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Weak solutions for a system of quasilinear elliptic equations

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Abstract

A system of quasilinear elliptic equations on an unbounded domain is considered. The existence of a sequence of radially symmetric weak solutions is proved via variational methods.

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1. Introduction

We consider the following problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda \alpha_1(x) f_1(v) & \text{in } \mathbb{R}^N, \\ -\Delta_q v + |v|^{q-2}v = \lambda \alpha_2(x) f_2(u) & \text{in } \mathbb{R}^N, \\ u, v \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1}$$

where $p, q > N > 1$. We assume that $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are continuous functions, $\alpha_1(x), \alpha_2(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ are nonnegative (not identically zero) radially symmetric maps, and λ is a real parameter. Also $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator.

Partial differential equations's is used to model a wide variety of physically significant problems arising in every different areas such as physics, engineering and other applied disciplines (see [7, 11, 12, 18, 25, 26, 28–31, 34–45]). Sobolev spaces play an important role in the theory of partial differential equations as well as Orlicz-Morrey space and $\dot{B}^{-1}_{\infty,\infty}$ space (see [2, 8–10, 32–34]). Laplace equation is the prototype for linear elliptic equations. This equation has a non-linear counterpart, the so-called p-Laplace equation (see [1, 6, 13, 14, 19, 21–24, 48]).

Here, by inspiration of [20], we prove the existence of a sequence of radially symmetric weak solutions for (1) in the unbounded domain $\mathbb{R}^N$. The solution of (1) belongs to the product space

$$W^{1,(p,q)}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N) \times W^{1,q}(\mathbb{R}^N)$$

equipped with the norm $\|(u,v)\|_{(p,q)} = \|u\|_p + \|u\|_q$.

Definition 1.1. *For fixed λ_1 and λ_2, $((u,v) : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is said to be a weak solution of (1), if $(u,,v) \in W^{1,(p,q)}(\mathbb{R}^N)$ and for every $(z,w) \in W^{1,(p,q)}(\mathbb{R}^N)$*

$$-\int_{\mathbb{R}^N} |\nabla u(x)|^{p-2}\nabla u(x).\nabla z(x)dx - \int_{\mathbb{R}^N} |\nabla u(x)|^{q-2}\nabla u(x).\nabla w(x)dx + \int_{\mathbb{R}^{\mathbb{N}}} |u(x)|^{p-2}u(x)z(x)dx + \int_{\mathbb{R}^{\mathbb{N}}} |v(x)|^{q-2}v(x)w(x)dx$$

$$-\lambda_1 \int_{\mathbb{R}^{\mathbb{N}}} \alpha_1(x) f_1(v(x))z(x)dx - \lambda_2 \int_{\mathbb{R}^{\mathbb{N}}} \alpha_2(x) f_2(u(x))w(x)dx = 0,$$

where

$$\|(u,v)\|_{W^{1,(p,q)}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^{\mathbb{N}}} |\nabla u(x)|^p dx + \int_{\mathbb{R}^{\mathbb{N}}} |u(x)|^p dx\right) + \left(\int_{\mathbb{R}^{\mathbb{N}}} |\nabla v(x)|^q dx + \int_{\mathbb{R}^{\mathbb{N}}} |v(x)|^q dx\right).$$

Note that the critical points of an energy functional are exactly the weak solutions of (1). Morrey's theorem, implies the continuous embedding

$$W^{1,(p,q)}(\mathbb{R}^{\mathbb{N}}) \hookrightarrow L^\infty(\mathbb{R}^{\mathbb{N}}) \times L^\infty(\mathbb{R}^{\mathbb{N}}), \tag{2}$$

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which says that there exists c (depends on p, q, N), such that

$$\|(u,v)\|_\infty \leq c\|(u,v)\|_{W^{1,(p,q)}(\mathbb{R}^N)},$$

for every $(u,v) \in W^{1,p}(\mathbb{R}^{\mathbb{N}}) \times W^{1,q}(\mathbb{R}^{\mathbb{N}})$, where $\|(u,v)\|_\infty := \max\{\|u\|_\infty, \|v\|_\infty\}$. Since in the low-dimensional case, every function $(u,v) \in W^{1,(p,q)}(\mathbb{R}^N)$ admits a continuous representation (see [4, p.166]). In the sequel we will replace (u,v) by this element. We need the following notations (see [5] or [17] for more details):

(I) $O(N)$ stands for the orthogonal group of $\mathbb{R}^N$.

(II) $B(0,s)$ denotes the open N-dimensional ball of center zero, radius $s > 0$ and standard Lebesgue measure, $meas(B(0,s))$.

(III) $\|\alpha\|_{B(0,\frac{s}{2})} := \int_{B(0,\frac{s}{2})} \alpha(x)dx$.

Definition 1.2.

- *A function $h : \mathbb{R}^N \to \mathbb{R}$ is radially symmetric if $h(gx) = h(x)$, for every $g \in O(N)$ and $x \in \mathbb{R}^N$.*
- *Let G be a topological group. A continuous map $\xi : G \times X \to X : (g,x) \to \xi(g,u) := gu$, is called the action of G on the Banach space $(X, \|.\|_X)$ if*

$$1u = u, \qquad (gm)u = g(mu), \qquad u \mapsto gu \text{ is linear}.$$

- *The action is said to be isometric if $\|gu\|_X = \|u\|_X$, for every $g \in G$.*
- *The space of G-invariant points is defined by*

$$Fix(G) := \{u \in X : gu = u, \text{for all } g \in G\}.$$

- *A map $m : X \to \mathbb{R}$ is said to be G-invariant if $mog = m$ for every $g \in G$.*

The following theorem is important to study the critical point of the functional (see [27]).

Theorem 1.1. *Assume that the action of the topological group G on the Banach space X is isometric. If $J \in C^1(X : \mathbb{R})$ is G-invariant and if u is a critical point of J restricted to $Fix(G)$, then u is a critical point of J.*

The action of the group $O(N)$ on $W^{1,p}(\mathbb{R}^N)$ can be defined by $(gu)(x) := u(g^{-1}x)$, for every $g \in W^{1,p}(\mathbb{R}^N)$ and $x \in \mathbb{R}^{\mathbb{N}}$. A computation shows that this group acts linearly and isometrically, which means $\|u\| = \|gu\|$, for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$.

Definition 1.3. *The subspace of radially symmetric functions of $W_r^{1,(p,q)}(\mathbb{R}^N)$ is defined by*

$$\begin{aligned} X := {} & W_r^{1,(p,q)}(\mathbb{R}^N) \\ := {} & \{(u,v) \in W^{1,(p,q)}(\mathbb{R}^N) : (g_1u, g_2v) = (u,v), \text{ for all } (g_1,g_2) \in O(N) \times O(N)\}, \end{aligned}$$

and endowed by the norm

$$\|(u,v)\|_{W_r^{1,(p,q)}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^{\mathbb{N}}} |\nabla u(x)|^p dx + \int_{\mathbb{R}^{\mathbb{N}}} |u(x)|^p dx\right) + \left(\int_{\mathbb{R}^{\mathbb{N}}} |\nabla v(x)|^q dx + \int_{\mathbb{R}^{\mathbb{N}}} |v(x)|^q dx\right).$$

In what follows: $\|(u,v)\|_r$ denotes $\|(u,v)\|_{W_r^{1,(p,q)}(\mathbb{R}^N)}$. The following crucial embedding result due to Kristály and principally based on a Strauss-type estimation (see [46]) (also see [15, Theorem 3.1], [16] and [47] for related subjects).

Theorem 1.2. *The embedding $W_r^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, is compact whenever $2 \leq N < p < +\infty$.*

Here we consider the following functionals:

- $F_i(\xi) := \int_0^\xi f_i(t)dt$ for every $\xi \in \mathbb{R}$.
- $\Phi(u,v) := \frac{\|u\|_r^p}{p} + \frac{\|v\|_r^q}{q}$ for every $(u,v) \in X$.
- $\Psi(u,v) := \int_{\mathbb{R}^{\mathbb{N}}} \alpha_1 F_1(v(x))dx + \int_{\mathbb{R}^{\mathbb{N}}} \alpha_2 F_2(u(x))dx$, for every $(u,v) \in X$.
- $I_\lambda(u,v) := \Phi(u,v) - \lambda\Psi(u,v)$ for every $(u,v) \in X$.

By standard arguments [5], we can show that Φ is Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous whose derivative at the point $(u,v) \in X$ is the functional $\Phi'(u,v) \in X^*$ given by

$$\Phi'(u,v)(z,w) = \quad (\int_{\mathbb{R}^{\mathbb{N}}} |\nabla u|^{p-2}\nabla u.\nabla z dx + \int_{\mathbb{R}^{\mathbb{N}}} |u|^{p-2}uz dx) + (\int_{\mathbb{R}^{\mathbb{N}}} |\nabla v|^{q-2}\nabla v.\nabla w dx + \int_{\mathbb{R}^{\mathbb{N}}} |v|^{q-2}vw dx),$$

for every $(z,w) \in X$. Also standard arguments show that the functional Ψ_i are well defined, sequentially weakly upper semicontinuous and Gâteaux differentiable whose Gâteaux derivative at the point $(u,v) \in X$ and for every $(z,w) \in X$ is given by

$$\Psi'(u,v)(z,w) = \int_{\mathbb{R}^{\mathbb{N}}} \alpha_1(x) f_1(u(x)) dx + \int_{\mathbb{R}^{\mathbb{N}}} \alpha_2(x) f_2(v(x)) dx.$$

2. Weak solutions

First, we recall the following theorem [3, Theorem 2.1].

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, set*

$$\varphi(r) := \inf_{\Phi(u)<r} \frac{\sup_{\Phi(v)<r} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$
$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \textit{and} \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) if $\gamma < +\infty$, then for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds either,

(b_1) I_λ possesses a global minimum, or

(b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

(c) if $\delta < +\infty$, then for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds either:

(c_1) there is a global minimum of Φ which is a local minimum of I_λ. or,

(c_2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_{u\in X} \Phi(u)$.

For fixed $D > 0$, set

$$m(D) := meas(B(0,D)) = D^N \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})},$$

where Γ is the Gamma function defined by $\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}dz$ for all $t > 0$. Moreover,

$$\Omega := \max\left\{ \frac{m(D)\left(\frac{\sigma(N,p)}{D^p} + g(p,N)\right)}{p\lambda B_1 \|\alpha_2\|_{B(0,\frac{D}{2})}}, \frac{m(D)\left(\frac{\sigma(N,q)}{D^q} + g(q,N)\right)}{q\lambda B_2 \|\alpha_1\|_{B(0,\frac{D}{2})}} \right\} > 0, \tag{3}$$

where $\sigma(N,p) := 2^{p-N}(2^N - 1)$, $c = \frac{2p}{2-N}$, m_1, m_0 are upper and lower bounds for $M(t)$ in (1) and

$$g(p,N) := \frac{1 + 2^{N+p} N \int_{\frac{1}{2}}^1 t^{N-1}(1-t)^p dt}{2^N}.$$

Assume $\|\cdot\|_1$ denotes the norm of $L^1(\Omega)$ and $F(\xi) := F_1(\xi) + F_2(\xi)$.

Theorem 2.2. *Let $f_i : \mathbb{R} \to \mathbb{R}$ be two continuous and radially symmetric functions. Set*

$$A := \liminf_{(\xi_1,\xi_2)\to+\infty} \frac{\max_{|t_1|\le\xi_1} F_2(t_1)}{|\xi_1|^p} + \frac{\max_{|t_2|\le\xi_2} F_1(t_2)}{|\xi_2|^q},$$
$$B_1 := \limsup_{\xi_2\to+\infty} \frac{F_1(\xi_2)}{|\xi_2|^p}, \textit{ and } B_2 := \limsup_{\xi_1\to+\infty} \frac{F_2(\xi_1)}{|\xi_1|^q}.$$

where $B := B_1 + B_2$, $\xi = (\xi_1, \xi_2)$. *If* $\inf_{(\xi_1,\xi_2)\geq 0} F_2(\xi_1) + F_1(\xi_2) = 0$ *and* $A < \Omega m_0 B$, *where* Ω *is given by* (3), *for every*

$$\lambda \in \Lambda := \left]\Omega, \frac{1}{(pc_1^p\|\alpha_2\|_1 + qc_2^q\|\alpha_1\|_1)\, A}\right[,$$

there exists an unbounded sequence of radially symmetric weak solutions for (1) *in* X.

Proof. For fixed $\lambda \in \Lambda$, we consider Φ, Ψ and I_λ as in the last section. Knowing that Φ and Ψ satisfy the regularity assumptions in Theorem 2.1. In order to study the critical points of I_λ in X, we show that $\lambda < \frac{1}{\gamma} < +\infty$, where $\gamma = \liminf_{r\to+\infty}\phi(r)$. Let $\{t_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} t_n = +\infty$,

$$r_{1n} := \frac{t_{1n}^p}{pc_1^p} \qquad \text{and} \qquad r_{2n} := \frac{t_{2n}^q}{qc_2^q},$$

for all $n \in \mathbb{N}$. Set $r_n = \min\{r_{n1}, r_{n2}\}$. Considering Theorem 1.2 (by relation (2)), a computation shows that

$$\begin{aligned} \Phi^{-1}(]-\infty, r_n[) =& \{(z,w) \in X : \Phi(z,w) < r_n\} \\ =& \{(z,w) \in X : \frac{\|z\|_r^p}{p} + \frac{\|w\|_r^q}{q} < r_n\} \\ \subset& \{(z,w) \in X; \|(z,w)\|_\infty < t_n\}, \end{aligned} \tag{4}$$

where $t_n = \min\{t_{n1}, t_{n2}\}$. Since $\Phi(0,0) = \Psi(0,0) = 0$, by a computation one can show

$$\begin{aligned} \varphi(r_n) =& \inf_{\Phi(u,v)<r_n} \frac{(\sup_{\Phi(z,w)<r_n} \Psi(z,w)) - \Psi(u,v)}{r_n - \Phi(u,v)} \\ &\leq (pc_1^p\|\alpha_2\|_1 + qc_2^q\|\alpha_1\|_1)\, A. \end{aligned}$$

Hence

$$\gamma \leq \liminf_{n\to+\infty} \varphi(r_n) \leq (pc_1^p\|\alpha_2\|_1 + qc_2^q\|\alpha_1\|_1)\, A < +\infty.$$

Now, we show that I_λ is unbounded from below. Let $\{d_{1n}\}$ and $\{d_{2n}\}$ be two sequences of positive numbers such that $\lim_{n\to+\infty} d_{1n} = \lim_{n\to+\infty} d_{2n} = +\infty$ and

$$B_1 = \lim_{n\to+\infty} \frac{F_1(d_{2n})}{d_{2n}^q}, \ B_2 = \lim_{n\to+\infty} \frac{F_2(d_{1n})}{d_{1n}^p}. \tag{5}$$

Define $\{(H_{1n}, H_{2n})\} \in X$ by

$$H_{in}(x) := \begin{cases} 0 & \mathbb{R}^{\mathbb{N}} \setminus B(0,D) \\ d_{in} & B(0, \frac{D}{2}) \\ \frac{2d_{in}}{D}(D - |x|) & B(0,D) \setminus B(0, \frac{D}{2}), \end{cases}$$

for every $n \in \mathbb{N}$ and $i = 1, 2$. By a similar argument and computations in [5, P.1017] one can show that

$$\begin{aligned} \|H_{2n}\|_r^q =& \ d_{2n}^q m(D)\left(\tfrac{\sigma(N,p)}{D^q} + g(q,N)\right), \text{ and} \\ \|H_{1n}\|_r^p =& \ d_{1n}^p m(D)\left(\tfrac{\sigma(N,p)}{D^p} + g(p,N)\right). \end{aligned}$$

Condition (i), implies

$$\begin{aligned} \int_{\mathbb{R}^{\mathbb{N}}} \alpha_1(x)F_1(H_{2n}(x))dx \geq& \ \int_{B(0,\frac{D}{2})} \alpha_1(x)F_1(d_{2n})dx = F_1(d_{2n})\|\alpha_1\|_{B(0,\frac{D}{2})}, \text{ and} \\ \int_{\mathbb{R}^{\mathbb{N}}} \alpha_2(x)F_2(H_{1n}(x))dx \geq& \ \int_{B(0,\frac{D}{2})} \alpha_2(x)F_2(d_{1n})dx = F_2(d_{1n})\|\alpha_2\|_{B(0,\frac{D}{2})}, \end{aligned}$$

for every $n \in \mathbb{N}$. Then

$$\begin{aligned} I_\lambda(H_{1n}, H_{2n}) =& \ \Phi(H_{1n}, H_{2n}) - \lambda\Psi(H_{1n}, H_{2n}) \\ =& \ \tfrac{\|H_{1n}\|_r^p}{p} + \tfrac{\|H_{2n}\|_r^q}{q} - \lambda\int_{\mathbb{R}^N} \alpha_1(x)F_1(H_{2n}(x))dx - \lambda\int_{\mathbb{R}^N} \alpha_2(x)F_2(H_{1n}(x))dx \\ \leq& \ \frac{d_{1n}^p m(D)\left(\frac{\sigma(N,p)}{D^P} + g(p,N)\right)}{p} + \frac{d_{2n}^q m(D)\left(\frac{\sigma(N,q)}{D^q} + g(q,N)\right)}{q} \\ & -\lambda\left(F_1(d_{2n})\|\alpha_1\|_{B(0,\frac{D}{2})} + F_2(d_{1n})\|\alpha_2\|_{B(0,\frac{D}{2})}.\right) \end{aligned}$$

If $B < +\infty$ ($B_1, B_2 < +\infty$), the conditions (5) implies that

there exists N_1 such that for all $n \geq N_1$ we have $F_1(d_{2n}) > \varepsilon B_1 d_{2n}^p$, and
there exists N_2 such that for all $n \geq N_2$ we have $F_2(d_{1n}) > \varepsilon B_2 d_{1n}^q$.

Then for every $n \geq N_\varepsilon := \max\{N_1, N_2\}$,

$$\begin{aligned} I_\lambda(H_{1n}, H_{2n}) \leq & \frac{d_{1n}^p m(D)\left(\frac{\sigma(N,p)}{D^P}+g(p,N)\right)}{p} + \frac{d_{2n}^q m(D)\left(\frac{\sigma(N,q)}{D^q}+g(q,N)\right)}{q} \\ & -\lambda\varepsilon \left(d_{1n}^p B_2 \|\alpha_1\|_{B(0,\frac{D}{2})} + d_{2n}^q B_2 \|\alpha_1\|_{B(0,\frac{D}{2})}.\right) \\ = & \; d_{1n}^p \left(\frac{m(D)\left(\frac{\sigma(N,p)}{D^P}+g(p,N)\right)}{p} - \lambda\varepsilon B_1 \|\alpha_2\|_{B(0,\frac{D}{2})}\right) \\ & + d_{2n}^q \left(\frac{m(D)\left(\frac{\sigma(N,q)}{D^q}+g(q,N)\right)}{q} - \lambda\varepsilon B_2 \|\alpha_1\|_{B(0,\frac{D}{2})}\right). \end{aligned}$$

If we set

$$\Omega := \max\left\{\frac{m(D)\left(\frac{\sigma(N,p)}{D^p}+g(p,N)\right)}{p\lambda B_1\|\alpha_2\|_{B(0,\frac{D}{2})}}, \frac{m(D)\left(\frac{\sigma(N,q)}{D^q}+g(q,N)\right)}{q\lambda B_2\|\alpha_1\|_{B(0,\frac{D}{2})}}\right\},$$

then for $\varepsilon \in (\Omega, 1)$ one can get

$$\lim_{n\to+\infty} I_\lambda(H_{1n}, H_{2n}) = -\infty.$$

If at least one of the B_1 or B_2 are $+\infty$. Let $B_1 = +\infty$, and consider $M_1 > \Omega$, then by (5) there exists N_{M_1} such that for every $n > N_{M_1}$, we have $F_1(d_{1n}) > M_1 d_{1n}^p$. Moreover, for every $n > N_{M_1}$

$$\begin{aligned} I_\lambda(H_{1n}, H_{2n}) \leq & \frac{d_{1n}^p m(D)\left(\frac{\sigma(N,p)}{D^P}+g(p,N)\right)}{p} + \frac{d_{2n}^q m(D)\left(\frac{\sigma(N,q)}{D^q}+g(q,N)\right)}{q} \\ & -\lambda \left(d_{1n}^p M_1 \|\alpha_2\|_{B(0,\frac{D}{2})} + d_{2n}^q M_1 \|\alpha_1\|_{B(0,\frac{D}{2})}.\right) \\ = & \; d_{1n}^p \left(\frac{m(D)\left(\frac{\sigma(N,p)}{D^P}+g(p,N)\right)}{p} - \lambda M_1 \|\alpha_2\|_{B(0,\frac{D}{2})}\right) \\ & + d_{2n}^q \left(\frac{m(D)\left(\frac{\sigma(N,q)}{D^q}+g(q,N)\right)}{q} - \lambda M_1 \|\alpha_1\|_{B(0,\frac{D}{2})}\right). \end{aligned}$$

This implies that $\lim_{n\to+\infty} I_\lambda(H_{1n}, H_{2n}) = -\infty$.

Now, Theorem 2.1 (*b*) implies, the functional I_λ admits an unbounded sequence $\{u_n\} \subset X$ of critical points. Considering Theorem 1.1, these critical points are also critical points for the smooth and $O(N)$-invariant functional $I_\lambda : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$. Therefore, there is a sequence of radially symmetric weak solutions for the problem (1), which are unbounded in $W^{1,p}(\mathbb{R}^N)$. □

Here we prove our second result which says that under different conditions the problem (1) has a sequence of weak solutions, which converges weakly to zero.

Theorem 2.3. *Let $f_i : \mathbb{R} \to \mathbb{R}$ be two continuous and radially symmetric functions. Set*

$$\begin{aligned} A' &:= \liminf_{(\xi_1,\xi_2)\to 0^+} \frac{\max_{|t_1|\leq\xi_1} F_2(t_1)}{|\xi_1|^p} + \frac{\max_{|t_2|\leq\xi_2} F_1(t_2)}{|\xi_2|^q}, \\ B_1' &:= \limsup_{\xi_2\to 0^+} \frac{F_1(\xi_2)}{|\xi_2|^p}, \text{ and } B_2' := \limsup_{\xi_1\to 0^+} \frac{F_2(\xi_1)}{|\xi_1|^q}, \end{aligned}$$

where $B' := B_1' + B_2'$, $\xi = (\xi_1, \xi_2)$. If $\inf_{(\xi_1,\xi_2)\geq 0} F_2(\xi_1) + F_1(\xi_2) = 0$ and $A' < \Omega m_0 B'$, where Ω is given by (3), for every

$$\lambda \in \Lambda' := \left]\Omega, \frac{1}{(pc_1^p\|\alpha_2\|_1 + qc_2^q\|\alpha_1\|_1)\, A'}\right[,$$

there exists an unbounded sequence of radially symmetric weak solutions for (1) in X.

Proof. For fixed $\lambda \in \Lambda'$, we consider Φ, Ψ and I_λ as in Section 2. Knowing that Φ and Ψ satisfy the regularity assumptions in Theorem (2.1), we show that $\lambda < \frac{1}{\delta}$. We know that $\inf_X \Phi = 0$. Set $\delta := \liminf_{r\to 0^+} \varphi(r)$. A computation similar to the one in the Theorem 2.2 implies $\delta < \infty$ and if $\lambda \in \Lambda'$ then $\lambda < \frac{1}{\delta}$. A compaction (similar in the Theorem 2.2) shows that $I_\lambda(H_{1n}, H_{2n}) < 0$ for n large enough and thus zero is not a local minimum of I_λ. Therefore, there exists a sequence $\{u_n\} \subset X$ of critical points of I_λ which converges weakly to zero in X as $\lim_{n\to+\infty} \Phi(u_n) = 0$. Again, considering Theorem 1.1, these critical points are also critical points for the smooth and $O(N)$-invariant functional $I_\lambda : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$. Therefore, there is a sequence of radially symmetric weak solutions for the problem (1), which converges weakly to zero in $W^{1,p}(\mathbb{R}^N)$. □

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