1.3].

Choose $\lambda \in]3, \mu[$, where $\mu > 3$ is given by (f_2). Then

$$(2\lambda - 3)\tilde{J}(s,u) - \partial_s \tilde{J}(s,u) = \frac{2\lambda-3}{2}s^2 - s + \frac{\lambda-3}{2}e^{3s}\int |\nabla u|^2\,\mathrm{d}x$$
$$+ \frac{\lambda-3}{2}e^{3s}\int \left[|\nabla u|^2 + \phi_u\, u^2\right]\,\mathrm{d}x + \frac{e^s}{2}\int \left[2(\lambda-2)V(xe^{-s}) + \nabla V(xe^{-s})\cdot xe^{-s}\right] u^2\,\mathrm{d}x$$
$$+ 2e^{-3s}\int \left[f(e^{2s}u)\,e^{2s}u - \lambda F(e^{2s}u)\right]\,\mathrm{d}x.$$

Using (2.5) on the second integral and (f_2) on the last, we thus obtain

$$(2\lambda - 3)\tilde{J}(s,u) - \partial_s \tilde{J}(s,u) \geq \frac{2\lambda-3}{2}s^2 - s + \frac{\lambda-3}{2}e^{3s}\int |\nabla u|^2\,\mathrm{d}x + (\lambda-3)e^{3s}\int |u|^3\,\mathrm{d}x$$
$$+ \frac{e^s}{2}\int \left[2(\lambda-2)V(xe^{-s}) + \nabla V(xe^{-s})\cdot xe^{-s}\right] u^2\,\mathrm{d}x + 2(\mu-\lambda)e^{-3s}\int F(e^{2s}u)\,\mathrm{d}x. \tag{3.12}$$

The third integral is bounded from below through (V_0)-(V_2) and Hölder's inequality. Indeed, set

$$v(x) = e^{3s/2}u(xe^s), \qquad W_\lambda(x) := 2(\lambda-2)V(x) + \nabla V(x)\cdot x.$$

Then, by a change of variables,

$$\int \left[2(\lambda-2)V(xe^{-s}) + \nabla V(xe^{-s})\cdot xe^{-s}\right] u^2\,\mathrm{d}x = \int W_\lambda\, v^2\,\mathrm{d}x. \tag{3.13}$$

As W_λ is bounded on bounded sets, we let $C_\lambda \in \mathbb{R}$ be such that $W_\lambda \geq -C_\lambda$ in B_R, R given in (V_1). Then

$$\int_{B_R} W_\lambda\, v^2\,\mathrm{d}x \geq -C_\lambda \int_{B_R} v^2\,\mathrm{d}x \geq -C_\lambda \left(\int |v|^3\,\mathrm{d}x\right)^{\frac{2}{3}}. \tag{3.14}$$

On $\mathbb{R}^3 \setminus B_R$, we split the integral on the two sets $\{V \geq 0\}$ and $\{V < 0\}$, the latter having finite measure by (V_0). Because $2(\lambda-2) > 2$, assumption (V_1) implies that

$$W_\lambda(x) = 2(\lambda-2)V(x) + \nabla V(x)\cdot x \geq 2V(x) + \nabla V(x)\cdot x \geq 0$$

for $x \in \{V \geq 0\} \setminus B_R$. On the other hand, by (V_2) and $V \geq -m$ we have

$$W_\lambda \geq 2(\lambda-2)V - \kappa(V+m) \geq -(2(\lambda-2)+\kappa)m \quad \text{on } \{V<0\}.$$

We thus have, for some possibily larger C_λ

$$\int_{\mathbb{R}^3\setminus B_R} W_\lambda\, v^2\,\mathrm{d}x \geq \int_{\{V<0\}\setminus B_R} W_\lambda\, v^2\,\mathrm{d}x \geq -C_\lambda \int_{\{V<0\}} v^2\,\mathrm{d}x \geq -C_\lambda\,|\{V<0\}|^{\frac{1}{3}} \left(\int |v|^3\,\mathrm{d}x\right)^{\frac{2}{3}}. \tag{3.15}$$

Combining (3.13), (3.14), (3.15) and computing $|v|_3$ in terms of $|u|_3$ by changing variable, we get

$$\int \left[2(\lambda-2)V(xe^{-s}) + \nabla V(xe^{-s})\cdot xe^{-s}\right] u^2\,\mathrm{d}x \geq -C_\lambda\, e^s \left(\int |u|^3\,\mathrm{d}x\right)^{\frac{2}{3}}. \tag{3.16}$$

Inserting the latter into (3.12), for our (PS)-sequence $\{(s_n, u_n)\}$, we have

$$O(1) \geq (2\lambda-3)\tilde{J}(s_n,u_n) - \partial_s \tilde{J}(s_n,u_n)$$
$$\geq \frac{2\lambda-3}{2}s_n^2 - s_n + \frac{\lambda-3}{2}e^{3s_n}\int |\nabla u_n|^2\,\mathrm{d}x + 2(\mu-\lambda)e^{-3s_n}\int F(e^{2s_n}u_n)\,\mathrm{d}x$$
$$+ (\lambda-3)e^{3s_n}\int |u_n|^3\,\mathrm{d}x - C_\lambda \left(e^{3s_n}\int |u_n|^3\,\mathrm{d}x\right)^{\frac{2}{3}}.$$

From $\lambda > 3$ we infer $(\lambda-3)\xi_n - C_\lambda \xi_n^{2/3} \to +\infty$ if $\xi_n = e^{3s_n}|u_n|_3^3 \to +\infty$, while also using $\mu - \lambda > 0$ and $F \geq 0$ we deduce from the previous estimate that

$$|s_n|, \quad \int |\nabla u_n|^2 \,\mathrm{d}x, \quad \int |u_n|^3 \,\mathrm{d}x, \quad \int F(e^{2s_n}u_n) \quad \text{are bounded}, \tag{3.17}$$

and recalling that $\tilde{J}(s_n, u_n) = O(1)$ we also get through the previous bounds

$$\int V(xe^{-s_n})\, u_n^2 \,\mathrm{d}x \leq O(1). \tag{3.18}$$

To complete the proof of the boundedness of $\|u_n\|$, let $S \geq 1$ be such that $|\kappa s_n| \leq S$. Applying Lemma 3.1 for x being xe^{-s_n} and t being e^{s_n}, we get

$$V(x) + m \leq e^S \left(V(xe^{-s_n}) + m \right). \tag{3.19}$$

Choose $k \geq m$ large enough such that

$$\frac{1}{2} e^{-S} k \geq \left(1 - e^{-S}\right) m.$$

Using (3.19), if $V(x) > k$, we have

$$V(xe^{-s_n}) \geq e^{-S}V(x) - \left(1 - e^{-S}\right) m \geq \frac{e^{-S}}{2} V(x).$$

Thus, also using $V \geq -m$ on $\{V \leq k\}$, we deduce

$$\begin{aligned}
\int V(xe^{-s_n}) u_n^2 \,\mathrm{d}x &= \int_{\{V>k\}} V(xe^{-s_n}) u_n^2 \,\mathrm{d}x + \int_{\{V \leq k\}} V(xe^{-s_n}) u_n^2 \,\mathrm{d}x \\
&\geq \frac{e^{-S}}{2} \int_{\{V>k\}} V u_n^2 \,\mathrm{d}x - m \int_{\{V \leq k\}} u_n^2 \,\mathrm{d}x \\
&\geq \frac{e^{-S}}{2} \int_{\{V>k\}} V u_n^2 \,\mathrm{d}x - m \, |\{V \leq k\}|^{1/3} \left(\int |u_n|^3 \right)^{2/3} \\
&\geq \frac{e^{-S}}{2} \int_{\{V>k\}} V u_n^2 \,\mathrm{d}x - O(1),
\end{aligned}$$

where we used (V_0) and (3.17) in the last inequality. From (3.18) we thus infer

$$\int_{\{V>k\}} V\, u_n^2 \,\mathrm{d}x \leq O(1).$$

Finally, due to $k > m$ it holds $V + m \leq 2V$ on the set $\{V > k\}$ and $V + m \leq 2k$ on $\{V \leq k\}$, thus

$$\begin{aligned}
\|u_n\|^2 &\leq \int |\nabla u_n|^2 \,\mathrm{d}x + 2 \int_{\{V>k\}} V\, u_n^2 \,\mathrm{d}x + 2k \int_{\{V \leq k\}} u_n^2 \,\mathrm{d}x \\
&\leq \int |\nabla u_n|^2 \,\mathrm{d}x + 2 \int_{\{V>k\}} V\, u_n^2 \,\mathrm{d}x + 2k \, |\{V \leq k\}|^{\frac{1}{3}} \left(\int |u_n|^3 \,\mathrm{d}x \right)^{\frac{2}{3}} \leq O(1)
\end{aligned}$$

by (3.17), proving the boundedness of $\{u_n\}$ in X. Finally, the proof of the strong compactness of $\{u_n\}$ again follows as in [6] thanks to the compactness of $X \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in [2, 6[$. $\square$

Lemma 3.7. *Assume* (f_1)-(f_2), (V_0)-(V_2). *For any* $\lambda \in \,]3, \mu[$, *there exists* $M_\lambda \geq 0$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{J}(\tau, u_t) \leq \frac{2\lambda - 3}{t} \left(\tilde{J}(\tau, u_t) + M_\lambda \right), \qquad t > 0, u \in X, \tau \in \mathbb{R}. \tag{3.20}$$

Proof. The estimate is independent of τ and u, so we let $v = u_{e^\tau}$ and observe that

$$\tilde{J}(\tau, u_t) = \frac{\tau^2}{2} + J(u_{te^\tau}) = \frac{\tau^2}{2} - \frac{\log^2 t}{2} + \tilde{J}(\log t, v) \geq \tilde{J}(\log t, v) - \frac{\log^2 t}{2} \tag{3.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{J}(\tau, u_t) = -\frac{\log t}{t} + \frac{1}{t}\partial_s \tilde{J}(\log t, v). \tag{3.22}$$

We claim that for given $\lambda \in]3, \mu[$, there exists $M_\lambda > 0$ such that

$$\partial_s \tilde{J}(s, v) - s \leq (2\lambda - 3)\left(\tilde{J}(s, v) - \frac{s^2}{2} + M_\lambda\right) \tag{3.23}$$

for all $s \in \mathbb{R}$ and $v \in X$. Then, using (3.21) and (3.22), with $s = \log t$ and $v = u_{e^\tau}$ in (3.23) we deduce

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{J}(\tau, u_t) &= -\frac{\log t}{t} + \frac{1}{t}\partial_s \tilde{J}(\log t, v) \\
&\leq \frac{2\lambda - 3}{t}\left(\tilde{J}(\log t, u_{e^\tau}) - \frac{\log^2 t}{2} + M_\lambda\right) = \frac{2\lambda - 3}{t}\left(J(u_{te^\tau}) + M_\lambda\right) \\
&\leq \frac{2\lambda - 3}{t}\left(\tilde{J}(\tau, u_t) + M_\lambda\right),
\end{aligned}$$

proving (3.20). To prove (3.23) ignore the nonnegative terms involving ∇u and F in (3.12) and use (3.16) to get

$$(2\lambda - 3)\tilde{J}(s, v) - \partial_s \tilde{J}(s, v) \geq \frac{2\lambda - 3}{2}s^2 - s + (\lambda - 3)\, e^{3s}\int |v|^3 \,\mathrm{d}x - C_\lambda \left(e^{3s}\int |v|^3 \,\mathrm{d}x\right)^{\frac{2}{3}}.$$

The last two terms are bounded from below thanks to $\lambda > 3$, thus (3.23) is proved. □

Lemma 3.8. *Suppose* (f_1)*-*(f_2) *and* (V_1) *hold true. Then, for any* $(s, u) \in \mathbb{R} \times X \setminus \{0\}$ *it holds*

$$\lim_{t \to +\infty} \tilde{J}(s, u_t) = -\infty.$$

Proof. Considering $v = u_{e^s}$ it suffices to prove that $J(v_t) \to -\infty$ as $t \to +\infty$. As in the proof of (3.5) we get

$$J(v_t) = \frac{t^3}{2}\int \left[|\nabla v|^2 + \frac{1}{2}\phi_v\, v^2\right] \mathrm{d}x + \frac{t}{2}\int V\, t^3 v^2(tx)\,\mathrm{d}x - t^{-3}\int F(t^2 v)\,\mathrm{d}x.$$

Since $v \neq 0$, we can suppose that for some $\varepsilon > 0$, $|\{|v| \geq \varepsilon\}|$ is finite and positive and by (3.1) we have

$$\int F(t^2 v)\,\mathrm{d}x \geq C\int_{\{|v|\geq\varepsilon\}} t^{2\mu}\,|v|^\mu\,\mathrm{d}x \geq C\,\varepsilon^\mu\,|\{|v| \geq \varepsilon\}|\, t^{2\mu} =: C_v\, t^{2\mu},$$

for some $C_v > 0$ and $t^2 \geq 1/\varepsilon$. V is bounded on B_R, therefore

$$\int_{B_R} V\, t^3 v^2(tx)\,\mathrm{d}x \leq \|V\|_{L^\infty(B_R)}\int t^3 v^2(tx)\,\mathrm{d}x = \|V\|_{L^\infty(B_R)}\int v^2\,\mathrm{d}x.$$

Assumption (V_1) implies that for any $|\omega| = R$ and $r \geq 1$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\tilde{V}(r\omega)r^2\right) = r\left(2\tilde{V}(r\omega) + \nabla V(r\omega)\cdot r\omega\right) \geq 0,$$

where as usual $\tilde{V} = V + m$, so that

$$H(x) = \tilde{V}(x)\,|x|^2\,\chi_{\mathbb{R}^3\setminus B_R}(x) \quad \text{is radially non-decreasing.}$$

Letting $w(x) = v(x)/|x|$, we have

$$\int_{\mathbb{R}^3\setminus B_R} \tilde{V}\,t^3 v^2(tx)\,\mathrm{d}x = t^2 \int H\,t^3 w^2(tx)\,\mathrm{d}x = t^2 \int H(x/t)\,w^2\,\mathrm{d}x,$$

and by the monotonicity of H, $H(x/t)\,w^2 \searrow 0$ as $t \to +\infty$, so that by monotone convergence the last integral vanishes as $t \to +\infty$. Therefore

$$\int V\,t^3 v^2(tx)\,\mathrm{d}x \le \int_{B_R} V\,t^3 v^2(tx)\,\mathrm{d}x + \int_{\mathbb{R}^3\setminus B_R} \tilde{V}\,t^3 v^2(tx)\,\mathrm{d}x \le \|V\|_{L^\infty(B_R)} \int v^2\,\mathrm{d}x + o(t^2).$$

Summing up,

$$J(v_t) \le \frac{t^3}{2} \int \left[|\nabla v|^2 + \phi_v\,v^2\right]\,\mathrm{d}x + \frac{t\,\|V\|_{L^\infty(B_R)}}{2} \int v^2\,\mathrm{d}x + o(t^3) - C_v\,t^{2\mu-3} \to -\infty,$$

as $t \to +\infty$, because $2\mu - 3 > 3$. $\square$

Theorem 3.9. *Assume* (f_1)-(f_2) *and* (V_0)-(V_2)*. Then for any sufficiently negative* $a \in \mathbb{R}$*, it holds*

$$H_q(\mathbb{R} \times X, \{\tilde{J} \le a\}) = 0 \qquad \textit{for any } q \in \{0, 1, 2, \ldots, \}.$$

Proof. Let $\dot{X} = X \setminus \{0\}$ and consider the continuous map

$$\mathbb{R} \times \dot{X} \times \mathbb{R}_+ \ni (s, u, t) \mapsto (s, u_t) \in \mathbb{R} \times \dot{X}.$$

Fix $\lambda \in\,]3, \mu[$ and $a < -M_\lambda$, where $M_\lambda \ge 0$ is given in Lemma 3.7. Then, by the previous Lemma, for any $(s, u) \in \mathbb{R} \times \dot{X}$

$$\lim_{t\to+\infty} \tilde{J}(s, u_t) = -\infty.$$

Therefore, we infer from (3.20) that the implicit equation $\tilde{J}(s, u_t) = a$ has a unique solution $t = \varphi(s, u)$ for any $(s, u) \in \mathbb{R} \times \dot{X}$ such that $\tilde{J}(s, u) > a$, and

$$\varphi : \{(s, u) \in \mathbb{R} \times \dot{X} : \tilde{J}(s, u) > a\} \to \mathbb{R}_+$$

is continuous by a standard application of the implicit function theorem. The map

$$\Phi : [0, 1] \times \mathbb{R} \times \dot{X} \to \mathbb{R} \times \dot{X}, \qquad \Phi\,(\xi, (s, u)) = \begin{cases} \left(s, u_{1-\xi+\xi\varphi(s,u)}\right) & \text{if } \tilde{J}(s, u) > a, \\ (s, u) & \text{if } \tilde{J}(s, u) \le a, \end{cases}$$

is a deformation retract of $\mathbb{R} \times \dot{X}$ onto $\{\tilde{J} \le a\}$, so that by homotopy invariance

$$H_*(\mathbb{R} \times X, \{\tilde{J} \le a\}) = H_*(\mathbb{R} \times X, \mathbb{R} \times \dot{X}).$$

Since $\mathbb{R} \times \dot{X}$ deformation retracts to $\{0\} \times S^\infty$, which is contractible in itself, we get the claim. $\square$

Recall that X_+, X_- and X_0 are the negative, positive and null eigenspaces of the bilinear form defined in (2.2).

Lemma 3.10. *Assume* (f_1)*-*(f_2)*,* (V_0)*-*(V_2) *and consider the decomposition* $\mathbb{R}\times X = \tilde{X}_- \oplus \tilde{X}_+$ *where*

$$\tilde{X}_- = X_- \oplus X_0, \qquad \tilde{X}_+ = \mathbb{R} \oplus X_+.$$

Then the functional $\tilde{J}$ *has a local linking in the following cases*

1. $\dim X_- > 0$*,* $\dim X_0 = 0$*.*
2. $\dim X_0 > 0$ *and* $F(t) \geq c\,|t|^\nu$ *for some* $\nu < 4$*.*

Proof. We first show that for some $r > 0$

$$\tilde{J}(s,u) > 0 \qquad \text{for } |s| < r, \quad u \in X_+, \quad \|u\| < r. \tag{3.24}$$

We have

$$\begin{aligned}
\int V(xe^{-s})\,u^2\,\mathrm{d}x &= \int V\,u^2\,\mathrm{d}x + \int\left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}V(e^{-s\tau}x)\,\mathrm{d}\tau\right)u^2\,\mathrm{d}x \\
&= \int V\,u^2\,\mathrm{d}x - s\int\left(\int_0^1 \nabla V(e^{-s\tau}x)\cdot(e^{-s\tau}x)\,\mathrm{d}\tau\right)u^2\,\mathrm{d}x,
\end{aligned}$$

therefore

$$\begin{aligned}
\left|\int V(xe^{-s})\,u^2\,\mathrm{d}x - \int V\,u^2\,\mathrm{d}x\right| &\leq |s|\int\left(\int_0^1 \left|\nabla V(e^{-s\tau}x)\cdot(e^{-s\tau}x)\right|\,\mathrm{d}\tau\right)u^2\,\mathrm{d}x \\
&\leq \kappa\,|s|\int\left(\int_0^1 \tilde{V}(e^{-s\tau}x)\,\mathrm{d}\tau\right)u^2\,\mathrm{d}x && \text{by } (V_2) \\
&\leq \kappa\,|s|\int\left(\int_0^1 e^{\kappa|s|\tau}\tilde{V}(x)\,\mathrm{d}\tau\right)u^2\,\mathrm{d}x && \text{by (3.2)} \\
&\leq \left(e^{\kappa|s|}-1\right)\int \tilde{V}\,u^2\,\mathrm{d}x \leq O(s)\,\|u\|^2 = o(\|(s,u)\|^2)
\end{aligned}$$

since, as $(s,b)\to(0,0)$, $sb^2 = o(|(s,b)|^2)$. Moreover, (f_1) and (3.1) imply that

$$|F(t)| \leq C\left(|t|^\mu + |t|^{p+1}\right),$$

so that, for $\|(s,u)\| \to 0$, we have

$$e^{-3s}\left|\int F(e^{2s}u)\,\mathrm{d}x\right| \leq C\int |u|^\mu + |u|^{p+1}\,\mathrm{d}x \leq o(\|u\|^2),$$

while by (2.1) we easily have $|V| \leq 2\tilde{V}$, hence

$$|s|\int |V|\,u^2\,\mathrm{d}x \leq 2|s|\int \tilde{V}\,u^2\,\mathrm{d}x = o(\|(s,u)\|^2).$$

Gathering these estimates and recalling (2.4), we get

$$\begin{aligned}
\tilde{J}(s,u) &= \frac{s^2}{2} + \frac{e^{3s}}{2}\int\left[|\nabla u|^2 + \frac{1}{2}\phi_u\,u^2\right]\mathrm{d}x + \frac{e^s}{2}\int V(xe^{-s})\,u^2\,\mathrm{d}x - e^{-3s}\int F(e^{2s}u)\,\mathrm{d}x \\
&= \frac{s^2}{2} + \frac{1+O(s)}{2}\int |\nabla u|^2\,\mathrm{d}x + \frac{1+O(s)}{2}\int V\,u^2\,\mathrm{d}x \\
&\qquad\qquad + O(\|u\|^4) + o(\|(s,u)\|^2) + o(\|u\|^2) \\
&= \frac{1}{2}\int\left[|\nabla u|^2 + V\,u^2\right]\,\mathrm{d}x + \frac{s^2}{2} + O(\|u\|^4) + o(\|(s,u)\|^2).
\end{aligned}$$

The latter readily yelds (3.24) and, for $s = 0$,

$$\tilde{J}(0, u) < 0 \qquad \text{for } u \in X_-, \quad \|u\| < r,$$

proving the claimed local linking in case (1).

In case (2) we proceed as in Theorem 1.1: the previous computations yield

$$\tilde{J}(0, u) \le \frac{1}{2} \int \left[|\nabla u|^2 + V\, u^2 \right] \mathrm{d}x + O(\|u\|^4) - c \int |u|^\nu \, \mathrm{d}x,$$

and being all norms in $X_- \oplus X_0$ equivalent we deduce

$$\tilde{J}(0, u) \le -\lambda_- \|u_-\|^2 + O(\|u\|^4) - c' \, \|u\|^\nu \qquad \text{for } u \in X_- \oplus X_0.$$

Thanks to $\nu < 4$, we infer

$$\tilde{J}(0, u) < 0 \qquad \text{for } u \in X_- \oplus X_0, \quad \|u\| < r,$$

concluding the proof in this case. □

Theorem 3.11. *Assume (f_1)-(f_2), (V_0)-(V_2) and either*

1. $\dim X_- > 0$, $\dim X_0 = 0$.
2. $\dim X_0 > 0$ *and* $F(t) \ge c\,|t|^\nu$ *for some* $\nu < 4$.

Then problem (1.2) has at least a nontrivial solution.

Proof. Theorems 3.6, 3.9 and the previous Lemma allow to apply [45, Corollary 2.3], giving a critical point $(\bar{s}, \bar{u}) \ne (0, 0)$ for $\tilde{J}$. But then Lemma 3.5 forces $\bar{s} = 0$, $\bar{u} \ne 0$ and $DJ(\bar{u}) = 0$. □

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