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On the algebraic boundaries among typical ranks for real binary forms

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ABSTRACT

We describe the algebraic boundaries of the regions of real binary forms with fixed typical rank and of degree at most eight, showing that they are dual varieties of suitable coincident root loci.

1. Introduction

The study of symmetric tensors, of their rank, decomposition and identifiability is a classical problem, which received great attention recently in both pure and applied mathematics; see e.g. [18] and references therein, see also [2,3,21,8,20,1].

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Symmetric tensors can be interpreted as homogeneous polynomials, also called forms. The rank of a degree d form f is the minimum integer r such that there exists a decomposition $f = \sum_{i=1}^{r} c_i(l_i)^d$, where l_i are linear forms and c_i are scalars.

In this paper we focus on the case of binary forms over the field of real numbers $\mathbb{R}$. In this case it is known that the (real) rank of a general form satisfies the inequalities $\frac{d+1}{2} \leq r \leq d$. Moreover all the ranks in this range are typical, that is, they occur in open subsets (with respect to the Euclidean topology) of the real vector space of degree d forms; see [4].

A natural problem is to understand the geometry of the sets $\Omega_{d,r}$ of forms of degree d and rank r. In particular we would like to describe the boundaries among the various sets of forms of given typical rank; more precisely, we are interested in understanding the algebraic boundaries, i.e., the Zariski closures of the topological boundaries (see Section 3 for the precise definitions).

The easiest case is the maximal one, that is when the rank is equal to the degree. Indeed it is proved in [10,9] that a binary form of degree d with distinct roots has rank d if and only if all its roots are real. Hence its algebraic boundary is the discriminant hypersurface of forms with two coincident roots.

The geometric description of the sets $\Omega_{d,r}$ becomes much more intricate for $r < d$. Indeed, although the rank of a form is always greater than or equal to the number of its real distinct roots, in general the number of real distinct roots is not invariant in the region $\Omega_{d,r}$.

In [19] the authors study the boundary of the set of forms of rank $\lceil \frac{d+1}{2} \rceil$, which is the minimal typical rank. They prove that the components of the boundary are dual varieties of suitable coincident root loci.

We tackle the problem of describing all the intermediate boundaries in general, as proposed by Lee and Sturmfels in [19, Remark 4.5]. Our approach provides a unified description of all the boundaries in terms of dual varieties of coincident root loci. We recall that the cases of degree $d \leq 5$ have been described in [9], while the case $d = 6$ follows by [19,10] (see Proposition 3.1 for more details). In this paper we focus on the cases $d = 7$ and $d = 8$, and we postpone a general description to future work.

The paper is organized as follows. Sections 2 and 3 are devoted to preliminary results; in particular, in Proposition 3.1 we recall the known results concerning algebraic boundaries for real binary forms of degree less than or equal to 6. Section 4 and 5 contain our main results, which are Theorem 4.1 and Theorem 5.1, describing the algebraic boundaries for real binary forms of degrees respectively 7 and 8. They turn out to be dual varieties of suitable coincident root loci. Finally in Section 6 we explain some of the computational methods of which we take advantage in our study.

2. Coincident root loci

We recall here some known results on coincident root loci, referring to [26,15,6,7,17] for details.

We regard a degree d binary form $f = \sum_{i=0}^{d} \binom{d}{i} a_i x^{d-i} y^i$ over the complex field $\mathbb{C}$ as a point of the projective space $\mathbb{P}(\mathbb{C}[x,y]_d)$, where $\mathbb{C}[x,y]_d = \mathrm{Sym}^d(\mathbb{C}^2)$. This space is identified with $\mathbb{P}^d$ using homogeneous coordinates $(a_0, \dots, a_d)$.

A partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of d is a list of integers $\lambda_1 \geq \cdots \geq \lambda_n \geq 1$ such that $\sum_{i=1}^{n} \lambda_i = d$. Given a partition λ, the *coincident root locus* $\Delta_\lambda \subset \mathbb{P}^d$ is the set of binary forms f of degree d which admit a factorization $f = \prod_{i=1}^{n} \ell_i^{\lambda_i}$ for some linear forms $\ell_1, \dots, \ell_n \in \mathbb{C}[x,y]_1$. A partition λ can be also represented by the list of integers $m_1, \dots, m_k$ defined as $m_j = |\{i : \lambda_i = j\}|$, and clearly $\sum_{j=1}^{k} j m_j = d$. Then the coincident root locus Δ_λ is given by the binary forms of degree d which have m_j roots of multiplicity at least j. It is classically known (see [13]) that $\Delta_\lambda \subset \mathbb{P}^d$ is a variety of dimension n and degree

$$\deg(\Delta_\lambda) = \frac{n!}{m_1! m_2! \cdots m_k!} \lambda_1 \lambda_2 \cdots \lambda_n. \tag{2.1}$$

If $\lambda = (2, 1^{d-2})$, the corresponding coincident root locus $\Delta_\lambda = \Delta$ is the classical discriminant hypersurface. In the opposite case, if $\lambda = (d)$ then Δ_λ is the rational normal curve $C_d \subset \mathbb{P}^d$. When $\lambda = (a, 1^{d-a})$, the partition is called *hook*, and the associated coincident root locus Δ_λ represents the tangential developable of $\Delta_{(a+1,1^{d-a-1})}$.

2.1. Singularities of Δ_λ

The singular loci of coincident root loci have been studied by Chipalkatti [6] and Kurmann [17].

Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, the singular locus $\mathrm{Sing}(\Delta_\lambda)$ is given by the union of Δ_μ for some suitable coarsenings μ of λ. See either [6, Definition 5.2], or [17, Proposition 2.1] for the precise description. In particular Δ_λ is smooth if and only if $\lambda_1 = \cdots = \lambda_n$. Otherwise the singular locus is of (not necessarily pure) codimension 1.

Example 2.1. For future use, we now compute the *iterate singular locus* of Δ_λ, for $\lambda = (2,1,1,1)$ and $\lambda = (2,1,1,1,1)$.

$$\mathrm{Sing}(\Delta_{(2,1,1,1)}) = \Delta_{(3,1,1)} \cup \Delta_{(2,2,1)};$$
$$\mathrm{Sing}(\Delta_{(3,1,1)}) = \Delta_{(4,1)}, \quad \mathrm{Sing}(\Delta_{(2,2,1)}) = \Delta_{(3,2)};$$
$$\mathrm{Sing}(\Delta_{(4,1)}) = \mathrm{Sing}(\Delta_{(3,2)}) = \Delta_{(5)}.$$

$$\mathrm{Sing}(\Delta_{(2,1,1,1,1)}) = \Delta_{(3,1,1,1)} \cup \Delta_{(2,2,1,1)};$$
$$\mathrm{Sing}(\Delta_{(3,1,1,1)}) = \Delta_{(4,1,1)} \cup \Delta_{(3,3)}, \quad \mathrm{Sing}(\Delta_{(2,2,1,1)}) = \Delta_{(3,2,1)} \cup \Delta_{(2,2,2)};$$
$$\mathrm{Sing}(\Delta_{(4,1,1)}) = \Delta_{(5,1)}, \quad \mathrm{Sing}(\Delta_{(3,2,1)}) = \Delta_{(3,3)} \cup \Delta_{(4,2)} \cup \Delta_{(5,1)};$$
$$\mathrm{Sing}(\Delta_{(4,2)}) = \mathrm{Sing}(\Delta_{(5,1)}) = \Delta_{(6)}.$$

2.2. Duality

Consider the dual ring of differential operators $\mathbb{C}[\partial_x, \partial_y] = \mathbb{C}[u, v]$, which acts on $\mathbb{C}[x, y]$ with the usual rules of differentiations and gives the pairing with respect to the degrees,

$$\mathbb{C}[x, y]_d \otimes \mathbb{C}[u, v]_k \to \mathbb{C}[x, y]_{d-k}.$$

The conormal variety of a coincident root locus Δ_λ is the Zariski closure of the set

$$\{(f, g) : f \text{ is a smooth point of } \Delta_\lambda, g \perp T_f\Delta_\lambda\} \subset \mathbb{P}(\mathbb{C}[x, y]_d) \times \mathbb{P}(\mathbb{C}[u, v]_d),$$

where $T_f\Delta_\lambda$ denotes the tangent space to Δ_λ at a point f. The dual variety $(\Delta_\lambda)^\vee$ of Δ_λ is the projection onto $\mathbb{P}(\mathbb{C}[u, v]_d)$ of the conormal variety of Δ_λ. The biduality theorem (see [11]) implies that $(\Delta_\lambda^\vee)^\vee = \Delta_\lambda$.

Lee and Sturmfels study duality for binary forms in [19]. We recall here some results which we will use in the sequel.

Proposition 2.2 *([19]). Given $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\Delta_\lambda \subset \mathbb{P}(\mathbb{C}[u, v]_d)$, the points of the dual variety $\Delta_\lambda^\vee \subset \mathbb{P}(\mathbb{C}[x, y]_d)$ are given by the binary forms $f(x, y)$ that are annihilated by some order $d - n$ operator of the form $\Pi_{i=1}^n \ell_i^{\lambda_i - 1}(\partial_x, \partial_y)$ where $\ell_i \in \mathbb{C}[u, v]_1$.*

Proposition 2.3 *([19]). Given $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\Delta_\lambda \subset \mathbb{P}(\mathbb{C}[u, v]_d)$, the dual variety $\Delta_\lambda^\vee \subset \mathbb{P}(\mathbb{C}[x, y]_d)$ has codimension $m_1 + 1$, and it is given by the join of the $(n - m_1)$ coincident root loci $\Delta_{(d-\lambda_i+2, 1^{\lambda_i-2})}$ for $1 \leq i \leq n$ with $\lambda_i \geq 2$.*

If $\lambda_i \geq 2$ for all i, then $\Delta_\lambda^\vee$ is a hypersurface of degree (see [22])

$$\frac{(n + 1)!}{m_2! \cdots m_k!}(\lambda_1 - 1)(\lambda_2 - 1) \cdots (\lambda_n - 1). \tag{2.2}$$

2.3. Chow forms and higher associated varieties

Let $\mathbb{G}(h, m)$ denote the Grassmannian of projective subspaces of dimension h in $\mathbb{P}^m$. Let $X \subset \mathbb{P}^m$ be a projective variety of dimension k.

The *i-th higher associated variety* $\mathrm{CH}_i(X)$ of X is defined as the closure of the set of all $(m-k-1+i)$-dimensional subspaces $L \subset \mathbb{P}^m$ such that $L \cap X \neq \emptyset$ and $\dim(L \cap T_xX) \geq i$ for some smooth point $x \in L \cap X$ (where T_xX denotes the embedded tangent space to X at x), see [11] for details.

For $i = 0$, the associated variety $\mathrm{CH}_0(X) \subset \mathbb{G}(m-k-1, m)$ is the Chow hypersurface, while for $i = k$, we have that $\mathrm{CH}_k(X) \subset \mathbb{G}(m-1, m)$ corresponds to the dual variety $X^\vee$ via the Grassmannian duality $\mathbb{G}(m - 1, \mathbb{P}^m) \simeq \mathbb{G}(0, (\mathbb{P}^m)^\vee)$. If $i = 1$ and $\deg(X) \geq 2$, the associated variety $\mathrm{CH}_1(X)$ is the Hurwitz hypersurface, see [25].

The variety $\mathrm{CH}_i(X)$ is a hypersurface if and only if $i \leq \dim(X) - (m-1-\dim(X^\vee))$, see [16]. In particular, if $X = \Delta_{(\lambda_1,\dots,\lambda_n)}$ is a coincident root locus, the higher associated variety $CH_i(X)$ is a hypersurface if and only if $i \leq |\{j : \lambda_j \geq 2\}|$.

3. Real rank of binary forms

3.1. Typical ranks for binary forms

Given a binary form f of degree d with complex (or real) coefficients, its *complex rank* is the minimum integer r such that f admits a decomposition $f = \sum_{i=1}^{r}(\ell_i)^d$ where ℓ_i are linear forms with complex coefficients. The *generic complex rank* for binary forms of degree d (that is the rank of a general binary form of degree d) is $\lceil \frac{d+1}{2} \rceil$. Sylvester Theorem says that a general binary form admits a unique minimal decomposition if the degree is odd, infinitely many (parametrized by a line) if the degree is even.

Consider now the polynomial ring $R = \mathbb{R}[x,y]$ of real binary forms. Given $f \in R_d$, the *real rank* of f (denoted by $\mathrm{rk}(f)$) is the minimum integer r such that f admits a decomposition $f = \sum_{i=1}^{r} c_i(l_i)^d$ where $l_i \in R_1$ and $c_i \in \mathbb{R}$; we can impose $c_i \in \{1,-1\}$ if d is even, and $c_i = 1$ if d is odd.

In the real field the notion of generic rank is replaced by the notion of *typical ranks*. A rank is called typical for binary forms of degree d if it occurs in an open subset of R_d, with respect to the Euclidean topology.

Define $\Omega_{d,r} = \{f \in R_d : \mathrm{rk}(f) = r\}$, and denote by $\mathcal{R}_{d,r}$ the interior of $\Omega_{d,r}$. Then $\mathcal{R}_{d,r}$ is a semi-algebraic set in the real vector space R_d, and a rank is typical exactly when $\mathcal{R}_{d,r}$ is not empty. From the main result of [4], a rank r is typical if and only if $\frac{d+1}{2} \leq r \leq d$. Thus, from now on we assume that $\frac{d+1}{2} \leq r \leq d$. We define the *topological boundary* $\partial(\mathcal{R}_{d,r})$ as the set-theoretic difference of the closure of $\mathcal{R}_{d,r}$ and the interior of the closure of $\mathcal{R}_{d,r}$. It is a semi-algebraic subset of R_d of pure codimension one. We define the *real rank boundary* $\partial_{\mathrm{alg}}(\mathcal{R}_{d,r})$ as the Zariski closure of the topological boundary $\partial(\mathcal{R}_{d,r})$ (see also [19, Section 4]). The real rank boundaries $\partial_{\mathrm{alg}}(\mathcal{R}_{d,r})$ are hypersurfaces of the real space R_d, that we consider as hypersurfaces of the complex projective space $\mathbb{P}(\mathbb{C}[x,y]_d) = \mathbb{P}^d_{\mathbb{C}}$. Let us remark that these hypersurfaces are invariant under the natural action of SL_2 on $\mathbb{P}^d$.

Real rank boundaries have been studied only in the two extreme cases, that is for maximum rank d and minimum rank $\overline{r} = \lceil \frac{d+1}{2} \rceil$. In the first case $\partial_{\mathrm{alg}}(\mathcal{R}_{d,d})$ is the discriminant hypersurface $\Delta_{(2,1^{d-2})}$ (see [9, Proposition 3.1] and [10, Corollary 1]); in the second case the real rank boundary $\partial_{\mathrm{alg}}(\mathcal{R}_{d,\overline{r}})$ is described in [19, Theorem 4.1]. Hence, for $d \leq 6$ we have a complete description of all the real rank boundaries, that we recall in the following:

Proposition 3.1 *([9,10,19]). The real rank boundaries for binary forms of degree ≤ 6 are the following hypersurfaces:*

$$\partial_{alg}(\mathcal{R}_{3,2}) = \partial_{alg}(\mathcal{R}_{3,3}) = (\Delta_{(3)})^\vee;$$
$$\partial_{alg}(\mathcal{R}_{4,3}) = \partial_{alg}(\mathcal{R}_{4,4}) = (\Delta_{(4)})^\vee;$$
$$\partial_{alg}(\mathcal{R}_{5,3}) = (\Delta_{(3,2)})^\vee,$$
$$\partial_{alg}(\mathcal{R}_{5,4}) = (\Delta_{(3,2)})^\vee \cup (\Delta_{(5)})^\vee,$$
$$\partial_{alg}(\mathcal{R}_{5,5}) = (\Delta_{(5)})^\vee;$$
$$\partial_{alg}(\mathcal{R}_{6,4}) = (\Delta_{(3,3)})^\vee \cup (\Delta_{(4,2)})^\vee,$$
$$\partial_{alg}(\mathcal{R}_{6,5}) = (\Delta_{(3,3)})^\vee \cup (\Delta_{(4,2)})^\vee \cup (\Delta_{(6)})^\vee,$$
$$\partial_{alg}(\mathcal{R}_{6,6}) = (\Delta_{(6)})^\vee.$$

Remark 3.2. The hypersurfaces $(\Delta_{(3)})^\vee$, $(\Delta_{(4)})^\vee$, $(\Delta_{(5)})^\vee$, $(\Delta_{(6)})^\vee$ coincide with the discriminant hypersurfaces for binary forms of degrees $3, 4, 5, 6$ and have degrees $4, 6, 8, 10$, respectively. For the other components, we have

- $(\Delta_{(3,2)})^\vee = \mathrm{Join}(\Delta_{(4,1)}, \Delta_{(5)})$ is a hypersurface of degree 12 (this is the *apple invariant* I_{12} considered in [9]);
- $(\Delta_{(3,3)})^\vee = \mathrm{Join}(\Delta_{(5,1)}, \Delta_{(5,1)})$ is a hypersurface of degree 12;
- $(\Delta_{(4,2)})^\vee = \mathrm{Join}(\Delta_{(4,1,1)}, \Delta_{(6)})$ is a hypersurface of degree 18.

3.2. Apolarity

We recall here classical techniques, going back to Sylvester. Even if the results of this section are more general, we present them in the case of real numbers. Let $R = \mathbb{R}[x, y]$ be the polynomial ring of real binary forms and let $D = \mathbb{R}[\partial_x, \partial_y] = \mathbb{R}[u, v]$ be the corresponding dual ring. Given $l = ax + by \in R_1$, the *apolar operator* is $l^\perp = -b\partial_x + a\partial_y \in D_1$. Given a form f in R_d, the *apolar ideal* $f^\perp \subset D$ is given by all the operators which annihilates f, that is: $f^\perp = \{g(\partial_x, \partial_y) \in D : g \perp f\}$. A basic tool is the following:

Lemma 3.3 *(Apolarity lemma). Assume $f \in R_d$ and let $l_i \in R_1$ be distinct linear forms for $1 \le i \le r$. There are coefficients $c_i \in \mathbb{R}$ such that $f = \sum_{i=1}^r c_i(l_i)^d$ if and only if the operator $l_1^\perp \circ \cdots \circ l_r^\perp$ is in the apolar ideal $f^\perp$.*

We will say that a form of degree d is *real-rooted* if it admits d distinct real roots. From Lemma 3.3, it follows that a form f has rank less than or equal to r if and only if $(f^\perp)_r = f^\perp \cap D_r$ contains a real-rooted form. So the rank of f is the smallest degree r such that $(f^\perp)_r$ contains a real-rooted form. The following result is an elementary consequence of Lemma 3.3.

Corollary 3.4. *Let f be a real binary form, and let r be an integer. Then $\mathrm{rk}(F) < r$ if and only if $(f^\perp)_r \subset D_r$ contains a special line whose generic member is a real-rooted form. Here, we say that a line $\langle g, g' \rangle \subset D_r$ is* special *if $\gcd(g, g')$ is a form of degree $r - 1$.*

The space of operators of degree r contained in $f^\perp$ is the kernel of the linear map $A_f : D_r \to R_{d-r}$. The *catalecticant (or Hankel) matrix* of f is the matrix $A_f^{d,r}$ of size $(d-r+1) \times (r+1)$ that represents A_f with respect to the standard basis. We denote by $A^{d,r}$ the generic catalecticant matrix of size $(d-r+1) \times (r+1)$.

The following result is well-known (see e.g. [14]):

Proposition 3.5. *Assume that $f \in R_d$ has rank greater than or equal to 2. Then its apolar ideal $f^\perp$ is generated by two real forms g, g' such that* $\deg g + \deg g' = d + 2$ *and* $\gcd(g, g') = 1$. *Conversely, any two such forms generate an ideal $f^\perp$ for some $f \in R$ with degree* $\deg g + \deg g' - 2$.

We say that $f \in R_d$ is *generated in generic degrees* if $(\deg g, \deg g') = (\lceil \frac{d+1}{2} \rceil, \lfloor \frac{d+3}{2} \rfloor)$. The forms that are not generated in generic degrees form a subvariety of R_d. More precisely, when the degree $d = 2k$ is even, it is the hypersurface defined by the determinant of the intermediate $(k+1) \times (k+1)$ catalecticant matrix $A^{d,k}$; when the degree $d = 2k+1$ is odd, it is the subvariety of codimension 2 defined by the maximal minors of the intermediate $(k+1) \times (k+2)$ catalecticant matrix $A^{d,k+1}$.

If f is a binary form of degree d, with $\frac{d}{2} \le r \le d$ and having catalecticant matrix $A_f^{d,r}$ of maximal rank, then $\dim(f^\perp)_r = 2r - d$. Thus, we can consider the *apolar map*

$$\Psi_{d,r} : \mathbb{P}^d \dashrightarrow \mathbb{G}(d-r, r) \simeq \mathbb{G}(2r-d-1, r)$$

which associates to a general binary form f of degree d the projective $(2r-d-1)$-dimensional subspace $\Pi_f = \mathbb{P}((f^\perp)_r) \subset \mathbb{P}(D_r)$ obtained from the degree r component of the apolar ideal. In coordinates the map $\Psi_{d,r}$ is defined by the maximal minors of the matrix $A^{d,r}$. We denote by $Z_{d,r} = \overline{\Psi_{d,r}(\mathbb{P}^d)} \subset \mathbb{G}(2r-d-1, r)$ the closure of the image of $\Psi_{d,r}$.

4. Real rank boundaries of degree 7 binary forms

In this section, we prove the following:

Theorem 4.1. *The real rank boundaries for degree* 7 *binary real forms are the following hypersurfaces:*

$$\begin{aligned}
\partial_{alg}(\mathcal{R}_{7,4}) &= (\Delta_{(3,2,2)})^\vee; \\
\partial_{alg}(\mathcal{R}_{7,5}) &= (\Delta_{(3,2,2)})^\vee \cup (\Delta_{(4,3)})^\vee \cup (\Delta_{(5,2)})^\vee; \\
\partial_{alg}(\mathcal{R}_{7,6}) &= (\Delta_{(4,3)})^\vee \cup (\Delta_{(5,2)})^\vee \cup (\Delta_{(7)})^\vee; \\
\partial_{alg}(\mathcal{R}_{7,7}) &= (\Delta_{(7)})^\vee.
\end{aligned}$$

Remark 4.2. From Proposition 2.3 and formula (2.2) we obtain:

- $(\Delta_{(3,2,2)})^\vee = \mathrm{Join}(\Delta_{(6,1)}, \Delta_{(7)}, \Delta_{(7)})$ is a hypersurface of degree 24;
- $(\Delta_{(4,3)})^\vee = \mathrm{Join}(\Delta_{(5,1,1)}, \Delta_{(6,1)})$ is a hypersurface of degree 36;
- $(\Delta_{(5,2)})^\vee = \mathrm{Join}(\Delta_{(4,1,1,1)}, \Delta_{(7)})$ is a hypersurface of degree 24;
- $(\Delta_{(7)})^\vee = \Delta_{(2,1,1,1,1,1)}$ is a hypersurface of degree 12.

Proof. We divide the proof in several steps.

The boundary $\partial_{alg}(\mathcal{R}_{7,7})$ between ranks 7 and ≤ 6.

From [9] and [10], it is known that the real rank boundary $\partial_{\mathrm{alg}}(\mathcal{R}_{7,7})$ is the discriminant hypersurface $\Delta_{(2,1^5)}$ in $\mathbb{P}^7$. Note that

$$\Delta_{(2,1^5)} = (\Delta_{(7)})^\vee = \overline{\Psi_{7,6}^{-1}(CH_0(\Delta_{(6)}))}$$

where $\Psi_{7,6} : \mathbb{P}^7 \dashrightarrow Z_{7,6} \subset \mathbb{G}(4,6) \subset \mathbb{P}^{20}$.

The boundary $\partial_{alg}(\mathcal{R}_{7,4})$ between ranks 4 and ≥ 5.

For the reader's convenience, we sketch briefly the proof given in [19] of the fact that $\partial_{\mathrm{alg}}(\mathcal{R}_{7,4}) = (\Delta_{(3,2,2)})^\vee$. Consider a binary form f of degree 7 with apolar ideal $f^\perp = (g_4, g_5)$, where $\deg(g_i) = i$. By Lemma 3.3, we have that $f \in \mathcal{R}_{7,4}$ if and only if g_4 is real-rooted. When f moves toward $\mathcal{R}_{7,5} \cup \mathcal{R}_{7,6} \cup \mathcal{R}_{7,7}$ and passes through the boundary $\partial_{\mathrm{alg}}(\mathcal{R}_{7,4})$, then (at least) two roots of g_4 must collapse and become a double root. Hence at the transition point the generator g_4 belongs to the discriminant locus $\Delta_{(2,1,1)} \subset \mathbb{P}(D_4)$, and by Proposition 2.2, we get $\partial_{\mathrm{alg}}(\mathcal{R}_{7,4}) \subseteq (\Delta_{(3,2,2)})^\vee$. Now since $\partial_{\mathrm{alg}}(\mathcal{R}_{7,4}) \neq \emptyset$ because 4 and 5 are typical ranks, and $\Delta_{(3,2,2)})^\vee$ is irreducible, it follows that $\partial_{\mathrm{alg}}(\mathcal{R}_{7,4}) = (\Delta_{(3,2,2)})^\vee$.

The boundary $\partial_{alg}(\mathcal{R}_{7,5})$ between ranks 5 and $\neq 5$.

We describe now the boundary between $\mathcal{R}_{7,5}$ and $\mathcal{R}_{7,6} \cup \mathcal{R}_{7,7}$. Let f_ε be a continuous family of forms crossing the boundary $\partial_{\mathrm{alg}}(\mathcal{R}_{7,5})$ at the point $f_0 = f$, going from $\mathcal{R}_{7,5}$ to $\mathcal{R}_{7,6} \cup \mathcal{R}_{7,7}$. Namely, we assume that $f_{-\varepsilon} \in \mathcal{R}_{7,5}$ and $f_\varepsilon \in \mathcal{R}_{7,6} \cup \mathcal{R}_{7,7}$ for any small ε with $\varepsilon > 0$. We can assume that for any ε the form f_ε is generated in generic degree, since the locus of non generated in generic degree forms has codimension 2. In particular, we can assume $f^\perp = (g_4, g_5)$, where $\deg(g_4) = 4$ and $\deg(g_5) = 5$.

Let $\mathbb{G}(2,5)$ be the Grassmannian of planes in $\mathbb{P}(D_5)$, and consider the apolar map

$$\Psi_{7,5} : \mathbb{P}^7 \dashrightarrow Z_{7,5} \subset \mathbb{G}(2,5) \subset \mathbb{P}^{19}, \tag{4.1}$$

which is a cubic birational map onto a subvariety $Z_{7,5} \subset \mathbb{P}^{19}$ of degree 84 and cut out by 42 quadric hypersurfaces. The map $\Psi_{7,5}$ sends the family f_ε into a continuous family of apolar planes Π_ε; in particular, $\Psi_{7,5}(f) = \Pi_0 = \langle ug_4, vg_4, g_5 \rangle$ is the apolar plane of f. From Lemma 3.3, we obtain that the plane Π_ε, with $\varepsilon < 0$, contains a real-rooted form $h_\varepsilon = l_1(\varepsilon)l_2(\varepsilon)l_3(\varepsilon)l_4(\varepsilon)l_5(\varepsilon)$ (where $l_i(\varepsilon) \in D_1$), while Π_ε, with $\varepsilon > 0$, does not contain any real-rooted form. The set of real-rooted forms is a full-dimensional connected semi-algebraic subset of $\mathbb{P}^5$, and the Zariski closure of its topological boundary is the discriminant hypersurface $\Delta = \Delta_{(2,1,1,1)}$. Thus the limit $h_0 = \lim_{\varepsilon \to 0^-} h_\varepsilon = l_1^2 l_2 l_3 l_4$ must belong to Δ. We now analyze, taking into account also Example 2.1, the possible positions of Π_0 with respect to Δ:

(1) The point h_0 is smooth and the tangent space $T_{h_0}(\Delta) = \langle l_1 u^i v^j : i + j = 4\rangle$ contains Π_0. This implies that $\Pi_0 \in CH_2(\Delta)$.
(2) The point h_0 is smooth in a component of $\Delta_{(3,1,1)} \cup \Delta_{(2,2,1)}$. We have the following subcases:
   (a) $h_0 = l_1^3 l_2 l_3 \in \Delta_{(3,1,1)}$ and $T_{h_0}(\Delta_{(3,1,1)}) = \langle l_1^2 u^i v^j : i + j = 3\rangle$ intersects Π_0 in a line L through h_0. This implies that $\Pi_0 \in CH_1(\Delta_{(3,1,1)})$.
   (b) $h_0 = l_1^2 l_2^2 l_3 \in \Delta_{(2,2,1)}$ and $T_{h_0}(\Delta_{(2,2,1)}) = \langle l_1 l_2 u^i v^j : i + j = 3\rangle$ intersects Π_0 in a line L through h_0. This implies that $\Pi_0 \in CH_1(\Delta_{(2,2,1)})$.
(3) The point h_0 belongs to a component of $\Delta_{(4,1)} \cup \Delta_{(3,2)}$, hence $\Pi_0 \in CH_0(\Delta_{(4,1)}) \cup CH_0(\Delta_{(3,2)})$.

Case (1). Clearly this case cannot occur. Indeed g_4 and g_5 would have l_1 as common divisor, and this is against our assumptions.

Case (2). We show that case (2a) cannot occur. With the same argument, one sees that neither case (2b) occurs.

If $h_0 \notin \langle ug_4, vg_4\rangle$, then we can take as degree 5 generator of the apolar ideal $g_5 = h_0$. Now, every point of L is a form divisible by l_1^2 and we have that $L \cap \langle ug_4, vg_4\rangle \neq \emptyset$. This implies that l_1 is a common divisor of g_4 and of g_5, which is impossible. It follows that $h_0 \in \langle ug_4, vg_4\rangle$ and in particular l_1^2 divides g_4. More precisely, this implies that g_4 is of the form $l_1^2 l_2 l_3$, or $l_1^3 l_2$, or $l_1^3 l_3$. In any cases it is obvious that f is limit of generic forms of degree 4. This implies that f is a singular point of the hypersurface $\partial_{\mathrm{alg}}(\mathcal{R}_{7,5})$. Hence f does not vary in a codimension 1 locus of $\mathbb{P}^7$, and $\overline{\Psi_{7,5}^{-1}(CH_1(\Delta_{3,1,1}))}$ cannot be a component of the boundary $\partial_{\mathrm{alg}}(\mathcal{R}_{7,5})$.

Case (3). We show now that both components corresponding to this case are in the boundary. Indeed it is enough to find an example of a binary form which lies exclusively on each component and is limit of a sequence of general forms of rank 5 and a sequence of general forms of rank 6. This is done in Example 4.3 below. Recall that by Proposition 2.2, we have $\overline{\Psi_{7,5}^{-1}(CH_0(\Delta_{(4,1)}))} = (\Delta_{(5,2)})^\vee$ and $\overline{\Psi_{7,5}^{-1}(CH_0(\Delta_{(3,2)}))} = (\Delta_{(4,3)})^\vee$. Hence we have proved that $\overline{\partial_{\mathrm{alg}}(\mathcal{R}_{7,5}) \setminus \partial_{\mathrm{alg}}(\mathcal{R}_{7,4})} = (\Delta_{(4,3)})^\vee \cup (\Delta_{(5,2)})^\vee$.

The boundary $\partial_{alg}(\mathcal{R}_{7,6})$ between ranks 6 and $\neq 6$.

At this point we know that

$$\begin{aligned}\overline{\partial_{\mathrm{alg}}(\mathcal{R}_{7,6}) \setminus \partial_{\mathrm{alg}}(\mathcal{R}_{7,4})} &= \left(\overline{\partial_{\mathrm{alg}}(\mathcal{R}_{7,5}) \setminus \partial_{\mathrm{alg}}(\mathcal{R}_{7,4})}\right) \cup \partial_{\mathrm{alg}}(\mathcal{R}_{7,7}) \\ &= (\Delta_{(4,3)})^\vee \cup (\Delta_{(5,2)})^\vee \cup (\Delta_{(7)})^\vee.\end{aligned}$$

So, we only need to show that the boundary between $\mathcal{R}_{7,4}$ and $\mathcal{R}_{7,6}$ is not of codimension 1 in $\mathbb{P}^7$. Let f_ε be a continuous family of forms such that $f_{-\varepsilon} \in \mathcal{R}_{7,4}$ and $f_\varepsilon \in \mathcal{R}_{7,6}$ for any small ε with $\varepsilon > 0$. The corresponding apolar plane $\Pi_\varepsilon = \Psi_{7,5}(f_\varepsilon)$ does not contain any real-rooted form for any $\varepsilon > 0$. On the other hand, from Corollary 3.4, we deduce that Π_ε must contain a special line L_ε which is generically contained in the locus of real-rooted forms for any $\varepsilon < 0$. Now the limit $L_0 = \lim_{\varepsilon \to 0^-} L_\varepsilon$ is a special line contained in the intersection of the plane Π_0 and of the discriminant $\Delta = \Delta_{(2,1,1,1)}$. By the previous analysis we deduce that the line $L_0 = \langle ug_4, vg_4 \rangle$ must be contained in $\Delta_{(4,1)} \cup \Delta_{(3,2)}$. This implies that $g_4 \in \Delta_{(4)}$, and this forces f_0 to move in some locus of codimension ≥ 2 in $\mathbb{P}^7$, which cannot be a component of the boundary. $\square$

Example 4.3. Given

$$g_4 = (u^2 + v^2)(u^2 - v^2), \quad g_5(\varepsilon) = (u^2 + \varepsilon v^2)uv(\varepsilon u + v),$$

the degree 7 form f_ε associated to the apolar ideal $(g_4, g_5(\varepsilon))$ is:

$$\begin{aligned} &\varepsilon^2 x^7 + 7(\varepsilon^2 + \varepsilon + 1)x^6 y - 21\varepsilon(\varepsilon^2 + \varepsilon + 1)x^5 y^2 - 35\varepsilon x^4 y^3 + 35\varepsilon^2 x^3 y^4 \\ &\quad + 21(\varepsilon^2 + \varepsilon + 1)x^2 y^5 - 7\varepsilon(\varepsilon^2 + \varepsilon + 1)xy^6 - \varepsilon y^7. \end{aligned}$$

We have $\mathrm{rk}(f_\varepsilon) = 6$ for any small $\varepsilon \geq 0$ and $\mathrm{rk}(f_{-\varepsilon}) = 5$ for any small $\varepsilon > 0$. Moreover $f_0 = x^6 y + 3x^2 y^5$ belongs to $(\Delta_{(4,3)})^\vee$ and it does not belong to $(\Delta_{(5,2)})^\vee \cup (\Delta_{(3,2,2)})^\vee$.

On the other hand, taking

$$g_4 = (u^2 + v^2)(2u^2 - v^2), \quad g_5(\varepsilon) = (\varepsilon u^2 + v^2)uv(\varepsilon u + v),$$

we consider the associated form f_ε:

$$\begin{aligned} &\varepsilon(\varepsilon^3 + \varepsilon^2 - \varepsilon - 3)x^7 + 14(\varepsilon^2 - \varepsilon - 1)x^6 y - 42\varepsilon(\varepsilon^2 - \varepsilon - 1)x^5 y^2 \\ &\quad - 70(\varepsilon^3 - 2)x^4 y^3 + 70\varepsilon(\varepsilon^3 - 2)x^3 y^4 - 42\varepsilon(\varepsilon^2 - 2\varepsilon + 2)x^2 y^5 \\ &\quad + 14\varepsilon^2(\varepsilon^2 - 2\varepsilon + 2)xy^6 - 2(3\varepsilon^3 - 2\varepsilon^2 + 2\varepsilon - 4)y^7. \end{aligned}$$

Again we have $\mathrm{rk}(f_\varepsilon) = 6$ for any small $\varepsilon \geq 0$ and $\mathrm{rk}(f_{-\varepsilon}) = 5$ for any small $\varepsilon > 0$. Moreover $f_0 = 7x^6 y - 70x^4 y^3 - 4y^7$ belongs to $(\Delta_{(5,2)})^\vee$ and it does not belong to $(\Delta_{(4,3)})^\vee \cup (\Delta_{(3,2,2)})^\vee$. For computational details, see Section 6.

5. Real rank boundaries of degree 8 binary forms

In this section, we prove the following:

Theorem 5.1. *The real rank boundaries for degree* 8 *binary real forms are the following hypersurfaces:*

$$\partial_{alg}(\mathcal{R}_{8,5}) = (\Delta_{(3,3,2)})^\vee \cup (\Delta_{(4,2,2)})^\vee;$$

$$\partial_{alg}(\mathcal{R}_{8,6}) = (\Delta_{(3,3,2)})^\vee \cup (\Delta_{(4,2,2)})^\vee \cup (\Delta_{(4,4)})^\vee \cup (\Delta_{(5,3)})^\vee \cup (\Delta_{(6,2)})^\vee;$$

$$\partial_{alg}(\mathcal{R}_{8,7}) = (\Delta_{(4,4)})^\vee \cup (\Delta_{(5,3)})^\vee \cup (\Delta_{(6,2)})^\vee \cup (\Delta_{(8)})^\vee;$$

$$\partial_{alg}(\mathcal{R}_{8,8}) = (\Delta_{(8)})^\vee.$$

Remark 5.2. From Proposition 2.3 and formula (2.2) we obtain:

- $(\Delta_{(3,3,2)})^\vee = \mathrm{Join}(\Delta_{(7,1)}, \Delta_{(7,1)}, \Delta_{(8)})$ is a hypersurface of degree 48;
- $(\Delta_{(4,2,2)})^\vee = \mathrm{Join}(\Delta_{(6,1,1)}, \Delta_{(8)}, \Delta_{(8)})$ is a hypersurface of degree 36;
- $(\Delta_{(4,4)})^\vee = \mathrm{Join}(\Delta_{(6,1,1)}, \Delta_{(6,1,1)})$ is a hypersurface of degree 27;
- $(\Delta_{(5,3)})^\vee = \mathrm{Join}(\Delta_{(5,1,1,1)}, \Delta_{(7,1)})$ is a hypersurface of degree 48;
- $(\Delta_{(6,2)})^\vee = \mathrm{Join}(\Delta_{(4,1,1,1,1)}, \Delta_{(8)})$ is a hypersurface of degree 30;
- $(\Delta_{(8)})^\vee = \Delta_{(2,1,1,1,1,1,1)}$ is a hypersurface of degree 14.

Proof. From [9] and [10], we have $\partial_{\mathrm{alg}}(\mathcal{R}_{8,8}) = (\Delta_{(8)})^\vee$. On the other hand, from [19], we have $\partial_{\mathrm{alg}}(\mathcal{R}_{8,5}) = (\Delta_{(3,3,2)})^\vee \cup (\Delta_{(4,2,2)})^\vee$.

We study now the boundary between ranks 6 and ≥ 6. Let f_ε be a continuous family of forms crossing the boundary $\partial_{\mathrm{alg}}(\mathcal{R}_{8,6})$ at the point $f_0 = f$, going from $\mathcal{R}_{8,6}$ to $\mathcal{R}_{8,7} \cup \mathcal{R}_{8,8}$. Namely, we assume that $f_{-\varepsilon} \in \mathcal{R}_{8,6}$ and $f_\varepsilon \in \mathcal{R}_{8,7} \cup \mathcal{R}_{8,8}$ for any small ε with $\varepsilon > 0$. We can also assume that for any $\varepsilon \neq 0$ the form f_ε is generated in generic degree, *i.e.* the apolar ideal $f_\varepsilon^\perp$ is generated by two quintic forms g_ε and g'_ε. Moreover, since f moves in a codimension 1 locus, we may assume that $f^\perp$ is generated by two forms g_0 and g'_0 either with $\deg(g_0) = \deg(g'_0) = 5$, or with $\deg(g_0) = 4$, $\deg(g'_0) = 6$, and moreover $g_0 \notin \Delta_{(2,1,1)}$. Note that in the former case we have $\mathbb{P}((f^\perp)_6) = \langle u g_0, v g_0, u g'_0, v g'_0\rangle$, while in the latter case we have $\mathbb{P}((f^\perp)_6) = \langle u^2 g_0, uv g_0, v^2 g_0, g'_0\rangle$.

Consider the apolar map

$$\Psi_{8,6} : \mathbb{P}^8 \dashrightarrow Z_{8,6} \subset \mathbb{G}(3,6) \subset \mathbb{P}^{34}, \tag{5.1}$$

which is a cubic birational map onto a subvariety $Z_{8,6} \subset \mathbb{P}^{34}$ of degree 686 and cut out by 186 quadric hypersurfaces. The map $\Psi_{8,6}$ sends the family f_ε into the continuous family of the 3-dimensional linear spaces $\Pi_\varepsilon = \mathbb{P}((f_\varepsilon^\perp)_6)$. From Lemma 3.3, we obtain that Π_ε, with $\varepsilon < 0$, contains a real-rooted form $h_\varepsilon = \prod_{i=1}^6 l_i(\varepsilon)$ (where $l_i \in D_1$), while Π_ε, with $\varepsilon > 0$, does not contain any real-rooted form. Thus the limit $h_0 = \lim_{\varepsilon \to 0^-} h_\varepsilon$ must belong to the discriminant hypersurface $\Delta = \Delta_{(2,1,1,1,1)}$. We now analyze, recalling Example 2.1, the possible positions of Π_0 with respect to Δ:

(1) The point h_0 is smooth and the tangent space $T_{h_0}(\Delta) = \langle l_1 u^i v^j : i + j = 5\rangle$ contains Π_0. This implies that $\Pi_0 \in CH_3(\Delta)$.

(2) The point h_0 is smooth in a component of $\Delta_{(3,1,1,1)} \cup \Delta_{(2,2,1,1)}$. We have the following subcases:
   (a) $h_0 = l_1^3 l_2 l_3 l_4 \in \Delta_{(3,1,1,1)}$ and $T_{h_0}(\Delta_{(3,1,1,1)}) = \langle l_1^2 u^i v^j : i+j=4\rangle$ intersects Π_0 in a plane P through h_0. This implies that $\Pi_0 \in CH_2(\Delta_{(3,1,1,1)})$.
   (b) $h_0 = l_1^2 l_2^2 l_3 l_4 \in \Delta_{(2,2,1,1)}$ and $T_{h_0}(\Delta_{(2,2,1,1)}) = \langle l_1 l_2 u^i v^j : i+j=4\rangle$ intersects Π_0 in a plane P through h_0. This implies that $\Pi_0 \in CH_2(\Delta_{(2,2,1,1)})$.

(3) The point h_0 is smooth in a component of $\Delta_{(3,2,1)} \cup \Delta_{(4,1,1)} \cup \Delta_{(2,2,2)}$. We have the following subcases:
   (a) $h_0 = l_1^3 l_2^2 l_3 \in \Delta_{(3,2,1)}$ and $T_{h_0}(\Delta_{(3,2,1)}) = \langle l_1^2 l_2 u^i v^j : i+j=3\rangle$ intersects Π_0 in a line L through h_0. This implies that $\Pi_0 \in CH_1(\Delta_{(3,2,1)})$.
   (b) $h_0 = l_1^4 l_2 l_3 \in \Delta_{(4,1,1)}$ and $T_{h_0}(\Delta_{(4,1,1)}) = \langle l_1^3 u^i v^j : i+j=3\rangle$ intersects Π_0 in a line L through h_0. This implies that $\Pi_0 \in CH_1(\Delta_{(4,1,1)})$.
   (c) $h_0 = l_1^2 l_2^2 l_3^2 \in \Delta_{(2,2,2)}$ and $T_{h_0}(\Delta_{(2,2,2)}) = \langle l_1 l_2 l_3 u^i v^j : i+j=3\rangle$ intersects Π_0 in a line L through h_0. This implies that $\Pi_0 \in CH_1(\Delta_{(2,2,2)})$.

(4) The point h_0 belongs to a component of $\Delta_{(3,3)} \cup \Delta_{(4,2)} \cup \Delta_{(5,1)}$, hence $\Pi_0 \in CH_0(\Delta_{(3,3)}) \cup CH_0(\Delta_{(4,2)}) \cup CH_0(\Delta_{(5,1)})$.

In the following, we show that only the last case occurs.

Case (1). This case cannot occur. Indeed from the fact that $\Pi_0 \subset T_{h_0}(\Delta)$, we would conclude that l_1 is a common divisor of g_0 and g_0'.

Case (2). Consider first the case when f is not generated in generic degree, so that we have $\Pi_0 = \langle u^2 g_0, uvg_0, v^2 g_0, g_0'\rangle$. If $h_0 \notin \langle u^2 g_0, uvg_0, v^2 g_0\rangle$, then we can take $g_0' = h_0$ and, since there are at least two points in $P \cap \langle u^2 g_0, uvg_0, v^2 g_0\rangle$, we deduce that g_0 and g_0' have a common divisor, which is a contradiction. Assume therefore that $h_0 \in \langle u^2 g_0, uvg_0, v^2 g_0\rangle$. Since $g_0 \notin \Delta_{(2,1,1)}$, the only possibility is that $g_0 = l_1 l_2 l_3 l_4$. This implies that $\mathrm{rk}(f) = 4$, and it is easy to see that f is limit of a general sequence of form of rank 5. This would imply that f is not a general point of the boundary between forms of rank 6 and rank ≥ 7. Hence we can assume $\deg(g_0) = \deg(g_0') = 5$, and consider the following subcases.

Case (2a). The plane P meets the special lines $\langle ug_0, vg_0\rangle$ and $\langle ug_0', vg_0'\rangle$ at points p_0 and p_0' respectively. Therefore, p_0 and p_0' are forms divisible by l_1^2, and then l_1 divides both g_0 and g_0', which is a contradiction.

Case (2b). Let us consider the surface

$$\mathcal{Q} = \bigcup_{\substack{m \in D_1, \\ g \in \langle g_0, g_0'\rangle}} mg \subset \Pi_0 \simeq \mathbb{P}^3$$

swept out by all the special apolar lines of f. Using that f is generated in generic degrees, one sees that $\mathcal{Q}$ is a smooth quadric surface, which we will call the *apolar quadric* of f. The intersection $P \cap \mathcal{Q}$ is a (possible reducible) plane conic, which in particular contains three noncollinear points: $p_0 = m\,g$, $p_0' = m'\,g'$ and $p_0'' = m''\,g''$. We can assume $g = g_0$, $g' = g_0'$, and since every point of P is a form divisible by $l_1\,l_2$, we conclude that g_0 and g_0' have a common factor, which is a contradiction.

Case (3). As above we consider first the case when $\deg(g_0) = 4$, $\deg(g_0') = 6$ and $\Pi_0 = \langle u^2 g_0, uvg_0, v^2 g_0, g_0'\rangle$. If $h_0 \notin \langle u^2 g_0, uvg_0, v^2 g_0\rangle$, then we can take $g_0' = h_0$. Moreover, since $L \cap \langle u^2 g_0, uvg_0, v^2 g_0\rangle \neq \emptyset$, we deduce that g_0 and g_0' have a common divisor, which is a contradiction. Thus we have that $h_0 \in \langle u^2 g_0, uvg_0, v^2 g_0\rangle$. This implies that $g_0 \in \Delta_{(2,1,1)}$, which contradicts our assumption. Hence we can assume $\deg(g_0) = \deg(g_0') = 5$, and consider the following subcases.

Case (3a). Let $\mathcal{Q}$ be again the apolar quadric of f. We have two cases: either the line L meets $\mathcal{Q}$ in two distinct points mg and $m'g'$, or there exists a point $mg \in L \cap \mathcal{Q}$ such that L is contained in the tangent plane $T_{mg}\mathcal{Q}$.

In the former case, since $l_1^2 l_2$ divides mg and $m'g'$, we deduce that l_1 divides g and g'. This is a contradiction, unless we have $g = g'$ and hence L is the special line $\langle gu, gv\rangle$. Now, since $h_0 \in L$, we obtain that $g \in \Delta_{(2,2,1)} \cup \Delta_{(3,1,1)}$, and thus $f \in (\Delta_{(3,3,2)})^\vee \cup (\Delta_{(4,2,2)})^\vee = \partial_{\mathrm{alg}}(\mathcal{R}_{8,5})$ is also limit of generic forms of rank 5. This implies that f belongs to the singular locus of the hypersurface $\partial_{\mathrm{alg}}(\mathcal{R}_{7,6})$ and then $\overline{\Psi_{8,6}^{-1}(CH_1(\Delta_{(3,2,1)}))}$ cannot be a component of the boundary $\partial_{\mathrm{alg}}(\mathcal{R}_{8,6})$.

In the latter case, we may assume $g = g_0$ and $T_{mg}\mathcal{Q} = \langle mg_0, mg_0', m'g_0\rangle$, for some $m' \in D_1$. We have $h_0 = l_1^3 l_2^2 l_3 = \alpha m g_0 + \beta m g_0' + \gamma m' g_0$, for some scalars α, β, γ, and we know that $l_1^2 l_2$ divides mg_0. Since $\gcd(g_0, g_0') = 1$, this implies $\beta = 0$, and then $l_1^3 l_2^2 l_3 = (\alpha m + \gamma m')g_0$. As above, from this it follows that $g_0 \in \Delta_{(2,2,1)} \cup \Delta_{(3,1,1)}$ and thus f does not vary in a codimension 1 locus of $\mathbb{P}^8$,

Cases (3b) and (3c). Arguing as above, we deduce that f must belong to $(\Delta_{(4,2,2)})^\vee$ and $(\Delta_{(3,3,2)})^\vee$, respectively, and furthermore we must have that f is a singular point of the hypersurface $\partial_{\mathrm{alg}}(\mathcal{R}_{8,6})$. This implies that $\overline{\Psi_{8,6}^{-1}(CH_1(\Delta_{(4,1,1)}))}$ and $\overline{\Psi_{8,6}^{-1}(CH_1(\Delta_{(2,2,2)}))}$ are not components of the boundary $\partial_{\mathrm{alg}}(\mathcal{R}_{8,6})$.

Case (4). We show in Example 5.3 below that each of the three components corresponding to this case are in the boundary. This proves that $\overline{\partial_{\mathrm{alg}}(\mathcal{R}_{8,6}) \setminus \partial_{\mathrm{alg}}(\mathcal{R}_{8,5})} = (\Delta_{(4,4)})^\vee \cup (\Delta_{(5,3)})^\vee \cup (\Delta_{(6,2)})^\vee$.

Finally we need to prove that there are no components of the boundary between $\mathcal{R}_5$ and $\mathcal{R}_7$. This can be done with the same argument used at the end of the proof of Theorem 4.1. □

Example 5.3. Given

$$g_0 = u^4 v - u^2 v^3 - 2v^5, \quad g_0' = -u^5 + 2u^3 v^2 + 2uv^4,$$

we have $ug_0 + vg_0' = u^3 v^3$ and the degree 8 form f_0 associated to the apolar ideal (g_0, g_0') is

$$f_0 = 8x^8 + 112x^6y^2 + 56x^2y^6 - y^8. \tag{5.2}$$

With the help of a computer, one can easily check (see Section 6) that $\mathrm{rk}(f_0) = 7$ and $f_0 \in (\Delta_{(4,4)})^\vee \setminus \big((\Delta_{(5,3)})^\vee \cup (\Delta_{(6,2)})^\vee\big)$. Moreover, we can construct near f_0 generic degree 8 forms $f_{\pm\varepsilon}$ having real ranks 6 and 7.

Analogously, given

$$g_0 = u^4v - u^2v^3 - 2v^5, \quad g_0' = -u^5 + u^4v + u^3v^2 + 2uv^4,$$

we have $ug_0 + vg_0' = u^4v^2$ and the associated degree 8 form is

$$f_0 = x^8 + 8x^7y + 28x^3y^5 - 2xy^7. \tag{5.3}$$

One verifies that $\mathrm{rk}(f_0) = 7$ and $f_0 \in (\Delta_{(5,3)})^\vee \setminus \big((\Delta_{(4,4)})^\vee \cup (\Delta_{(6,2)})^\vee\big)$.

Finally, given

$$g_0 = u^5 + u^4v + 3u^3v^2 + 3u^2v^3 + 2uv^4 + 2v^5, \quad g_0' = -3u^3v^2 - 2uv^4,$$

we have $ug_0 + (u+v)g_0' = u^5(u+v)$ and the associated degree 8 form is

$$\begin{aligned} f_0 = {} & 8x^8 - 64x^7y + 224x^6y^2 - 448x^5y^3 - 840x^4y^4 \\ & + 672x^3y^5 + 504x^2y^6 - 144xy^7 - 17y^8. \end{aligned} \tag{5.4}$$

One verifies that $\mathrm{rk}(f_0) = 7$ and $f_0 \in (\Delta_{(6,2)})^\vee \setminus \big((\Delta_{(4,4)})^\vee \cup (\Delta_{(5,3)})^\vee\big)$.

6. Computations

We provide a package for Macaulay2 [12], named `CoincidentRootLoci` and included with the current stable version of Macaulay2, which implements methods useful to check the correctness of Examples 4.3 and 5.3. This package depends on the packages `Cremona` and `Resultants` (see [23] and [24]). In the following, we illustrate briefly some of the methods available. For technical details and examples, we refer to the documentation of the package, which can be shown using the command `viewHelp`.

The method `realrank` computes the real rank of a binary form with rational coefficients. Indeed, Lemma 3.3 reduces the problem of computing the real rank of a binary form to that of establishing whether certain semi-algebraic sets are nonempty. The Tarski formulas defining these semi-algebraic sets can be obtained via the computation of kernels of appropriate catalecticant matrices. The problem of deciding the truth of a Tarski formula can be handled by Qepcad B via a quantifier elimination by partial cylindrical algebraic decomposition (see [5]). The method calls automatically Qepcad B without requiring user intervention (provided it is installed on the system). Below, we compute the real rank of the binary form (5.2) (the run time is about 30 seconds).

```
Macaulay2, version 1.12
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "CoincidentRootLoci";
i2 : R := QQ[x,y];
i3 : F = 8*x^8+112*x^6*y^2+56*x^2*y^6-y^8;
i4 : realrank F
o4 = 7
```

The method `member` tests membership of a binary form in the dual variety of a coincident root locus (or in a coincident root locus). It does not pass through the hard computation of the equations but uses Proposition 2.2. Below, we verify that the binary form (5.2) lies in $(\Delta_{(4,4)})^\vee$ but not in $(\Delta_{(5,3)})^\vee \cup (\Delta_{(6,2)})^\vee$ (the run time is less than one second).

```
i5 : X = dual coincidentRootLocus(4,4)
o5 = CRL(6,1,1) * CRL(6,1,1) (dual of CRL(4,4))
o5 : JoinOfCoincidentRootLoci
i6 : member(F,X)
o6 = true
i7 : member(F,dual coincidentRootLocus(5,3)) or member(F,dual coincidentRootLocus(6,2))
o7 = false
```

The method `apolar` computes the apolar ideal of a binary form, while `recover`, as the name suggests, recovers the binary form from its apolar ideal. Basically, these two methods translate to problems of computing the image or the inverse image of a point via a (bi)rational map, and then the computation is performed using tools of the package `Cremona`. For example, the following calculation involves the birational map (5.1) (the run time is less than one second).

```
i8 : F == recover apolar F
o8 = true
```

For the convenience of the user, the method `realRankBoundary` implements Theorems 4.1 and 5.1. For example, below we get immediately the degree of the first component of $\partial_{\mathrm{alg}}(\mathcal{R}_{8,5})$.

```
i9 : Y = first realRankBoundary(8,5)
o9 = CRL(7,1) * CRL(7,1) * CRL(8) (dual of CRL(3,3,2))
o9 : JoinOfCoincidentRootLoci
i10 : degree Y
o10 = 48
```

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