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Universal minimal flow in the theory of topological groupoids

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Abstract. We extend the notion of Universal Minimal Flows to groupoid actions of locally trivial groupoids. We also prove that any G-bundle with compact fibers has a global section if G is extremely amenable.

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1. Introduction

In this paper we investigate some connections between topological dynamics, the theory of principal bundles, and the theory of locally trivial groupoids.

Topological dynamics is the study of continuous actions of Hausdorff topological groups G on Hausdorff compact spaces (G-flows). For notations and results of this theory we mainly follow and refer to [1], [14], and [7].

Every topological group G has some natural compactifications. They can be described as the maximal ideal spaces of certain function algebras and using the particular structures of those spaces it is possible to grasp new information about the group itself. In particular, the greatest ambit $S(G)$ is the compactification corresponding to the algebra of all right uniformly continuous bounded functions on G. There is a natural action of G on $S(G)$ which has a universal property with respect to all actions of G on compact spaces. There also exists the universal minimal compact G-space $M(G)$, which is a minimal G-flow that can be homomorphically and equivariantly mapped onto any other minimal G-flow. The flow $M(G)$ can be derived from $S(G)$. Indeed, it can be constructed as any minimal sub-flow of $S(G)$. Therefore $S(G)$ universally describes, in a sense, the dynamics of G.

There is a natural correspondence, developed by C. Ehresmann in [2], between G-principal bundles over locally simply connected bases and locally trivial groupoids; see [2, Section 3, Definition 4]. In order to exploit this correspondence, we move from groups to locally trivial groupoids and from group actions on compact spaces to groupoid actions on spaces with proper maps onto a locally compact base. This allows us to extend part of topological dynamic to groupoid actions. For example, as we said above, every topological group G has a universal minimal flow $M(G)$. Here we prove the existence of an analogous one for groupoids $\mathcal{G} = (G_0, G_1, s, t, u, (\cdot)^{-1})$, with locally compact unit space G_0 and which act on spaces Y with a proper map $\rho: Y \to G_0$ (see Section 2 for the definition of a topological groupoid and Section 5 for the definition of a groupoid action). This reveals a correspondence between the fixed point on compacta property (e.g. extreme amenability) and the existence of global sections in the theory of fiber bundles.

This paper is organized as follows. In Section 2 we review the concept of topological groupoids. We also review the definition of a groupoid action.

In Section 3 we give a self-contained treatment of Ehresmann's theory relating locally trivial groupoids with principal bundles. In this theory, to any connected locally trivial groupoid and any $x_0 \in G_0$, it is associated a *left* G-principal bundle $P = s^{-1}(x_0)$ over G_0 with structural group $G = \mathcal{G}[x_0]$, defined as the set of arrows $\tau \in G_1$ such that $s(\tau) = t(\tau) = x_0$. Conversely, to any *right* G-principal bundle $P \to G_0$ it is associated a locally trivial groupoid reverting the above construction, i.e. such that $s^{-1}(x_0)$ is the canonical left G-bundle associated to P.

In Sections 4 and 5 we generalize the notion of greatest G-ambit to the category of actions of locally trivial groupoids $\mathcal{G}$, by defining $S(\mathcal{G}, x_0)$ as a *relative* compactification of the $G = \mathcal{G}[x_0]$-principal bundle $P = s^{-1}(x_0)$. This means that we introduce a proper map $S(\mathcal{G}, x_0) \to G_0$ such that the principal bundle $P = s^{-1}(x_0)$ can be densely embedded in $S(\mathcal{G}, x_0)$ in a compatible way with the projection map to the base G_0.

In Section 6 we show how the locally trivial groupoid associated to P acts on $S(\mathcal{G}, x_0)$.

In Section 7 we show that $S(\mathcal{G}, x_0)$ plays the same role for groupoid actions as $S(G)$ plays in the usual theory of G-actions on compact spaces, by showing that it enjoys a similar universal property.

In Section 8 we show the existence and uniqueness up to isomorphisms of the universal minimal flows $M(\mathcal{G}, x_0)$ for locally trivial groupoids $\mathcal{G}$, which are defined as a minimal subflows of $S(\mathcal{G}, x_0)$, see Theorem 8.7. In Subsection 8.1 we show how our theory reduces to the known construction and properties of the universal minimal flow, in the case when the groupoid $\mathcal{G}$ is a group. In Theorem 8.8 we describe the universal minimal flows $M(\mathcal{G}, x_0)$ as locally trivial fibrations over G_0.

In Section 9 we use the conceptual framework of the previous sections to prove the following result. Let G be an extremely amenable group, like for example $U(H)$, for H a infinite dimensional separable Hilbert space and $U(H)$ endowed with the strong operator topology, or $\mathbb{P}U(H)$ endowed with the quotient topology from $U(H)$. Then for any locally trivial principal bundle $P \to X$, with X locally compact, and for any G-flow K, the bundle with fibers K associated to P has a global section, see Theorem 9.4.

2. Preliminaries and Definitions

In the following we will consider groupoids in the context of topological spaces. The most concise definition of a groupoid is the following.

Definition 2.1. A groupoid is a small category $\mathcal{G}$ in which every morphism is invertible.

In order to make concrete the above definition, it is common to introduce the following notations. The set of objects, or units, of $\mathcal{G}$ will be denoted by

$$G_0 = \mathrm{Ob}(\mathcal{G}).$$

The set of morphisms, or arrows, of $\mathcal{G}$ will be denoted by

$$G_1 = \mathrm{Mor}(\mathcal{G}).$$

We will denote by $s(g)$ the source, i.e. the domain (respectively $t(g)$ the target, i.e. the range) of the morphism g. We thus obtain functions

$$s, t \colon G_1 \longrightarrow G_0$$

The multiplication operator $m\colon (g,h) \to gh$ is defined on the set of composable pairs of arrows $G_1 \times_{r,s} G_1 := \{(g,h) \mid s(g) = r(h)\}$

$$m \colon G_1 \times_{r,s} G_1 \longrightarrow G_1$$

The inversion operation is a bijection $(\cdot)^{-1}\colon g \to g^{-1}$ of G_1. Denoting by $u(x)$ the unit map of the object $x \in G_0$, we obtain an inclusion of G_0 into G_1. We see that a groupoid $\mathcal{G}$ is completely determined by the spaces G_0 and G_1 and the structural morphisms s, t, m, u, $(\cdot)^{-1}$. The structural maps satisfy the following properties:

(i) $t(gh) = t(g), s(gh) = s(h)$ for any pair $(g,h) \in G_1 \times_{r,s} G_1$, and the partially defined multiplication m is associative;

(ii) $s(u(x)) = r(u(x)) = x$, for all $x \in G_0$, $u(t(g))g = g$ and $gu(s(g)) = g$, for all $g \in G_1$ and $u\colon G_0 \to G_1$ is one-to-one;

(iii) $t(g^{-1}) = s(g), s(g^{-1}) = t(g), gg^{-1} = u(t(g))$ and $g^{-1}g = u(s(g))$.

Then one can define a topological groupoid as follows.

Definition 2.2. A groupoid $\mathcal{G}$ is a topological groupoid if G_0 and G_1 are Hausdorff topological spaces and all the structure maps $s, t, m, u, (\cdot)^{-1}$ are continuous functions.

Notation 2.3. For $x, y \in G_0$ we define

$$\mathcal{G}[x, y] = \{g \in G_1 \mid s(g) = x,\ t(g) = y\}.$$

Finally, we will denote

$$\mathcal{G}[x] = \mathcal{G}[x, x].$$

It is important to observe that $\mathcal{G}[x]$ is actually a group.

Definition 2.4. A groupoid $\mathcal{G}$ is *transitive* if for all $x, y \in G_0$ there exists $g \in G_1$ such that $s(g) = x$ and $t(g) = y$.

Some authors define such a groupoid as *connected* groupoid, we avoid this term to prevent any confusion with its topological analogue. Henceforth we assume that a groupoid $\mathcal{G}$ is always a transitive topological groupoid and we easily get the following:

Proposition 2.5. *If $\mathcal{G}$ is transitive then all the groups $\mathcal{G}[x]$ are isomorphic.*

Notation 2.6. From now on we will adopt *the inverse notation* for the multiplication m of the elements $g, h \in G_1$, that is, we redefine the multiplication so that $s(gh) = s(g)$ and $t(gh) = t(h)$.

2.1. Groupoid actions.

We also recall the definition of right action of a topological groupoid on a topological space.

Definition 2.7. Let us consider two continuous maps

- $\rho\colon Y \to G_0$, which we will call the *anchor map*;
- $\alpha\colon Y \times_{\rho,s} G_1 \to Y$, where $Y \times_{\rho,s} G_1 = \{(y, g) \mid s(g) = \rho(y)\}$.

Assume that for the maps above the following conditions are met:

$$\rho(yg) = t(g), \quad (yh)g = y(hg), \quad yu(\rho(y)) = y.$$

Then we will say that the maps ρ, α *define a right action* of the topological groupoid $\mathcal{G} = (G_0, G_1)$ on the topological space Y. When the structure maps are fixed, we will use the simplified notation $\alpha(y, g) = yg$.

We can then give the natural analogue of the notion of G-flow of topological dynamics.

Definition 2.8. A $\mathcal{G}$-flow is a groupoid action as above with $\rho\colon Y \to G_0$ a proper map. A minimal $\mathcal{G}$-flow is a flow that does not strictly contain any subflow.

Definition 2.9. A universal minimal flow $\rho_M\colon M \to G_0$ for $\mathcal{G}$ actions is a flow such that for any minimal $\mathcal{G}$-flow $\rho\colon Y \to G_0$ there exists a $\mathcal{G}$-equivariant map $M \to Y$ (hence in particular compatible with the tho anchor maps)

It is immediate to observe that the map $M \to Y$ must be surjective, by minimality of Y.

3. Ehresmann Groupoids

We begin this section with a self-contained review of some results due to C. Ehresmann, see [2], who established the equivalence between G-principal bundles and locally trivial groupoids, see Definition 3.1 below.

Henceforth we will assume all topological spaces to be Hausdorff spaces.

Definition 3.1. Let $\mathcal{G}$ be a groupoid. $\mathcal{G}$ is said *locally trivial* if for all $x_0 \in G_0$ there exists an open neighborhood U of x_0 such that for any $x \in U$ there exists a continuous section $\sigma_x : U \to G_1$ of the map s such that $t \circ \sigma_x(y) = x$ for all $y \in U$, or equivalently, such that $t \circ \sigma_x$ is the constant map x. This property can be rephrased by requiring that $\sigma_x(y) \in \mathcal{G}[y, x]$ for all $y \in U$.

Remark 3.2. The definition above may be equivalently formulated by exchanging the roles of s and t.

Definition 3.3. A right principal bundle on a topological space X, with structural topological group G, is the datum of topological space P and a continuous map $p : P \to X$ such that

(1) there exists a right action $P \times G \to P$ which preserves fibers of p and G freely and transitively acts on $p^{-1}(x)$, for all $x \in X$;

(2) the map p identifies X with the quotient space P/G.

Similarly one can also define the notion of a *left* principal bundle with structural group G, in which case there is a left action $G \times P \to P$ with analogous properties as above.

Example 3.4. The simplest example of principal bundle on a space X is the trivial bundle $X \times G$ with p the projection on the first coordinate. If the space X is contractible then all the principal bundles on X are trivial.

Definition 3.5. A right principal bundle $p : P \to X$ is said *locally trivial* if for any $x \in X$ there exists some open set U containing x over which p has a section, i.e. there exists $\sigma : U \to P$ with $p \circ \sigma = \mathrm{id}_U$.

Then the following is well known.

Proposition 3.6. *Let $p : P \to X$ be a locally trivial bundle. Then for all $x \in X$ there exists a neighborhood $U \subset X$ and $x \in U$ such that $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is a trivial G-bundle. In particular there exists a homeomorphism $p^{-1}(U) \cong U \times G$, that identifies the right action of G on $p^{-1}(U)$ with the right action on $U \times G$ defined by the maps $U \times G \ni (x, h) \mapsto (x, hg)$, with $g \in G$.*

Remark 3.7. If X is a locally contractible (namely, all $x \in X$ has a contractible neighborhood), then all bundles $p: P \to X$ are locally trivial. In any case, if $p: P \to X$ is a locally trivial principal bundle (even if X is not locally contractible), if $\{U_i\}$ is a open covering of X such over any U_i there exists a section for $p: P \to X$, then the bundle is defined by gluing maps $\phi_{i,j}: (U_i \cap U_j) \times G \to (U_i \cap U_j) \times G$ of the form $\phi_{i,j}(x, h) = (x, g_{i,j}(x)h)$, with $g_{i,j}: U_{i,j} \to G$ suitable continuous maps. The fact that $g_{i,j}(x)$ acts on the factor G of $(U_i \cap U_j) \times G$ by multiplication on the left is meant to preserve the *right* action of G on $(U_i \cap U_j) \times G$. Of course if $p: P \to X$ is instead a left principal bundle, then one may identify $p^{-1}(U_i) = G \times U_i$ and, over $U_i \cap U_j$, the gluing maps will have the form $\phi_{i,j}(h, x) = (hg_{i,j}(x), x)$.

Remark 3.8. To any right principal bundle $p: P \to X$ one can associate the left principal bundle $p: P \to X$ by defining the left G-action as $g \cdot u = ug^{-1}$. Locally, if $p^{-1}(U_i) \cong U_i \times G$ one also has that $p^{-1}(U_i) \cong G \times U_i$ and the homeomorphism between the two is given by $(x, h) \mapsto (h^{-1}, x)$.

Example 3.9. Let $P = \mathbb{C}^{n+1} \setminus \{0\}$, $X = \mathbb{CP}^n$ be the n-dimensional complex projective space, let G be $\mathbb{C}^*$, the multiplicative group on the complex space, with the topology induced by $\mathbb{C}$, and $p: P \to X$, the canonical projection $v \mapsto [v]$, where $[v]$ is the equivalence class for the equivalence relation $v \sim \lambda v$ for $\lambda \in \mathbb{C}^*$. This is an example of non trivial principal bundle with structural group $G = \mathbb{C}^*$. The action of G on P is the multiplication $v \mapsto \lambda v$, for $v \in \mathbb{C}^{n+1} \setminus \{0\}$ and $\lambda \in \mathbb{C}^*$. In this case it does not matter which action we decide to assume, whether right or left, since the group is commutative. If $n \geq 1$ this is not a trivial bundle.

Definition 3.10. Given a (right) principal G-bundle $\pi: P \to X$ we introduce the following canonical sets.

- $G_0 = X$.
- For any $x, y \in G_0$ we denote

$$\mathcal{G}_P[x, y] = \{f: \pi^{-1}(x) \to \pi^{-1}(y) \mid f \text{ equivariant}\}.$$

  Recall that one says that f is *equivariant* whenever $f(zg) = f(z)g$ for any $z \in X$ and $g \in G$.
- $G_1 = \bigcup_{x,y \imath G_0} \mathcal{G}_P[x, y]$.

We will call $\mathcal{G}_P$ the datum (G_0, G_1). Note that the set G_1 can be equivalently regarded as the set of equivalence classes $(P \times P)/\sim$ under the equivalence relation where $(u, v) \sim (u', v')$ if and only if there is $g \in G$ such that $u' = ug$ and $v' = vg$. In other terms G_1 is the set of the orbits of the diagonal action of G on $P \times P$, defined by $(u, v)g = (ug, vg)$.

Notation 3.11. We denote by $[u, v]$ the orbit of (u, v) under the action of G.

In the following proposition we describe the structural maps of $\mathcal{G}_P$ that make it a groupoid.

Proposition 3.12. *The couple of sets $\mathcal{G}_P = (G_1, G_0)$ becomes a groupoid when endowed with the following structural maps.*

(i) *The maps $s, t\colon G_1 \to G_0$ are respectively defined by $s([u,v]) = p(u)$ and $t([u,v]) = p(v)$, where $p\colon P \to X = G_0$ is the bundle map.*

(ii) *The map $u\colon G_0 \to G_1$ is defined by $u(x) = [v,v]$, with $v \in p^{-1}(x) \subset P$ arbitrary.*

(iii) *The composition law $G_1 \times_{t,s} G_1 \to G_1$ is defined by $[u,v][v',w'] = [ug,w']$, if and only if there exists $g \in G$ such that $v' = vg$.*

(iv) *The inverse map $(\cdot)^{-1}\colon G_1 \to G_1$ is defined by $[u,v] \mapsto [v,u]$.*

Moreover, the groupoid $\mathcal{G}_P$ has the following properties.

(1) *For any fixed $u_0 \in P$, let $x_0 = p(u_0)$. There exists a unique isomorphism*

$$\phi_{u_0}\colon \mathcal{G}[x_0] = \{[u,v] \in G_1 \mid p(u) = p(v) = x_0\} \to G$$

defined by

$$\phi_{u_0}([w,u_0]) = g \iff w = u_0 g.$$

For any other choice $u_1 = u_0 h \in p^{-1}(x_0)$ one has

$$\phi_{u_1}([u,v]) = h^{-1}\phi_{u_0}([u,v])h,$$

for any $[u,v] \in \mathcal{G}[x_0]$.

(2) *For any fixed $u_0 \in P$, let $x_0 = p(u_0)$. There exists a unique isomorphism*

$$\psi_{u_0}\colon \mathcal{G}[x_0] = \{[u,v] \in G_1 \mid p(u) = p(v) = x_0\} \to G$$

defined by

$$\psi_{u_0}([u_0,w]) = g^{-1} \iff w = u_0 g.$$

For any other choice $u_1 = u_0 h \in p^{-1}(x_0)$ one has

$$\psi_{u_1}([u,v]) = h^{-1}\psi_{u_0}([u,v])h,$$

for any $[u,v] \in \mathcal{G}[x_0]$.

(3) *Fixing x_0, u_0 as above, there exists a unique bijective map $\tau_{u_0}\colon t^{-1}(x_0) \to P$ defined by $\tau_{u_0}([w,u_0]) = w$. If we change u_0 into $u_1 = u_0 h$ then one has $\tau_{u_1}(y) = \tau_{u_0}(y)h$ for any $y \in t^{-1}(x_0)$. By identifying $\mathcal{G}[x_0] = G$ through ψ_{u_0}, the map τ_{u_0} becomes a right G-invariant map. Moreover one has $p \circ \tau_{u_0} = s$.*

(4) *Similarly, there exists a unique bijective map* $\sigma_{u_0}\colon s^{-1}(x_0) \to P$ *defined by* $\sigma_{u_0}([u_0, w]) = w$ *which, by identification* $\mathcal{G}[x_0] = G$ *through the isomorphism* ϕ_{u_0}, *is* G*-invariant with respect to the left actions of* $\mathcal{G}[x_0]$ *on* $s^{-1}(x_0)$ *and* G *on* P, *and one obtains* $p \circ \sigma_{u_0} = t$.

Proof. The maps s, t are well defined since $p(zg) = p(z)$ for any $z \in P$ and $g \in G$. The map $u\colon G_0 \to G_1$ is well defined as well, since a different choice of $v' \in p^{-1}(x)$ yields $[v', v'] = [v, v]$. In order to prove that the product is well defined, let us consider $[u, v] = [uh, vh]$, therefore, for $v' = vg$, we get $[uh, vh][v', w] = [uh, vh][(vh)h^{-1}g, w] = [uh, wg^{-1}h] = [u, wg^{-1}]$. In a similar way if $[v'', w''] = [v', w]$ then $v'' = v'h = vgh$ and $w' = wh$, which in turns implies $[u, v][v'', w'] = [u, w'h^{-1}g^{-1}] = [u, wg^{-1}]$. The inverse map is, obviously, well defined. The properties to be a groupoid are easily verified. For example, since the product is well defined, we can deduce associativity from the following fact

$$([u, v][v, w])[w, z] = [u, z] = [u, v]([v, w][w, z]).$$

(1) The map ϕ_{u_0} is bijective and well defined, since it is the unique representation of $[u, v] \in \mathcal{G}[x]$ as $[u, v] = [w, u_0] = [u_0 g, u_0]$. Moreover ϕ_{u_0} is a group homomorphism. Indeed, one has

$$\begin{aligned} \phi_{u_0}([u_0 g, u_0][u_0 h, u_0]) &= \phi_{u_0}([u_0 gh, u_0]) \\ &= gh \\ &= \phi_{u_0}([u_0 g, u_0])\phi_{u_0}([u_0 h, u_0]). \end{aligned}$$

Since

$$\begin{aligned} \phi_{u_1}([u, v]) &= \phi_{u_1}([wh, u_0 h]) \\ &= \phi_{u_1}([u_0 gh, u_1]) \\ &= \phi_{u_1}([u_1 h^{-1} gh, u_1]) \\ &= h^{-1} gh = h^{-1}\phi_{u_0}([u, v])h. \end{aligned}$$

(2) Similar to the proof of (1).

(3) The map $\tau_{u_0}\colon t^{-1}(x_0) \to P$ is bijective and well defined. We observe that again by replacing u_0 with $u_1 = u_0 h$ we get $\tau_{u_1}([w, u_0]) = \tau_{u_1}([wh, u_1]) = wh = \tau_{u_0}([w, u_0])h$. Let $[u, v] \in \mathcal{G}[x]$, we may assume $[u, v] = [u_0, u_0 g]$, therefore we can identify this element with $g^{-1} \in G$ trough ψ_{u_0}. We thus obtain

$$\begin{aligned} \tau_{u_0}([w, u_0] \cdot g^{-1}) &= \tau_{u_0}([w, u_0][u_0, u_0 g]) = \tau_{u_0}([w, u_0 g]) \\ &= \tau_{u_0}[wg^{-1}, u_0] = wg^{-1} = \tau_{u_0}([w, u_0])g^{-1}. \end{aligned}$$

Finally $p(\tau_{u_0}([w, u_0])) = p(w) = s([w, u_0])$.

(4) Analogous to the proof of (3), taking into account the definition of the left action of G on P reviewed in Remark 3.8. □

Remark 3.13. Let us assume $X = G_0 = \{1\}$ and $P = G \to X$, with p equal to the constant map $G \to \{1\}$. Then $\mathcal{G}_P = G \times G/\sim$ is the collection of the orbits $[a, b]$ with $a, b \in G$. By choosing as canonical representatives $[ab^{-1}, 1]$, if we proceed in analogy with the construction of Proposition 3.12, we get $\mathcal{G}_P \cong G$, using the bijection $[g, 1] \mapsto g$. In order to verify that the map $[g, 1] \mapsto g$ is an isomorphism, it is sufficient to do the following calculation $[g, 1][h, 1] = [g, 1][1, h^{-1}] = [g, h^{-1}] = [gh, 1]$.

Theorem 3.14 (Ehresmann). *Let $p: P \to X$ be a locally trivial right G-principal bundle. Then there exists a natural topology on $\mathcal{G}_P$ induced by P such that $\mathcal{G}_P$ is a locally trivial topological groupoid.*

Proof. In order to define a topology on the quotient $(P \times P)/\sim$ in a meaningful way, we use the second version of the Definition 3.10.

Assuming $\pi: P \times P \to \mathcal{G}_P$ is the canonical projection, a set $\mathcal{A}$ is an open set if and only if, given an arbitrary $(u, v) \in \pi^{-1}(\mathcal{A})$, there exist two open neighborhoods U, V respectively of $u, v \in P$, such that $U \times V \subset \pi^{-1}(\mathcal{A})$, which implies $\pi(U \times V) \subset \mathcal{A}$, and in particular $\pi^{-1}(\pi(U \times V)) \subset \pi^{-1}(\mathcal{A})$.

Since $\pi^{-1}(\pi(U \times V)) = \{(u'g, v'g) \mid (u', v') \in U \times V, g \in G\} = \bigcup_{g \in G}(U \times V)g$, then $\pi^{-1}(\pi(U \times V))$ is an open set. Hence the collection of sets as $\pi(U \times V)$ is a basis of $\mathcal{G}_P$. Observe that, since U and V can be chosen in an arbitrary basis of open sets, by Proposition 3.6 they, in particular, can be of the following type $U \cong U_0 \times W_1$ and $V \cong V_0 \times W_1$, with U_0 and V_0 contractible neighborhood of $x = p(u)$ and $y = p(v)$ respectively, and W_1 in a basis of open neighborhoods of the identity element $1 \in G$. Now we consider the following sections $U_0 \to U \cong U_0 \times W_1$ and $V_0 \to V \cong V_0 \times W_1$ defined by the map $x \mapsto (x, 1)$ composed with the inverses of the above isomorphisms.

Notation 3.15. We denote by U_0', V_0' the images of U_0 and V_0 inside U and V through the above sections, and x', y' the images of elements $x \in U_0$ and $y \in V_0$, respectively.

Note that we can identify $U = U_0'W_1$ and $V = V_0'W_1$. Then

$$U \times V = \{(x'g, y'h) \mid x' \in U_0',\ y' \in V_0',\ g, h \in W_1\},$$
$$\pi^{-1}(\pi(U \times V)) = \{(x'gk, y'hk) \mid x' \in U_0',\ y' \in V_0',\ g, h \in W_1, k \in G\}.$$

Since $[x'gk, y'hk] = [x'gh^{-1}, y']$, we can find the bijection

$$W_1 W_1^{-1} \times U_0 \times V_0 \cong \pi(U \times V),$$

which can be defined explicitly as follows:

$$\psi: W_1 W_1^{-1} \times U_0 \times V_0 \longrightarrow \pi(U \times V), \quad \psi(r, x, y) = [x'r, y']. \tag{1}$$

The map ψ is clearly surjective. Moreover if $[x'r, y'] = [x''s, y'']$ then $y' = y''g$ and $x'r = x''sg$ for some $g \in G$, but, since $y', y'' \in V_0'$, with V_0' local section of $p\colon P \to X$ onto V_0, we deduce $g = 1$, hence $y' = y''$ and $x'r = x''s$, therefore $x' = x''sr^{-1}$. Since U_0' is a section of $p\colon P \to X$ onto U_0, this yields $s = r$, hence ψ is injective. Choosing U, V as above then we obtain a concrete description of an open basis of $\mathcal{G}_P$.

Finally we fix arbitrarily $x \in X = G_0$ and $[u, u] \in \mathcal{G}_P$ with $p(u) = x$. In this case for the above construction we can choose $U = V \cong U_0 \times W_1$.

Therefore we can define the section $\sigma\colon U_0 \to W_1 W_1^{-1} \times U_0 \times U_0 \cong \pi(U \times U)$ of the target map t through $y \mapsto (1, x, y)$ on the open set $U_0 \subset G_0$. This section has the properties requested by Definition 3.1. The continuity properties of the structural maps can be obtained by routine arguments, so $\mathcal{G}_P$ is a topological groupoid. □

Conversely, let $\mathcal{G}$ be a locally trivial topological transitive groupoid, then the following holds.

Theorem 3.16 (Ehresmann). *Let $\mathcal{G}$ a locally trivial transitive groupoid, then for any $x_0 \in G_0$ the source map $s\colon P = t^{-1}(x_0) \to G_0$ is a locally trivial right principal bundle with structural group $G = \mathcal{G}[x]$ and one obtains a canonical isomorphism $\mathcal{G} \cong \mathcal{G}_P$.*

Similarly, the target map $t\colon P = s^{-1}(x_0) \to G_0$ is a left principal bundle with structural group $G = \mathcal{G}[x_0]$.

Sketch of proof. The right $G = \mathcal{G}[x_0]$ action on $P = t^{-1}(x_0)$ is the obvious one, defined by $P \times G \ni (\tau, g) \mapsto \tau g$, using the inner product of G_1. For any $\tau_1, \tau_2 \in P$ with $s(\tau_1) = s(\tau_2) = x \in G_0$, it is clear that $\tau_2 = \tau_1 \tau_1^{-1} \tau_2$, with $\tau_1^{-1}\tau_2 \in G$. This allows one to prove show that the right action of G on the fiber of $s\colon P \to G_0$ over x is free, and moreover, at least set-theoretically, that one has $G_0 = P/G$. That this is also a topological quotient follows from the local triviality of the groupoid $\mathcal{G}$, from which it follows that the map $s\colon P = t^{-1}(x_0) \to G_0$ is not only continuous, but also open. Note moreover that by the transitivity of $\mathcal{G}$ we have $\mathcal{G}[x] \cong \mathcal{G}[x_0] = G$ for any $x \in G_0$. Hence $s\colon P = t^{-1}(x_0) \to G_0$ is a right G-principal bundle. The fact that $s\colon P = t^{-1}(x_0) \to G_0$ is a locally trivial principal bundle is an immediate consequence of Definition 3.1, since a local section of G_1 as in that definition is actually a local section for P.

Observe that for $x, y \in G_0$ the set of arrows $\mathcal{G}[x, y]$ with source x and target y can be obtained as the set of $\tau = \alpha\beta^{-1}$, with $\alpha \in \mathcal{G}[x, x_0]$ and $\beta \in \mathcal{G}[y, x_0]$. One easily sees that $\alpha_1\beta_1^{-1} = \alpha_2\beta_2^{-1}$ if and only if $\alpha_2 = \alpha_1 g$ and $\beta_2 = \beta_1 g$ for some $g \in G = \mathcal{G}[x_0]$, we recover the fact that, at least set-theoretically, the groupoid $\mathcal{G}$ is of the form $\mathcal{G}_P$ as in the construction of Definition 3.10. We omit the easy checks that also the algebraic and topological structures provided by Proposition 3.12 and Theorem 3.14 agree with the original ones on $\mathcal{G}$.

Finally, the fact that $t: s^{-1}(x_0) \to G_0$ is the left bundle structure canonically associated to the right bundle $P = t^{-1}(x_0)$ can now be achieved as a consequence of (4) of Proposition 3.12. □

Remark 3.17. For any given a transitive locally trivial groupoid $\mathcal{G}$, the results above allow to represent the arrow in G_1 as equivalence classes $[\tau, \tau']$ of couples of elements $\tau, \tau' \in P = t^{-1}(x_0)$, for any fixed $x_0 \in G_0$ as in Definition 3.10. Note moreover that the algebraic and topological structure of $\mathcal{G}$ will be those described in Proposition 3.12 and Theorem 3.14, that is we can represent $\mathcal{G}$ as $\mathcal{G}_P$. This will have many advantages when doing explicit computations on $\mathcal{G}$ and continuous functions on it.

Example 3.18. Assume that X is arcwise connected and locally contractible space and $P \to X$ is a universal covering of X, which one can see as a locally trivial $\pi_1(X, x_0)$ principal bundle. Then $\mathcal{G}_P$ with the topology constructed above is the *path groupoid* of X.

4. Universal relative compactifications of principal bundles

We begin with a brief review of a compactification of topological groups G endowed with a universal property with respect to actions of G on compact spaces, widely studied in the theory of Universal Minimal Flows (see [1] and [14] for a survey). Then we will explain how to generalize the construction to locally trivial groupoids.

Notation 4.1. Let G be a topological group, we recall that the C^*-algebra $R_G = RUC^b(G)$ of *right uniformly continuous* functions, is defined as the algebra of bounded continuous functions that satisfy the following:

$$\begin{aligned}&\text{for all } \epsilon > 0 \text{ there exists } \mathcal{V} \text{ such that}\\ &|f(y) - f(gy)| < \epsilon, \text{ for all } g \in \mathcal{V},\ y \in G,\end{aligned}$$

where the set $\mathcal{V}$ is an open neighborhood of the identity element $1 \in G$.

Endowed with pointwise addition, multiplication, complex conjugation and with the sup norm, R_G is an abelian C^*-algebra. The Gelfand duality associates to R_G the compact space $S(G) = \Omega(R_G)$, the space of non-zero $\mathbb{C}$-algebra homomorphisms $R_G \to \mathbb{C}$. The space $S(G)$ is also called the *Samuel compactification* of the topological group G. There is a natural left action of G on $S(G)$ that extends the left multiplication action on G itself, associated to the action of G on R_G defined by $h(f)(y) = f(h^{-1}y)$. All minimal subflows of $S(G)$, i.e. minimal compact subsets of $S(G)$ closed with respect to the action of G, are isomorphic to each other and it is known that they are *universal minimal flows*, that is they can be mapped equivariantly onto any other minimal flow, see for example [14] or [7].

Starting with the present section, we will extend the result above to the flows of any locally trivial groupoid $\mathcal{G}$ with G_0 a locally compact space. The notion of a $\mathcal{G}$-flow has already been introduced in Definitions 2.7 and 2.8. In the articles [14] and [7] the authors assume the topological group G metrizable, with right invariant compatible metric, in order to separate the elements of G. In this paper we will consider groups G with the same properties and we assume that $\mathcal{G}$ is a locally trivial groupoid with G_0 locally compact, hence associated to a G-principal bundle as shown in Section 3. The reason to assume G_0 locally compact is to be able to exploit the Gelfand construction. Hence we will be able to consider G_0 as the locally compact space associated to the C^*-algebra $C_0(G_0)$, the algebra of continuous functions $f\colon G_0 \to \mathbb{C}$ vanishing at infinity, endowed with sup-norm.

Recall that Gelfand construction provides a categorical duality between the category of compact Hausdorff spaces and the category of unital commutative C^*-algebras (see [11], Theorems 2.1.10 and 2.1.15). In the sequel we will need the one directional functorial version of Gelfand construction dealing with locally compact Hausdorff spaces instead of compact ones. Indeed, the space of characters $\Omega(A)$ of a commutative C^*-algebra A is locally compact, $A = C_0(\Omega(A))$, see [11] Theorems 1.3.5 and 2.1.10, and a homomorphism $\phi^*\colon A \to B$ corresponds to a continuous map $\phi\colon \Omega(B) \to \Omega(A)$. In particular the following holds:

Fact 4.2. *Let A and B be two isomorphic commutative C^*-algebras then the related locally compact spaces $\Omega(A)$ and $\Omega(B)$ are homeomorphic.*

Observe that the existence of a continuous map $f\colon X \to Y$ between locally compact spaces does not necessary implies the existence of a homomorphism $f^*\colon C_0(Y) \to C_0(X)$, since this happens only when the map f is proper.

4.1. Notational setup for locally trivial groupoids. Let $\mathcal{G} = (G_0, G_1, s, t, u, (\cdot)^{-1})$ be a transitive locally trivial groupoid with G_0 locally compact. Let us fix $x_0 \in G_0$, let G be the group $\mathcal{G}[x_0]$ and let us consider $s\colon P = t^{-1}(x_0) \to G_0$, which a right G-principal bundle, in view of the results of Theorem 3.16. As observed in Remark 3.17, we can represent $\mathcal{G}$ as the groupoid $\mathcal{G}_P$. Then, again by Theorem 3.16, one can identify the *left* G-principal bundle structure on P with $t\colon s^{-1}(x_0) \to G_0$. Some times, with a slight abuse of notation, we will call P both $t^{-1}(x_0)$ and $s^{-1}(x_0)$, leaving to the context to decide whether we are considering its right or left G-bundle structure, respectively. But we will denote by $p\colon P \to G_0$ only the bundle projection defined by $s\colon P = t^{-1}(x_0) \to G_0$. Then $p^{-1}(x) = s^{-1}(x) \cap t^{-1}(x_0) = G[x, x_0]$ and, for example $p^{-1}(x_0) = G[x_0] = G$.

As observed in Remark 3.17, the elements of G_1 can be represented as the equivalence classes $[\tau, \tau']$, with $\tau, \tau' \in P = t^{-1}(x_0)$ under the relation $(\tau, \tau') \sim (\tau g, \tau' g)$ for $g \in G$.

We will also set $u_0 = u(x_0) \in G[x_0]$, serving as the unit of G. Let us also notice that in our notational setup, any element $\tau \in s^{-1}(x_0)$ is uniquely represented as $\tau = [u_0, y']$, with $u_0 \in p^{-1}(x_0) = G[x_0]$ as defined above and $y' \in p^{-1}(y) = s^{-1}(y) \cap t^{-1}(x_0) = G[y, x_0]$. In this representation, the map $t\colon s^{-1}(x_0) \to G_0$ is clearly defined by $t([u_0, y']) = p(y') = y$.

The C^*-algebra $R(\mathcal{G}, x_0)$. The next definition generalizes $RUC^b(G)$ to the case of a locally trivial groupoid $\mathcal{G}$ with G_0 locally compact.

Definition 4.3. Denote $R(\mathcal{G}, x_0)$ the set of functions $f\colon s^{-1}(x_0) \to \mathbb{C}$ with the following properties:

(1) f bounded, continuous and there exists $g \in C_0(G_0)$ such that $|f(u)| \leq |g(t(u))|$ for any $u \in s^{-1}(x_0)$;

(2) for all local sections $x \mapsto x'$ of $p\colon P \to G_0$ defined on some open set $U \subset G_0$ with compact closure in G_0 and for every $\epsilon > 0$, there exists an open neighbourhood $\mathcal{V}$ of $1 \in G$ such that for every $g \in \mathcal{V}$, for every $h \in G$ and for every $x \in U$ we have $|f([u_0 gh, x']) - f([u_0 h, x'])| < \epsilon$.

It is not difficult to see that $R(\mathcal{G}, x_0)$ is a C^*-algebra with respect to complex conjugation and the sup norm. We omit the details.

Proposition 4.4. *The open set $\mathcal{V}$ in the definition above is independent from the choice of the section $x \to x'$.*

Proof. Let us assume $x \mapsto x''$ is a different section of p on U, then one can write x' as $x''k(x)$, with $k\colon U \to G$ a suitable continuous function. Hence $[u_0 h, x''] = [u_0 h, x'k(x)^{-1}] = [u_0 hk(x), x']$. Then a function f satisfying the properties of the definition above is such that $|f([u_0 ghk(x), x']) - f([u_0 hk(x), x'])| < \epsilon$, for any $x \in U$ and $g \in \mathcal{V}$, by the arbitrariness of h in the Definition 4.3. Therefore for any $g \in \mathcal{V}$ we have $|f([u_0 gh, x'']) - f([u_0 h, x''])| < \epsilon$. ☐

We will use the above argument to replace in Definition 4.3 the sentence “for all sections” with “there exists a section.” This modification has significant advantages, since, using local sections, one can introduce a more classical local notation for functions $f \in R(\mathcal{G}, x_0)$.

Proposition 4.5. *Let $U \subset G_0$ be an open set and assume that on U is given a section $x \mapsto x'$ of the projection $p\colon P \to G_0$. As above we identify $P \cong s^{-1}(x_0)$ as a left G-bundle. Then the map $\psi'\colon G \times U \to p^{-1}(U)$ defined by $(h, x) \mapsto [u_0 h, x']$ is a homeomorphism. In particular the restrictions of all functions $f\colon P \to \mathbb{C}$ to $p^{-1}(U)$ can be identified with the functions $F\colon G \times U \to \mathbb{C}$ by putting $F(h, x) = f([u_0 h, x'])$. Moreover, the condition (2) of Definition 4.3 can be rephrased with the inequality $|F(gh, x) - F(h, x)| < \epsilon$ for any $x \in U$, $h \in G$ and $g \in \mathcal{V}$.*

Proof. The map ψ' is a restriction of the homeomorphism ψ defined by (1) in the proof of Theorem 3.14. This in turns implies that ψ' is a homeomorphism and, by a straightforward calculation, it also implies the statement on the condition (2) of Definition 4.3. □

Using the following lemma one can construct many useful examples of functions in $R(\mathcal{G}, x_0)$.

Lemma 4.6. *Let $\eta: G_0 \to \mathbb{C}$ be a continuous function belonging to $C_0(G_0)$. Then $\eta \circ p$ belongs to $R(\mathcal{G}, x_0)$. Moreover let U be an open set of G_0 on which there exists a section of $p: P \to G_0$ and let $\theta: G \to \mathbb{C}$ be a function in R_G and η as above. Then, using the notations of Proposition 4.5, the function $F: G \times U \to \mathbb{C}$ defined by $F(h, x) = \eta(x)\theta(h)$ belongs to $R(\mathcal{G}, x_0)$.*

Proof. The property (1) of the Definition 4.3 is immediately verified. Property (2) follows by the inequality

$$|F(gh, x) - F(h, x)| \leq (\max|\eta|)|\theta(gh) - \theta(h)|$$

and the fact that $\theta \in R_G$. □

The construction of the greatest $\mathcal{G}$-ambit $S(\mathcal{G}, x_0)$

Definition 4.7. Denote by $S(\mathcal{G}, x_0) = \Omega(R(\mathcal{G}, x_0))$, the locally compact space corresponding, by Gelfand construction, to the commutative C^*-algebra $R(\mathcal{G}, x_0)$.

The space $S(\mathcal{G}, x_0)$ will be shown later to be an adequate generalization of the notion of greatest G-ambit $S(G)$ to the theory of groupoid flows, for the locally trivial groupoid $\mathcal{G}$ with locally compact G_0.

Recall that the space $S(\mathcal{G}, x_0) = \Omega(R(\mathcal{G}, x_0))$ is the set of the surjective ring homomorphisms $\xi: R(\mathcal{G}, x_0) \to \mathbb{C}$ and that there exists a natural map $i: P \to S(\mathcal{G}, x_0)$ defined as follows: $i(\tau) = \hat{\tau}$, where $\hat{\tau} \in S(\mathcal{G}, x_0)$ is the homeomorphism $\hat{\tau}: R(\mathcal{G}, x_0) \to \mathbb{C}$ such that $\hat{\tau}(f) = f(\tau)$.

Theorem 4.8. *$i: P \to S(\mathcal{G}, x_0)$ is a continuous injective map with dense image.*

Proof. Since the topology of $S(\mathcal{G}, x_0)$ has the subbasis made of subsets of the form $f^{-1}(B)$, with $f \in R(\mathcal{G}, x_0)$ and $B \subset \mathbb{C}$ open set, whose preimages by i are open sets of P, the continuity of i easily follows.

Moreover, a continuous function $f \in R(\mathcal{G}, x_0)$ which is zero in $i(P)$ is equal to zero since $f(\hat{\tau}) = \hat{\tau}(f) = f(\tau) = 0$ for any $\tau \in P$. Therefore the subspace $i(P) \subset S(\mathcal{G}, x_0)$ is dense.

Let$[u_0, v]$ and $[u_0, w]$ a pair of distinct elements of P. If $p(v) \neq p(w)$ then, since G_0 is locally compact, we can find a function $\eta \in R(\mathcal{G}, x_0)$ as in Lemma 4.6, such that $\eta(p(v)) \neq \eta(p(w))$, hence $i([u_0, v]) \neq i([u_0, w])$.

If $p(v) = p(w) = y$, then $[u_0, w] = [u_0, vh^{-1}] = [u_0 h, v]$ for some $h \in G$ with $h \neq 1$. Consider a function $\theta: G \to \mathbb{C}$ in R_G such that $\theta(1) \neq \theta(h)$. It follows that there exists a local section $x \mapsto x'$ defined in a neighborhood U of $y \in G_0$ such that $y' = v$. Let η be a function with compact support inside U such that $\eta(y) = 1$. Using notations of Proposition 4.5, the function $G: G \times U \to \mathbb{C}$ built in Lemma 4.6 is such that $F(h, y) = \theta(h) \neq \theta(1) = F(1, y)$. Therefore $i([u_0, v]) \neq i([u_0, v'])$. □

5. The space $S(\mathcal{G}, x_0)$ as a G-bundle on G_0

Let, as in the previous section, $\mathcal{G}$ be a transitive locally trivial groupoid with a locally compact unit space G_0. Recall that $\mathcal{G} = \mathcal{G}_P$ with $P = t^{-1}(x_0)$ considered as a right $G = G[x_0]$ bundle by means of the projection $s: t^{-1}(x_0) \to G_0$, which will be denoted also as $p: P \to G_0$. Recall that for any fixed $x_0 \in G_0$, the map $t: s^{-1}(x_0) \to G_0$ can also be denoted $p: P \to G_0$ and regarded as the left G-bundle structure on P. We set $u_0 = u(x_0)$.

We consider the locally compact space $S(\mathcal{G}, x_0)$ associated with the C^*-algebra $R(\mathcal{G}, x_0)$. There exists a natural inclusion $\rho_{\mathcal{G}}^*: C_0(G_0) \hookrightarrow R(\mathcal{G}, x_0)$ well defined by $\eta \mapsto \eta \circ t$, by Lemma 4.6.

Since G_0 is locally compact, by Gelfand construction, G_0 is the space associated to its C^*-algebra $C_0(G_0)$. It follows that the inclusion $\rho_{\mathcal{G}}^*: C_0(G_0) \hookrightarrow R(\mathcal{G}, x_0)$ is the dual of a continuous surjective map

$$\rho_{\mathcal{G}}: S(\mathcal{G}, x_0) \to G_0. \tag{2}$$

Theorem 5.1. *The map $\rho_{\mathcal{G}}$ is proper and it has all the fibers isomorphic to $S(G)$. Moreover there exists an open covering $\{U_i\}$ of G_0, whose elements are contractible open sets with compact closures, with the following properties.*

(1) *$\rho_{\mathcal{G}}^{-1}(U_i) \cong S(G) \times U_i$ and the left action of G on $\rho_{\mathcal{G}}^{-1}(U_i)$ is given by the left action of G on $S(G)$.*

(2) *Denoting $g_{i,j}(x)$ the transition functions of the left G-bundle P associated to the covering $\{U_i\}$, then the transition functions*

$$\theta_{ij}: \rho_{\mathcal{G}}^{-1}(U_i)|_{U_i \cap U_j} \to \rho_{\mathcal{G}}^{-1}(U_j)|_{U_i \cap U_j}$$

are defined by $\tau \mapsto \tau g_{i,j}(x)$, i.e. they are induced by the functions $g_{i,j}(x)$.

Proof. Let us first observe that $P|_{U_i}$ is dense in $\rho_{\mathcal{G}}^{-1}(U_i)$, hence $C_0(\rho_{\mathcal{G}}^{-1}(U_i))$ is isomorphic to the restriction of $R(\mathcal{G}, x_0)$ to $P|_{U_i}$. Since $\{U_i\}$ is an open covering of G_0 with $\bar{U}_i$ compact, then the image of $C_0(G_0)$ by the restriction map from G_0 to $\bar{U}_i$ is actually $C(\bar{U}_i)$. Moreover, up to a suitable refinement of the covering, we can assume $\bar{U}_i$ contractible, hence the bundle $P|_{\bar{U}_i} = t^{-1}(\bar{U}_i)$ has a section on $\bar{U}_i$.

Let us denote such a section by $x \mapsto x'$. Then there exists a homeomorphism $P|_{\bar{U}_i} \cong G \times \bar{U}_i$ as in the notations of Proposition 4.5. We calculate the left action of $G = \mathcal{G}[x_0]$ on $P|_{\bar{U}_i}$ as follows. Let us identify $g \in G$ with $[u_0 g, u_0] \in \mathcal{G}[x_0]$, acting as in the definition of the isomorphism ϕ_{x_0} introduced in (1) of Proposition 3.12. Then we have

$$\mathcal{G}[u_0, v] = [u_0 g, u_0][u_0, v] = [u_0 g, v] = [u_0 gh, x'] \longmapsto (gh, x) \in G \times U_i.$$

We can regard such action as the left action of G on $G \times \bar{U}_i$. Proposition 4.5 and Definition 4.3 show that the functions f on $P|_{\bar{U}_i}$ that are restrictions of functions in $R(\mathcal{G}, x_0)$ can be described as continuous functions $F(h, x)$ such that for any $\epsilon > 0$ there exists a neighborhood $\mathcal{V} \ni 1 \in G$ such that $|F(gh, x) - F(h, x)| < \epsilon$ for any $x \in \bar{U}_i$, $h \in G$ and any $g \in \mathcal{V}$. Hence the algebra of such functions coincides with $C(S(G) \times \bar{U}_i)$ which in turns implies that $S(\mathcal{G}, x_0)|_{\bar{U}_i}$ is homeomorphic to $S(G) \times \bar{U}_i$. The action of G on the functions $f(h, x)$ obtained from $R(\mathcal{G}, x_0)$ can be calculated through the formula $F^g(h, x) = F(g^{-1} \cdot (h, x)) = F(g^{-1}h, x)$, and this action corresponds to the left action of G on $S(G) \times U_i$ induced by the natural left action of G on $S(G) = \Omega(R_G)$. Then $\rho_{\mathcal{G}}^{-1}(U_i) \cong S(G) \times U_i$, with left action of G induced by the canonical one defined on $S(G)$. Let be $P|_{U_i} \cong G \times U_i$ and $P|_{U_j} \cong G \times U_j$ two trivializations, related by a transition isomorphism $(h, x) \mapsto (hg_{i,j}(x)^{-1}, x)$ on $U_i \cap U_j$.

This isomorphism corresponds to the relation $x' = x'' g_{i,j}(x)$, for x' section on U_i and x'' section on U_j. Hence it can be identified with the following composition of isomorphisms

$$(h, x) \longmapsto [u_0 h, x'] = [u_0 h, x'' g_{i,j}(x)] = [u_0 h g_{i,j}(x)^{-1}, x''] \longmapsto (hg_{i,j}(x)^{-1}, x).$$

As usual, the induced action on the functions $F(h, x)$ from $R(\mathcal{G}, x_0)$, with $x \in U_i \cap U_j$, is $F(h, x) \mapsto F(hg_{i,j}(x), x)$, and this maps $R(\mathcal{G}, x_0)$ to itself, as one can easily see. Therefore it induces an automorphism of $S(G) \times (U_i \cap U_j)$ compatible with the left G-action. This is enough to show that a gluing map exists between $\rho_{\mathcal{G}}^{-1}(U_i)$ and $\rho_{\mathcal{G}}^{-1}(U_j)$ along $\rho_{\mathcal{G}}^{-1}(U_i \cap U_j)$ which extends the given one for $P|_{U_i}$ and $P|_{U_j}$. □

6. The right action of $\mathcal{G}$ on $S(\mathcal{G}, x_0)$

Here we again identify $\mathcal{G} = \mathcal{G}_P$, hence its elements will be represented as classes $[v, w]$ with $v, w \in P$ as in Definition 3.10 and section 4.1.

Then we will also be able to identify $s^{-1}(x_0) = P$ and $t: s^{-1}(x_0) \to G_0$ will be identified with the left G-bundle structure for $p: P \to G_0$. The elements of $s^{-1}(x_0)$ will be represented as classes $[u_0, y']$.

Theorem 6.1. *The right groupoid action*

$$\alpha_{\mathcal{G}} \colon P \times_{t,s} G_1 \longrightarrow P$$

defined by $\alpha_{\mathcal{G}}([u_0, v], [v, w]) = [u_0, w]$ *has an extension to a continuous right groupoid action*

$$\alpha_{\mathcal{G}} \colon S(\mathcal{G}, x_0) \times_{\rho_{\mathcal{G}}, s} G_1 \longrightarrow S(\mathcal{G}, x_0)$$

with anchor map $\rho_{\mathcal{G}}$.

Proof. The groupoid action $\alpha_{\mathcal{G}} \colon P \times_{t,s} G_1 \to P$ defined by $\alpha_{\mathcal{G}}([u_0, v], [v, w]) = [u_0, w]$ is a restriction of the multiplication map of G_1, hence it is continuous. Let us now consider an open covering $\{U_i\}$ of contractible open sets of G_0 as in Theorem 5.1, consequently $P|_{U_i} \cong G \times U_i$ e $\rho_{\mathcal{G}}^{-1}(U_j) \cong S(G) \times U_i$. Let $G_{i,j} = \psi(G \times U_i \times U_j)$ the open covering of G_1, constructed by means of the map ψ defined by $\psi(k, x, y) = [x'k, y']$, using the notation (1) introduced in the proof of Theorem 3.14. Then for any i, j, the map $\alpha_{\mathcal{G}}$ restricts to a continuous map

$$\alpha_{\mathcal{G}} \colon P|_{U_i} \times_{t,s} G_{i,j} \longrightarrow P|_{U_j}.$$

Let us notice that $P|_{U_i} \times_{t,s} G_{i,j} \cong G \times G \times U_i \times U_j$ by the map

$$\tilde{\psi} \colon G \times G \times U_i \times U_j \longrightarrow P|_{U_i} \times_{t,s} G_{i,j}$$

defined by $\tilde{\psi}(h, k, x, y) = ([u_0 h, x'], [x'k, y'])$. Using the above identifications the map $\alpha_{\mathcal{G}}$ can be regarded as the map

$$\alpha_{\mathcal{G}} \colon (h, k, x, y) \longmapsto (hk, y).$$

We consider the pull-back of functions $f \in R(\mathcal{G}, x_0)$ through $\alpha_{\mathcal{G}}$. Recall that the functions $f \in R(\mathcal{G}, x_0)$ restricted to $P|_{U_j}$, in the identification $P|_{U_j} \cong G \times U_j$, are continuous functions $F(k, y)$ such that for any $\epsilon > 0$ there exists a neighborhood $\mathcal{V} \ni 1 \in G$ such that $|F(gk, y) - F(k, y)| < \epsilon$ for any k, y and any $g \in \mathcal{V}$. Then the function $F(h, k, x, y) = f(\alpha_{\mathcal{G}}(h, k, x, y)) = F(hk, y)$ satisfies the property $|F(gh, k, x, y) - F(h, k, x, y)| < \epsilon$ for any x, y, h, k and any $g \in \mathcal{V}$. Consequently the algebra $\alpha_{\mathcal{G}}^*(R(\mathcal{G}, x_0))$ restricted to $P|_{U_i} \times_{t,s} G_{i,j}$ is contained in $C_0(S(G) \times U_i \times G_{i,j})$, and therefore in $C_0(S(\mathcal{G}, x_0)|_{U_i} \times G_{i,j})$. Hence the action $\alpha_{\mathcal{G}}$ extends a continuous map

$$\alpha_{\mathcal{G}} \colon S(\mathcal{G}, x_0)|_{U_i} \times_{\rho_{\mathcal{G}}, s} G_{i,j} \longrightarrow S(\mathcal{G}, x_0)|_{U_j} \tag{3}$$

for any i, j, by Fact 1. Since those actions restricted to $P \times G_1$ glue each other to form the given action $\alpha_{\mathcal{G}} \colon P \times_{t,s} G_1 \to P$, and by density of $P|_{U_i}$ in $S(\mathcal{G}, x_0)|_{U_i}$ for any i, then the functions (3) can be glued together to form a global action $\alpha_{\mathcal{G}} \colon S(\mathcal{G}, x_0) \times_{\rho_{\mathcal{G}}, s} G_1 \to S(\mathcal{G}, x_0)$. The required properties for the groupoid action are easy to prove, for example by using the density of P in $S(\mathcal{G}, x_0)$ and the same properties of the action of G_1 on P. □

7. Universal property of $S(\mathcal{G}, x_0)$

Let $\beta: Y \times_{\rho,s} G_1 \to Y$ be a right action of a transitive locally trivial topological groupoid $\mathcal{G} = (G_0, G_1)$ on a space Y with proper anchor map $\rho: Y \to G_0$.

Theorem 7.1 (Universal Property of $S(\mathcal{G}, x_0)$). *For any action*

$$\beta: Y \times_{\rho,s} G_1 \longrightarrow Y$$

with proper anchor map $\rho: Y \to G_0$ and for any fixed $y \in \rho^{-1}(x_0)$ there exists a unique continuous map $l_y: S(\mathcal{G}, x_0) \to Y$ such that $l_y(u_0) = y$ and compatible with the actions $\alpha_{\mathcal{G}}$ and β.

Proof. It is sufficient to prove that the homomorphism of algebras $l_y^*: C_0(Y) \to C(P)$ defined by $l_y^* f([u_0, v]) = f(\beta(y, [u_0, v]))$, where $C(P)$ is the algebra of *all* continuous functions on P, actually has its image in $R(\mathcal{G}, x_0)$. We will adopt the multiplicative notation $\beta(y[u_0, v]) = y[u_0, v]$ for the action of G_1 on Y. Let $y_v = y[u_0, v]$ and, since $l_y^* f([u_0, v]) = f(y_v)$, we see that if $|f| < \varepsilon$ outside $\rho^{-1}(K)$, for some $K \subset G_0$ compact, then, since $\rho(y_v) = p(v)$, one has $|l_y^*(f)| < \varepsilon$ outside $t^{-1}(K)$. Hence condition (1) of the Definition 4.3 is verified.

For an open set $U \subset G_0$ with local section $x \mapsto x'$ assume

$$F(h, x) = l_y^* f([u_0 h, x']) = l_y^* f([u_0, x' h^{-1}]) = f(y_{x' h^{-1}}).$$

Let us fix $\varepsilon > 0$. Observe that $|f| < \varepsilon/2$ outside some compact $\rho^{-1}(K)$. Then there exists $\mathcal{V}$, a neighborhood of $1 \in G$, such that for any $x \in K$, any $h \in G$ and any $g \in \mathcal{V}$ we have

$$|F(gh, x) - F(h, x)| < \epsilon.$$

By Proposition 4.5, this implies condition (2) of the Definition 4.3.

Since $l_y^*: C_0(Y) \to R(\mathcal{G}, x_0)$ is a homomorphism of C^*-algebras, then there exists a map $l_y: S(\mathcal{G}, x_0) \to Y$ as desired. Moreover, this map is uniquely determined by its restriction to P, which is the map $[u_0, v] \to y[u_0, v]$. This shows the uniqueness of l_y by density of P in $S(\mathcal{G}, x_0)$. Finally, using again the density of P and the compatibility of the restriction of l_y to P, we deduce the compatibility of l_y with the given actions on $S(\mathcal{G}, x_0)$ and on Y. □

8. Existence and uniqueness up to $\mathcal{G}$-invariant isomorphisms of the universal minimal $\mathcal{G}$-flows

In this section we will show the existence and uniqueness up to isomorphisms of universal minimal flows $\mathcal{G}$-actions, with $\mathcal{G}$ a locally trivial groupoid with G_0 locally compact. We refer to Section 2.1 for the definitions of flows, minimal flows and universal minimal flows for topological groupoids.

For the reader's convenience we briefly recall some notations and results in the classical case of topological groups. Let G be a topological group, a G-space is a topological space Y with a continuous action of G. A G-space Y is minimal if the orbit Gy is dense in Y. The universal minimal compact G-space $M(G)$ is characterized by the following property: $M(G)$ is a minimal compact G-space, and for every compact minimal G-space Y there exists a G-map of $M(G)$ onto Y. It is well known that any two universal minimal compact G spaces are isomorphic, see for example [1] or [14].

In this section we will show that any minimal $\mathcal{G}$-subflow of $S(\mathcal{G}, x_0)$ has the analogous universal property for minimal $\mathcal{G}$-flows. Our proof is a slight modification of the one contained in [14]. Let us consider $P = s^{-1}(x_0)$ and the anchor map $\rho_{\mathcal{G}}: S(\mathcal{G}, x_0) \to G_0$, which extends the map $t: P \to G_0$. By Theorem 6.1 for any $y \in \rho_{\mathcal{G}}^{-1}(x_0)$ there exists a unique map $l_y: S(\mathcal{G}, x_0) \to S(\mathcal{G}, x_0)$ such that $l_y(u_0) = y$. If $y \neq y'$ then $l_y \neq l_{y'}$, moreover, by Theorem 5.1, $\rho_{\mathcal{G}}^{-1}(x_0) \cong S(G)$ and, by such isomorphism, the element $u_0 \in p^{-1}(x_0)$, identified with $[u_0, u_0] \in t^{-1}(x_0) \subset \rho_{\mathcal{G}}^{-1}(x_0)$, corresponds to $1 \in G$. We now can define a left action $\Phi: S(G) \times S(\mathcal{G}, x_0) \to S(\mathcal{G}, x_0)$ such that $\Phi(y, z) = l_y(z)$. Notice that, by the universal property of $S(G)$, for y, z in $S(G) = \rho_{\mathcal{G}}^{-1}(x_0)$, the product $l_y(z) = yz$ is the same as the one that gives $S(G)$ a semigroup structure, defined in Theorem 2.1 of [14]. However we warn the reader that in [14] the product in $S(G)$ is defined as $yz = r_z(y)$, that is, by means of similar maps r_z as the l_y defined here, such that r_z is invariant for the *left* action of G on $S(G)$, whereas we have a right G_1 invariance for the maps l_y.

Notation 8.1. For $y, z \in \rho_{\mathcal{G}}^{-1}(x_0) \cong S(G)$, we denote $l_y(z) = yz$.

Proposition 8.2. *For $y, z \in \rho_{\mathcal{G}}^{-1}(x_0)$ we have $l_y l_z = l_{yz}$.*

Proof. The maps $l_y l_z$ and l_{yz} are G_1-maps that send u_0 to the same element yz. By uniqueness the thesis follows. □

Proposition 8.3. *Let $f: S(\mathcal{G}, x_0) \to S(\mathcal{G}, x_0)$ be a G_1-map (in particular a G-map) then $f = l_y$ for some $y \in \rho_{\mathcal{G}}^{-1}(x_0)$*

Proof. Denote $f(u_0) = y$ and consider the map l_y. Then f and l_y send u_0 to the same point and by uniqueness of l_y we have $f = l_y$. □

Proposition 8.4. *Let $M(\mathcal{G}, x_0)$ be a $\mathcal{G}$-minimal flow of $S(\mathcal{G}, x_0)$ then $M(\mathcal{G}, x_0) = l_y(S(\mathcal{G}, x_0))$ for some idempotent $y \in \rho_{\mathcal{G}}^{-1}(x_0) \cap M(\mathcal{G}, x_0)$. Moreover l_y is the identity on $M(\mathcal{G}, x_0)$.*

Proof. First observe that $M(\mathcal{G}, x_0) \cap \rho_{\mathcal{G}}^{-1}(x_0)$ is necessarily a G-minimal subflow of $\rho_{\mathcal{G}}^{-1}(x_0) = S(G)$, otherwise $(M(\mathcal{G}, x_0) \cap \rho_{\mathcal{G}}^{-1}(x_0))G_1$ would be a proper subflow of $M(\mathcal{G}, x_0)$, against the minimality of $M(\mathcal{G}, x_0)$. By the results of Section 3 of [14] there exists a idempotent $y \in M(\mathcal{G}, x_0) \cap \rho_{\mathcal{G}}^{-1}(x_0)$ and one can consider the map $l_y : S(\mathcal{G}, x_0) \to S(\mathcal{G}, x_0)$, which, by minimality, is such that $l_y(M(\mathcal{G}, x_0)) = M(\mathcal{G}, x_0)$. More precisely, from $y^2 = y$ one finds $yyz = yz$ for any $z \in M(\mathcal{G}, x_0)$ and, since the yz span $M(\mathcal{G}, x_0)$, the map l_y is the identity on $M(\mathcal{G}, x_0)$. □

Proposition 8.5. *Every $\mathcal{G}$-map $f : M(\mathcal{G}, x_0) \to M(\mathcal{G}, x_0)$ is bijective.*

Proof. Composing f with $l_y : S(\mathcal{G}, x_0) \to M(\mathcal{G}, x_0)$ of the Proposition 8.4, and using Proposition 8.3, we obtain $f \circ l_y = l_z$ for some $z \in M(\mathcal{G}, x_0)$. As $l_y|_{M(\mathcal{G},x_0)}$ is the identity map of $M(\mathcal{G}, x_0)$, we obtain $f = l_z|_{M(\mathcal{G},x_0)}$. The rest of the proof runs exactly as the proof of Proposition 3.4 of [14]. □

Theorem 8.6. *$M(\mathcal{G}, x_0)$ is unique up to $\mathcal{G}$-invariant isomorphism.*

Proof. Let M and M' be two minimal flows of $S(\mathcal{G}, x_0)$, then, by universality of $S(\mathcal{G}, x_0)$, there exists a G_1-map $f : S(\mathcal{G}, x_0) \to M'$, so that $f|_M : M \to M'$. By reversing roles we obtain a similar map from M' to M. By minimality of M and M' they are both surjective functions, moreover their composition is G_1-map from M to M, and, according to Proposition 8.5 is actually a bijection, hence f is injective, from which, using the properness over G_0 of M and M' and the compatibility of these maps with the anchor map, one sees that they are homeomorphic. □

Theorem 8.7. *Any minimal $\mathcal{G}$-subflow $M(\mathcal{G}, x_0)$ of $S(\mathcal{G}, x_0)$ is a* ***universal minimal flow*** *for right $\mathcal{G}$-actions with proper anchor map.*

Proof. If M is a minimal flow, consider a $\mathcal{G}$-invariant map $f : S(\mathcal{G}, x_0) \to M$, whose existence is guaranteed by the universal property of $S(\mathcal{G}, x_0)$. By minimality, $f(M(\mathcal{G}, x_0)) = M$. □

8.1. Reduction to the classical case when the groupoid $\mathcal{G}$ is a group. A groupoid $\mathcal{G}$ is a group when $G_0 = \{e\}$, in which case $G_1 = G$. For any action of G as groupoid, as defined in section 2.1, the anchor map ρ is a constant and the continuous action α is automatically a function defined on the whole of $Y \times G$. From this we easily conclude that α is a continuous action of G on Y.

Then we show that $S(\mathcal{G}, x_0) = S(G)$, for $\mathcal{G}$ equal to a group G. Indeed Definition 4.3 in this case says that the C^*-algebra $R(\mathcal{G}, x_0)$ is made of right uniformly continuous functions on G, i.e. it coincides with $RUC_b(G)$. This implies that $S(\mathcal{G}, x_0)$ and $S(G)$ coincide. Finally all the statements of the universal properties for $S(\mathcal{G}, x_0)$ and $M(\mathcal{G}, x_0)$ reduce to those already known for $S(G)$ and $M(G)$, see for example [14].

8.2. Description of the universal minimal flows for $\mathcal{G}$ as locally trivial fibrations. One can easily describe a universal minimal flow $M(\mathcal{G}, x_0)$ for $\mathcal{G}$ in the following way.

Theorem 8.8. *Any minimal subflow $M(\mathcal{G}, x_0)$ of $S(\mathcal{G}, x_0)$ has the form $M \cdot G_1$, i.e. it is the orbit of M under the $\mathcal{G}$ action, with $M \subset \rho_{\mathcal{G}}^{-1}(x) \cong S(G)$ some G-minimal subflow. In particular $M(\mathcal{G}, x_0) \to G_0$ is a locally trivial fibration with fibers isomorphic to $M(G)$, the universal minimal flow for the group G.*

Proof. Indeed, since $\rho_{\mathcal{G}}^{-1}(x) \cap M(\mathcal{G}, x_0)$ is $G = G[x]$-invariant, it is easy to see that $\rho_{\mathcal{G}}^{-1}(x) \cap M(\mathcal{G}, x_0)$ contains some $M \cdot G_1$, with M a minimal G-flow for $\rho_{\mathcal{G}}^{-1}(x)$, hence $M(\mathcal{G}, x_0) = M \cdot G_1$, by $\mathcal{G}$-invariance and minimality. Recall from Theorem 5.1 that $S(\mathcal{G}, x_0)$ is locally of the form $S(G) \times U$ and that its fibers $\rho_{\mathcal{G}}^{-1}(x)$ are isomorphic to $S(G)$, the Samuel compactification of G. It immediately follows that $M \cong M(G)$, the universal minimal flow of G. Recall also that in Theorem 5.1 the projection $\rho_{\mathcal{G}}\colon S(\mathcal{G}, x_0) \to G_0$ has been proved to be a locally trivial fibration with gluing maps defined by unique extension of the gluing maps between $P|_{U_i} \cong G \times U_i$ and $P|_{U_j} \cong G \times U_j$ given by the a transition isomorphism $(h, x) \mapsto (hg_{i,j}(x)^{-1}, x)$ on $U_i \cap U_j$. Then the extension to $S(G) \times (U_i \cap U_j)$ must preserve the chosen minimal flow $M \subset S(G)$ over some point $x \in U_i \cap U_j$. This shows that $M(\mathcal{G}, x_0) \to G_0$ is a locally trivial fibration. □

9. Considerations in the case of extremely amenable groups

It turns out that most well known groups, e.g. discrete groups, Lie groups, or in general locally compact groups, have very big compactifications $S(G)$ and also very big universal minimal flows $M(G)$. For example when G is discrete then $S(G) = M(G) = \beta G$, the Stone–Cech compactification of G, a fact originally proved by Ellis and also a consequence of Veech's theorem [15]. On the other hand some very huge groups, like $\mathrm{Homeo}_+(S^1)$, the group of orientation preserving self-homeomorphism of S^1, with the compact open topology, or $U(H)$, the unitary group of a infinite dimensional separable Hilbert space, with the strong operator topology, have small $M(G)$, i.e. $M(G) = S^1$ for the first and $M(G) = \{*\}$ is a singleton in the latter case. We refer to section 4 of [13] for a detailed discussion of these and many other examples. In particular $U(H)$ is a prototypical example of a *extremely amenable group*, a fact proved in [6]. We recall the general definition of this class of groups.

Definition 9.1. A topological group G is said *extremely amenable* if any continuous action of G on a compact Hausdorff space has a fixed point. This is equivalent to saying that $M(G)$ is a point.

Assume that $\mathcal{G}$ is a transitive groupoid with $\mathcal{G}[x] = G$ an extremely amenable group. If G_0 is locally compact, then all the results of the preceding sections apply. In particular the result of Theorem 8.8 holds, and, since G extremely amenable, one has $M(G) = \{*\}$. This means that $\rho_{\mathcal{G}}: S(\mathcal{G}, x_0) \to G_0$ has a $\mathcal{G}$-invariant section. Then, by the universal property of $S(\mathcal{G}, x_0)$, any $\rho: Y \to G_0$ with a right $\mathcal{G}$-action has a $\mathcal{G}$-invariant section.

But actually the same result holds even when G_0 it is not locally compact, and hence one cannot apply Theorem 8.8.

Theorem 9.2. *Let G be a topological group. Then G is extremely amenable if and only if for any locally trivial transitive groupoid $\mathcal{G}$ with structural group G and any action of $\mathcal{G}$ on a topological space Y with proper anchor map $\rho: Y \to G_0$ there exists a continuous $\mathcal{G}$-invariant section $G_0 \to Y$ of ρ.*

Proof. ($\Longrightarrow$) Let $\alpha: Y \times_{\rho,s} G_1 \to Y$ be an action of $\mathcal{G}$ on Y with $\rho: Y \to G_0$ a proper map. Then for any $x \in G_0$ there exists an induced action $\mathcal{G}[x] = G$ on $\rho^{-1}(x)$, which is a compact space.

Since G is extremely amenable there exists a fixed point $z \in \rho^{-1}(x)$. Let us consider the orbit $zG_1 \subset Y$ and let us show that it is the image of a continuous section $\sigma: G_0 \to Y$. The fact the zG_1 is the image of a set-theoretical section is a consequence of the following.

Claim. *For any $x' \in G_0$ one has $|\rho^{-1}(x') \cap zG_1| = 1$.*

Indeed, by the transitivity of $\mathcal{G}$ there exists $g \in G_1$ such that $s(g) = x$ and $tg = x'$. Then $t(zg) = x'$, therefore $|\rho^{-1}(x') \cap zG_1| \geq 1$. If $z', z'' \in \rho^{-1}(x') \cap zG_1$, then $z' = zg$ and $z'' = zh$ for suitable $g, h \in G_1$ and one has $t(g) = t(h) = \rho(z') = \rho(z'') = x'$. Hence $gh^{-1} \in \mathcal{G}[x]$ and one has $z = zgh^{-1}$, which implies $z' = zg = zh = z''$.

Now let us show that the section $\sigma: G_0 \to Y$ with $\mathrm{Im}(\sigma) = zG_1$ is continuous. As σ is a bijection between G_0 and $\mathrm{Im}(\sigma) = zG_1$, with inverse equal to the restriction $\rho|_{zG_1}$, then it is sufficient to show that $\rho|_{zG_1}$ is an open map. Let $W \cap zG_1$ be an open set in the induced topology on zG_1, with W open in Y. If $z' \in zG_1 \cap W$ then $z' = zh$ for some h and $\rho(z') = t(h)$. It follows that $\rho(zG_1 \cap W) = \{t(h) \mid zh \in W\}$. The set $\{(z, h) \mid zh \in W\}$ is equal to $\alpha^{-1}(W) \cap (z \times s^{-1}(x))$, which is homeomorphic to an open set $W_0 \subset s^{-1}(x)$, since $\alpha^{-1}(W) \subset Y \times_{\rho,s} G_1$, by the continuity of the action α, is an open set, as well. Now let us consider the restriction of the target map $t: s^{-1}(x) \to G_0$. It is easy to see that $\mathcal{G}$ locally trivial implies that t is open. Then we have $\rho(zG_1 \cap W) = \{t(h) \mid zh \in W\} = t(W_0)$ open.

($\Longleftarrow$) Let $Y \times G \to Y$ an action G on a compact Y. We consider the group as a groupoid, i.e. $G_1 = G$, $G_0 = \{1\}$ and the group action as a groupoid action $Y \times_{\rho,s} G_1 \to Y$, with trivial anchor map $\rho: Y \to \{1\}$. Then ρ is proper and an invariant section corresponds to a fixed point $x = \sigma(1) \in Y$ for the action of G. $\square$

The theorem above has an interesting consequence for the theory G-principal bundles, in the case when G is an extremely amenable group. First we set some notation.

Notation 9.3. Let G be a topological group and $p\colon P \to X$ a locally trivial right G-principal bundle. Let K be any G-flow, i.e. a compact space for which there exists a continuous right action of $K \times G \to K$. Let $Y \to X$ be the bundle with fibers K induced by P. We recall that the bundle $Y \to X$ can be defined as follows. One sets $Y = (K \times P)/\sim$, where $\sim$ is the equivalence relation $(k, u) \sim (k', u')$ if and only if there exists $g \in G$ such that $k' = kg$ and $u' = ug$. The projection map $Y \to X$ is defined by means of $p([k, u]) = p(u)$. Note that if P is locally defined by trivializations $P|_{U_i} \cong U_i \times G$ and transition functions $(x, h) \mapsto (x, g_{i,j}(x)h)$ for $x \in U_i \cap U_j$, then $Y|_{U_i} \cong K \times U_i$, with transition functions $(k, x) \mapsto (kg_{i,j}(x)^{-1}, x)$ for $x \in U_i \cap U_j$. But we will not use transition functions in the proof of the next result.

Theorem 9.4. *Under the notations above, if G is extremely amenable, then for any choice of a locally trivial principal bundle $P \to X$ and for any G-flow K, the associated bundle $Y \to X$ has a global section.*

Sketch of proof. There is a right $\mathcal{G}_P$ action on Y defined by $[k, u]\cdot[u', v'] = [kg, v]$ if and only if $u' = ug$. Then one can apply Theorem 9.2 to obtain a $\mathcal{G}$-invariant section of $Y \to X$, in particular a global section. $\square$

Remark 9.5. The theorem above is definitely false for locally compact groups. Indeed even for the *compact* group S^1 there exist non trivial principal S^1-bundles, that therefore do not admit any global section. In this case the flow K is the group itself. For example the famous Hopf bundle $S^3 \to S^2$ with fibers S^1 is a non-trivial S^1-principal bundle, that can be defined as the restriction to $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ of the $\mathbb{C}^*$-principal bundle $\mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ of Example 3.9, setting $n = 1$ and recalling that $\mathbb{CP}^1 \cong S^2$.

Remark 9.6. Recall that if a principal bundle $P \to X$ has a global section, then it is trivial. On the other hand, if the group G is extremely amenable, then for any principal bundle $P \to X$, the bundle $S(\mathcal{G}, x_0) \to X = G_0$ constructed in this paper does have global sections, by the theorem above. Recall that $S(\mathcal{G}, x_0) \to X$ has fibers homeomorphic to the compactifications $S(G)$ of the fibers of P, homeomorphic to G. One might wonder if this is sufficient to ensure that P is trivial. The answer is negative. Indeed the existence of non-trivial bundles for a group G is related to the homotopy theory of G. Recall that if for an arcwise connected and compactly generated group G any principal G-bundle is trivial, then G is *aspherical* or *quasi-contractible*, that is all homotopy groups $\pi_i(G)$ are trivial. Indeed, from the theory of fiber bundles, it is well known that the set $\mathrm{Prin}_G(S^n)$ of isomorphism classes of principal G-bundles with base $X = S^n$ is in bijective correspondence with $\pi_{n-1}(G)$, for any $n \geq 1$. But *there exist arcwise connected and compactly generated non-aspherical extremely amenable groups*. For example consider a separable infinite dimensional Hilbert space H, its unitary group $U(H)$ with the strong topology, and the projective linear group $\mathbb{P}U(H)$ obtained as the quotient of $U(H)/U(1)$, being $U(1)$ the center of $U(H)$. Then $\mathbb{P}U(H)$ is extremely amenable as well (any of flow of $\mathbb{P}U(H)$ is also a $U(H)$ flow and therefore it has fixed points). On the other hand, it is well known that $\pi_2(\mathbb{P}U(H)) \cong \pi_1(U(1)) = \mathbb{Z}$, as one can see for example considering the long homotopy exact sequence (see for example [9] chapter 9 section 3) derived from the exact sequence of groups $(1) \to U(1) \to U(H) \to \mathbb{P}U(H) \to (1)$ and using the fact that $U(H)$ is aspherical, by Kuiper's theorem [8]. In particular there exist non-trivial $\mathbb{P}U(H)$ principal bundles over S^3.

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