Journal of Nonlinear and Convex Analysis
Volume 20, Number 8, 2019, 1–

# APPLYING TWICE A MINIMAX THEOREM

BIAGIO RICCERI

*To Professor Wataru Takahashi, with esteem and friendship, on his 75th birthday*

Abstract. Here is one of the results obtained in this paper: Let $X, Y$ be two convex sets each in a real vector space, let $J : X \times Y \to \mathbf{R}$ be convex and without global minima in $X$ and concave in $Y$, and let $\Phi : X \to \mathbf{R}$ be strictly convex. Also, assume that, for some topology on $X$, $\Phi$ is lower semicontinuous and, for each $y \in Y$ and $\lambda > 0$, $J(\cdot, y)$ is lower semicontinuous and $J(\cdot, y) + \lambda\Phi(\cdot)$ is inf-compact.

Then, for each $r \in ]\inf_X \Phi, \sup_X \Phi[$ and for each closed set $S \subseteq X$ satisfying

$$\Phi^{-1}(r) \subseteq S \subseteq \Phi^{-1}(]-\infty, r]) ,$$

one has

$$\sup_Y \inf_S J = \inf_S \sup_Y J .$$

## 1. Introduction

A real-valued function $f$ on a topological space is said to be inf-compact (resp. sup-compact) if $f^{-1}(]-\infty, r])$ (resp. $f^{-1}([r, +\infty[)$ is compact for all $r \in \mathbf{R}$.

A real-valued function $f$ on a convex set is said to be quasi-concave if $f^{-1}([r, +\infty[)$ is convex for all $r \in \mathbf{R}$.

In [3], we proved two general minimax theorems which, grouped together, can be stated as follows:

**Theorem 1.1** ([3], Theorems 1.1 and 1.2)**.** *Let $X$ be a topological space, $Y$ a convex set in a Hausdorff real topological vector space and $f : X \times Y \to \mathbf{R}$ a function such that $f(\cdot, y)$ is lower semicontinuous, inf-compact and has a unique global minimum for all $y \in Y$. Moreover, assume that either, for each $x \in X$, $f(x, \cdot)$ is continuous and quasi-concave or, for each $x \in X$, $f(x, \cdot)$ is concave.*

*Then, one has*

$$\sup_Y \inf_X f = \inf_X \sup_Y f .$$

Theorem 1.1 was first proved in the case where $Y$ is a real interval ([1], [2]) and successively extended to the general case by means of a suitable inductive argument.

In [1], we applied Theorem 1.1 (with $Y$ a real interval) to obtain a result ([1], Theorem 1) about the following problem: given two functions $f, g : X \to \mathbf{R}$, find

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2010 *Mathematics Subject Classification.* 49J35, 49K35, 49K27, 90C47.

*Key words and phrases.* Minimax theorem, global minimum; iniqueness.

The author has been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by the Università degli Studi di Catania, "Piano della Ricerca 2016/2018 Linea di intervento 2".

a interval $I \subseteq g(X)$ such that, for each $r \in I$, the restriction of $f$ to $g^{-1}(r)$ has a unique global minimum.

The aim of the present paper is to establish a new minimax theorem (Theorem 2.1) which is the fruit of a joint application of Theorem 1.1 and Theorem 1 of [1]. So, it follows, essentially, from a double application of Theorem 1.1, as the title stresses.

We then show some consequences of Theorem 2.1.

## 2. Results

In the sequel, $X$ is a topological space, $Y$ is a non-empty set, $J : X \times Y \to \mathbf{R}$, $\Phi : X \to \mathbf{R}$, $a, b$ are two numbers in $[0, +\infty]$, with $a < b$.

For $y \in Y$ and $\lambda \in [0, +\infty]$, we denote by $M_{\lambda,y}$ the set of all global minima of the function $J(\cdot, y) + \lambda\Phi(\cdot)$ if $\lambda < +\infty$, while if $\lambda = +\infty$, $M_{\lambda,y}$ stands for the empty set. We adopt the conventions $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$. We also set

$$\alpha := \sup_{y\in Y} \max \left\{ \inf_X \Phi, \sup_{M_{b,y}} \Phi \right\} ,$$

$$\beta := \inf_{y\in Y} \min \left\{ \sup_X \Phi, \inf_{M_{a,y}} \Phi \right\} .$$

The following assumption will be adopted:

$(a)$ $Y$ is a convex set in a Hausdorff real topological vector space and either, for each $x \in X$, the function $J(x, \cdot)$ is continuous and quasi-concave, or, for each $x \in X$, the function $J(x, \cdot)$ is concave.

Our main result is as follows:

**Theorem 2.1.** *Besides $(a)$, assume that:*
$(a_1)$ $\alpha < \beta$ *;*
$(a_2)$ $\Phi$ *is lower semicontinuous ;*
$(a_3)$ *for each $\lambda \in ]a, b[$ and each $y \in Y$, the function $J(\cdot, y)$ is lower semicontinuous and the function $J(\cdot, y) + \lambda\Phi(\cdot)$ is inf-compact and admits a unique global minimum in $X$.*

*Then, for each $r \in ]\alpha, \beta[$ and for each closed set $S \subseteq X$ satisfying*

$$\Phi^{-1}(r) \subseteq S \subseteq \Phi^{-1}(]-\infty, r]) , \tag{2.1}$$

*one has*

$$\sup_Y \inf_S J = \inf_S \sup_Y J . \tag{2.2}$$

*Proof.* Since $r \in ]\alpha, \beta[$, for each $y \in Y$, Theorem 1 of [1] (see Remark 1 of [1]) ensures the existence of $\lambda_{r,y} \in ]a, b[$ such that the unique global minimum of $J(\cdot, y) + \lambda_{r,y}\Phi(\cdot)$, say $x_{r,y}$, lies in $\Phi^{-1}(r)$. Notice that $x_{r,y}$ is the only global minimum of the restriction of the function $J(\cdot, y)$ to $\Phi^{-1}(]-\infty, r])$. Indeed, if not, there would exist $u \in \Phi^{-1}(]-\infty, r])$, with $u \neq x_{r,y}$, such that $J(u, y) \leq J(x_{r,y}, y)$. Then, (since $\lambda_{r,y} > 0$) we would have

$$J(u, y)+\lambda_{r,y}\Phi(u) \leq J(x_{r,y}, y)+\lambda_{r,y}\Phi(u) \leq J(x_{r,y}, y)+\lambda_{r,y}r = J(x_{r,y}, y)+\lambda_{r,y}\Phi(x_{r,y})$$

which is absurd. Therefore, since $S$ satisfies (2.1), the restriction of $J(\cdot, y)$ to $S$ has a unique global minimum. Now, observe that, for each $y \in Y$, $\rho \in \mathbf{R}$, $\lambda \in ]a, b[$, one has

$$\{x \in S : J(x, y) \leq \rho\} \subseteq \{x \in X : J(x, y) + \lambda\Phi(x) \leq \rho + \lambda r\} .$$

By assumption, the set on the right-hand side is compact. Hence, the set $\{x \in S : J(x, y) \leq \rho\}$, being closed, is compact too. Summarizing: for each $y \in Y$, the restriction of the function $J(\cdot, y)$ to $S$ is lower semicontinuous, inf-compact and has a unique global minimum. So, $J_{|S\times Y}$ satisfies the hypoteses of Theorem 1.1 and hence (2.2) follows. □

**Remark 2.2.** From the above proof, it follows that, when $X$ is Hausdorff and each sequentially compact subset of $X$ is compact, Theorem 2.1 is still valid if we replace "lower semicontinuous", "inf-compact", "closed" with "sequentially lower semicontinuous", "sequentially inf-compact", "sequentially closed", respectively.

We now draw a series of consequences from Theorem 2.1

**Corollary 2.3.** *In addition to the assumptions of Theorem 2.1, suppose that $\beta = \sup_X \Phi$ and that $\Phi$ has no global maxima. Moreover, suppose that the function $J(x, \cdot)$ is upper semicontinuous for all $x \in X$ and $J(x_0, \cdot)$ is sup-compact for some $x_0 \in X$.*

*Then, one has*

$$\sup_Y \inf_X J = \inf_X \sup_Y J .$$

*Proof.* Clearly, the assumptions imply that

$$X = \bigcup_{\alpha<r<\beta} \Phi^{-1}(]-\infty, r]) .$$

Since the family $\{\Phi^{-1}(]-\infty, r])\}_{r\in]\alpha,\beta[}$ is filtering with respect to inclusion, the conclusion follows from a joint application of Theorem 2.1 and Proposition 2.1 of [3]. □

Another corollary of Theorem 2.1 is as follows:

**Corollary 2.4.** *Besides $(a)$, assume that $X$ is a convex set in a real vector space and that:*
$(b_1)$ *$\Phi$ is lower semicontinuous and strictly convex ;*
$(b_2)$ *for each $\lambda > 0$ and each $y \in Y$, the function $J(\cdot, y)$ is convex, lower semicontinuous and has no global minima, and the function $J(\cdot, y) + \lambda\Phi(\cdot)$ is inf-compact.*

*Then, for each $r \in ]\inf_X \Phi, \sup_X \Phi[$ and for each closed set $S \subseteq X$ satisfying*

$$\Phi^{-1}(r) \subseteq S \subseteq \Phi^{-1}(]-\infty, r]) ,$$

*one has*

$$\sup_Y \inf_S J = \inf_S \sup_Y J .$$

*Proof.* We apply Theorem 2.1 taking $a = 0$ and $b = +\infty$. So, we have

$$\alpha = \inf_X \Phi$$

as well as

$$\beta = \sup_X \Phi$$

since $M_{0,y} = \emptyset$ for all $y \in Y$. By strict convexity, the function $J(\cdot, y) + \lambda\Phi(\cdot)$ has a unique global minimum for all $y \in Y$, $\lambda > 0$. So, each assumption of Theorem 2.1 is satisfied and the conclusion follows. $\square$

**Remark 2.5.** We stress that, in Corollary 2.4, no relation is required between the considered topology on $X$ and the algebraic structure of the vector space which contains it.

**Remark 2.6.** In the setting of Corollary 2.4, although $J$ is convex in $X$, we cannot apply the classical Fan-Sion theorem when $S$ is not convex.

If $E, F$ are Banach spaces and $A \subseteq E$, a function $\psi : A \to F$ is said to be $C^1$ if it is the restriction to $A$ of a $C^1$ function on an open convex set containing $A$.

A further remarkable corollary of Theorem 2.1 is as follows:

**Corollary 2.7.** *Besides $(a)$, assume that $X$ is a closed and convex set in a reflexive real Banach space $E$ and that:*
$(c_1)$ $\Phi$ *is of class $C^1$ and there is $\nu > 0$ such that*

$$(\Phi'(x) - \Phi'(u))(x - u) \geq \nu\|x - u\|^2$$

*for all $x, u \in X$ ;*
$(c_2)$ *for each $y \in Y$, the function $J(\cdot, y)$ is $C^1$, sequentially weakly lower semicontinuous and $J'_x(\cdot, y)$ is Lipschitzian with constant L (independent of $y$) ;*
$(c_3)$ $\inf_{y\in Y} \inf_{M_{\frac{L}{\nu},y}} \Phi > \inf_X \Phi$ .

*Then, for each $r \in \left]\inf_X \Phi, \inf_{y\in Y} \inf_{M_{\frac{L}{\nu},y}} \Phi\right[$ and for each sequentially weakly closed set $S \subseteq X$ satisfying*

$$\Phi^{-1}(r) \subseteq S \subseteq \Phi^{-1}(]-\infty, r]) ,$$

*one has*

$$\sup_Y \inf_S J = \inf_S \sup_Y J .$$

*Proof.* For each $x, u \in X$, $y \in Y$, $\lambda \geq \frac{L}{\nu}$, we have

$$(J'_x(x, y) + \lambda\Phi'(x) - J'_x(u, y) - \lambda\Phi'(u))(x - u)$$
$$\geq \lambda\nu\|x - u\|^2 - \|J'_x(x, y) - J'_x(u, y)\|_{E^*}\|x - u\| \geq (\lambda\nu - L)\|x - u\|^2 .$$

Hence, the function $J(\cdot, y) + \lambda\Phi(\cdot)$, if $\lambda > \frac{L}{\nu}$, is strictly convex and coercive when $X$ is unbounded ([4], pp. 247-249). Hence, if we consider $X$ with the relative weak topology, we can apply Theorem 2.1 (in the sequential form pointed out in Remark 2.2) taking $a = \frac{L}{\nu}$ and $b = +\infty$, and the conclusion follows. $\square$

If $E$ is a normed space, for each $r > 0$, we put

$$B_r = \{x \in E : \|x\| \leq r\} .$$

If $A \subseteq E$, a function $f : A \to E$ is said to be sequentially weakly-strongly continuous if, for each $x \in A$ and for each sequence $\{x_k\}$ in $A$ weakly converging to $x$, the sequence $\{f(x_k)\}$ converges strongly to $f(x)$.

**Corollary 2.8.** *Let $E$ be a real Hilbert space and let $X = B_\rho$ for some $\rho > 0$. Besides (a) and ($c_2$), assume that*

$$\delta := \inf_{y\in Y} \|J'_x(0,y)\| > 0 \ .$$

*Then, for each $r \in \left]0, \min\left\{\rho, \frac{\delta}{2L}\right\}\right[$, one has*

$$\sup_Y \inf_{B_r} J = \inf_{B_r} \sup_Y J \ .$$

*Proof.* Apply Corollary 2.7, taking $\Phi(x) = \|x\|^2$. Let $y \in Y$ and $\tilde{x} \in M_{\frac{L}{2},y}$, with $\|\tilde{x}\| < \rho$. Then, we have

$$J'_x(\tilde{x}, y) + L\tilde{x} = 0 \ .$$

Consequently, in view of ($c_2$), we have

$$\|L\tilde{x} + J'_x(0,y)\| \leq \|L\tilde{x}\| \ .$$

In turn, using the Cauchy-Schwarz inequality, this readily implies that

$$\|\tilde{x}\| \geq \frac{\|J'_x(0,y)\|}{2L} \geq \frac{\delta}{2L} \ .$$

Therefore, we have the estimate

$$\inf_{y\in Y} \inf_{x\in M_{\frac{L}{2},y}} \|x\| \geq \min\left\{\rho, \frac{\delta}{2L}\right\}$$

and the conclusion follows from Corollary 2.7. □

We now apply Corollary 2.8 to a particular function $J$.

**Corollary 2.9.** *Let $E, X$ be as in Corollary 2.8, let $Y \subseteq E$ be a closed bounded convex set and let $f : X \to E$ be a sequentially weakly-strongly continuous $C^1$ function whose derivative is Lipschitzian with constant $\gamma$. Moreover, let $\eta$ be the Lipschitz constant of the function $x \to x - f(x)$, set*

$$\theta := \sup_{x\in X} \|f'(x)\|_{\mathcal{L}(E)} \ ,$$

$$L := 2\left(\eta + \theta + \gamma\left(\rho + \sup_{y\in Y}\|y\|\right)\right)$$

*and assume that*

$$\sigma := \inf_{y\in Y} \sup_{\|u\|=1} |\langle f'(0)(u), y\rangle - \langle f(0), u\rangle| > 0 \ .$$

*Then, for each $r \in \left]0, \min\left\{\rho, \frac{\sigma}{L}\right\}\right[$ and for each non-empty closed convex set $T \subseteq Y$, there exist $x^* \in \partial B_r$ and $y^* \in T$ such that*

$$\|x^* - f(x^*)\|^2 + \|f(x) - y^*\|^2 - \|x - f(x)\|^2 \leq \|f(x^*) - y^*\|^2 = (\mathrm{dist}(f(x^*), T))^2$$

*for all $x \in B_r$ .*

*Proof.* Consider the function $J : X \times Y \to \mathbf{R}$ defined by

$$J(x, y) = \|f(x) - x\|^2 - \|f(x) - y\|^2$$

for all $x \in X$, $y \in Y$. Clearly, for each $y \in Y$, $J(\cdot, y)$ is sequentially weakly lower semicontinuos and $C^1$. Moreover, one has

$$\langle J'_x(x, y), u\rangle = 2\langle x - f(x), u\rangle - 2\langle f'(x)(u), x - y\rangle$$

for all $x \in X$, $u \in E$. Fix $x, v \in X$ and $u \in E$, with $\|u\| = 1$. We have

$$\begin{aligned}
&\frac{1}{2}|\langle J'_x(x, y) - J'_x(v, y), u\rangle| \\
&= |\langle x - f(x) - v + f(v), u\rangle - \langle f'(x)(u), x - y\rangle + \langle f'(v)(u), v - y\rangle| \\
&\leq \eta\|x - v\| + |\langle f'(x)(u), x - v\rangle + \langle f'(x)(u) - f'(v)(u), v - y\rangle| \\
&\leq \eta\|x - v\| + \|f'(x)(u)\|\|x - v\| + \|f'(x)(u) - f'(v)(u)\|\|v - y\| \\
&\leq \left(\eta + \theta + \gamma\left(\rho + \sup_{y\in Y}\|y\|\right)\right)\|x - v\| \ .
\end{aligned}$$

Therefore, the function $J'(\cdot, y)$ is Lipschitzian with constant $L$.
Fix $r \in \left]0, \min\left\{\rho, \frac{\sigma}{L}\right\}\right[$ and a non-empty closed convex set $T \subseteq Y$. Clearly

$$\inf_{y\in T} \||J'_x(0, y)\| \geq \inf_{y\in Y} \|J'_x(0, y)\| = 2\sigma$$

and

$$\frac{\inf_{y\in T} \||J'_x(0, y)\|}{2L} > r \ .$$

Then, applying Corollary 2.8 to the restriction of $J$ to $B_r \times T$, we get

$$\sup_T \inf_{B_r} J = \inf_{B_r} \sup_T J \ .$$

By the weak compactness of $B_r$ and $T$, we then infer the existence of $x^* \in B_r$ and $y^* \in T$ such that

$$J(x^*, y) \leq J(x^*, y^*) \leq J(x, y^*)$$

for all $x \in B_r$, $y \in T$ which is equivalent to the conclusion. To show that $x^* \in \partial B_r$, notice that if $\|x^*\| < r$ then we would have $J'_x(x^*, y^*) = 0$ and so

$$r < \frac{\sigma}{L} \leq \frac{\|J'_x(0, y^*)\|}{2L} \leq \frac{L\|x^*\|}{2L} < r \ ,$$

an absurd. □

From Corollary 2.9, in turn, we draw the following characterization about the existence and uniqueness of fixed points:

**Corollary 2.10.** *Let the assumptions of Corollary 2.9 be satisfied.*

*Then, for each* $r \in \left]0, \min\left\{\rho, \frac{\sigma}{L}\right\}\right[$ *such that* $f(B_r) \subseteq Y$, *the following assertions are equivalent:*

(*i*) *the function* $f$ *has a unique fixed point in* $B_r$ *and this lies in* $\partial B_r$ ;
(*ii*) *the function* $f$ *has a fixed point in* $\partial B_r$ ;
(*iii*) *for each* $x \in \partial B_r$ *for which* $f(x) \notin B_r$, *there exists* $v \in B_r$ *such that*

$$\|f(x) - x\|^2 > \|f(v) - v\|^2 - \|f(v) - f(x)\|^2 \ .$$

*Proof.* The implications $(i) \to (ii) \to (iii)$ are obvious. So, suppose that $(iii)$ holds. Apply Corollary 2.9 taking $T = \overline{\text{conv}}(f(B_r))$. Let $x^*, y^*$ be as in the conclusion of Corollary 2.9. Then, we have

$$\|f(x^*) - y^*\| = \text{dist}(f(x^*), T) = 0$$

and

$$\|x^* - f(x^*)\|^2 + \|f(x) - f(x^*)\|^2 - \|x - f(x)\|^2 \leq 0 \qquad (2.3)$$

for all $x \in B_r$. Clearly, in view of $(iii)$, we have $f(x^*) \in B_r$. So, in particular, (2.3) holds for $x = f(x^*)$ and this implies that

$$\|x^* - f(x^*)\| \leq 0$$

that is $x^*$ is a fixed point of $f$ in $B_r$. Finally, if $\tilde{x} \in B_r$ and $\tilde{x} = f(\tilde{x})$, from (2.3) it follows that $f(\tilde{x}) = f(x^*)$, and so $\tilde{x} = x^*$. That is, $x^*$ is the unique fixed point of $f$ in $B_r$. □

**Remark 2.11.** It is important to notice that, when $\dim(E) < \infty$, Corollaries 2.4, 2.5 and 2.6 are still valid replacing $B_r$ with any closed set $S$ satisfying $\partial B_r \subseteq S \subseteq B_r$.

## References

[1] B. Ricceri, *Well-posedness of constrained minimization problems via saddle-points*, J. Global Optim. **40** (2008), 389–397.
[2] B. Ricceri, *Multiplicity of global minima for parametrized functions*, Rend. Lincei Mat. Appl. **21** (2010), 47–57.
[3] B. Ricceri, *On a minimax theorem: an improvement, a new proof and an overview of its applications*, Minimax Theory Appl. **2** (2017), 99–152.
[4] E. Zeidler, *Nonlinear functional analysis and its applications*, vol. III, Springer, 1985.

*Manuscript received* ,
*revised* ,

B. Ricceri
Department of Mathematics and Informatics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy
*E-mail address*: ricceri@dmi.unict.it