A package for computations with classical resultants

GIOVANNI STAGLIANÒ

```
gap> g:= SymmetricGroup( 4 );
Sym( [ 1 .. 4 ] )
gap> tbl:= CharacterTable( g );;  HasIrr( tbl );
false
gap> tblmod2:= CharacterTable( tbl, 2 );
BrauerTable( Sym( [ 1 .. 4 ] ), 2 )
gap> tblmod2 = CharacterTable( tbl, 2 );
true
gap> tblmod2 = BrauerTable( tbl, 2 );
true
gap> libtbl:= CharacterTable( "M" );
CharacterTable( "M" )
gap> CharacterTableRegular( libtbl, 2 );
BrauerTable( "M"
gap> BrauerTable( libtbl, 2 );
fail
gap> CharacterTable( "Symmetric", 4 );
CharacterTable( "Sym(4)" )
gap> ComputedBrauerTables( tbl );
[ , BrauerTable( Sym( [ 1 .. 4 ] ), 2 ) ]
```

```
i5 : betti(t,Weights=>{1,0})

          0 1  2  3 4
o5 = total: 1 4 13 14 4
       0: 1 .  .  . .
       1: . 2  2  4 2
       2: . 2  5  6 .
       3: . .  4  . .
       4: . .  .  . 4
       5: . .  2  . .

o5 : BettiTally
i6 : betti(t,Weights=>{0,1})

          0 1  2  3 4
o6 = total: 1 4 13 14 4
       0: 1 .  .  . .
       1: . 2  2  4 2
       2: . 2  5  6 .
       3: . .  4  . .
       4: . .  .  . 4
       5: . .  2  . .

o6 : BettiTally
i7 : t1 = betti(t,Weights=>{1,1})

          0 1  2  3 4
o7 = total: 1 4 13 14 4
       0: 1 .  .  . .
       1: . .  .  . .
       2: . .  .  . .
       3: . 2  .  . .
       4: . .  .  . .
       5: . 2  .  . .
       6: . .  1  . .
       7: . .  8  6 .
       8: . .  4  8 4

o7 : BettiTally
i8 : peek t1

o8 = BettiTally{(0, {0, 0}, 0) => 1 }
               (1, {2, 2}, 4) => 2
               (1, {3, 3}, 6) => 2
               (2, {3, 7}, 10) => 2
               (2, {4, 4}, 8) => 1
               (2, {4, 5}, 9) => 4
               (2, {5, 4}, 9) => 4
               (2, {7, 3}, 10) => 2
               (3, {4, 7}, 11) => 4
               (3, {4, 11}, 6) =>
               (3, {7, 4}, 11) => 4
               (4, {5, 7}, 12) => 2
               (4, {7, 5}, 12) => 2
```

```
ring r1 = 32003,(x,y,z),ds;
int a,b,c,t=11,5,3,0;
poly f = x^a+y^b+z^(3*c)+x^(c+2)*y^(c-1)+x^
         x^(c-2)*y^c*(y^2+t*x)^2;
option(noprot);
timer=1;
ring r2 = 32003,(x,y,z),dp;
poly f=imap(r1,f);
ideal j=jacob(f);
vdim(std(j));
==> 536
vdim(std(j+f));
==> 195
timer=0;  // reset timer
```

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GIOVANNI STAGLIANÒ

ABSTRACT: We present the Macaulay2 package *Resultants*, which provides
commands for the effective computation of multivariate resultants, discriminants,
and Chow forms. We provide some background for the algorithms implemented
and show, with a few examples, how the package works.

INTRODUCTION. The *resultant* characterizes the existence of nontrivial solutions
for a square system of homogeneous polynomial equations as a condition on the co-
efficients. One of its important features is that it can be used to compute elimination
ideals and to solve polynomial equations. Indeed, it provides one of the two main
tools in elimination theory, along with Gröbner bases. The resultant of the system
of equations given by the partial derivatives of a complex homogeneous polyno-
mial F is called (up to a constant factor) the *discriminant* of F. It characterizes
the existence of singular points in the projective hypersurface $V(F)$ as a condition
on the coefficients of F. In this special case, all polynomial equations have the
same total degree. Every time the system of equations consists of $n + 1$ polynomial
equations of the same total degree d, the resultant has a further interesting property:
it can be expressed as a polynomial of degree d^n in the $(n+1) \times (n+1)$ minors of an
$(n+1) \times \binom{n+d}{n}$ matrix, the coefficient matrix of the system of equations. This allows
us to write down a *generic* resultant in a more compact form. The polynomial of
degree d^n so obtained is geometrically interpreted as the *Chow form* of the d-th
Veronese embedding of $\mathbb{P}^n$.

The package *Resultants*, included with [Macaulay2], provides commands for
the explicit computation of resultants and discriminants. The main algorithm used
is based on the so-called *Poisson formula*, which reduces the computation of the
resultant of $n + 1$ equations to the product of the resultant of n equations with the
determinant of an appropriate matrix. This algorithm requires a certain genericity
condition on the input polynomials, achievable with a generic change of coordi-
nates. The package also includes tools for working with Chow forms and more
generally with *tangential Chow forms*.

In Section 1, from a more computational point of view, we give some background information on the general theory of resultants, discriminants, and Chow forms. In Section 2, we briefly illustrate how to use the package with the help of some examples; more detailed information and examples can be found in its documentation.

1. OVERVIEW OF CLASSICAL RESULTANTS.   We present an overview of some classically well known facts on the theory of resultants for forms in several variables. For details and proofs, we refer mainly to [Gelfand et al. 1994; Cox et al. 2005]; other references are [Jouanolou 1991; 1997; van der Waerden 1950; Demazure 2012; Emiris and Mourrain 1999; Bajaj et al. 1988; Busé and Jouanolou 2014], and [Cox et al. 2007] for the case of two bivariate polynomials.

Resultants. Suppose we are given $n+1$ homogeneous polynomials $F_0, \ldots, F_n$ in $n+1$ variables $x_0, \ldots, x_n$ over the complex field $\mathbb{C}$. For $i = 0, \ldots, n$, let d_i denote the total degree of F_i so that we can write $F_i = \sum_{|\alpha|=d_i} c_{i,\alpha} x^\alpha$, where x^α denotes $x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. For each pair of indices i, α, we introduce a variable $u_{i,\alpha}$ and form the *universal ring of coefficients* $\mathbb{U}_{d_0,\ldots,d_n} := \mathbb{Z}[u_{i,\alpha} : i = 0, \ldots, n, |\alpha| = d_i]$. If $P \in \mathbb{U}_{d_0,\ldots,d_n}$, we denote by $P(F_0, \ldots, F_n)$ the element in $\mathbb{C}$ obtained by replacing each variable $u_{i,\alpha}$ with the corresponding coefficient $c_{i,\alpha}$.

Theorem 1.1 [Gelfand et al. 1994; Cox et al. 2005]. *If we fix positive degrees* $d_0, \ldots, d_n$, *then there is a unique polynomial* $\mathrm{Res} = \mathrm{Res}_{d_0,\ldots,d_n} \in \mathbb{U}_{d_0,\ldots,d_n}$ *which has the following properties*:

(1) *If* $F_0, \ldots, F_n \in \mathbb{C}[x_0, \ldots, x_n]$ *are homogeneous of degrees* $d_0, \ldots, d_n$, *then the equations*

$$F_0 = 0, \ \ldots, \ F_n = 0$$

have a nontrivial solution over $\mathbb{C}$ *(i.e.,* $\varnothing \neq V(F_0, \ldots, F_n) \subset \mathbb{P}^n_{\mathbb{C}}$) *if and only if* $\mathrm{Res}(F_0, \ldots, F_n) = 0$.

(2) Res *is irreducible, even when regarded as a polynomial over* $\mathbb{C}$.

(3) $\mathrm{Res}(x_0^{d_0}, \ldots, x_n^{d_n}) = 1$.

Definition 1.2. We call $\mathrm{Res}(F_0, \ldots, F_n)$ the *resultant* of $F_0, \ldots, F_n$.

Remark 1.3. If A is any commutative ring, we define the resultant of $n+1$ homogeneous polynomials $F_0, \ldots, F_n \in A[x_0, \ldots, x_n]$ again as $\mathrm{Res}(F_0, \ldots, F_n) \in A$, i.e., by specializing the coefficients of the integer polynomial Res. Thus, the formation of resultants commutes with specialization.

Example 1.4. The resultant is a direct generalization of the determinant. Indeed, if $d_0 = \cdots = d_n = 1$, then $\mathrm{Res}(F_0, \ldots, F_n)$ equals the determinant of the $(n+1) \times (n+1)$ coefficient matrix.

Proposition 1.5 [Jouanolou 1991; Jouanolou 1997]. *The following hold*:

(1) (homogeneity) *For a fixed j between 0 and n, Res is homogeneous in the variables $u_{j,\alpha}$, $|\alpha| = d_j$, of degree $d_0 \cdots d_{j-1} d_{j+1} \cdots d_n$; hence its total degree is $\sum_{j=0}^{n} d_0 \cdots d_{j-1} d_{j+1} \cdots d_n$.*

(2) (symmetry) *If σ is a permutation of $\{0, \ldots, n\}$, then*

$$\mathrm{Res}(F_{\sigma(0)}, \ldots, F_{\sigma(n)}) = \mathrm{sign}(\sigma)^{d_0 \cdots d_n} \, \mathrm{Res}(F_0, \ldots, F_n).$$

(3) (multiplicativity) *If $F_j = F'_j F''_j$, then we have*

$$\mathrm{Res}(F_0, \ldots, F_j, \ldots, F_n) = \mathrm{Res}(F_0, \ldots, F'_j, \ldots, F_n) \, \mathrm{Res}(F_0, \ldots, F''_j, \ldots, F_n).$$

(4) (SL($n+1$)-invariance) *For each $(n+1) \times (n+1)$ matrix A over $\mathbb{C}$, we have*

$$\mathrm{Res}(F_0(Ax), \ldots, F_n(Ax)) = \det(A)^{d_0 \cdots d_n} \, \mathrm{Res}(F_0(x), \ldots, F_n(x)),$$

where Ax denotes the product of A with the column vector $(x_0, \ldots, x_n)^t$.

(5) (elementary transformation) *If H_i is homogeneous of degree $d_j - d_i$, then*

$$\mathrm{Res}\Big(F_0, \ldots, F_j + \sum_{i \neq j} H_i F_i, \ldots, F_n\Big) = \mathrm{Res}(F_0, \ldots, F_j, \ldots, F_n).$$

Remark 1.6. On the product $\mathbb{A}^M \times \mathbb{P}^n = \mathrm{Spec}(\mathbb{C}[u_{i,\alpha}]) \times \mathrm{Proj}(\mathbb{C}[x_0, \ldots, x_n])$, where $M = \sum_{i=0}^{n} \binom{n+d_i}{n}$, we have an *incidence variety*

$$W := \left\{ ((c_{i,\alpha}), p) \in \mathbb{A}^M \times \mathbb{P}^n : p \in V\Big(\sum_{|\alpha|=d_0} c_{0,\alpha} x^\alpha, \ldots, \sum_{|\alpha|=d_n} c_{n,\alpha} x^\alpha \Big) \right\}.$$

The first projection $\pi_1 : W \to \mathbb{A}^M$ is birational onto its image, whereas all the fibers of the second projection $\pi_2 : W \to \mathbb{P}^n$ are linear subspaces of dimension $M - n - 1$. It follows that W is a smooth irreducible variety which is birational to $\overline{\pi_1(W)} = \pi_1(W) = V(\mathrm{Res}_{d_0, \ldots, d_n}) \subset \mathbb{A}^M$.

The following result is called the *Poisson formula* and allows one to compute resultants inductively.

Theorem 1.7 [Jouanolou 1991; Cox et al. 2005]. *Let*

$$f_i(x_0, \ldots, x_{n-1}) := F_i(x_0, \ldots, x_{n-1}, 1)$$

and $\overline{F}_i(x_0, \ldots, x_{n-1}) := F_i(x_0, \ldots, x_{n-1}, 0)$. If $\mathrm{Res}(\overline{F}_0, \ldots, \overline{F}_{n-1}) \neq 0$, then the quotient ring $A = \mathbb{C}[x_0, \ldots, x_{n-1}]/(f_0, \ldots, f_{n-1})$ has dimension $d_0 \cdots d_{n-1}$ as a vector space over $\mathbb{C}$, and

$$\mathrm{Res}(F_0, \ldots, F_n) = \mathrm{Res}(\overline{F}_0, \ldots, \overline{F}_{n-1})^{d_n} \det(m_{f_n} : A \to A), \qquad (1\text{-}1)$$

where $m_{f_n} : A \to A$ is the linear map given by multiplication by f_n.

With the same hypotheses as Theorem 1.7, a monomial basis for A over $\mathbb{C}$ (useful in the implementation) can be constructed as explained in [Cox et al. 2005, Chapter 2, §2]. Note also that we have

$$\det(m_{f_n} : A \to A) = \prod_{p \in V} f_n(p)^{\mathrm{mult}_p(V)}, \tag{1-2}$$

where $V = V(f_0, \ldots, f_{n-1})$.

We now describe the most popular way to compute resultants, which is due to Macaulay [1903]. Let

$$\delta = \sum_{i=0}^{n} d_i - n \quad \text{and} \quad N = \binom{n+\delta}{n}.$$

We can divide the monomials x^α of total degree δ into the $n+1$ mutually disjoint sets

$$S_i := \{x^\alpha : |\alpha| = \delta, \ \min\{j : x_j^{d_j} | x^\alpha\} = i\}, \quad \text{for } i = 0, \ldots, n.$$

A monomial x^α of total degree δ is called *reduced* if $x_i^{d_i}$ divides x^α for exactly one i. Consider the following N homogeneous polynomials of degree δ:

$$x^\alpha / x_i^{d_i} F_i, \quad \text{for } i = 0, \ldots n \text{ and } x^\alpha \in S_i. \tag{1-3}$$

By regarding the monomials of total degree δ as unknowns, the polynomials in (1-3) form a system of N linear equations in N unknowns. Let

$$\mathbb{D} = \mathbb{D}(F_0, \ldots, F_n)$$

denote the coefficient matrix of this linear system, and let $\mathbb{D}'(F_0, \ldots, F_n)$ denote the submatrix of $\mathbb{D}$ obtained by deleting all rows and columns corresponding to reduced monomials. The following result is called the *Macaulay formula* and allows one to compute the resultant as a quotient of two determinants.

Theorem 1.8 [MacAulay 1903; Jouanolou 1997; Cox et al. 2005]. *The following formula holds*:

$$\det(\mathbb{D}(F_0, \ldots, F_n)) = \mathrm{Res}(F_0, \ldots, F_n) \det(\mathbb{D}'(F_0, \ldots, F_n)). \tag{1-4}$$

In several special cases, the resultant can be expressed as a single determinant (see [Gelfand et al. 1994, Chapter 13, Proposition 1.6]). We also mention that besides (1-4), there are other ways to represent resultants as quotients: these include *Bezoutians* [Elkadi and Mourrain 1998] and *Dixon matrices* [Kapur et al. 1994]; see also [Emiris and Mourrain 1999] and [Cox et al. 2005, p. 110]. However, all these matrices are usually of much larger size than those involved by the Poisson formula (1-1), as shown in the following simple example (see [Emiris and Mourrain 1999], for a comparison between Macaulay and other resultant matrices).

Example 1.9. Let $F_0 = x^3 + y^2 z$, $F_1 = xy + y^2 + xz + yz$, $F_2 = y^4 + z^4 \in \mathbb{C}[x, y, z]$. The Poisson formula expresses $\mathrm{Res}(F_0, F_1, F_2)$ as the following product of determinants:

$$\mathrm{Res}(F_0, F_1, F_2) = \left(1^2 \det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\right)^4 \cdot \det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 & 0 & 2 \end{pmatrix} = 16.$$

The Macaulay formula yields the same result as a quotient $\det(\mathbb{D})/\det(\mathbb{D}')$, where $\mathbb{D}$ and $\mathbb{D}'$ are square matrices of size 36×36 and 10×10, respectively.

Discriminants. Let $F = \sum_{|\alpha|=d} c_\alpha x^\alpha \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial of a certain degree d. As above, for each index α we introduce a variable u_α and form the universal ring of coefficients $\mathbb{U}_d := \mathbb{C}[u_\alpha : |\alpha| = d]$. Then one can show that, up to sign, there is a unique polynomial $\mathrm{Disc} = \mathrm{Disc}_d \in \mathbb{U}_d$ which has the following properties:

(1) If $F \in \mathbb{C}[x_0, \dots, x_n]$ is homogeneous of degrees d, then the equations

$$\partial F/\partial x_0 = 0, \dots, \partial F/\partial x_n = 0$$

have a nontrivial solution over $\mathbb{C}$ (i.e., the hypersurface defined by F is singular) if and only if $\mathrm{Disc}(F) = 0$;

(2) Disc is irreducible, even when regarded as a polynomial over $\mathbb{C}$.

Proposition 1.10 [Gelfand et al. 1994]. *Up to sign, we have the formula*

$$\mathrm{Disc}(F) = c_{d,n} \, \mathrm{Res}\left(\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}\right), \text{ where } c_{d,n} = d^{\frac{(-1)^{n+1} - (d-1)^{n+1}}{d}}. \quad (1\text{-}5)$$

Definition 1.11. We call the polynomial defined by (1-5) the *discriminant* of F.

Proposition 1.12 [Gelfand et al. 1994]. *The following hold*:

(1) *The polynomial* Disc *is homogeneous of degree* $(n+1)(d-1)^n$.

(2) *For each* $(n+1) \times (n+1)$ *matrix A over $\mathbb{C}$, we have*

$$\mathrm{Disc}(F(Ax)) = \det(A)^{d(d-1)^n} \mathrm{Disc}(F(x)),$$

where Ax denotes the product of A with the column vector $(x_0, \dots, x_n)^t$.

Geometrically, we have the following interpretation.

Proposition 1.13 [Gelfand et al. 1994]. *The discriminant hypersurface $V(\mathrm{Disc}_d)$ in the space of forms of degree d on $\mathbb{P}^n$ coincides with the dual variety of the d-th Veronese embedding of $\mathbb{P}^n$.*

Chow forms. Let $X \subset \mathbb{P}^n$ be an irreducible subvariety of dimension k and degree d. Consider the subvariety $Z(X)$ in the Grassmannian $\mathbb{G}(n-k-1, \mathbb{P}^n)$ of all $(n-k-1)$-dimensional projective subspaces of $\mathbb{P}^n$ that intersect X. It turns out that $Z(X)$ is an irreducible hypersurface of degree d; thus $Z(X)$ is defined by the vanishing of some element R_X, unique up to a constant factor, in the homogeneous component of degree d of the coordinate ring of the Grassmannian $\mathbb{G}(n - k - 1, \mathbb{P}^n)$ in the Plücker embedding. This element is called the *Chow form* of X. It is notable that X can be recovered from its Chow form. See [Gelfand et al. 1994, Chapter 3, §2] for details.

Consider the product $\mathbb{P}^k \times X$ as a subvariety of $\mathbb{P}^{(k+1)(n+1)-1}$ via the Segre embedding. Identify $\mathbb{P}^{(k+1)(n+1)-1}$ with the projectivization $\mathbb{P}(\mathrm{Mat}(k + 1, n + 1))$ of the space of $(k + 1) \times (n + 1)$ matrices and consider the natural projection $\rho : \mathbb{P}(\mathrm{Mat}(k + 1, n + 1)) \dashrightarrow \mathbb{G}(k, n) \simeq \mathbb{G}(n - k - 1, n)$. The following result is called the *Cayley trick*.

Theorem 1.14 [Gelfand et al. 1994; Weyman and Zelevinsky 1994]. *The dual variety of $\mathbb{P}^k \times X$ coincides with the closure $\overline{\rho^{-1}(Z(X))}$, where*

$$Z(X) \subset \mathbb{G}(n - k - 1, n)$$

is the hypersurface defined by the Chow form of X.

The defining polynomial of the hypersurface $\overline{\rho^{-1}(Z(X))} \subset \mathbb{P}(\mathrm{Mat}(k+1, n+1))$ is called *X-resultant*; it provides another way of writing the Chow form of X.

Now, let $F_0, \ldots, F_n$ be $n + 1$ generic homogeneous polynomials on $\mathbb{P}^n$ of the same degree $d > 0$, and let $\mathbb{M} = \mathbb{M}(F_0, \ldots, F_n)$ be the $(n + 1) \times N$ matrix of the coefficients of these polynomials, $N = \binom{n+d}{n}$. We consider the projection $\rho_{n,d} : \mathbb{P}(\mathrm{Mat}(n + 1, N)) \dashrightarrow \mathbb{G}(n, N - 1) \simeq \mathbb{G}(N - n - 2, N - 1)$ defined by the maximal minors of $\mathbb{M}$.

Proposition 1.15 [Gelfand et al. 1994; Cox et al. 2005]. *The hypersurface of degree $(n + 1)d^n$ in $\mathbb{P}(\mathrm{Mat}(n + 1, N))$ defined by the resultant $\mathrm{Res}(F_0, \ldots, F_n)$ coincides with the closure $\overline{\rho_{n,d}^{-1}(V(R_{n,d}))}$, where $R_{n,d}$ denotes the Chow form of the d-th Veronese embedding of $\mathbb{P}^n$. In particular, $\mathrm{Res}(F_0, \ldots, F_n)$ is a polynomial in the maximal minors of $\mathbb{M}$.*

2. IMPLEMENTATION.   In this section, we illustrate briefly some of the methods available in the package *Resultants*, included with [Macaulay2]. We refer to the package documentation (which can be viewed with `viewHelp Resultants`) for more details and examples.

One of the main methods is `resultant`, which accepts as input a list of $n + 1$ homogeneous polynomials in $n + 1$ variables with coefficients in some commutative ring A and returns an element of A, the resultant of the polynomials. There are

no limitations on the ring A because of Remark 1.3. The algorithms implemented are the Poisson formula (Theorem 1.7) and the Macaulay formula (Theorem 1.8). The former is used by default since it is typically faster, while for the latter one has to set the Algorithm option: `resultant(...,Algorithm=>"Macaulay")`. The method can also be configured to involve interpolation of multivariate polynomials (see [Manocha and Canny 1993]), i.e., it can reconstruct the polynomial resultant from its values at a sufficiently large number of points, which in turn are evaluated using the same formulas. The main derived method is `discriminant`, which applies the formula (1-5) to compute discriminants of homogeneous polynomials.

Example 2.1. In the following code, we take two forms F, G of degree 6 on $\mathbb{P}^3$. We first verify that $\mathrm{Disc}(F) = 0$ and $\mathrm{Disc}(G) \neq 0$ and then we compute the intersection of the pencil generated by F and G with the discriminant hypersurface in the space of forms of degree 6 on $\mathbb{P}^3$, which is a hypersurface of degree 500 in $\mathbb{P}^{83}$. (The algorithm behind these calculations is the Poisson formula; this is one of the cases where the Macaulay formula is much slower).

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : loadPackage "Resultants";
i2 : ZZ[w,x,y,z]; (F,G) = (w^6+x^6+y^6+w*x*y^4,w^6+x^6+y^6+z^6)
      6   6        4   6   6    6   6   6
o3 = (w + x + w*x*y + y , w + x + y + z )
o3 : Sequence
i4 : time discriminant F
     -- used 0.0179806 seconds
o4 = 0
i5 : time discriminant G
     -- used 0.0310744 seconds
o5 = 1405708114831691994706380179327883585442821877173978446563248267695521602784763325
640650214512085523667681169748888243576021771407839966410501967238133874822857638880169
042329841357623161361759778624522173244483459194112043602458289220741512289591637737144
663616815976480976587530707398334499978646836010657856
i6 : R := ZZ[t,u][w,x,y,z]; pencil = t*sub(F,R) + u*sub(G,R)
          6         6        4       6        6
o7 = (t + u)w + (t + u)x + t*w*x*y + (t + u)y + u*z
o7 : ZZ[t, u][w, x, y, z]
i8 : time D = discriminant pencil
     -- used 7.05101 seconds
                                375 125                         374 126 ...
     11918167904272470982401...000t  u   + 44811489377450403137211...000t  u   ...
o8 : ZZ[t, u]
i9 : factor D
        125       195  3    2       2     3 30   3    2       2     3 30
o9 = (u)   (t + u)   (25t + 81t u + 81t*u + 27u ) (29t + 81t u + 81t*u + 27u ) (
18453098603344854356045130076201433820906084922117987408631404035314583354936784858690
19666668055428407222803144055042891867966935429959336227999512218285981355846846846364
62680139762581395705805883401098082876658292464056)
o9 : Expression of class Product
```

In particular, we deduce that the pencil $\langle F, G \rangle$ intersects the discriminant hypersurface in F with multiplicity 125, in $F - G$ with multiplicity 195, and in another six distinct points with multiplicity 30.

The package also provides methods for working with Chow forms and more generally tangential Chow forms of projective varieties (see [Gelfand et al. 1994, p. 104] and [Green and Morrison 1986]). In the following example, we apply some of these methods.

Example 2.2. Take $C \subset \mathbb{P}^3$ to be the twisted cubic curve.

```
i10 : C = kernel veronese(1,3)
              2                       2
o10 = ideal (x  - x x , x x  - x x , x  - x x )
              2    1 3   1 2    0 3   1    0 2
o10 : Ideal of QQ[x , x , x , x ]
                   0   1   2   3
```

The Chow form of C in $\mathbb{G}(1, 3)$ can be obtained as follows:

```
i11 : w = chowForm C
       3                           2       2
o11 = x     - x   x   x   + x   x   + x   x   - 2x   x   x   - x   x   x
       1,2    0,2 1,2 1,3   0,1 1,3   0,2 2,3    0,1 1,2 2,3   0,1 0,3 2,3

       QQ[x   , x   , x   , x   , x   , x   ]
          0,1   0,2   1,2   0,3   1,3   2,3
o11 : -----------------------------------
          x   x   - x   x   + x   x
          1,2 0,3   0,2 1,3   0,1 2,3
```

We can recover C from its Chow form by taking the so-called Chow equations; see [Gelfand et al. 1994, p. 102; Catanese 1992].

```
i12 : C == saturate chowEquations w
o12 = true
```

The X-resultant of C can be obtained applying first the duality isomorphism $\mathbb{G}(1, \mathbb{P}^3) = \mathbb{G}(1, \mathbb{P}^{3*})$ and then passing from the Plücker to the Stiefel coordinates.

```
i13 : w' = dualize w
       3                           2       2
o13 = x     - x   x   x   + x   x   + x   x   - x   x   x   - 2x   x   x
       0,3    0,2 0,3 1,3   0,1 1,3   0,2 2,3   0,1 1,2 2,3    0,1 0,3 2,3

       QQ[x   , x   , x   , x   , x   , x   ]
          0,1   0,2   1,2   0,3   1,3   2,3
o13 : -----------------------------------
          x   x   - x   x   + x   x
          1,2 0,3   0,2 1,3   0,1 2,3
i14 : fromPluckerToStiefel w'
```

```
        3   3          2   2                    2       2          2   3      ...
o14 = - x   x    + x   x   x   x    - x   x   x   x    + x   x   x     - ...
       0,3 1,0    0,2 0,3 1,0 1,1    0,1 0,3 1,0 1,1    0,0 0,3 1,1     ...
o14 : QQ[x   , x   , x   , x   , x   , x   , x   , x   ]
        0,0   0,1   0,2   0,3   1,0   1,1   1,2   1,3
```

The method `cayleyTrick` returns a pair consisting of the defining ideal of $\mathbb{P}^1 \times C \subset \mathbb{P}^7 \simeq \mathbb{P}(\mathrm{Mat}(2,4))$ and the X-resultant of C, considered as a hypersurface $Z \subset \mathbb{P}(\mathrm{Mat}(2,4))$. Theorem 1.14 ensures that Z is the dual variety of $\mathbb{P}^1 \times C$. We can check this using the method `dualVariety`.

```
i15 : (P1xC,Z) = cayleyTrick C;
i16 : dualVariety(P1xC) == Z
o16 = true
```

Some overlapping packages. There are two further packages related to resultant computations, which are included in Macaulay2: [Elimination] by M. E. Stillman, and [EliminationMatrices] by N. Botbol, L. Busé and M. Dubinsky. The former contains functions to compute Sylvester resultants. The latter can compute different resultant matrices; in particular, it contains an implementation of the Macaulay formula.

A further package for working with Chow forms is *Coisotropy*, by K. Kohn (see [Kohn 2016]), which, in particular, contains a useful function to compute the degrees of all tangential Chow forms of a given projective variety.

For all these overlapping functions, it does not seem easy to rank implementations in terms of efficiency because this generally depends on the problem. They also differ in how they handle input and output. For instance, the discriminant of a binary form computed using the package *Elimination* lies again in the same ring, rather than in the ring of coefficients, and the Chow form of a projective variety computed using *Coisotropy* lies in a polynomial ring, rather than in a quotient ring.

SUPPLEMENT. The online supplement contains version 1.2.1 of *Resultants*.

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