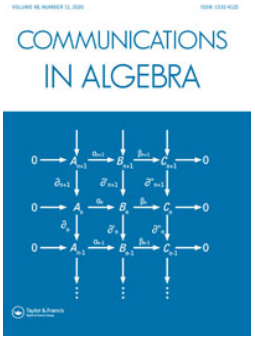

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ABSTRACT
Let I_X be the saturated homogeneous ideal defining a codimension two arithmetically Cohen-Macaulay scheme $X \subseteq \mathbb{P}^n$, and let $I_X^{(m)}$ denote its m-th symbolic power. We are interested in when $I_X^{(m)} = I_X^m$. We survey what is known about this problem when X is locally a complete intersection, and in particular, we review the classification of when $I_X^{(m)} = I_X^m$ for all $m \geq 1$. We then discuss how one might weaken these hypotheses, but still obtain equality between the symbolic and ordinary powers. Finally, we show that this classification allows one to: (1) simplify known results about symbolic powers of ideals of points in $\mathbb{P}^1 \times \mathbb{P}^1$; (2) verify a conjecture of Guardo, Harbourne, and Van Tuyl, and (3) provide additional evidence to a conjecture of Römer.

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1. Introduction

Throughout this paper, $R = k[x_0, \ldots, x_n]$ where k is an algebraically closed field of characteristic zero. For any non-zero homogeneous ideal $I \subseteq R$, the *m-th symbolic power* of I, denoted $I^{(m)}$, is the ideal

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} (I^m R_{\mathfrak{p}} \cap R),$$

where $R_{\mathfrak{p}}$ denotes the localization of R at the prime ideal $\mathfrak{p}$, and $\mathrm{Ass}(I)$ is the set of associated primes of I. In general, $I^m \subseteq I^{(m)}$, but the reverse containment may fail. The *ideal containment problem* consists of determining the values of r and m for which $I^{(r)} \subseteq I^m$. holds. The papers of Ein, Lazarsfeld, and Smith [12], Hochster and Huneke [25], and Bocci and Harbourne [5, 6] are among the first papers to systematically study this problem. Recent work includes [4, 11, 23, 32, 37]; see also the surveys [9, 41] and book [8].

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A complementary problem, and one which we consider in this paper, is to ask for conditions on I that force $I^m = I^{(m)}$ for all $m \geq 1$. This problem is equivalent to asking when $r=m$ in the ideal containment problem, the smallest value that r could have. A classical result in this direction is a result of Zariski and Samuel [45, Lemma 5, Appendix 6] that states $I^m = I^{(m)}$ for all $m \geq 1$ if I is generated by a regular sequence, or equivalently, a complete intersection. Ideals that have the property $I^m = I^{(m)}$ for all $m \geq 1$ are called normally torsion free because their Rees algebra is normal. The normally torsion free squarefree monomial ideals were classified by Gitler, Valencia and Villarreal [16]. They showed that a squarefree monomial ideal is normally torsion free if and only if the corresponding hypergraph satisfies the max-flow min-cut property. Simis, Vasconcelos and Villarreal [38] and separately Sullivant [40] showed that edge ideals of graphs are normally torsion free if and only if the graph is bipartite. Furthermore, Olteanu [33] characterizes normally torsion free ideals that are lexsegment. More recent work on the equality between symbolic and ordinary powers includes Morey's paper [31] on a local version of this question, Guardo, Harbourne, and Van Tuyl's paper [19] which identifies all the ideals of general points in $\mathbb{P}^1 \times \mathbb{P}^1$ that satisfy $I^m = I^{(m)}$ for all $m \geq 1$, and Hosry, Kim, and Validashti's work [26] which identifies some families of prime ideals P such that $P^m \neq P^{(m)}$.

One family of ideals for which a complete answer is known to the question of when $I^m = I^{(m)}$ is the family of ideals defining a codimension two arithmetically Cohen-Macaulay subscheme of $\mathbb{P}^n$ that is also locally a complete intersection.

Theorem 1.1. *Let $I=I_X$ be the saturated homogeneous ideal defining a subscheme $X \subset \mathbb{P}^n$ such that*

- $\operatorname{codim}(X) = 2$;
- *X is arithmetically Cohen-Macaulay;*
- *X is locally a complete intersection.*

Then the following conditions are equivalent:

(a) $I^{(n)} = I^n$;
(b) $I^{(m)} = I^m$ *for all* $m \geq 1$;
(c) *I has at most n minimal generators.*

Furthermore, if $m < n$, then $I_X^{(m)} = I_X^m$ regardless of the number of generators.

Theorem 1.1 is a graded version of work of Ulrich [43] and Morey [31] which considered local versions of this problem. Given the current interest in the containment problem for homogeneous ideals, the purpose of Section 2 is to provide a proof of Theorem 1.1 in the graded case. In fact, we give a slightly more general result by also proving a similar statement for codimension three arithmetically Gorenstein schemes.

We are additionally interested in the graded minimal free resolutions (i.e., Betti numbers) of ideals that meet the hypotheses of Theorem 1.1. Using the graded strands of a certain Koszul complex, we show that under suitable technical hypotheses, one can construct the graded minimal free resolutions of powers of perfect homogeneous ideals of codimension two (see Theorem 2.9). As we describe in Remarks 2.10 and 2.11, these resolutions could also be constructed using [2, 24] or [42]; we present a more targeted proof of this known result.

Our new contributions are in the remaining sections. In Section 3, we discuss the relative importance of the different hypotheses in Theorem 1.1 by providing a menagerie of examples. Among other things, we show that the assumption that X be arithmetically Cohen-Macaulay is essential in codimension two. If it is dropped, then it might happen that all symbolic and ordinary powers of I are equal but I has more than n generators – see Example 3.1 and Theorem 3.3. In contrast, we provide some evidence that the condition on the codimension can be extended to codimension three in the ACM situation – see Examples 3.5 and 3.6. All of these considerations lead us to ask Question 3.7 as an indication of possible future directions for this investigation. Finally, we examine the hypothesis that X be locally a complete intersection and propose Conjecture 3.8. A collection of examples concerning ideals of fat points in $\mathbb{P}^2$ (see Examples 3.9, 3.10 and 3.11) which satisfy $I^{(m)} = I^m$ is also provided.

In Section 4 we present an application of Theorem 1.1 to points in $\mathbb{P}^1 \times \mathbb{P}^1$. Specifically, we verify the following statement, which was conjectured in [18, Conjecture 4.1]. (See Corollary 4.4 for a slightly more precise statement.)

Corollary 1.2. *Let $I = I_X$ be the saturated defining ideal of an arithmetically Cohen-Macaulay set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then $I^{(3)} = I^3$ if and only if I is a complete intersection or I is an almost complete intersection (i.e., it has exactly three minimal generators).*

In addition, we show how Theorem 1.1 significantly simplifies earlier arguments of Guardo and Van Tuyl [21] and Guardo, Harbourne, and Van Tuyl [18]. In fact, the original motivation of this project was to prove [18, Conjecture 4.1]. We were initially able to verify this conjecture using Peterson's [35] results on quasi-complete intersections, which first suggested the importance of being locally a complete intersection. Generalizing our specialized proof lead to the much stronger results of this paper.

In the final section, we use the minimal graded free resolution given in Theorem 2.9 to verify that for small powers of codimension two perfect ideals that are locally complete intersections, a question of Römer [36] has an affirmative answer. In the case of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$, we also have a new proof of a result of Guardo and Van Tuyl [21].

2. Background results

The purpose of this section is two-fold. We first weave together the various strands in the literature to present a proof of Theorem 1.1, which is a graded version of known results. We then give a description of the graded minimal free resolution of I^m under suitable hypotheses on I, again using known results in the literature.

For the convenience of the reader, we first recall the relevant definitions. Let $R = k[x_0, \ldots, x_n]$ and denote by $\mathfrak{m}$ the homogeneous maximal ideal $\mathfrak{m} = (x_0, \ldots, x_n)$. For any homogeneous ideal $I \subseteq R$, the *saturation* of I is the ideal defined by $I^{\mathrm{sat}} = \cup_{k=1}^{\infty} I : \mathfrak{m}^k$. We say that an ideal I is saturated if $I = I^{\mathrm{sat}}$. A homogeneous ideal $I \subseteq R$ is *Cohen-Macaulay* (or *perfect*) if $\operatorname{depth}(R/I) = \dim(R/I)$.

In the following, we denote by I_X the saturated homogeneous ideal defining a projective scheme X. A subscheme X of $\mathbb{P}^n$ is *arithmetically Cohen-Macaulay (ACM)* or *arithmetically Gorenstein* if R/I_X is a Cohen-Macaulay ring or a Gorenstein ring, respectively. The *codimension* of X is $\operatorname{codim}(\mathrm{X}) = n - \dim(X) = \operatorname{ht}(I_X)$. The subscheme X is an *almost complete intersection* if the number of minimal generators is one more than the codimension. A subscheme X is *locally a complete intersection* if the localization of I_X at any prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \neq \mathfrak{m}$ and $I_X \subseteq \mathfrak{p}$ is a complete intersection of codimension equal to the codimension of X. A subscheme X is a *generic complete intersection* if the localization of I_X at any minimal associated prime ideal of I_X is a complete intersection of codimension equal to the codimension of X. Finally, a scheme X is *equidimensional* if I_X is an unmixed ideal, that is, all of the associated primes of I_X have the same height.

The next result is part of the folklore; we include a short proof for completeness.

Lemma 2.1. *Let $X \subset \mathbb{P}^n$ be locally a complete intersection scheme of any codimension. Then $I_X^{(m)}$ is equal to the saturation of I_X^m.*

Proof. We denote for brevity $I = I_X$. Let m be a positive integer and $\mathfrak{p} \in \operatorname{Ass}(R/I^m) \setminus \{\mathfrak{m}\}$. Then $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}(R_\mathfrak{p}/(I^m)_\mathfrak{p}) = \operatorname{Ass}(R_\mathfrak{p}/(I_\mathfrak{p})^m) = \operatorname{Ass}(R_\mathfrak{p}/I_\mathfrak{p})$, where the first equality holds because localization commutes with taking powers of ideals and the second holds because $I_\mathfrak{p}$ is a complete intersection and powers of complete intersections are unmixed ([45, Appendix 6, Lemma 5]). Thus $\mathfrak{p} \in \operatorname{Ass}(R/I)$ and the desired conclusion follows because the argument above yields that $\operatorname{Ass}(R/I^m)$ is in $\operatorname{Ass}(R/I) \cup \{\mathfrak{m}\}$. □

Remark 2.2. Lemma 2.1 can be rephrased to say that for a locally complete intersection scheme X with ideal sheaf $\mathcal{I}_X$, we have

$$I_X^{(m)} = \bigoplus_{t \geq 0} H^0(\mathcal{I}_X^m(t)).$$

Another way to interpret Lemma 2.1 is that the only possible embedded prime of I^m is the maximal ideal $\mathfrak{m}$. Hence $I^m = I^{(m)}$ if and only if $\operatorname{depth}(R/I^m) > 0$.

In the following, $\mu(J)$ stands for the cardinality of a minimal set of generators for an ideal J. The next theorem gives necessary and sufficient conditions for the equality of ordinary and symbolic powers for two families of ideals in terms of the number of generators of an ideal. Our statement is more general than the one presented in the introduction since we can also say something about codimension three arithmetically Gorenstein schemes that are also locally a complete intersection.

Theorem 2.3. *Let $I = I_X$ be the saturated homogeneous ideal defining a subscheme $X \subset \mathbb{P}^n$ such that one of the two sets of assumptions listed below holds:*

Assumptions I:		*Assumptions II:*
• $\operatorname{codim}(X) = 2$;	*or*	$\operatorname{codim}(X) = 3$;
• *X is arithmetically Cohen-Macaulay;*		*X is arithmetically Gorenstein;*
• *X is locally a complete intersection*		*X is locally a complete intersection.*

Then the following conditions are equivalent:

(a) $I^{(n)} = I^n$;
(b) $I^{(m)} = I^m$ *for all* $m \geq 1$;
(c) *I has at most n minimal generators.*

Furthermore, if $m < n-1$ for Assumptions II and n even, or if $m < n$ in all other cases, then $I_X^{(m)} = I_X^m$ regardless of the number of generators.

Proof. Recall that for any graded homogeneous ideal J of $R = k[x_0, ..., x_n]$, if we localize at the maximal ideal $\mathfrak{m}$, then $\mu(J) = \mu(J_\mathfrak{m})$. Furthermore, $\operatorname{depth}(R/J) = \operatorname{depth}(R_\mathfrak{m}/J_\mathfrak{m})$ by [7, Proposition 1.5.15]. Therefore all of our assumptions localize and thus one can reduce to the case where I is an ideal in a regular local ring R.

The implication $(a) \Rightarrow (b)$ follows by the proofs of Morey's results [31, Theorems 3.2 and 3.3] and $(b) \Rightarrow (c)$ follows by [31, Corollary 3.4]. Next we explain how the last conclusion and the remaining implications follow in a more general context. Note that by [27, Theorem 2.3] the classes of ideals considered in this theorem are licci (i.e., in the linkage class of a complete intersection).

For the remaining implications we reason as follows. Assume that I is a licci ideal that is locally a complete intersection on the punctured spectrum of R and has $\mu(I) \leq n$. By [27, Theorem 1.14] licci ideals have Cohen-Macaulay Koszul homology for their generating sequences. Then [43, Remark 2.10 and Corollary 2.13] yield

$$\operatorname{depth}(R/I^m) \geq n+1-\mu(I), \forall m \geq 1.$$

Since $\mu(I) \leq n$, this estimate gives that $\operatorname{depth}(R/I^m) \geq 1, \forall m \geq 1$. Therefore the ideal I^m is saturated, and by Lemma 2.1 we conclude that $I^{(m)} = I^m$. This gives that $(c) \Rightarrow (b)$. Finally, $(b) \Rightarrow (a)$ is clear.

Working still under the assumption that I is a licci ideal that is locally a complete intersection on the punctured spectrum of R, then [43, Remark 2.10] gives that $\operatorname{depth}(R/I^j) \geq n+2-\operatorname{ht}(I)-j$ for all j such that $1 \leq j \leq n+2-\operatorname{ht}(I)$. Assume that $\operatorname{ht}(I)=2$. Then for $m < n = n+2-\operatorname{ht}(I)$, the depth estimate above gives $\operatorname{depth}(R/I^m) \geq n-m > 0$. Again, the ideal I^m is saturated, and by Lemma 2.1 we conclude that $I^{(m)} = I^m$. Now assume that $\operatorname{ht}(I)=3$. In this case the depth estimate above yields that $I^{(m)} = I^m$ for $m < n-1$. When X is an arithmetically Gorenstein variety of codimension 3 (Assumptions II), we may also consider the complex $\mathfrak{D}_\bullet$ defined in [30]. The strand $\mathfrak{D}_m$ of the complex will be a resolution for I^m by [30, Theorem 6.25], which gives $\operatorname{depth}(R/I^m) = n+1-\min\{\mu(I), 1+2\lfloor\frac{m+1}{2}\rfloor\} \geq n-2\lfloor\frac{m+1}{2}\rfloor$. For $m=n-1$ we obtain $\operatorname{depth}(R/I^{n-1}) > 0$ if n is odd. This proves the last statement in our theorem. □

Remark 2.4. The two classes of ideals considered in Theorem 2.3 are licci (linked to complete intersections). The equivalence of parts (b) and (c) of Theorem 2.3 remains valid in the case of licci ideals I. Indeed, the implication $(c) \Rightarrow (b)$ is shown above and $(b) \Rightarrow (c)$ follows from the estimate $\operatorname{depth}(R/I^m) = n+1-\mu(I)$ for $m \gg 0$ discussed in the preceding proof. It would be interesting to investigate whether the equivalence between these statements and condition (a) remains valid for licci ideals. We pose this problem as Question 3.7 (iii).

Remark 2.5. Cases of Theorem 2.3 were previously known.

(i) If X is a set of points in $\mathbb{P}^2$, then $I_X^{(m)} \neq I_X^m$ for any $m \geq 2$ if and only if X is not a complete intersection. This follows from [29, Theorem 2.8], which gives that R/I_X^m is not Cohen-Macaulay for any $m \geq 2$ unless X is a complete intersection.
(ii) If $C \subseteq \mathbb{P}^3$ is a curve that is locally a complete intersection and is an almost complete intersection, then Theorem 2.3 gives [35, Corollary 2.7].
(iii) A connection between the number of generators and the equality of the regular and symbolic powers in a special case can be found in [28, Corollary 2.5]. In particular, if R is a regular local ring with $\dim R = 3$, and if P is a height two prime ideal with three or more generators, it is shown that $P^m \neq P^{(m)}$ for all $m \geq 1$. See also the discussion of [17, Remark 1.27].
(iv) As we will show in Section 4, any ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ is also locally a complete intersection. Theorem 2.3 thus implies that if X is any ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, then we get $I_X^{(2)} = I_X^2$. This was first shown in [18]. See also Corollary 4.4.

Remark 2.6. The *big height* of an ideal I, denoted $\operatorname{bigheight}(I)$, is the maximum among the heights of the minimal primes of I. Huneke asked if $I^{(m)} = I^m$ for all $m \leq \operatorname{bigheight}(I)$, then is it true that $I^{(m)} = I^m$ for all $m \geq 1$? In [18] it was shown that the answer to this question is negative by showing that when X is an ACM set of reduced points in $\mathbb{P}^1 \times \mathbb{P}^1$, then $\operatorname{bigheight}(I_X) = 2$, but one needs to check if $I_X^{(3)} = I_X^3$ to guarantee that $I_X^{(m)} = I_X^m$ for all $m \geq 1$. In fact, Theorem 2.3 shows that we may have to check powers arbitrarily larger than bigheight(I) to guarantee that $I^{(m)} = I^m$ for all $m \geq 1$.

The remainder of this section is devoted to describing the graded minimal free resolution of I^m when I is a perfect ideal of codimension two under some additional hypotheses. The length of these complexes determines whether $I^m = I^{(m)}$ and hence are closely connected to Theorem 2.3.

Let us start with some preparation.

Lemma 2.7. *Consider a complex of finitely generated R-modules*

$$0 \to F_p \xrightarrow{\partial_p} F_{p-1} \to \cdots \to F_0 \xrightarrow{\partial_0} M \to 0,$$

where the modules $F_0, \ldots, F_p$ are free R-modules. If the complex has homology of finite length, ∂_0 is surjective, and $p \leq n$, then the complex is exact.

Proof. Break the complex into short exact sequences and compute local cohomology, or apply the New Intersection Theorem [34]. □

Recall that a homomorphism $\varphi : F \to G$ of free R-modules is called *minimal* if its image is contained in $\mathfrak{m}G$. This means that any coordinate matrix of φ has no unit entries.

Lemma 2.8. *Let $I \subset R$ be a homogeneous ideal admitting a free graded presentation*

$$F \xrightarrow{\varphi} G \to I \to 0.$$

Then, for every integer $m > 0$, there is a complex of graded R-modules

$$0 \to \bigwedge^m F \to \bigwedge^{m-1} F \otimes \mathrm{Sym}^1 G \to \bigwedge^{m-2} F \otimes \mathrm{Sym}^2 G \to \cdots$$

$$\to \bigwedge^2 F \otimes \mathrm{Sym}^{m-2} G \to F \otimes \mathrm{Sym}^{m-1} G \to \mathrm{Sym}^m G \to \mathrm{Sym}^m I \to 0$$

whose right-most map is surjective. Moreover, if φ is a minimal map, then all the maps in the complex above are minimal.

Proof. Let Sym$I$ denote the symmetric algebra of I. A presentation for Sym$I$ can be obtained from the given presentation of I by applying the symmetric algebra functor, which yields the exact sequence

$$F \otimes \mathrm{Sym}\, G \to \mathrm{Sym}\, G \to \mathrm{Sym}\, I \to 0.$$

Concretely, if $\operatorname{rank} G = r$ and $t_1, \ldots, t_r$ are new indeterminates, then the surjection $\mathrm{Sym}\, G \cong R[t_1, \ldots, t_r] \to \mathrm{Sym} I$ is obtained by mapping each of the variables t_i to a generator f_i of I. We view this as a bigraded map, by assigning $\deg(t_i) = (1, \deg(f_i))$ and declaring that each element $g \in R$ has bidegree $(0, \deg(g))$ in $\mathrm{Sym}\, G$. We shall refer to the first component of this bigrading as the t-degree. If $\operatorname{rank} F = s$, then the kernel of the map $\mathrm{Sym} G \to \mathrm{Sym}\, I$ is the ideal $C = (\sum_{i=1}^r \varphi_{ij} t_i \mid 1 \leq j \leq s)$, where φ_{ij} are the entries of a coordinate matrix representing φ. In this notation, the short exact sequence above gives $\mathrm{Sym} I \cong (\mathrm{Sym}\, G)/C$.

The Koszul complex $\mathbf{K}_\bullet$ on the generators of the ideal C of $\mathrm{Sym}\, G$ takes the following form:

$$0 \to \bigwedge^s F(-s) \otimes \mathrm{Sym}\, G \to \bigwedge^{s-1} F(-s+1) \otimes \mathrm{Sym}\, G \to \cdots \to \bigwedge^1 F(-1) \otimes \mathrm{Sym}\, G \to \mathrm{Sym}\, G \to 0.$$

It is a complex of free $\mathrm{Sym}\, G$-modules and the graded twists refer to the t-grading. The linear strand in t-degree m of the Koszul complex is the following complex of free R-modules, which also appears in [13, p. 597]:

$$0 \to \bigwedge^{m} F \to \bigwedge^{m-1} F \otimes_R \mathrm{Sym}^1 G \to \bigwedge^{m-2} F \otimes_R \mathrm{Sym}^2 G \to \cdots$$
$$\to \bigwedge^{2} F \otimes_R \mathrm{Sym}^{m-2} G \to F \otimes_R \mathrm{Sym}^{m-1} G \to \mathrm{Sym}^m G.$$

Moreover, the given presentation for $\mathrm{Sym}\, I$ induces the exact sequence

$$F \otimes \mathrm{Sym}^{m-1} G \to \mathrm{Sym}^m G \to \mathrm{Sym}^m I \to 0.$$

Combining the two complexes gives the desired conclusion. The differentials in the family of complexes described above involve only the elements φ_{ij}, thus minimality for any of these complexes is equivalent to the minimality of φ. □

The above family of complexes can be used to extract information on the minimal free resolution for the powers of I in several cases. We introduce the notation $V(J) = \{\mathfrak{p} \in \mathrm{Proj}(R) | J \subseteq \mathfrak{p}, \mathfrak{p} \neq \mathfrak{m}\}$ for the elements of the punctured spectrum of R containing J.

Theorem 2.9. *Consider a graded minimal free resolution of a homogeneous perfect ideal $I \subset R = k[x_0, ..., x_n]$ of codimension two*

$$0 \to F \xrightarrow{\varphi} G \to I \to 0.$$

Let m be a positive integer and assume further that I is locally a complete intersection and $\min\{\mathrm{rank}\, G - 1, m\} \leq n$. Then $\mathrm{Sym}^m I \cong I^m$ and the complex in Lemma 2.8 is a graded minimal free resolution of I^m.

Proof. In view of our Lemma 2.8, it is sufficient to verify that the complex therein is acyclic and that the canonical surjection $\mathrm{Sym}^m I \to I^m$ is in fact an isomorphism $\mathrm{Sym}^m I \cong I^m$. We continue with the notation introduced in the proof of Lemma 2.8.

Localizing the short exact sequence $0 \to F \to G \to I \to 0$ at $\mathfrak{p} \in V(I)$ yields the direct sum of a minimal free resolution for the height two complete intersection $I_\mathfrak{p} = (f, g)$ and an isomorphism $0 \to R_\mathfrak{p}^{s-1} \to R_\mathfrak{p}^{s-1} \to 0$. This gives $C_{\mathfrak{p}R[t_1, ..., t_s]} = (ft_1 - gt_2, t_3, ..., t_s)$, thus C is a complete intersection in $R[t_1, ... t_s]$. Similarly, when localizing at primes $\mathfrak{p}$ not containing I, $\mathbf{K}_\bullet$ becomes the Koszul complex on the variables $t_1, ..., t_s$ and thus $\mathbf{K}_\bullet$ and all of its graded strands are exact complexes when localized at $\mathfrak{p} \neq \mathfrak{m}$. Therefore the homology of the graded strands of $\mathbf{K}_\bullet$ has finite length. Applying Lemma 2.7, we conclude that the complex in Lemma 2.8 is exact when $m \leq n$.

Furthermore, by [42, Theorem 5.1], the isomorphism $\mathrm{Sym}^m I \cong I^m$ holds true if $\mu(I_\mathfrak{p}) \leq \mathrm{depth}\, R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ containing I such that $\mathrm{depth}\, R_\mathfrak{p} \leq \min\{\mu(I), m\}$. For $\mathfrak{p} \in V(I)$, using the fact that I is locally a complete intersection, we have $\mu(I_\mathfrak{p}) = \mathrm{ht}(I_\mathfrak{p}) = \mathrm{depth}\, I_\mathfrak{p} \leq \mathrm{depth}\, R_\mathfrak{p}$. Next we analyze the possibility that $\mathfrak{p} = \mathfrak{m}$ is among the primes that satisfy $\mathrm{depth}\, R_\mathfrak{p} \leq \min\{\mu(I), m\}$. This occurs when $n + 1 = \mathrm{depth}\, R_\mathfrak{m} \leq \min\{\mu(I), m\} = \min\{\mathrm{rank}\, G, m\}$. Since by hypothesis we have $\min\{\mathrm{rank}\, G - 1, m\} \leq n$, it must be the case that $\mu(I) = \mathrm{rank}\, G = n + 1 \leq m$. But in this case, $\mu(I_\mathfrak{m}) = \mathrm{rank}\, G = n + 1 = \mathrm{depth}\, R_\mathfrak{m}$, thus the desired conclusion that $\mathrm{Sym}^m I \cong I^m$ follows. □

Remark 2.10. The proof of [42, Theorem 5.1] shows that the locally complete intersection hypothesis in Theorem 2.9 can be weakened to $\mu(I_\mathfrak{p}) \leq \mathrm{depth}\, R_\mathfrak{p}$ for all primes containing I and such that $\mathrm{depth}\, R_\mathfrak{p} \leq \min\{\mu(I), m\}$.

Remark 2.11. Previously known approaches to obtaining Theorem 2.9:

(i) Lemma 2.8 and Theorem 2.9 could also be obtained by applying the results of [24]. In particular, the complex of Lemma 2.8 is the *approximation complex* of [24] constructed from the minimal free resolution of a perfect ideal of codimension two. We prefer to give a more explicit description of this complex for future use in Section 5.

(ii) In [2, Theorem 5.4], the resolutions of powers of an ideal that satisfies $\operatorname{ht} I_j(\varphi) \geq \mu(I) + 1 - j$ are given. This provides an alternate proof of our Theorem 2.9.

It is of interest to record here the minimal free resolution that we obtain in the special cases of $m = 2$ and $m = 3$ for ACM curves in $\mathbb{P}^3$. This will be useful in Section 4.

Corollary 2.12. *Let X be an arithmetically Cohen-Macaulay curve in $\mathbb{P}^3$ that is locally a complete intersection, and let $0 \to F \xrightarrow{\varphi} G \to I_X \to 0$ be a graded minimal free resolution.*

(a) The graded minimal free resolution of I_X^2 is

$$0 \to \bigwedge^2 F \to F \otimes G \to \operatorname{Sym}^2 G \to I_X^2 \to 0.$$

(b)The graded minimal free resolution of I_X^3 is

$$0 \to \bigwedge^3 F \to \bigwedge^2 F \otimes G \to F \otimes \operatorname{Sym}^2 G \to \operatorname{Sym}^3 G \to I_X^3 \to 0.$$

Remark 2.13. (i) The above result applies in particular to any smooth space curve that is arithmetically Cohen-Macaulay.

(ii) Note that any arithmetically Cohen-Macaulay union of lines in $\mathbb{P}^3$ such that no three lines meet in a point is locally a complete intersection. Again, the above result applies to configurations of this type.

3. Remarks on the hypotheses

In this section we comment on the importance of the various hypotheses in our results in Section 2, especially Theorem 2.3, and we give examples to indicate various ways that they might be weakened.

3.1. The ACM hypothesis

The assumption that X is arithmetically Cohen-Macaulay is essential in Theorem 2.3. If it is dropped, then it might happen that all symbolic and ordinary powers of I are equal but I has more than n generators. This is illustrated by Example 3.1.

Example 3.1. Let X be the union of 5 lines in $\mathbb{P}^3$ defined by 5 general points in $\mathbb{P}^1 \times \mathbb{P}^1$ (see Section 4 for more on points in $\mathbb{P}^1 \times \mathbb{P}^1$). This configuration has codimension two and is locally a complete intersection, but is not ACM. However, from [19, Theorem 3.1.4], $I = I_X$ has 6 minimal generators and $I^{(m)} = I^m$ for all $m \geq 1$. A similar phenomenon holds for $s = 2$, or 3 general points in $\mathbb{P}^1 \times \mathbb{P}^1$. The case of $s = 2$ general points (i.e., two skew lines in $\mathbb{P}^3$ or even in $\mathbb{P}^n$) is also discussed in [20, Remark 4.2]. Note that all the cases of this example are non-ACM cases.

It is interesting to notice that for 5 general lines in $\mathbb{P}^3$ the picture is quite different, but still shows that the ACM hypothesis is necessary for the final statement of Theorem 2.3.

Example 3.2. Let X be the union of 5 general lines $L_1, \ldots, L_5$ in $\mathbb{P}^3$. Again, X is not ACM, but $\operatorname{codim}(X) = 2$ and X is locally a complete intersection. Then for $I = I_X$ we have

$$I^{(2)} \neq I^2.$$

Indeed, for any three mutually distinct indices $\{i,j,k\} \subset \{1,2,3,4,5\}$ with $i<j<k$ let Q_{ijk} be the quadric containing the lines L_i, L_j and L_k. Altogether there are 10 such quadrics. The ideal I is then generated by the following products of the quadrics

$$F_1 = Q_{123}Q_{145}, F_2 = Q_{145}Q_{235}, F_3 = Q_{235}Q_{124}, F_4 = Q_{124}Q_{345}, F_5 = Q_{345}Q_{123},$$
$$F_6 = Q_{125}Q_{234}, F_7 = Q_{234}Q_{135}, F_8 = Q_{135}Q_{124}, F_9 = Q_{134}Q_{245}, F_{10} = Q_{345}Q_{125}.$$

These generators not only vanish along all lines but they vanish along one of the lines to order 2. Since there is a cubic vanishing along 4 general lines in $\mathbb{P}^3$, this gives rise to 10 septics vanishing to order 2 along all five lines. Hence there are elements of degree 7 in $I^{(2)}$ but the initial degree of I^2 is 8. This example shows that the final statement of Theorem2.3 also requires the ACM hypothesis.

Nevertheless, the condition that X be ACM in Theorem 2.3 seems not to be essential if we allow the codimension to go up. Indeed, it is of interest to seek results about symbolic powers for non-ACM subschemes, possibly of higher codimension. We first give a simple such result.

Theorem 3.3. *Let $C = C_1 \cup C_2$ be a disjoint union of two complete intersections of dimension r in $\mathbb{P}^{2r+1}$. Then $I_C^m = I_C^{(m)}$ for all positive integers m.*

Proof. For $i = 1, 2$, we have that $I_{C_i}^m = I_{C_i}^{(m)}$ for all m since I_{C_i} is generated by a regular sequence (see [45, Lemma 5, Appendix 6]). Thus

$$I_C^m = (I_{C_1}I_{C_2})^m = I_{C_1}^m \cdot I_{C_2}^m = I_{C_1}^m \cap I_{C_2}^m = I_{C_1}^{(m)} \cap I_{C_2}^{(m)} = I_C^{(m)}.$$

The first and third equalities are true because for disjoint ACM subschemes of dimension r in $\mathbb{P}^{2r+1}$, the intersection of the ideals is equal to the product thanks to a special case of Théorème 4 and the subsequent Corollaire in [39, pp. 142–143]. □

The following example shows that even the assumption that C be a disjoint union, in Theorem 3.3, is not always needed.

Example 3.4. Consider the union, C, of two planes in $\mathbb{P}^4$ meeting at a point P. At P, C not only fails to be locally a complete intersection, but in fact it fails to be locally Cohen-Macaulay, since C is a cone over two skew lines in $\mathbb{P}^3$ with vertex at P. Yet a similar argument as given in the proof of Theorem 3.3 shows that $I_C^m = I_C^{(m)}$ for all $m \geq 1$. In particular, it is worth noting that the powers of I_C do not pick up an embedded point at P.

3.2. The hypothesis on low codimension

As noted in Remark 2.4 It is conceivable that the equivalence of the three statements (a), (b) and (c) in Theorem 2.3 may hold in higher codimension as long as the ideal is locally a complete intersection and linked to a complete intersection. We offer some evidence toward this conjecture below. In the following example we give "essentially" the same licci ideal, viewed in $\mathbb{P}^3$ and in $\mathbb{P}^4$. In both cases the ideal is locally a complete intersection (in fact smooth) and is ACM. It has four minimal generators, so the condition that I has at most n minimal generators fails in the case of $\mathbb{P}^3$ but is satisfied in the case of $\mathbb{P}^4$. This example was obtained using CoCoA [1].

Example 3.5. Consider a sufficiently general complete intersection of type (1,1,2) in $\mathbb{P}^3$ and (separately) also in $\mathbb{P}^4$. In each case, link this ideal using a sufficiently general complete intersection of type (2, 2, 2). Both in $\mathbb{P}^3$ and in $\mathbb{P}^4$, the residual will be ACM of codimension 3, and an easy mapping cone argument shows that the residual has 4 minimal generators and Hilbert function

$(1,4,6,6,\ldots)$. In $\mathbb{P}^3$ the residual is a set of 6 distinct points, and in $\mathbb{P}^4$ the residual is a smooth curve of degree 6 and genus 2, so both are locally complete intersections. Indexing by the ambient space, we denote these ideals by I_3 and I_4 respectively. One can check by hand or with a computer algebra program that the Betti diagram of both I_3 and I_4 is

	0	1	2	3
0 :	1	–	–	–
1 :	–	4	2	–
2 :	–	–	3	2
Tot :	1	4	5	2.

Then one can check with a computer algebra program that the Betti diagram of both I_3^2 and I_4^2 is

	0	1	2	3	4
0 :	1	–	–	–	–
1 :	–	–	–	–	–
2 :	–	–	–	–	–
3 :	–	10	8	1	–
4 :	–	–	9	8	1
Tot :	1	10	17	9	1.

By the Auslander-Buchsbaum formula, this means that I_3^2 is not saturated, hence $I_3^2 \neq I_3^{(2)}$. On the other hand, it means that I_4^2 is saturated; then since I_4 is locally a complete intersection, we get $I_4^2 = I_4^{(2)}$ (although I_4^2 is not ACM). We have verified on CoCoA that in fact $I_4^k = I_4^{(k)}$ for $1 \leq k \leq 7$.

An additional piece of evidence that the hypothesis on the codimension might be weakened is provided by ideals of scrolls $\mathbb{P}^1 \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{2n+1}$. Note that the ideals defining these scrolls are not even licci, as can be shown by applying the criterion in [44, Theorem 2.8].

Example 3.6 (Scrolls). Let X be the scroll $\mathbb{P}^1 \times \mathbb{P}^n \to \mathbb{P}^{2n+1}$ embedded by the Segre embedding. The ideal J of X is determined by the 2×2 minors of a generic $2 \times (n+1)$ matrix. By [10, Corollary 7.3] we have $J^{(m)} = J^m$ for all $m \geq 1$. Indeed, in the notation of the just cited corollary, we have $J = I_2$, $n = 2$ and $\ell = 2$.

Notice that the ideal J has $\binom{n+1}{2}$ minimal generators and the corresponding scroll lies in $\mathbb{P}^{2n+1}$. The variety we get has codimension n. When $n=2$ or $n=3$, it is true that the number of minimal generators is smaller than the dimension of the projective space. The latter, in particular, gives further evidence for Question 3.7 below. But as soon as $n \geq 4$, we get an example where the number of minimal generators is larger than the dimension of the projective space (violating part (c) of Theorem 2.3) but nevertheless the statements of (a) and (b) are true.

With the above examples and comments in mind, we pose the following questions:

Question 3.7. *Let $X \subseteq \mathbb{P}^n$ be an ACM subscheme that is locally a complete intersection.*

(i) If $\operatorname{codim}(X) = 3$, are conditions (a), (b) and (c) of Theorem 2.3 still equivalent?

(ii) If $\operatorname{codim}(X) > 3$, are conditions (a) and (b) of Theorem 2.3 still equivalent?

(iii) If I_X is linked to a complete intersection, are conditions (a), (b) and (c) of Theorem 2.3 still equivalent?

3.3. The hypothesis of being locally a complete intersection

We now address the assumption that X is locally a complete intersection. For many of the results in this paper, Lemma 2.1 has been important. It says that the property that a reduced subscheme $X \subset \mathbb{P}^n$ is locally a complete intersection implies that its powers do not pick up non-irrelevant embedded components, so that the failure of a power to be a symbolic power comes only from the failure to be saturated. It is natural to ask if the converse holds; that is, one might ask if the following statement is true: If X is not locally a complete intersection, then I_X^m defines a scheme with embedded components.

We have seen in Example 3.4 that this is not the case. On the other hand, it is not hard to verify that if X consists of three lines in $\mathbb{P}^3$ meeting at a point, then X is ACM but I_X^2 *does* have an embedded point; so even though I_X^2 is saturated, it is not equal to $I_X^{(2)}$. We make the following conjecture, which is also based on other computer experiments. Example 3.4 shows that it is not true without the assumption that $X \subset \mathbb{P}^3$.

Conjecture 3.8. *Let $X \subset \mathbb{P}^3$ be a subvariety (reduced and unmixed) of codimension 2. Assume that there is a point $P \in X$ such that the localization of I_X at P is not a complete intersection. Then for any integer $m \geq 2$, the saturation of I_X^m has an embedded component at P. In particular,*

$$I_X^{(m)} \neq I_X^m \quad \text{for all } m \geq 2.$$

The above conjecture has been proved in [3, Theorem 4.4] for irreducible varieties X. We now give some examples related to this conjecture.

Example 3.9. Recall that a fat points scheme in $\mathbb{P}^n$ is defined by an ideal of the form $J = I(P_1)^{m_1} \cap \cdots \cap I(P_s)^{m_s}$, where $\{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$ is a finite set of distinct points and $m_1, \ldots, m_s$ are non-negative integers. We denote the fat points scheme by $m_1P_1 + \cdots + m_sP_s$. By Remark 2.5(i), if J is the ideal of a reduced set of points in $\mathbb{P}^2$ (so J is radical), then $J^{(m)} = J^m$ for all $m \geq 1$ (if and) only if J is a complete intersection. However, the situation is more subtle for fat points. There are examples of nonreduced ideals $J = I(P_1)^{m_1} \cap \cdots \cap I(P_s)^{m_s}$ of fat points all of whose powers are symbolic. Since a nonradical fat points ideal is never locally a complete intersection, this shows that Conjecture 3.8 is false without the assumption that X is a variety, where variety here means any reduced subscheme of $\mathbb{P}^n$ (not necessarily irreducible).

It is an interesting but open problem to classify those ideals I of fat points in $\mathbb{P}^2$ whose powers are all symbolic. Of course, if $J = I^{(t)}$ where I is a radical complete intersection ideal of points in $\mathbb{P}^n$, then $J^{(m)} = I^{(tm)} = I^{tm} = (I^t)^m = J^m$, so J is a fat points ideal and all powers of J are symbolic. A sufficient condition for $J^{(m)} = J^m$ for all $m \geq 1$ to hold for an ideal J of fat points in $\mathbb{P}^2$ is given in [23, Proposition 3.5]: if J is of the form $J = I^{(t)}$ where I is a radical ideal of n points in $\mathbb{P}^2$, and if $\alpha(J)\beta(J) - t^2n = 0$, where $\alpha(J)$ is the least degree of a non-zero form in J and $\beta(J)$ is the least degree d such that the base locus of the linear system J_d is 0-dimensional, then all powers of J are symbolic. So for example, if I is either the ideal of five general points in $\mathbb{P}^2$ or the ideal of a star configuration in $\mathbb{P}^2$, then $J^{(m)} = J^m$ holds for all $m \geq 1$ for $J = I^{(2)}$ (see [23, Corollary 3.9 and Lemma 3.11]; see also [15]). In neither case is I a complete intersection; in particular $I^2 \neq I^{(2)}$ for these examples.

Example 3.10. We now recall examples of radical ideals $I(m)$ of points in $\mathbb{P}^2$ given in [32]. The results of [32] show that the powers of $I(m)^{(m)}$ are all symbolic but $I(m)^{(m)}$ is not a power of a complete intersection and (as long as $m > 3$) the criterion given in [23, Proposition 3.5] does not apply. (For example, for $m = 4$, we have $\alpha(I(4)) = 16$ and $\beta(I(4)) = 20$, but $m^2n = 16(19)$.) Fix some integer $m \geq 3$ and consider the ideal

$$I(m) = (x(y^m - z^m), y(z^m - x^m), z(x^m - y^m)) \subset R = k[x, y, z].$$

This ideal is the homogeneous ideal of a set of $m^2 + 3$ points, which has been dubbed a Fermat point configuration. By Theorem 2.3 (c), we know $I(m)^{(2)} \neq I(m)^2$. However, for each positive integer j one has by [32, Proposition 4.1]

$$I(m)^{(mj)} = (I(m)^{(m)})^j.$$

Example 3.11. The examples given above of ideals $J = I(P_1)^{m_1} \cap \cdots \cap I(P_s)^{m_s}$ of fat points whose powers are all symbolic all have the property that $\gcd(m_1, ..., m_s) \neq 1$, but this is not essential. Examples with $\gcd(m_1, ..., m_s) = 1$ are given in [6, Example 5.1]. In particular, let J be the ideal of $Z = (d-1)P_1 + P_2 + \cdots + P_{2d}$ where $P_1, ..., P_{2d}$ are general points in $\mathbb{P}^2$ and $d > 2$. It is shown in [6] that $J^m = J^{(m)}$ and noted there that there is no $m \geq 1$ such that $J^{(m)}$ is a power of any ideal which is prime, radical, or a complete intersection. As noted in [6], further examples can be obtained from this using the action of the Cremona group. These examples, Example 3.9 above, and the case $m = 3$ of Example 3.10 have the property that $\alpha(J)\beta(J) - \sum_i m_i^2 = 0$, where $\alpha(J)$ and $\beta(J)$ are as defined in Example 3.9. Thus, perhaps [23, Proposition 3.5] can be generalized to ideals of fat points rather than just for certain powers of certain radical ideals. However Example 3.10 shows this would still not cover all cases of ideals of fat points whose powers are all symbolic.

4. Application 1: points in $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we apply the main results of Section 2, namely Theorem 2.3, to ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, we show how our new results give new short proofs to results in [18, 21].

We begin with a quick review of the relevant definitions and notation. For a more thorough introduction to this topic, see [22]. The polynomial ring $k[x_0, x_1, x_2, x_3]$ with the bigrading given by $\deg x_0 = \deg x_1 = (1, 0)$ and $\deg x_2 = \deg x_3 = (0, 1)$ is the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$. A point $P = [a_0 : a_1] \times [b_0 : b_1]$ in $\mathbb{P}^1 \times \mathbb{P}^1$ has a bihomogeneous ideal $I_P = (a_1x_0 - a_0x_1, b_1x_2 - b_0x_3)$. A set of points $X = \{P_1, ..., P_s\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is associated to the bihomogeneous ideal $I_X = \cap_{P \in X} I_P$. If we only consider the standard grading of this ideal, then I_X defines a union X of lines in $\mathbb{P}^3$. In order to apply the results of the previous sections, we first require the following lemma.

Lemma 4.1. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be any set of points. Then I_X is locally a complete intersection.*

Proof. We will consider a point $P \in \mathbb{P}^1 \times \mathbb{P}^1$ as the line in $\mathbb{P}^3$ that is defined by the ideal $I_P = (a_1x_0 - a_0x_1, b_1x_2 - b_0x_3)$. We now show that the union of lines in $\mathbb{P}^3$ coming from a union of points in $\mathbb{P}^1 \times \mathbb{P}^1$ in this way is locally a complete intersection. There is no problem at a smooth point, so we must determine how two or more such lines can meet at a single point. Let C be a union of such lines.

The planes of the form $a_1x_0 - a_0x_1$ form the pencil of planes, Λ_1, through the line λ_1 defined by $x_0 = x_1 = 0$. Similarly, the planes of the form $b_1x_2 - b_0x_3$ form the pencil of planes, Λ_2, through the line λ_2 defined by $x_2 = x_3 = 0$. Notice that λ_1 and λ_2 are disjoint. A point P in $\mathbb{P}^1 \times \mathbb{P}^1$, then, corresponds to the line of intersection of a plane in Λ_1 and a plane in Λ_2. Given any point A in $\mathbb{P}^3$ not on either λ_1 or λ_2, there is a unique element of Λ_1 and a unique element of Λ_2 passing through A. Hence two lines of C cannot meet at a point not on one of the two lines, λ_1 or λ_2. Now assume that $A \in \lambda_1$ (the case $A \in \lambda_2$ is identical). In order for two or more lines of C to meet at A, it must be that the plane $H_A \in \Lambda_2$ containing A is fixed, while the plane in Λ_1 is not. Hence lines meeting at A all lie on H_A, and so are coplanar. But any plane curve is a complete intersection, so the same holds for any localization. Therefore, any set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ defines a union of lines in $\mathbb{P}^3$ that is locally a complete intersection. □

First we give a short proof of [18, Theorem 1.1].

Theorem 4.2. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be an ACM set of points. Then $I_X^m = I_X^{(m)}$ for all $m \geq 1$ if and only if $I_X^3 = I_X^{(3)}$.*

Proof. We will view X as a union of lines in $\mathbb{P}^3$. By Lemma 4.1, X is locally a complete intersection. Now apply Theorem 2.3. □

In [18], it was also asked what sets of points $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ satisfy $I_X^3 = I_X^{(3)}$ and, in particular, if there is a geometric classification of such points. The authors proposed such a classification. We require the following notation. Let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ denote the natural projection onto the first factor. If $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a finite set of reduced points, $\pi_1(X) = \{A_1, ..., A_h\}$ is the set of distinct first coordinates that appear in X. For $i = 1, ..., h,$ set $\alpha_i = |X \cap \pi_1^{-1}(A_i)|,$ i.e., the number of points in X whose first coordinate is A_i. After relabeling the A_i's so that $\alpha_i \geq \alpha_{i+1}$ for $i = 1, ..., h-1,$ we set $\alpha_X = (\alpha_1, ..., \alpha_h)$.

Remark 4.3. One of the themes of the monograph [22] is to demonstrate that when X is an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, many of the homological invariants of I_X, e.g., bigraded Betti numbers, Hilbert function, can be computed directly from the tuple α_X. As shown in the next corollary, α_X can also be used to determine when $I_X^{(m)} = I_X^m$.

We now prove [18, Conjecture 4.1]. In fact, we give a slightly stronger version.

Corollary 4.4. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be any ACM set of points. Then*

(a) $I_X^2 = I_X^{(2)}$.
(b) *The following are equivalent:*
  (i) I_X^2 *defines an ACM scheme;*
  (ii) $I_X^3 = I_X^{(3)}$ *is the saturated ideal of an ACM scheme;*
  (iii) X *is a complete intersection;*
  (iv) $\alpha_X = (a, a, ..., a)$ *for some integer* $a \geq 1$.
(c) *The following are equivalent:*
  (i) $I_X^3 = I_X^{(3)}$ *is the saturated ideal of a non-ACM scheme;*
  (ii) I_X *is an almost complete intersection;*
  (iii) $\alpha_X = (a, ..., a, b, ..., b)$ *for integers* $a > b \geq 1$.

Proof. Since X (viewed as a union of lines in $\mathbb{P}^3$) is locally a complete intersection, for any m the condition that $I_X^m = I_X^{(m)}$ is equivalent to the condition that I_X^m is saturated., Equivalently, proj-dim$(I_X^m) \leq 2$. In the former case the scheme it defines is ACM; in the latter case it is not. Part (a) was first proved in [18, Theorem 2.6], but it also follows from Corollary 2.12 (a). From Corollary 2.12 (a) we see that the scheme defined by I_X^2 is ACM if and only if $\bigwedge^2 F = 0,$ i.e., rank$(F) = 1,$ meaning that X is a complete intersection.

That I_X^3 is the saturated ideal of an ACM scheme is equivalent by Corollary 2.12 (b) to $\bigwedge^2 F = 0$, which means again that X is a complete intersection. The equivalence of (iii) and (iv) in (b) is [22, Theorem 5.13]. This proves (b).

To prove (c) we need the resolution to be one step longer, which is equivalent to $\bigwedge^2 F \neq 0$ and $\bigwedge^3 F = 0,$ i.e., F has rank 2, meaning that I_X has three minimal generators. By [22, Corollary 5.6], I_X has three minimal generators if and only if there exist integers a, b with $a > b$ such that $\alpha_X = (a, ..., a, b, ..., b)$. □

Guardo and Van Tuyl give an algorithm [21, Algorithm 5.1] to compute the bigraded Betti numbers for any set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ provided that the support is ACM. Although we will not reproduce the algorithm here, it was shown that the bigraded Betti numbers of a set of double points only depend upon the tuple $\alpha_X = (\alpha_1, \ldots, \alpha_h)$ describing the support X. We can now deduce this result from our new work.

Corollary 4.5. *Let $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of fat points where every point has multiplicity two. If X is the support of Z and if X is ACM, then there exists an algorithm to compute the bigraded Betti numbers of I_Z using only α_X.*

Proof. Because X is ACM, the bigraded minimal free resolution of I_X can be computed directly from α_X (see [22, Theorem 5.3]). By Corollary 4.4, $I_Z = I_X^{(2)} = I_X^2$. We can then use Corollary 2.12 to compute the bigraded resolution of I_X^2 using the bigraded resolution of I_X. In particular, the bigraded Betti numbers of I_Z only depend upon knowing α_X. □

Note that one can use Corollary 2.12 to write out all the bigraded Betti numbers. Although we will not do this here, we will show how to compute the bigraded Betti numbers of triple points whose support is an ACM set of points that is an almost complete intersection. Except for the result in the remark below, we are not aware of any similar results of this type.

Corollary 4.6. *Let $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a homogeneous set of triple points (i.e., where every point has multiplicity three) and let X denote the support of Z. If I_X is an almost complete intersection with $\alpha_X = (\underbrace{a, \ldots, a}_{c}, \underbrace{b, \ldots, b}_{d})$, then I_Z has a bigraded minimal free resolution of the form*

$$0 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I_Z \longrightarrow 0$$

where

$$\begin{aligned}
F_0 = {} & R(-3c-3d, 0) \oplus R(-3c-2d, -b) \oplus R(-2c-2d, -a) \oplus R(-3c-d, -2b) \oplus R(-2c-d, -b-a) \oplus \\
& R(-c-d, -2a) \oplus R(-3c, -3b) \oplus R(-2c, -2b-a) \oplus R(-c, -b-2a) \oplus R(0, -3a) \\
F_1 = {} & R(-c, -3a) \oplus R(-2c, -2a-b) \oplus R(-3c, -a-2b) \oplus R(-c-d, -2a-b) \oplus R(-2c-d, -a-2b) \oplus \\
& R(-3c-d, -3b) \oplus R(-2c-d, -2a) \oplus R(-3c-d, -a-b) \oplus R(-2c-2d, -a-b) \oplus \\
& R(-3c-2d, -2b) \oplus R(-3c-2d, -a) \oplus R(-3c-3d, -b) \\
F_2 = {} & R(-3c-2d, -b-a) \oplus R(-3c-d, -a-2b) \oplus R(-2c-d, -2a-b).
\end{aligned}$$

Proof. Because I_X is an almost complete intersection, Corollary 4.4 implies that $I_Z = I_X^{(3)} = I_X^3$. So the bigraded resolution of I_Z can be computed using Corollary 2.12 if we know the bigraded resolution of I_X. But by [22, Theorem 5.3] the bigraded resolution of an almost complete intersection I_X with $\alpha_X = (\underbrace{a, \ldots, a}_{c}, \underbrace{b, \ldots, b}_{d})$ is

$$0 \rightarrow R(-c-d, -b) \oplus R(-c, -a) \rightarrow R(-c-d, 0) \oplus R(-c, -b) \oplus R(0, -a) \rightarrow I_X \rightarrow 0.$$

□

Remark 4.7. In the previous statement, the set of fat points all have multiplicity three. Favacchio and Guardo [14] have generalized this result. In particular, they have shown that if Z is set of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose support is an almost complete intersection, one can weaken the hypothesis that the fat points are all homogeneous of degree three to construct a set of nonhomogeneous fat points (in a controlled fashion), and still prove that $I_Z^{(m)} = I_Z^m$ for all $m \geq 1$.

5. Application 2: a question of Römer

In this last section we show how one can use Theorem 2.9 to give further evidence for a question of Römer [36]. We begin by defining and introducing the relevant notation.

Let I be a homogeneous ideal of $R = k[x_0, \ldots, x_n]$. The graded minimal free resolution of R/I has the form

$$0 \to F_p \to F_{p-1} \to \cdots \to F_1 \to R \to R/I \to 0$$

where $F_i = \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{i,j}(R/I)}$. The number $p = \text{proj} - dim(R/I)$ is the *projective dimension*, and the numbers $\beta_{i,j}(R/I)$ are the $(i,\, j)$-th *graded Betti numbers* of R/I. The *i-th Betti number of R/I* is $\beta_i(R/I) = \sum_{j\in\mathbb{Z}} \beta_{i,j}(R/I)$.

Römer [36] initiated an investigation into the relationship between the i-th Betti numbers of I and the shifts that appear in the graded minimal free resolution. In particular, Römer asked the following question.

Question 5.1. *Let I be a homogeneous ideal of $R = k[x_0, \ldots, x_n]$. Does the following bound hold for all $i = 1, \ldots, p$:*

$$\beta_i(R/I) \leq \frac{1}{(i-1)!(p-i)!} \prod_{j\neq i} M_j$$

where $M_i := \max\{j \mid \beta_{i,j}(R/I) \neq 0\}$?

Römer showed that Question 5.1 is true for all codimension two Cohen-Macaulay ideals, while Guardo and Van Tuyl [21] verified the question for any set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose support is ACM. We now use Theorem 2.9 to extend the family of known positive answers to Question 5.1. In the statement below, recall that $\alpha(I) = \min\{i \mid (I)_i \neq 0\}$.

Theorem 5.2. *Consider a homogeneous perfect ideal $I \subset R$ of codimension two that is also locally a complete intersection. Fix an integer $m \in \{1, \ldots, n\}$. Then*

$$\beta_i(R/I^m) \leq \frac{1}{(i-1)!(m+1-i)!} \prod_{j\neq i} M_j \quad \text{for all } 1 \leq i \leq m+1$$

where $M_i := \max\{j \mid \beta_{i,j}(R/I) \neq 0\}$.

Proof. If $m = 1$, then this result follows from [36, Corollary 5.2]. Let $d = \mu(I)$. By the Hilbert-Burch Theorem, the ideal I has a minimal resolution of the form:

$$0 \to R^{d-1} \to R^d \to I \to 0.$$

Furthermore, the minimal generators of I are given by the $(d-1)\times(d-1)$ minors of the Hilbert-Burch matrix. Since every entry of this matrix is either 0 or has degree ≥ 1, we have $\alpha(I) \geq d-1$. In particular, for all $m \geq 2$, we have $d \leq m\alpha(I)$.

Because of our hypotheses on I and m, Theorem 2.9 implies that the complex of Lemma 2.8 is a graded minimal free resolution of I^m, and consequently, proj-dim$(R/I^m) = m+1$ and for all $1 \leq i \leq m+1$,

$$\beta_i(R/I^m) = \dim_k\left(\bigwedge^{i-1} R^{d-1} \otimes \mathrm{Sym}^{m-i+1}(R^d)\right) = \binom{d-1}{i-1}\binom{d+m-i}{d-1}.$$

We have $\alpha(I^m) = m\alpha(I)$. As a result, for each $i = 1, \dots, m+1$, $\beta_{i,j}(R/I^m) = 0$ for all $j < m\alpha(I) + (i-1)$. So, because $d \le m\alpha(I)$, we have

$$d + (i-1) \le m\alpha(I) + (i-1) \le M_i \quad \text{for all } i = 1, \dots, m+1.$$

Combining these pieces together now gives

$$\begin{aligned}
\beta_i(R/I^m) &= \frac{(d-1)(d-2)\cdots(d-i+1)(d-i)!}{(i-1)!(d-i)!}\frac{(d+m-i)\cdots(d)(d-1)!}{(m+1-i)!(d-1)!} \\
&= \frac{((d-i+1)(d-i+2)\cdots(d-1))((d)(d+1)\cdots(d+m-i))}{(i-1)!(m+1-i)!} \\
&\le \frac{((d)(d+1)\cdots(d-i-2))((d+i)(d+i+1)\cdots(d+m))}{(i-1)!(m+1-i)!} \\
&\le \frac{M_1M_2\cdots M_{i-1}M_{i+1}\cdots M_{m+1}}{(i-1)!(m+1-i)!}.
\end{aligned}$$

This now verifies the inequality. □

We can reprove and extend [21, Theorem 6.1] which first proved the case $m = 2$.

Corollary 5.3. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be an ACM set of points. If $I = I_X$, then Question 5.1 has an affirmative answer for I^m with $m = 1$, 2 and 3.*

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