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Effective identifiability criteria for tensors and polynomials

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ABSTRACT

A tensor T, in a given tensor space, is said to be h-identifiable if it admits a unique decomposition as a sum of h rank one tensors. A criterion for h-identifiability is called effective if it is satisfied in a dense, open subset of the set of rank h tensors. In this paper we give effective h-identifiability criteria for a large class of tensors. We then improve these criteria for some symmetric tensors. For instance, this allows us to give a complete set of effective identifiability criteria for ternary quintic polynomials. Finally, we implement our identifiability algorithms in Macaulay2.

1. Introduction

A tensor rank decomposition of a tensor T, lying in a given tensor space over a field k, is an expression of the type

$$T = \lambda_1 U_1 + \ldots + \lambda_h U_h \tag{1.1}$$

where the U_i's are linearly independent rank one tensors, $\lambda_i \in k^*$, and k is either the real or complex field. The *rank* of T, denoted by $\mathrm{rank}(T)$, is the minimal positive integer h such that T admits a decomposition as in (1.1).

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Tensor decomposition problems and techniques are of relevance in both pure and applied mathematics. For instance, tensor decomposition algorithms have applications in psycho-metrics, chemometrics, signal processing, numerical linear algebra, computer vision, numerical analysis, neuroscience and graph analysis (Kolda and Bader, 2009; Comon and Mourrain, 1996; Comon et al., 2008; Landsberg and Ottaviani, 2015; Massarenti and Raviolo, 2013, 2014). In pure mathematics tensor decomposition issues naturally arise in constructing and studying moduli spaces of all possible additive decompositions of a general tensor into a given number of rank one tensors (Dolgachev, 2004; Dolgachev and Kanev, 1993; Massarenti and Mella, 2013; Massarenti, 2016; Ranestad and Schreyer, 2000; Takagi and Zucconi, 2011).

We say that a tensor rank one decomposition has the *generic identifiability property* if the expression (1.1) is unique, up to permutations and scaling of the factors, on a dense open subset of the set of tensors admitting an expression as in (1.1). This uniqueness property is useful in several application, we refer to Chiantini et al. (2017a) and Hauenstein et al. (2016) for an account.

We would like to mention that in Hauenstein et al. (2016), using new numerical methods and higher order flattenings the authors discovered several new cases of identifiability, and furthermore they proposed a conjecture on generic identifiability.

Given a tensor rank one decomposition of length h as in (1.1) the problem of *specific identifiability* consists in proving that such a decomposition is unique. Following Chiantini et al. (2017a) we call an algorithm for specific identifiability *effective* if it is sufficient to prove identifiability on a dense open subset of the set of tensors admitting a decomposition as in (1.1). Therefore, an algorithm is effective if its constraints are satisfied generically, in other words if the same algorithm proves generic identifiability as well.

In this paper we consider symmetric tensors, mixed skew-symmetric tensors, and mixed symmetric tensors. The corresponding rank one tensors are parametrized respectively by Veronese varieties, Segre–Grassmann varieties, and Segre–Veronese varieties. We provide h-identifiability effective criteria for these spaces, under suitable numerical assumptions on h. Our algorithms are based on the existence of suitable flattenings of a given tensor admitting a decomposition as in (1.1). We would like to stress that we do not need to know an explicit decomposition but just the fact that such a decomposition exists.

Recall that the border rank $\underline{\mathrm{rank}}(T)$ of a tensor T is the smallest integer $r > 0$ such that T is in the Zariski closure, in the tensor space where T belongs, of the set of tensors of rank r. In particular $\underline{\mathrm{rank}}(T) \leq \mathrm{rank}(T)$. Roughly speaking, our methods require that suitable linear spaces, defined in terms of flattenings, intersect the relevant varieties parametrizing rank one tensors in a zero-dimension scheme of a given length. Such a zero dimensional scheme is not required to be reduced and then our criteria can be applied also in border rank identifiability problems, see Remark 3.7.

Symmetric tensors can also be interpreted as homogeneous polynomials. By rephrasing (1.1) in the symmetric case we say that a polynomial rank one decomposition of a homogeneous degree d polynomial $F \in k[x_0, \dots, x_n]_d$ is an expression of the type

$$F = \lambda_1 L_1^d + \dots + \lambda_h L_h^d \tag{1.2}$$

where L_i are linearly independent degree 1 polynomials, $\lambda_i \in k^*$, and k is either the real or complex field. Let $h(n,d)$ be the minimum integer such that a general $F \in k[x_0, \dots, x_n]_d$ admits a decomposition as in (1.2). The number $h(n,d)$ has been determined in Alexander and Hirschowitz (1995) and $h(n,d)$-identifiability very seldom holds (Mella, 2006, 2009; Galuppi and Mella, 2017). Indeed, by Galuppi and Mella (2017, Theorem 1) a general polynomial $F \in k[x_0, \dots, x_n]_d$ is $h(n,d)$-identifiable only in the following cases:

- $n = 1$, $d = 2m+1$, $h(n,d) = m$ (Sylvester, 1904),
- $n = d = 3$, $h(3,3) = 5$ (Sylvester, 1904),
- $n = 2$, $d = 5$, $h(2,5) = 7$ (Hilbert, 1888).

In Theorem 3.8 we provide effective h-identifiability criteria for these polynomials and combined with the previous results this furnishes a complete set of identifiability criteria for these, and few more, polynomials. We would like to stress that the identifiability criteria in Theorem 3.8 give new proves of the uniqueness of the decomposition for the general polynomial in the three cases listed above. Finally, in Section 3.1 we implemented our identifiability algorithms in Macaulay2 (1992).

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2. Tensors and flattenings

Let $\underline{n} = (n_1, \ldots, n_p)$ and $\underline{d} = (d_1, \ldots, d_p)$ be two p-uples of positive integers. Set

$$d = d_1 + \cdots + d_p, \; n = n_1 + \cdots + n_p, \; \text{and } N(\underline{n}, \underline{d}) = \prod_{i=1}^{p} \binom{n_i + d_i}{n_i}.$$

Let $V_1, \ldots, V_p$ be vector spaces of dimensions $n_1 + 1 \le n_2 + 1 \le \cdots \le n_p + 1$, and consider the product

$$\mathbb{P}^{\underline{n}} = \mathbb{P}(V_1^*) \times \cdots \times \mathbb{P}(V_p^*).$$

The line bundle

$$\mathcal{O}_{\mathbb{P}^{\underline{n}}}(d_1, \ldots, d_p) = \mathcal{O}_{\mathbb{P}(V_1^*)}(d_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}(V_1^*)}(d_p)$$

induces an embedding

$$\begin{array}{rccc} \sigma\nu_{\underline{d}}^{\underline{n}}: & \mathbb{P}(V_1^*) \times \cdots \times \mathbb{P}(V_p^*) & \longrightarrow & \mathbb{P}(\mathrm{Sym}^{d_1} V_1^* \otimes \cdots \otimes \mathrm{Sym}^{d_p} V_p^*) = \mathbb{P}^{N(\underline{n},\underline{d})-1}, \\ & ([v_1], \ldots, [v_p]) & \longmapsto & [v_1^{d_1} \otimes \cdots \otimes v_p^{d_p}] \end{array}$$

where $v_i \in V_i$. We call the image

$$\mathcal{SV}_{\underline{d}}^{\underline{n}} = \sigma\nu_{\underline{d}}^{\underline{n}}(\mathbb{P}^{\underline{n}}) \subset \mathbb{P}^{N(\underline{n},\underline{d})-1}$$

a *Segre–Veronese variety*. It is a smooth variety of dimension n and degree $\frac{(n_1+\cdots+n_p)!}{n_1!\ldots n_p!} d_1^{n_1} \ldots d_p^{n_p}$ in $\mathbb{P}^{N(\underline{n},\underline{d})-1}$.

When $p = 1$, $\mathcal{SV}_d^n$ is a Veronese variety. In this case we write $\mathcal{V}_d^n$ for $\mathcal{SV}_d^n$, and ν_d^n for the Veronese embedding. When $d_1 = \cdots = d_p = 1$, $\mathcal{SV}_{1,\ldots,1}^{\underline{n}}$ is a Segre variety. In this case we write $\mathcal{S}^{\underline{n}}$ for $\mathcal{SV}_{1,\ldots,1}^{\underline{n}}$, and $\sigma^{\underline{n}}$ for the Segre embedding. Note that

$$\sigma\nu_{\underline{d}}^{\underline{n}} = \sigma^{\underline{n}'} \circ \left(\nu_{d_1}^{n_1} \times \cdots \times \nu_{d_p}^{n_p}\right),$$

where $\underline{n}' = (N(n_1, d_1) - 1, \ldots, N(n_p, d_p) - 1)$.

Similarly, given a p-uple of k-vector spaces $(V_1^{n_1}, \ldots, V_p^{n_p})$ and p-uple of positive integers $\underline{d} = (d_1, \ldots, d_p)$ we may consider the Segre–Plücker embedding

$$\begin{array}{rccc} \sigma\pi_{\underline{d}}^{\underline{n}}: & Gr(d_1, n_1) \times \cdots \times Gr(d_p, n_p) & \longrightarrow & \mathbb{P}(\bigwedge^{d_1} V_1^{n_1} \otimes \cdots \otimes \bigwedge^{d_p} V_p^{n_p}) = \mathbb{P}^{N(\underline{n},\underline{d})-1}, \\ & ([H_1], \ldots, [H_p]) & \longmapsto & [H_1 \otimes \cdots \otimes H_p] \end{array}$$

where $N(\underline{n},\underline{d})=\prod_{i=1}^{p}\binom{n_i}{d_i}$. We call the image

$$\mathcal{SG}_{\underline{d}}^{\underline{n}}=\sigma\pi_{\underline{d}}^{\underline{n}}(Gr(d_1,n_1)\times\cdots\times Gr(d_p,n_p))\subset\mathbb{P}^{N(\underline{n},\underline{d})}$$

a *Segre–Grassmann variety*.

The *h-secant variety* $\mathbb{S}ec_h(X)$, of an irreducible, non-degenerate n-dimensional variety $X\subset\mathbb{P}^N$, is the Zariski closure of the union of the linear spaces spanned by collections of h points on X. The *expected dimension* of $\mathbb{S}ec_h(X)$ is

$$\operatorname{expdim}(\mathbb{S}ec_h(X)):=\min\{nh+h-1,N\}.$$

However, the actual dimension of $\mathbb{S}ec_h(X)$ might be smaller than the expected one. Indeed, this happens when through a general point of $\mathbb{S}ec_h(X)$ there are infinitely many $(h-1)$-planes h-secant to X. We will say that X is *h-defective* if $\dim(\mathbb{S}ec_h(X))<\operatorname{expdim}(\mathbb{S}ec_h(X))$.

The following remark was the starting point of the investigation in Massarenti and Mella (2013).

Remark 2.1. If a polynomial $F\in k[x_0,\dots,x_n]_d$ admits a decomposition as in (1.2) then $F\in\mathbb{S}ec_h(\mathcal{V}_d^n)$, and conversely a general $F\in\mathbb{S}ec_h(\mathcal{V}_d^n)$ can be written as in (1.2). If $F=\lambda_1L_1^d+\dots+\lambda_hL_h^d$ is a decomposition then the partial derivatives of order s of F can be decomposed as linear combinations of $L_1^{d-s},\dots,L_h^{d-s}$ as well.

These partial derivatives are $\binom{n+s}{n}$ homogeneous polynomials of degree $d-s$ spanning a linear space $H_{\partial,s}\subseteq\mathbb{P}(k[x_0,\dots,x_n]_{d-s})$. Therefore, the linear space $\left\langle L_1^{d-s},\dots,L_h^{d-s}\right\rangle$ contains $H_{\partial,s}$.

Our first aim is to generalize Remark 2.1 to tensors. The natural tools to replace partial derivatives are flattenings.

2.1. Flattenings

Let $V_1,\dots,V_p$ be k-vector spaces of finite dimension, and consider the tensor product $V_1\otimes\dots\otimes V_p=(V_{a_1}\otimes\dots\otimes V_{a_s})\otimes(V_{b_1}\otimes\dots\otimes V_{b_{p-s}})=V_A\otimes V_B$ with $A\cup B=\{1,\dots,p\}$, $B=A^c$. Then we may interpret a tensor

$$T\in V_1\otimes\dots\otimes V_p=V_A\otimes V_B$$

as a linear map $\widetilde{T}:V_A^*\to V_{A^c}$. Clearly, if the rank of T is at most r then the rank of $\widetilde{T}$ is at most r as well. Indeed, a decomposition of T as a linear combination of r rank one tensors yields a linear subspace of V_{A^c}, generated by the corresponding rank one tensors, containing $\widetilde{T}(V_A^*)\subseteq V_{A^c}$. The matrix associated to the linear map $\widetilde{T}$ is called an (A,B)*-flattening* of T.

In the case of mixed tensors we can consider the embedding

$$\operatorname{Sym}^{d_1}V_1\otimes\dots\otimes\operatorname{Sym}^{d_p}V_p\hookrightarrow V_A\otimes V_B$$

where $V_A=\operatorname{Sym}^{a_1}V_1\otimes\dots\otimes\operatorname{Sym}^{a_p}V_p$, $V_B=\operatorname{Sym}^{b_1}V_1\otimes\dots\otimes\operatorname{Sym}^{b_p}V_p$, with $d_i=a_i+b_i$ for any $i=1,\dots,p$. In particular, if $n=1$ we may interpret a tensor $F\in\operatorname{Sym}^{d_1}V_1$ as a degree d_1 homogeneous polynomial on $\mathbb{P}(V_1^*)$. In this case the matrix associated to the linear map $\widetilde{F}:V_A^*\to V_B$ is nothing but the a_1-th *catalecticant matrix* of F, that is the matrix whose lines are the coefficient of the partial derivatives of order a_1 of F. This identifies the linear space $H_{\partial,s}$ in Remark 2.1 with $\mathbb{P}(\widetilde{F}(V_A^*))\subseteq\mathbb{P}(V_B)$, where $a_1=s$, $b_1=d-a_1=d-s$.

Similarly, by considering the inclusion

$$\bigwedge^{d_1}V_1\otimes\dots\otimes\bigwedge^{d_p}V_p\hookrightarrow V_A\otimes V_B$$

where $V_A = \bigwedge^{a_1} V_1 \otimes \ldots \otimes \bigwedge^{a_p} V_p$, $V_B = \bigwedge^{b_1} V_1 \otimes \ldots \otimes \bigwedge^{b_p} V_p$, with $d_i = a_i + b_i$ for any $i = 1, \ldots, p$, we get the so called *skew-flattenings*. We refer to Landsberg (2012) for details on the subject.

3. Effective identifiability

In this section we give h-identifiability criteria for tensors, and we derive effective h-identifiability criteria, under some constraints on h.

Proposition 3.1. *Let $T \in \mathrm{Sym}^{d_1} V_1 \otimes \ldots \otimes \mathrm{Sym}^{d_n} V_n$ be a tensor admitting a decomposition $T = \sum_{i=1}^{h} \lambda_i U_i$ as in (1.1). Fix an (A, B)-flattening $\widetilde{T} : V_A^* \to V_B$ of T such that $N(\underline{n}, \underline{a}) \geq h$, and assume that*

i) *the linear space $\mathbb{P}(\widetilde{T}(V_A^*))$ has dimension $h-1$,*
ii) $\dim(\mathbb{P}(\widetilde{T}(V_A^*)) \cap \mathcal{SV}^{\underline{n}}_{\underline{b}}) = 0$,
iii) $\deg(\mathbb{P}(\widetilde{T}(V_A^*)) \cap \mathcal{SV}^{\underline{n}}_{\underline{b}}) = h$,

where $\underline{b} = (b_1, \ldots, b_n)$. Then T is h-identifiable and it has rank h.

In particular, in the symmetric case we have the following. Let $F \in k[x_0, \ldots, x_n]_d$ be a polynomial admitting a decomposition $F = \sum_{i=1}^{h} \lambda_i L_i^d$. Fix and integer s such that $\binom{n+s}{n} \geq h > \binom{n+s-1}{n}$. Assume that

i) *the linear space $H_{\partial,s}$ generated by the partial derivatives of order s of F has dimension $h-1$,*
ii) $\dim(H_{\partial,s} \cap \mathcal{V}^n_{d-s}) = 0$,
iii) $\deg(H_{\partial,s} \cap \mathcal{V}^n_{d-s}) = h$.

Then F is h-identifiable and it has rank h.

Proof. Assume that $T = \sum_{i=1}^{h} \lambda_i U_i = \sum_{i=1}^{h} \mu_i W_i$ admits two different decompositions. Since $\dim(\mathbb{P}(\widetilde{T}(V_A^*))) = h-1$ by Section 2.1 we have $\mathbb{P}(\widetilde{T}(V_A^*)) = \langle \widetilde{U}_1, \ldots, \widetilde{U}_h \rangle = \langle \widetilde{W}_1, \ldots, \widetilde{W}_h \rangle$, where $\widetilde{U}_i, \widetilde{W}_i$ are the rank one tensors in $\mathbb{P}(V_B)$ induced by U_i and W_i respectively. Hence there are at least $h+1$ points in the intersection $\mathbb{P}(\widetilde{T}(V_A^*)) \cap \mathcal{SV}^{\underline{n}}_{\underline{b}}$, contradicting iii). □

Next, we check when the conditions in Proposition 3.1 define effective criteria.

Proposition 3.2. *The criterion in Proposition 3.1 is effective when $N(\underline{n}, \underline{b}) > h + \dim(\mathcal{SV}^{\underline{n}}_{\underline{b}})$ in the mixed symmetric case. In particular, in the symmetric case the criterion is effective when $\binom{n+d-s}{n} > h + n$.*

Proof. Let $[T] \in \mathbb{S}ec_h(\mathcal{SV}^{\underline{n}}_{\underline{d}})$ be a general point. Assume that $\dim(\mathbb{P}(\widetilde{T}(V_A^*))) \leq h-2$. This condition forces the (A, B)-flattening matrix to have rank at most $h-1$. On the other hand, by Simis and Ulrich (2000, Proposition 4.1) these minors do not vanish on $\mathbb{S}ec_h(\mathcal{SV}^{\underline{n}}_{\underline{d}})$ and therefore define a closed subset of $\mathbb{S}ec_h(\mathcal{SV}^{\underline{n}}_{\underline{d}})$. To conclude observe that by the Trisecant Lemma (Chiantini and Ciliberto, 2002, Proposition 2.6), the general h-secant $(h-1)$-linear space intersects $\mathcal{SV}^{\underline{n}}_{\underline{b}}$ in h points as long as $N(\underline{n}, \underline{b}) > h + n$. □

We may slightly improve Proposition 3.2, under suitable numerical assumption.

Proposition 3.3. *Let $T \in \mathrm{Sym}^{d_1} V_1 \otimes \ldots \otimes \mathrm{Sym}^{d_p} V_p$ be a tensor admitting a decomposition $T = \sum_{i=1}^{h} \lambda_i U_i$. Fix an (A, B)-flattening $\widetilde{T} : V_A^* \to V_B$ of T such that $N(\underline{n}, \underline{a}) \geq h$, and assume that*

i) *the linear space $\mathbb{P}(\widetilde{T}(V_A^*))$ has dimension $h-1$,*
ii) $\dim(\mathbb{P}(\widetilde{T}(V_A^*)) \cap \mathcal{SV}^{\underline{n}}_{\underline{b}}) = 0$,

iii) $h + n = N(\underline{n}, \underline{b})$,
iv) $\deg(\mathcal{SV}^{\underline{n}}_{\underline{b}}) \le h+1$,
v) $\deg(\langle [U_1], \dots, [U_h] \rangle \cap \mathcal{SV}^{\underline{n}}_{\underline{d}}) = h$.

Then T is h-identifiable and the criterion is effective.

In particular, in the symmetric case we have the following. Let $F \in k[x_0, \dots, x_n]_d$ be a polynomial admitting a decomposition $F = \sum_{i=1}^{h} \lambda_i L_i^d$. Fix an integer s such that $\binom{n+s}{n} \ge h > \binom{n+s-1}{n}$. Assume that:

i) *the linear space $H_{\partial,s}$ generated by the partial derivatives of order s of F has dimension $h-1$,*
ii) $\dim(H_{\partial,s} \cap \mathcal{V}^n_{d-s}) = 0$,
iii) $h + n = \binom{n+d-s}{n}$,
iv) $(d-s)^n \le h+1$,
v) $\deg(\langle [L_1^d], \dots, [L_h^d] \rangle \cap \mathcal{V}^n_d) = h$.

Then F is h-identifiable and the criterion is effective.

Proof. Assume that $T = \sum_{i=1}^{h} \lambda_i U_i = \sum_{i=1}^{h} \mu_i W_i$ admits two different decompositions. Since $\dim(\mathbb{P}(\widetilde{T}(V_A^*))) = h-1$ by Section 2.1 we have $\mathbb{P}(\widetilde{T}(V_A^*)) = \langle \widetilde{U}_1, \dots, \widetilde{U}_h \rangle = \langle \widetilde{W}_1, \dots, \widetilde{W}_h \rangle$, where $\widetilde{U}_i, \widetilde{W}_i$ are the rank one tensors in $\mathbb{P}(V_B)$ induced by U_i and W_i respectively. Assumptions ii), iii), and iv) show that $\mathbb{P}(\widetilde{T}(V_A^*))$ intersects $\mathcal{SV}^{\underline{n}}_{\underline{b}}$ in at most $h+1$ points. Therefore, without loss of generality we may assume that $U_i = W_i$, for $i = 1, \dots, h-1$. By construction we have

$$\langle W_1, \dots, W_h \rangle = \langle W_1, \dots, W_{h-1}, T \rangle = \langle U_1, \dots, U_{h-1}, T \rangle = \langle U_1, \dots, U_h \rangle$$

hence $\deg(\langle U_1, \dots, U_h \rangle \cap \mathcal{SV}^{\underline{n}}_{\underline{d}}) \ge h+1$ contradicting assumption v). The criterion is effective again by the Trisecant Lemma (Chiantini and Ciliberto, 2002, Proposition 2.6). □

Remark 3.4. Propositions 3.1, 3.2, 3.3 can be easily extended to the skew-symmetric case, using the skew-flattenings in Section 2.1, and the Segre–Grassmann variety instead of the Segre–Veronese variety. We leave the details to the reader.

Next, we work out our criterion in some interesting cases, for the readers' convenience we report also the skew-symmetric case.

Corollary 3.5. *Let us consider the tensor space $\mathrm{Sym}^{d_1} V_1^{n_1} \otimes \dots \otimes \mathrm{Sym}^{d_p} V_p^{n_p}$ with $n_1 = \dots = n_p = n$, and set $m_i = \lfloor \frac{d_i}{2} \rfloor$. If*

$$h < \prod_{i=1}^{p} \binom{n-1+m_i}{n-1} - p(n-1)$$

then the criterion in Proposition 3.1 is effective, while for tensors in $\bigwedge^{d_1} V_1^{n_1} \otimes \dots \otimes \bigwedge^{d_p} V_p^{n_1}$ with $n_1 = \dots = n_p = n$ the criterion in Proposition 3.1 is effective when

$$h < \prod_{i=1}^{p} \binom{n}{m_i} - \prod_{i=1}^{p} m_i(n-m_i).$$

Now, consider $V_1^{n_1} \otimes \dots . \otimes V_p^{n_p}$ with $n_1 = \dots = n_p = n$ and set $m = \lfloor \frac{p}{2} \rfloor$. If

$$h < n^m - m(n-1)$$

then the criterion in Proposition 3.1 is effective.

Finally, let $V_1^{n_1} \otimes \dots . \otimes V_p^{n_p}$ be an unbalanced product, that is $n_1 > 1 + \prod_{i=2}^p n_i - \sum_{i=2}^p (n_i - 1)$. If

$$h < \prod_{i=2}^p n_i - \sum_{i=2}^p (n_i - 1)$$

then the criterion in Proposition 3.1 is effective.

Proof. In the mixed symmetric case consider the flattening

$$\left(\bigotimes_{i=1}^p \mathrm{Sym}^{\lceil \frac{d_i}{2} \rceil} V_i^n\right)^* \to \bigotimes_{i=1}^p \mathrm{Sym}^{\lfloor \frac{d_i}{2} \rfloor} V_i^n$$

and apply Proposition 3.2.

In the mixed skew-symmetric case it is enough to consider the analogous skew-flattening and to argue as in the proofs of Propositions 3.1, 3.2 with the Segre–Grassmann variety instead of the Segre–Veronese variety.

Similarly, in the second case we choose the flattening

$$\left(\bigotimes_{i=1}^{\lceil \frac{p}{2} \rceil} V_i^n\right)^* \to \bigotimes_{i=\lceil \frac{p}{2} \rceil+1}^{p} V_i^n$$

and apply Proposition 3.2.

Finally, in the unbalanced case we consider the flattening

$$(V_1^{n_1})^* \to \bigotimes_{i=2}^p V_i^{n_i}$$

and again we apply Proposition 3.2. □

Remark 3.6. For Veronese varieties our results are equivalent to the identifiability criterion given in Iarrobino and Kanev (1999, Theorem 2.6). Recently, L. Chiantini, G. Ottaviani and N. Vannieuwenhoven (Chiantini et al., 2017a) improved Kruskal criterion (Kruskal, 1977) by means of the reshaped Kruskal criterion (Chiantini et al., 2017a, Section 4).

In the p-factor Segre case our results are weaker than reshaped Kruskal (Chiantini et al., 2017a, Proposition 16) for p odd but they perform better for p even. For unbalanced Segre our criteria perform better than Chiantini et al. (2017a, Proposition 17).

Remark 3.7. The algorithm in Proposition 3.1 works for the border rank as well. Indeed, let T be a tensor, and $P_t = U_{1,t} + \dots + U_{r,t}$, $Q_t = W_{1,t} + \dots + W_{r,t}$ be two sequence of rank r tensors such that $\lim_{t \mapsto 0} P_t = \lim_{t \mapsto 0} Q_t = T$, and $\lim_{t \mapsto 0}\{U_{1,t}, \dots, U_{r,t}\} \neq \lim_{t \mapsto 0}\{W_{1,t}, \dots, W_{r,t}\}$. Fix an (A, B)-flattening $\widetilde{T} : V_A^* \to V_B$ of T such that $N(\underline{n}, \underline{a}) \geq r$, and let us denote by $\widetilde{U}_{i,t}, \widetilde{W}_{j,t}$, $\widetilde{P}_t$, $\widetilde{Q}_t$ the corresponding flattenings of $U_{i,t}, W_{j,t}, P_t, Q_t$. Then $\mathbb{P}(\widetilde{P}_t(V_A^*)) \subseteq \langle \widetilde{U}_{1,t}, \dots, \widetilde{U}_{r,t} \rangle$ and $\mathbb{P}(\widetilde{Q}_t(V_A^*)) \subseteq \langle \widetilde{W}_{1,t}, \dots, \widetilde{W}_{r,t} \rangle$ yield $\lim_{t \mapsto 0} \mathbb{P}(\widetilde{P}_t(V_A^*)) \subset \Gamma_U$, $\lim_{t \mapsto 0} \mathbb{P}(\widetilde{Q}_t(V_A^*)) \subset \Gamma_V$, where $\Gamma_U = \lim_{t \mapsto 0} \langle \widetilde{U}_{1,t}, \dots, \widetilde{U}_{r,t} \rangle$ and $\Gamma_V = \lim_{t \mapsto 0} \langle \widetilde{W}_{1,t}, \dots, \widetilde{W}_{r,t} \rangle$.

Now, let $X \subset \mathbb{P}(V_B)$ be the variety parametrizing rank one tensors. Since by hypothesis $\dim(\mathbb{P}(\widetilde{T}(V_A^*))) = r - 1$ we have that $\mathbb{P}(\widetilde{T}(V_A^*)) = \lim_{t \mapsto 0} \mathbb{P}(\widetilde{P}_t(V_A^*)) = \lim_{t \mapsto 0} \mathbb{P}(\widetilde{Q}_t(V_A^*))$ forces $\mathbb{P}(\widetilde{T}(V_A^*)) = \Gamma_U = \Gamma_V$. Finally, since

$$\lim_{t \mapsto 0}\{\widetilde{U}_{1,t}, \dots, \widetilde{U}_{r,t}\} \subseteq X \cap \Gamma_U = X \cap \mathbb{P}(\widetilde{T}), \ \lim_{t \mapsto 0}\{\widetilde{W}_{1,t}, \dots, \widetilde{W}_{r,t}\} \subseteq X \cap \Gamma_V = X \cap \mathbb{P}(\widetilde{T}(V_A^*))$$

and $\lim_{t\mapsto 0}\{\widetilde{U}_{1,t},\dots,\widetilde{U}_{r,t}\}\neq \lim_{t\mapsto 0}\{\widetilde{W}_{1,t},\dots,\widetilde{W}_{r,t}\}$ we get that $\deg(\mathbb{P}(\widetilde{T}(V_A^*))\cap X)\geq r+1$, a contradiction with hypothesis iii) of Proposition 3.1.

Finally, we give an effective 7-identifiability criterion for plane quintics, and we extend it to the cases listed in Section 1 when the uniqueness of decomposition holds for a general polynomial.

Theorem 3.8. *Let $F\in\mathbb{C}[x_0,\dots,x_n]_d$ be a polynomial, and $H_{\partial,s}$ the linear span of its partial derivatives of order s in $\mathbb{P}(k[x_0,\dots,x_n]_{d-s})$.*

Assume that:

- $(n,d,h,s)\in\{(1,2h-1,h,h-2),(2,5,7,2),(3,3,5,1)\}$,
- *$H_{\partial,s}$ has dimension $\binom{n+s}{n}-1$,*
- *$H_{\partial,s}\cap\mathcal{V}^n_{d-s}$ is empty.*

Then F is h-identifiable.

Proof. Let us consider the case $(n,d,h,s)=(2,5,7,2)$. Assume that F admits two different decompositions $F=\sum_{i=1}^7\lambda_iL_i^5=\sum_{i=1}^7\mu_il_i^5$. Consider the second partial derivatives of F and their span $H_{\partial,2}\subseteq\mathbb{P}^9$. By Remark 2.1 a decomposition of F induces a decomposition of its partial derivatives, hence we have

$$H_L:=\langle L_1^3,\dots,L_7^3\rangle\supset H_{\partial,2}\subset\langle l_1^3,\dots,l_7^3\rangle=:H_l.$$

By hypothesis $\dim H_{\partial,2}=5$ and $H_{\partial,2}\cap\mathcal{V}^3_2=\emptyset$, these yield:

i) $H_{\partial,2}=H_L\cap H_l$. Indeed, $\dim H_{\partial,2}=5$ and $H_{\partial,2}\cap\mathcal{V}_2^3=\emptyset$ yield $\dim(H_L)=\dim(H_l)=6$. Then $H_{\partial,2}\subsetneqq H_L\cap H_l$ would imply $H_L=H_l$, and since $\dim(\mathcal{V}^2_3)+\dim(H_L)<9$ this would force $\{L_1,\dots,L_7\}=\{l_1,\dots,l_7\}$. A contradiction.

ii) $L_i\neq l_j$ for any $i,j\in\{1,\dots,7\}$,

iii) $H_L\cap\mathcal{V}^2_3$ and $H_l\cap\mathcal{V}^2_3$ are zero dimensional and $\sharp(H_L\cap\mathcal{V}^2_3)=\sharp(H_l\cap\mathcal{V}^2_3)=7$.

Let $H:=\langle H_L,H_l\rangle$ then H intersects $\mathcal{V}^2_3$ in at least 14 points and therefore $H\cap\mathcal{V}^2_3$ contains a curve Γ of degree $3\gamma\leq 6$. Let $|\Lambda|$ be the pencil of hyperplanes containing H. Then any element of the linear system $|\Lambda_{|\mathcal{V}^3_2}|$ is of the form $\Gamma\cup\Sigma$, where Σ is an element of a pencil of curves $|\Sigma|$. Let s be the degree of the base locus of $|\Sigma|$. The hypothesis $H_{\partial,2}\cap\mathcal{V}^3_2=\emptyset$ and iii) yield

$$s+6\gamma=14.$$

On the other hand we only have the following possibilities:

- $\gamma=1$ and $s=4$,
- $\gamma=2$ and $s=1$.

This contradiction proves the statement.

For 4-uples $(n,d,h,s)=(1,2h-1,h,h-2),(3,3,5,1)$ we may argue similarly to derive h-identifiability criteria and we leave the details to the reader. □

For some special values our methods yield a complete set of identifiability criteria.

Corollary 3.9. *Let $V(n,d):=k[x_0,\dots,x_n]_d$ be the vector space of homogeneous polynomial of degree d, with k a field of characteristic zero. Assume that the pair (n,d) is in the following list*

$$(1,d),(2,3),(2,4),(2,5),(2,6),(3,3),(3,4).$$

Then there is an effective criteria for specific s-identifiability for $V(n,d)$ for every s where generic s-identifiability holds.

Proof. Let $k = \mathbb{C}$ be the complex field. For pairs $(1,d)$, d odd, $(2,5)$, $(3,3)$ we apply the identifiability conditions expressed in Theorem 3.8 for the generic rank and Proposition 3.2 for subgeneric ranks. For $(2,4)$ Proposition 3.2 applies to ranks less then or equal to 4, and for rank 5 there is not generic identifiability due to defectivity. For $(3,4)$ Proposition 3.2 applies to ranks less than or equal to 6 and Proposition 3.3 applies to rank 7, while rank 8 is not generically identifiable (Chiantini et al., 2017b). For $(2,6)$ we apply Proposition 3.2 for $s \leq 7$ and Proposition 3.3 for $s = 8$, while rank 9 is not generically identifiable, due to weak defectivity (Chiantini et al., 2017b).

To conclude we only need to extend the results to a general field k of characteristic zero. For this let $F = \sum_1^{k_i} \lambda_i L_i^d$ be a polynomial rank one decomposition over k. Then since $\mathrm{char}(k) = 0$ via a field extension we may consider it over $\mathbb{C}$ and apply the criterion to prove identifiability over $\mathbb{C}$ and hence over k. □

3.1. Macaulay2 implementation

Finally, we implement our identifiability algorithms in Macaulay2 (1992). The package is in the ancillary file `Identifiability.m2`. After loading this package in Macaulay2, the main method available is `certifyIdentifiability`.

The easiest ways to use this method are either by inputting a mixed symmetric tensor T, represented by a multihomogeneous polynomial, and a positive integer h, or by inputting one of its decompositions $T = T_1 + \cdots + T_h$ into h rank one mixed symmetric tensors. Then the method returns the boolean value `true` if the constraints of the correspondent h-identifiability criterion are satisfied for T. For more details see the documentation (`viewHelp certifyIdentifiability`).

By Hillar and Lim (2013) we know that tensor problems are usually NP-Hard and we should not expect any reasonably fast algorithm for computing tensor decompositions. On the other hand, in restricted settings, the existing algorithms like TensorLab (Vervliet et al., 2016), TensorToolbox in Matlab (Bader and Kolda, 2015) and the homotopy technique in Hauenstein et al. (2016), work in a reasonable amount of time. The aim of our algorithm is to have a fast identifiability test avoiding such a computation. In what follows we show how it works in some cases.

```
Macaulay2, version 1.9.2
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "Identifiability";
--** Identifiability (v0.3) loaded **--
-- Example 1 -- Random degree 5 polynomial in 3 variables
i2 : P2 = QQ[x,y,z];
i3 : T = for i in 1..7 list (random(1,P2))^5;
i4 : time certifyIdentifiability(sum T,7)
-- got symmetric tensor of dimension 3 and degree 5
-- applying Theorem 3.8 (7-identifiability for 3-forms of degree 5)...
-- 7-identifiability certified
     -- used 0.257789 seconds
o4 = true
i5 : time certifyIdentifiability matrix{T}
-- got symmetric tensor of dimension 3 and degree 5
-- applying Theorem 3.8 (7-identifiability for 3-forms of degree 5)...
-- 7-identifiability certified
     -- used 0.228473 seconds
o5 = true
```

```
i6 : -- first 6 summands of T
     T' = T_{0..5};
i7 : time certifyIdentifiability(sum T',6)
-- got symmetric tensor of dimension 3 and degree 5
-- specific 6-identifiability certified
     -- used 0.0363902 seconds
o7 = true
i8 : time certifyIdentifiability matrix{T'}
-- got symmetric tensor of dimension 3 and degree 5
-- 6-identifiability certified
     -- used 0.0511795 seconds
o8 = true
-- Example 2 -- the command below creates a random mixed symmetric
-- tensor of dimensions {2,5,4}, multidegree {3,2,3}, rank<=5
i9 : T = multirandom({2,5,4},{3,2,3},5);
i10 : -- number terms of the tensor T
      # terms T
o10 = 1200
i11 : time certifyIdentifiability(T,5)
-- got mixed symmetric tensor of dimensions {2, 5, 4}
   and multidegree {3, 2, 3}
-- specific 5-identifiability certified
     -- used 4.54164 seconds
o11 = true
-- Example 3 -- Random 1 x 7 matrix of degree 4 polynomials in 4 variables
i12 : decomposition = multirandom'({4},{4},7);
i13 : time certifyIdentifiability decomposition
-- got symmetric tensor of dimension 4 and degree 4
-- applying Proposition 3.3...
-- 7-identifiability certified
     -- used 1.03492 seconds
o13 = true
-- Example 4 -- Random 1 x 8 matrix of degree 6 polynomials in 3 variables
i14 : decomposition = multirandom'({3},{6},8);
i15 : time certifyIdentifiability decomposition
-- got symmetric tensor of dimension 3 and degree 6
-- applying Proposition 3.3...
-- 8-identifiability certified
     -- used 0.440192 seconds
o15 = true
-- Example 5 -- Random degree 3 polynomial in 4 variables of rank<=5
i16 : F = multirandom({4},{3},5);
i17 : time certifyIdentifiability(F,5)
-- got symmetric tensor of dimension 4 and degree 3
-- applying Theorem 3.8 (5-identifiability for 4-forms of degree 3)...
-- 5-identifiability certified
     -- used 0.098442 seconds
o18 = true
-- Example 6 -- Random degree 69 polynomial in 2 variables
i19 : P1 = QQ[x,y];
i20 : F = random(69,P1);
i21 : time certifyIdentifiability(F,35)
-- got symmetric tensor of dimension 2 and degree 69
-- applying Theorem 3.8 (35-identifiability for 2-forms of degree 69)...
-- 35-identifiability certified
     -- used 469.406 seconds
o21 = true
```

Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jsc.2017.11.006.

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