A Presentation of the Kähler Differential Module for a Fat Point Scheme in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$

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Abstract. Let $\mathbb{Y}$ be a fat point scheme in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ over a field K of characteristic zero. In this paper we introduce the multi-graded Kähler differential module for $\mathbb{Y}$ and we establish a short exact sequence of this module in terms of the thickening of $\mathbb{Y}$.

1 Introduction

In [1], G. de Dominicis and M. Kreuzer introduced some methods using algebraic differential forms into the study of finite sets of points in the projective n-space $\mathbb{P}^n$ over a field K of characteristic zero. Explicitly, given a finite set of points $\mathbb{X}$ in $\mathbb{P}^n$ with homogeneous vanishing ideal $I_{\mathbb{X}}$ in $R = K[X_0, \dots, X_n]$ and homogeneous coordinate ring $R_{\mathbb{X}} = R/I_{\mathbb{X}}$, the *Kähler differential module* for $\mathbb{X}$ is the $R_{\mathbb{X}}$-module $\Omega^1_{R_{\mathbb{X}}/K} = J/J^2$ where J is the kernel of the multiplication map $\mu : R_{\mathbb{X}} \otimes_K R_{\mathbb{X}} \to R_{\mathbb{X}}$. One of interesting results in [1] is the canonical exact sequence for the Kähler differential module

$$0 \to I_{\mathbb{X}}^{(2)}/I_{\mathbb{X}}^2 \to I_{\mathbb{X}}/I_{\mathbb{X}}^2 \to R_{\mathbb{X}}^{n+1}(-1) \to \Omega^1_{R_{\mathbb{X}}/K} \to 0.$$

Based on this exact sequence, the structure of this module can be precisely described in several special cases, for instance, if $\mathbb{X}$ is the complete intersection of hypersurfaces of degrees $d_1, \dots, d_n$ then it follows that the Hilbert function of $\Omega^1_{R_{\mathbb{X}}/K}$ is given by $\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K}}(i) = (n+1)\,\mathrm{HF}_{\mathbb{X}}(i-1) - \sum_{j=1}^{n} \mathrm{HF}_{\mathbb{X}}(i - d_j)$ for all $i \in \mathbb{Z}$. Later, in [2, 4], the differential algebra techniques were extended to fat point schemes in $\mathbb{P}^n$ and in $\mathbb{P}^1 \times \mathbb{P}^1$.

In this paper we will consider the natural question of whether these differential algebraic methods can be applied to study fat point schemes of a multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and, especially, we look closely at the generalization of the above canonical exact sequence to fat point schemes in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $\mathbb{Y}$ be a fat point scheme in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with multihomogeneous vanishing ideal $I_{\mathbb{Y}} \subseteq S = K[X_{10}, \dots, X_{1n_1}, \dots, X_{k0}, \dots, X_{kn_k}]$ and let $R_{\mathbb{Y}} = S/I_{\mathbb{Y}}$ be the multi-graded coordinate ring of $\mathbb{Y}$. We show that the Kähler differential module $\Omega^1_{R_{\mathbb{Y}}/K}$ admits the following exact sequence.

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Theorem 1.1. *(Theorem 3.5) Let $\mathbb{Y} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$, and let $\mathbb{V} = (m_1+1)P_1 + \cdots + (m_s+1)P_s$ be the thickening of $\mathbb{Y}$. Then we have the exact sequence of $\mathbb{Z}^k$-graded $R_{\mathbb{Y}}$-modules*

$$0 \longrightarrow I_{\mathbb{Y}}/I_{\mathbb{V}} \longrightarrow R_{\mathbb{Y}}^{n_1+1}(-e_1) \oplus \cdots \oplus R_{\mathbb{Y}}^{n_k+1}(-e_k) \longrightarrow \Omega^1_{R_{\mathbb{Y}}/K} \longrightarrow 0.$$

This result shows that one can compute the Hilbert function of the Kähler differential module for $\mathbb{Y}$ from the Hilbert functions of $\mathbb{Y}$ and $\mathbb{V}$, in particularly, to compute the Hilbert function $\mathrm{HF}_{\Omega^1_{R_{\mathbb{Y}}/K}}(\underline{i})$, we need to compute $\mathrm{HF}_{\Omega^1_{R_{\mathbb{Y}}/K}}(\underline{i})$ for only a finite number of $\underline{i} \in \mathbb{Z}^k$.

2 Basic Facts and Notation

Let K be a field of characteristic zero. Let $k \geq 2$ be a positive integer, let $\underline{i}$ denote the tuple $(i_1, \ldots, i_k) \in \mathbb{Z}^k$, and let $|\underline{i}| = \sum_l i_l$. We write $\underline{i} \leq \underline{j}$ if $i_l \leq j_l$ for every $l = 1, \ldots, k$. Also, let $\{e_1, \ldots, e_k\}$, $e_i = (0, \ldots, 1, \ldots, 0)$, be the canonical basis of $\mathbb{Z}^k$.

The multi-graded coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is the the polynomial ring $S = K[X_{10}, \ldots, X_{1n_1}, \ldots, X_{k0}, \ldots, X_{kn_k}]$ equipped with the $\mathbb{Z}^k$-grading defined by $\deg X_{i0} = \cdots = \deg X_{in_i} = e_i$ for $i = 1, \ldots, k$. For $\underline{i} = (i_1, ..., i_k) \in \mathbb{Z}^k$, we let $S_{\underline{i}}$ be the homogeneous component of degree $\underline{i}$ of S, i.e., the K-vector space with basis

$$\{X_{10}^{\alpha_{10}} \cdots X_{1n_1}^{\alpha_{1n_1}} \cdots X_{k0}^{\alpha_{k0}} \cdots X_{kn_k}^{\alpha_{kn_k}} \mid \textstyle\sum_{l=1}^{n_j} \alpha_{jl} = i_j,\ \alpha_{jl} \in \mathbb{Z}\}.$$

Given an ideal $I \subseteq S$, we set $I_{\underline{i}} := I \cap S_{\underline{i}}$ for all $\underline{i} \in \mathbb{Z}^k$. Clearly, $I_{\underline{i}}$ is a K-vector subspace of $S_{\underline{i}}$ and $I \supseteq \bigoplus_{\underline{i}\in\mathbb{Z}^k} I_{\underline{i}}$. The ideal I is called $\mathbb{Z}^k$-*homogeneous* if $I = \bigoplus_{\underline{i}\in\mathbb{Z}^k} I_{\underline{i}}$. If I is a $\mathbb{Z}^k$-homogeneous ideal of S then the quotient ring S/I also inherits the structure of a multi-graded ring via $(S/I)_{\underline{i}} := S_{\underline{i}}/I_{\underline{i}}$ for all $\underline{i} \in \mathbb{Z}^k$.

A finitely generated S-module M is a $\mathbb{Z}^k$-*graded S-module* if it has a direct sum decomposition

$$M = \bigoplus_{\underline{i}\in\mathbb{Z}^k} M_{\underline{i}}$$

with the property that $S_{\underline{i}} M_{\underline{j}} \subseteq M_{\underline{i}+\underline{j}}$ for all $\underline{i}, \underline{j} \in \mathbb{Z}^k$.

Definition 2.1. Let M be a finitely generated $\mathbb{Z}^k$-graded S-module. The *Hilbert function* of M is the numerical function $\mathrm{HF}_M : \mathbb{Z}^k \to \mathbb{Z}$ defined by

$$\mathrm{HF}_M(\underline{i}) := \dim_K M_{\underline{i}} \quad \text{for all } \underline{i} \in \mathbb{Z}^k.$$

In particular, for a $\mathbb{Z}^k$-homogeneous ideal I of S, the Hilbert function of S/I satisfies

$$\mathrm{HF}_{S/I}(\underline{i}) := \dim_k (S/I)_{\underline{i}} = \dim_k S_{\underline{i}} - \dim_k I_{\underline{i}} \quad \text{for all } \underline{i} \in \mathbb{Z}^k.$$

If M is a finitely generated $\mathbb{Z}^k$-graded S-module such that $\mathrm{HF}_M(\underline{i}) = 0$ for $\underline{i} \not\geq (0, \ldots, 0)$, we write the Hilbert function of M as an infinite matrix, where the initial row and column are indexed by 0.

A point in the space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ has the form

$$P = [a_{10} : a_{11} : \cdots : a_{1n_1}] \times \cdots \times [a_{k0} : a_{k1} : \cdots : a_{kn_k}] \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$$

where $[a_{j0} : a_{j1} : \cdots : a_{jn_j}] \in \mathbb{P}^{n_j}$. Its vanishing ideal is the bihomogeneous prime ideal of the form

$$I_P = \langle L_{11}, \ldots, L_{1n_1}, \ldots, L_{k1}, \ldots, L_{kn_k} \rangle$$

where $L_{jl} = a_{jl}X_{j0} - a_{j0}X_{jl}$ and $\deg(L_{jl}) = e_j$ for $l = 1, \ldots, n_j$.

When $\mathbb{X} = \{P_1, \ldots, P_s\}$ is a set of s distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then the vanishing ideal of $\mathbb{X}$ is $I_{\mathbb{X}} = I_{P_1} \cap \cdots \cap I_{P_s}$ and its $\mathbb{Z}^k$-graded coordinate ring is $R_{\mathbb{X}} = S/I_{\mathbb{X}}$.

Definition 2.2. Let $\mathbb{X} = \{P_1, \ldots, P_s\}$ is a set of s distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and let $m_1, \ldots, m_s$ are positive integers. The ideal $I_{\mathbb{Y}} = I_{P_1}^{m_1} \cap \cdots \cap I_{P_s}^{m_s}$ defines a 0-dimensional subscheme $\mathbb{Y}$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

(a) The scheme $\mathbb{Y}$ is called a *fat point scheme* of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and is denoted by $\mathbb{Y} = m_1P_1 + \cdots + m_sP_s$.

(b) The number m_j is called the *multiplicity* of the point P_j.

(c) The fat point scheme $\mathbb{V} = (m_1 + 1)P_1 + \cdots + (m_s + 1)P_s$ in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is called the *thickening* of $\mathbb{Y}$.

The support of $\mathbb{Y}$ is $\mathrm{Supp}(\mathbb{Y}) = \mathbb{X}$, the vanishing ideal of $\mathbb{Y}$ is $I_{\mathbb{Y}}$ and its $\mathbb{Z}^k$-graded coordinate ring is the residue class ring $R_{\mathbb{Y}} = S/I_{\mathbb{Y}}$.

If $\mathbb{Y}$ is a fat point scheme in $\mathbb{P}^n$, then there exists a linear form that is a non-zerodivisor for the homogeneous coordinate ring of $\mathbb{Y}$. Using this property, the following lemma for fat point schemes in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ follows similarly from [3, Lemma 1.2].

Lemma 2.3. *Let $\mathbb{Y}$ be a fat point scheme of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Then for $i \in \{1, \ldots, k\}$ there exists $L_i \in S_{e_i}$ such that the image of L_i in the residue class ring $R_{\mathbb{Y}} = S/I_{\mathbb{Y}}$ is a non-zerodivisor of $R_{\mathbb{Y}}$.*

Remark 2.4. After a suitable change of coordinates, we can assume that $L_i = X_{i0}$ for $i = 1, \ldots, k$. This also implies that X_{i0} does not vanish at any point of $\mathrm{Supp}(\mathbb{Y}) = \mathbb{X}$. In this case if x_{ij} denotes the image of X_{ij} in $R_{\mathbb{Y}}$, then $x_{10}, \ldots, x_{k0}$ are non-zerodivisors of $R_{\mathbb{Y}}$.

As a consequence of the lemma and [8, Proposition 1.9] we get several basis properties of the Hilbert function of $\mathbb{Y}$.

Theorem 2.5. *Let $\mathbb{Y} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with Hilbert function $\mathrm{HF}_{\mathbb{Y}}$, and let $j \in \{1, ..., k\}$ and $N = n_1 + \cdots + n_k$.*

(a) For all $\underline{i} \in \mathbb{Z}^k$ we have $\mathrm{HF}_{\mathbb{Y}}(\underline{i}) \leq \mathrm{HF}_{\mathbb{Y}}(\underline{i} + e_j) \leq \sum_{k=1}^{s} \binom{N+m_k-1}{m_k-1}$.

(b) If $\mathrm{HF}_{\mathbb{Y}}(\underline{i}) = \mathrm{HF}_{\mathbb{Y}}(\underline{i} + e_j)$ then $\mathrm{HF}_{\mathbb{Y}}(\underline{i}) = \mathrm{HF}_{\mathbb{Y}}(\underline{i} + 2e_j)$.

3 A Presentation of the Kähler Differential Module

In the following we let $\mathbb{Y}$ be a fat point scheme in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ supported at $\mathbb{X}$. We also denote the image of X_{ij} in $R_{\mathbb{Y}}$ by x_{ij}. According to Remark 2.4, we may always assume that $x_{10}, \ldots, x_{k0}$ are non-zerodivisors of $R_{\mathbb{Y}}$.

In the multi-graded algebra

$$R_{\mathbb{Y}} \otimes_K R_{\mathbb{Y}} = \bigoplus_{\underline{i} \in \mathbb{Z}^k} \Big(\bigoplus_{\underline{j}+\underline{h}=\underline{i}} (R_{\mathbb{Y}})_{\underline{j}} \otimes (R_{\mathbb{Y}})_{\underline{h}} \Big)$$

we have the $\mathbb{Z}^k$-homogeneous ideal $J = \ker(\mu)$, where $\mu : R_{\mathbb{Y}} \otimes_K R_{\mathbb{Y}} \to R_{\mathbb{Y}}$ is the multi-homogeneous $R_{\mathbb{Y}}$-linear map given by $\mu(f \otimes g) = fg$. A $\mathbb{Z}^k$-homogeneous system of generators of J is

$$\{ x_{ij_i} \otimes 1 - 1 \otimes x_{ij_i} \mid 1 \leq i \leq k, 0 \leq j_i \leq n_i \}.$$

Definition 3.1. The $\mathbb{Z}^k$-graded $R_{\mathbb{Y}}$-module $\Omega^1_{R_{\mathbb{Y}}/K} = J/J^2$ is called the *module of Kähler differentials* of $R_{\mathbb{Y}}/K$. The $\mathbb{Z}^k$-homogeneous K-linear map $d_{R_{\mathbb{Y}}/K} : R_{\mathbb{Y}} \to \Omega^1_{R_{\mathbb{Y}}/K}$ given by $f \mapsto f \otimes 1 - 1 \otimes f + J^2$ satisfies universal property. We call d the *universal derivation* of $R_{\mathbb{Y}}/K$.

More generally, for any $\mathbb{Z}^k$-graded K-algebra T/S we can define in the same way the Kähler differential module $\Omega^1_{T/S}$, and the universal derivation of T/S (cf. [6, Section 2]).

Remark 3.2. (a) The $\mathbb{Z}^k$-graded S-module $\Omega^1_{S/K}$ has the representation

$$\Omega^1_{S/K} = \bigoplus_{i=1}^{k} \bigoplus_{j=0}^{n_i} S\, dX_{ij} \cong S^{n_1+1}(-e_1) \oplus \cdots \oplus S^{n_k+1}(-e_k).$$

(b) If I_1, I_2 are $\mathbb{Z}^k$-homogeneous ideals of S, then

$$I_1\Omega^1_{S/K} \cap I_2\Omega^1_{S/K} = (I_1 \cap I_2)\Omega^1_{S/K}$$

(see [7, Chapter 3, §7, Theorem 7.4(i)]).

Some following properties of the module of Kähler differentials follows from [6, Propositions 4.12 and 4.13].

Proposition 3.3. *Let $dI_{\mathbb{Y}}$ be the submodule of $\Omega^1_{S/K}$ generated by $\{dF \mid F \in I_{\mathbb{Y}}\}$.*

(a) There is an isomorphism of $\mathbb{Z}^k$-graded $R_{\mathbb{Y}}$-modules

$$\Omega^1_{R_{\mathbb{Y}}/K} \cong \Omega^1_{S/K}/(dI_{\mathbb{Y}} + I_{\mathbb{Y}}\Omega^1_{S/K}).$$

(b) The set $\{dx_{ij_i} \mid 1 \le i \le k, 0 \le j_i \le n_i\}$ is a $\mathbb{Z}^k$-homogeneous system of generators of $\Omega^1_{R_{\mathbb{Y}}/K}$.

Example 3.4. Let $\mathbb{X} = \{P\}$ be the set of only one point in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ where $P = [1 : 0 : \cdots : 0] \times \cdots \times [1 : 0 : \cdots : 0]$. We have $I_P = \langle X_{ij_i} \mid 1 \le i \le k, 1 \le j \le n_i \rangle$ and $R_{\mathbb{X}} = K[x_{10}, \ldots, x_{k0}]$. Also, we see that

$$I_P/I_P^2 \cong R_{\mathbb{X}}^{n_1}(-e_1) \oplus \cdots \oplus R_{\mathbb{X}}^{n_k}(-e_k)$$

and

$$\Omega^1_{R_{\mathbb{X}}/K} = \langle dx_{i0} \mid 1 \le i \le k \rangle \cong R_{\mathbb{X}}(-e_1) \oplus \cdots \oplus R_{\mathbb{X}}(-e_k).$$

Moreover, we get the exact sequence of $\mathbb{Z}^k$-graded $R_{\mathbb{X}}$-modules

$$0 \longrightarrow I_P/I_P^2 \longrightarrow R_{\mathbb{X}}^{n_1+1}(-e_1) \oplus \cdots \oplus R_{\mathbb{X}}^{n_k+1}(-e_k) \longrightarrow \Omega^1_{R_{\mathbb{X}}/K} \longrightarrow 0.$$

In this case the Hilbert function of $\Omega^1_{R_{\mathbb{X}}/K}$ satisfies

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K}}(\underline{i}) = \mathrm{HF}_{\mathbb{X}}(\underline{i} - e_1) + \cdots + \mathrm{HF}_{\mathbb{X}}(\underline{i} - e_k).$$

In general, we have a presentation of module of Kähler differential as follows.

Theorem 3.5. *Let $\mathbb{Y} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and let $\mathbb{V}$ be the thickening of $\mathbb{Y}$. Then we have the exact sequence of $\mathbb{Z}^k$-graded $R_{\mathbb{Y}}$-modules*

$$0 \longrightarrow I_{\mathbb{Y}}/I_{\mathbb{V}} \longrightarrow R_{\mathbb{Y}}^{n_1+1}(-e_1) \oplus \cdots \oplus R_{\mathbb{Y}}^{n_k+1}(-e_k) \longrightarrow \Omega^1_{R_{\mathbb{Y}}/K} \longrightarrow 0.$$

To prove Theorem 3.5, we will first require the following lemma.

Lemma 3.6. *Let* $P = [1 : 0 : \cdots : 0] \times \cdots \times [1 : 0 : \cdots : 0]$ *be a point of* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$*, let* $m \geq 1$*, and let* $F \in I_P^m \setminus I_P^{m+1}$ *be a* $\mathbb{Z}^k$*-homogeneous polynomial of degree* $\underline{i} \in \mathbb{Z}^k$*. Then we have* $dF \notin I_P^m \Omega^1_{S/K}$*.*

Proof. Observe that $I_P^{m+1} = \langle X_{ij_i} \mid 1 \leq i \leq k, 1 \leq j \leq n_i \rangle^{m+1}$. Let $N = n_1 + \cdots + n_k$. For $\underline{j} \in \mathbb{N}^k$ and $t \geq 1$, we set

$$\Gamma_{\underline{j},t} = \left\{ \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^N \,\middle|\, \begin{array}{l} |\alpha| = t,\ (|\alpha_1|, \ldots, |\alpha_k|) \leq \underline{j}, \\ \alpha_l = (\alpha_{l1}, \ldots, \alpha_{ln_l}), 1 \leq l \leq k \end{array} \right\}$$

Since $F \in I_P^m$, the polynomial F can be written as

$$F = \sum_{\alpha \in \Gamma_{\underline{i},m+1}} X_{11}^{\alpha_{11}} \cdots X_{1n_1}^{\alpha_{1n_1}} \cdots X_{k1}^{\alpha_{k1}} \cdots X_{kn_k}^{\alpha_{kn_k}} F_\alpha + \sum_{\beta \in \Gamma_{\underline{i},m}} X_{11}^{\beta_{11}} \cdots X_{1n_1}^{\beta_{1n_1}} \cdots X_{k1}^{\beta_{k1}} \cdots X_{kn_k}^{\beta_{kn_k}} G_\beta$$

where $F_\alpha \in S_{\underline{i}-(|\alpha_1|,\ldots,|\alpha_k|)}$, and $G_\beta = a_\beta X_{10}^{c_{\beta 1}} \cdots X_{k0}^{c_{\beta k}}$ with $(c_{\beta 1}, \ldots, c_{\beta k}) = \underline{i} - (|\beta_1|, \ldots, |\beta_k|)$ and $a_\beta \in K$. It follows from $F \notin I_P^{m+1}$ that there exists $\widetilde{\beta} \in \Gamma_{\underline{i},m}$ such that $a_{\widetilde{\beta}} \neq 0$.

In order to prove $dF \notin I_P^m \Omega^1_{S/K}$, it suffices to show that

$$\omega = \sum_{\beta \in \Gamma_{\underline{i},m}} G_\beta d(X_{11}^{\beta_{11}} \cdots X_{1n_1}^{\beta_{1n_1}} \cdots X_{k1}^{\beta_{k1}} \cdots X_{kn_k}^{\beta_{kn_k}}) \notin I_P^m \Omega^1_{S/K}.$$

Write $G_{\widetilde{\beta}} = a_{\widetilde{\beta}} \cdot T$ with the term $T = X_{10}^{c_{\widetilde{\beta}1}} \cdots X_{k0}^{c_{\widetilde{\beta}k}}$, and set

$$\widetilde{\Gamma} = \{\beta \in \Gamma_{\underline{i},m} \mid a_\beta \neq 0,\ G_\beta = a_\beta T\}.$$

Clearly, $\widetilde{\beta} \in \widetilde{\Gamma} \neq \emptyset$. Let (i, j) be the smallest tuple w.r.t. Lex such that there exists $\beta \in \widetilde{\Gamma}$ such that $b_{ij} \neq 0$. Write $\widetilde{\Gamma} = \{\widetilde{\beta}^1, \ldots, \widetilde{\beta}^s\}$ such that $\widetilde{\beta}^1 \leq_{\mathtt{Lex}} \cdots \leq_{\mathtt{Lex}} \widetilde{\beta}^s$. Then $\widetilde{\beta}^1_{ij} \leq \cdots \leq \widetilde{\beta}^s_{ij}$. Set $\varrho = \min\{\widetilde{\beta}^l_{ij} \mid \widetilde{\beta}^l_{ij} > 0, 1 \leq l \leq s\}$. We have

$$\begin{aligned}
\widetilde{\omega} :&= \sum_{\beta \in \widetilde{\Gamma}} a_\beta T d(X_{11}^{\beta_{11}} \cdots X_{1n_1}^{\beta_{1n_1}} \cdots X_{k1}^{\beta_{k1}} \cdots X_{kn_k}^{\beta_{kn_k}}) \\
&= \sum_{\beta \in \widetilde{\Gamma}} a_\beta T d(X_{ij}^{\beta_{ij}} \cdots X_{in_i}^{\beta_{in_i}} \cdots X_{k1}^{\beta_{k1}} \cdots X_{kn_k}^{\beta_{kn_k}}) \\
&= \sum_{\beta \in \widetilde{\Gamma}, \beta_{ij} \neq \varrho} a_\beta T d(X_{ij}^{\beta_{ij}} \cdots X_{in_i}^{\beta_{in_i}} \cdots X_{k1}^{\beta_{k1}} \cdots X_{kn_k}^{\beta_{kn_k}}) + \sum_{\beta \in \widetilde{\Gamma}, \beta_{ij} = \varrho} a_\beta T d(X_{ij}^{\varrho} \cdots X_{in_i}^{\beta_{in_i}} \cdots X_{k1}^{\beta_{k1}} \cdots X_{kn_k}^{\beta_{kn_k}}) \\
&\overset{(\star)}{=} \sum_{\beta \in \widetilde{\Gamma}, \beta_{ij} \neq \varrho} a_\beta T d(X_{ij}^{\beta_{ij}} \cdots X_{in_i}^{\beta_{in_i}} \cdots X_{k1}^{\beta_{k1}} \cdots X_{kn_k}^{\beta_{kn_k}}) \\
&\quad + \sum_{\beta \in \widetilde{\Gamma}, \beta_{ij} = \varrho} a_\beta T \varrho X_{ij}^{\varrho - 1} X_{ij+1}^{\beta_{ij+1}} \cdots X_{in_i}^{\beta_{in_i}} \cdots X_{k1}^{\beta_{k1}} \cdots X_{kn_k}^{\beta_{kn_k}} dX_{ij} \\
&\quad + \sum_{\beta \in \widetilde{\Gamma}, \beta_{ij} = \varrho} a_\beta T X_{ij}^{\varrho} d(X_{ij+1}^{\beta_{ij+1}} \cdots X_{in_i}^{\beta_{in_i}} \cdots X_{k1}^{\beta_{k1}} \cdots X_{kn_k}^{\beta_{kn_k}}).
\end{aligned}$$

Note that if $\beta, \beta' \in \widetilde{\Gamma}$, $\beta \neq \beta'$ and $\beta_{ij} = \beta'_{ij} = \varrho$, then $(\beta_{ij+1}, \ldots, \beta_{kn_k}) \neq (\beta'_{ij+1}, \ldots, \beta'_{kn_k})$. By Macaulay's Basis Theorem (cf. [5, Theorem 1.5.7]), we have

$$\sum_{\beta \in \widetilde{\Gamma}, \beta_{ij} = \varrho} a_\beta T \varrho X_{ij}^{\varrho - 1} X_{ij+1}^{\beta_{ij+1}} \cdots X_{in_i}^{\beta_{in_i}} \cdots X_{k1}^{\beta_{k1}} \cdots X_{kn_k}^{\beta_{kn_k}} dX_{ij} \notin I_P^m \Omega^1_{S/K}.$$

It follows that $\widetilde{\omega} \notin I_P^m \Omega^1_{S/K}$ (since in the right-hand side of ($\star$) the first summand has $\beta_{ij} \neq \varrho$ and the last summand does not contain dX_{ij}). Therefore we get $\omega \notin I_P^m \Omega^1_{S/K}$, as we wanted to show. □

We are now ready to state and prove the main result of this section.

Proof. Let $\varphi : I_{\mathbb{Y}}/I_{\mathbb{V}} \to \Omega^1_{S/K}/I_{\mathbb{Y}}\Omega^1_{S/K}$ be the map given by $\varphi(F + I_{\mathbb{V}}) = dF + I_{\mathbb{Y}}\Omega^1_{S/K}$ for all $F \in I_{\mathbb{Y}}$. It is easy to check that the map φ is well-defined, $\mathbb{Z}^k$-homogeneous of degree $(0, \dots, 0)$, and $R_{\mathbb{Y}}$-linear. For any $\mathbb{Z}^k$-homogeneous element $F \in I_{\mathbb{Y}} \setminus I_{\mathbb{V}}$, we have $F \in I_{P_j}^{m_j} \setminus I_{P_j}^{m_j+1}$ for some $j \in \{1, \dots, s\}$. W.l.o.g. we may assume that $P_j = [1 : 0 : \cdots : 0] \times \cdots \times [1 : 0 : \cdots : 0]$. So, Lemma 3.6 yields $dF \notin I_{P_j}^{m_j}\Omega^1_{S/K}$. Since $I_{\mathbb{Y}}\Omega^1_{S/K} = \bigcap_{j=1}^s I_{P_j}^{m_j}\Omega^1_{S/K}$ by Remark 3.2(b), we obtain $dF \notin I_{\mathbb{Y}}\Omega^1_{S/K}$. Consequently, the map φ is injective. Moreover, by Remark 3.2(a) and Proposition 3.3, we see that

$$\Omega^1_{S/K}/I_{\mathbb{Y}}\Omega^1_{S/K} \cong R_{\mathbb{Y}}^{n_1+1}(-e_1) \oplus \cdots \oplus R_{\mathbb{Y}}^{n_k+1}(-e_k)$$

and

$$(\Omega^1_{S/K}/I_{\mathbb{Y}}\Omega^1_{S/K})/\operatorname{im}(\varphi) \cong \Omega^1_{S/K}/(dI_{\mathbb{Y}} + I_{\mathbb{Y}}\Omega^1_{S/K}) \cong \Omega^1_{R_{\mathbb{Y}}/K}.$$

Therefore the conclusion follows. □

The following relation between the Hilbert function of $\Omega^1_{R_{\mathbb{Y}}/K}$ and of $\mathbb{Y}$ and $\mathbb{V}$ is an immediate consequence of Theorem 3.5

Corollary 3.7. *$\mathbb{Y} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and let $\mathbb{V}$ be the thickening of $\mathbb{Y}$. Then the Hilbert function of $\Omega^1_{R_{\mathbb{Y}}/K}$ satisfies*

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{Y}}/K}}(\underline{i}) = (n_1 + 1)\,\mathrm{HF}_{\mathbb{Y}}(\underline{i} - e_1) + \cdots + (n_k + 1)\,\mathrm{HF}_{\mathbb{Y}}(\underline{i} - e_k) + \mathrm{HF}_{\mathbb{Y}}(\underline{i}) - \mathrm{HF}_{\mathbb{V}}(\underline{i})$$

for all $\underline{i} \in \mathbb{Z}^k$.

References

[1] G. de Dominicis and M. Kreuzer, Kähler differentials for points in $\mathbb{P}^n$, J. Pure Appl. Alg. **141**, 153–173 (1999).

[2] E. Guardo, M. Kreuzer, T. N. K. Linh, and L.N. Long, Kähler differentials for fat point schemes in $\mathbb{P}^1 \times \mathbb{P}^1$, Submitted (2018).

[3] E. Guardo and A. Van Tuyl, Fat Points in $\mathbb{P}^1 \times \mathbb{P}^1$ and their Hilbert functions, Canad. J. Math. **56** , no. 4, 716–741 (2004).

[4] M. Kreuzer, T. N. K. Linh, and L. N. Long, Kähler differentials and Kähler differents for fat point schemes, J. Pure Appl. Algebra **219**, 4479–4509 (2015).

[5] M. Kreuzer and L. Robbiano, *Computational Commutative Algebra 1*, (Springer Verlag, Heildelberg, 2000).

[6] E. Kunz, *Kähler Differentials* (Adv. Lectures Math., Wieweg Verlag, Braunschweig, 1986).

[7] H. Matsumura, *Commutative Ring Theory* (Cambridge University Press, Cambrige, 1986).

[8] J. Sidman and A. Van Tuyl, Multigraded regularity: Syzygies and fat points, Beiträge zur Algebra und Geometrie **47**, 1–22 (2006).