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Positive solutions for anisotropic singular (p, q)-equations

Nikolaos S. Papageorgiou and Andrea Scapellato

Abstract. We consider a nonlinear elliptic Dirichlet problem driven by the anisotropic (p, q)-Laplacian and with a reaction which is nonparametric and has the combined effects of a singular and of a superlinear terms. Using variational tools together with truncation and comparison techniques, we show that the problem has at least two positive smooth solutions.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following anisotropic singular (p, q)-equation (double phase problem)

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) = u(z)^{-\eta(z)} + f(z, u(z)) & \text{in } \Omega \\ u\Big|_{\partial\Omega} = 0, \ u > 0 \end{cases}, \tag{1.1}$$

Given $r \in C(\overline{\Omega})$ with $1 < \min_{\overline{\Omega}} r$, by $\Delta_{r(z)}$ we denote the $r(z)$-Laplace differential operator defined by

$$\Delta_{r(z)} = \operatorname{div}\left(|Du|^{r(z)-2}Du\right) \text{ for all } u \in W_0^{1,r(z)}(\Omega).$$

In Problem (1.1), we have the sum of two such operators (double phase problem). In the reaction (right-hand side of (1.1)), we have the competing effects of two different terms of different nature. One is the singular term $u^{-\eta(z)}$, and the other term is a Carathéodory perturbation $f(z, x)$ (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z, x)$ is continuous) which exhibits $(p_+ - 1)$-superlinear growth as $x \to +\infty$ (here $p_+ = \max_{\overline{\Omega}} p$). We point out that problem (1.1) is nonparametric. Our aim is to prove the existence and the multiplicity of positive solutions for problem (1.1).

Usually, singular problems are studied with a parameter involved in the reaction. By varying and restricting the parameter, we are able to satisfy the geometry of the minimax theorems of critical point theory and then use them to produce a positive solution. Indicatively, we mention the works of Bai-Motreanu-Zeng [3], Candito-Gasiński-Livrea [5], Gasiński-Papageorgiou [13], Ghergu-Rădulescu [17,18], Giacomoni-Schindler-Takáč [19], Haitao [21], Kyritsi-Papageorgiou [22], Papageorgiou-Rădulescu-Repovš [26–28], Papageorgiou-Repovš-Vetro [31], Papageorgiou-Smyrlis [32], Papageorgiou-Vetro-Vetro [35], Sun-Wu-Long [40]. All the aforementioned works consider parametric isotropic singular semilinear or nonlinear problems. Nonparametric isotropic singular problems were considered by Bai-Gasiński-Papageorgiou [2], Papageorgiou-Rădulescu-Repovš [25] and Papageorgiou-Vetro-Vetro [34]. Papers [2,25] deal with equations driven by the p-Laplacian and in [2] the perturbation $f(z, \cdot)$ is $(p - 1)$-superlinear, while in [25] the perturbation $f(z, \cdot)$ is $(p-1)$-linear and resonant. In [34], the authors consider a $(p, 2)$-equation with

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superlinear perturbation. In contrast, the study of anisotropic singular problems is lagging behind. To the best of our knowledge, there is only the recent work of Byun-Ko [4], who study an equation driven by the $p(z)$-Laplacian and with a reaction of the form $\lambda u^{-\eta(z)} + u^{r(z)}$, where $\lambda > 0$ is a parameter and $r \in C(\overline{\Omega})$, $p(z) < r(z) + 1$ for all $z \in \overline{\Omega}$. We also mention the works of Gasiński-Papageorgiou [12,14], Gasiński-Winkert [15,16], Papageorgiou-Rădulescu-Repovš [29] and Papageorgiou-Vetro [33], which also deal with anisotropic equations with a superlinear reaction, but no singular term.

We mention that partial differential equations with variable exponents arise in several models of electrorheological fluids (see Qian [37], Ruzicka [39]) and in image processing and image restoration (see Chen-Levine-Rao [6]).

Further applications can be found in the book of Rădulescu-Repovš [38].

2. Preliminaries—auxiliary results and hypotheses

Let $C^{0,1}(\overline{\Omega})$ denote the space of Lipschitz continuous functions. If $r \in C^{0,1}(\overline{\Omega})$, we set $r_- = \min_{\overline{\Omega}} r$ and $r_+ = \max_{\overline{\Omega}} r$. We introduce the sets

$$E_1 = \left\{ r \in C^{0,1}(\overline{\Omega}) : 1 \le r_- \right\},$$
$$M(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable} \}.$$

We identify two elements in $M(\Omega)$ if they differ only on a set of zero Lebesgue measure.

Given $r \in E_1$, we define the variable exponent Lebesgue space $L^{r(z)}(\Omega)$ by

$$L^{r(z)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u|^{r(z)} \, \mathrm{d}z < \infty \right\}.$$

We furnish $L^{r(z)}(\Omega)$ with the following norm (known as the Luxemburg norm):

$$\|u\|_{r(z)} = \inf \left[\lambda > 0 : \int_\Omega \left| \frac{u(z)}{\lambda} \right|^{r(z)} \mathrm{d}z < \infty \right].$$

With this norm $L^{r(z)}(\Omega)$ becomes a separable Banach space. If $1 < r_-$, then $L^{r(z)}(\Omega)$ is also uniformly convex, thus reflexive. If $r_1, r_2 \in E_1$ and $r_1(z) \le r_2(z)$ for all $z \in \overline{\Omega}$, then $L^{r_2(z)}(\Omega) \hookrightarrow L^{r_1(z)}(\Omega)$ continuously. Moreover, if $r \in E_1$ with $1 < r_-$, then $L^{r(z)}(\Omega)^* = L^{r'(z)}(\Omega)$ where $r' \in E_1$ and satisfies $\frac{1}{r(z)} + \frac{1}{r'(z)} = 1$ for all $z \in \overline{\Omega}$. If $u \in L^{r(z)}(\Omega)$ and $v \in L^{r'(z)}(\Omega)$, then we have the following Hölder-type inequality

$$\left| \int_\Omega uv \, \mathrm{d}z \right| \le \left(\frac{1}{r_-} + \frac{1}{r'_-} \right) \|u\|_{r(z)} \|v\|_{r'(z)}.$$

The following modular function is important in the study of variable exponent Lebesgue spaces,

$$\rho_r(u) = \int_\Omega |u|^{r(z)} \, \mathrm{d}z \quad \text{for all } u \in L^{r(z)}(\Omega).$$

The next proposition shows that there is a close relation between the modular function $\rho_r(\cdot)$ and the norm $\| \cdot \|_{r(z)}$.

Proposition 2.1. *If $r \in E_1$ and $1 < r_-$, then*

(a) *for $u \ne 0$, $\|u\|_{r(z)} = \lambda \Leftrightarrow \rho_r\left(\frac{u}{\lambda}\right) = 1$;*
(b) *$\|u\|_{r(z)} < 1$ (resp. $= 1, > 1$) $\Leftrightarrow$ $\rho_r(u) < 1$ (resp. $= 1, > 1$);*

(c) $\|u\|_{r(z)} \le 1 \Rightarrow \|u\|_{r(z)}^{r_+} \le \rho_r(u) \le \|u\|_{r(z)}^{r_-}$;
(d) $\|u\|_{r(z)} \ge 1 \Rightarrow \|u\|_{r(z)}^{r_-} \le \rho_r(u) \le \|u\|_{r(z)}^{r_+}$;
(e) $\|u_n\|_{r(z)} \to 0 \Leftrightarrow \rho_r(u_n) \to 0$ *as* $n \to \infty$;
(f) $\|u_n\|_{r(z)} \to +\infty \Leftrightarrow \rho_r(u_n) \to +\infty$ *as* $n \to \infty$.

Using the variable exponent Lebesgue spaces, we can define the corresponding variable exponent Sobolev spaces.

So, let $r \in E_1$ with $1 < r_-$. The anisotropic Sobolev space $W^{1,r(z)}(\Omega)$ is defined by

$$W^{1,r(z)}(\Omega) = \left\{ u \in L^{r(z)}(\Omega) : |Du| \in L^{r(z)}(\Omega) \right\}$$

(here the gradient Du is understood in the weak sense).

We equip $W^{1,r(z)}(\Omega)$ with the following norm:

$$\|u\|_{1,r(z)} = \|u\|_{r(z)} + \big\| |Du| \big\|_{r(z)} \quad \text{for all } u \in W^{1,r(z)}(\Omega).$$

We set $W_0^{1,r(z)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,r(z)}}$ and define

$$r^*(z) = \begin{cases} \frac{Nr(z)}{N-r(z)} & \text{if } r(z) < N \\ +\infty & \text{if } N \le r(z) \end{cases} \quad \text{for all } z \in \overline{\Omega}.$$

We know that:

(a) Both $W^{1,r(z)}(\Omega)$ and $W_0^{1,r(z)}(\Omega)$ are separable and uniformly convex (thus reflexive) Banach spaces.
(b) If $s \in E_1$ with $1 < s_-$ and $s(z) \le r^*(z)$ (resp. $s(z) < r^*(z)$) for all $z \in \overline{\Omega}$, then $W^{1,s(z)}(\Omega) \hookrightarrow L^{r^*(z)}(\Omega)$ continuously (resp. compactly); similarly for the space $W_0^{1,r(z)}(\Omega)$.
(c) The Poincaré inequality holds, namely

$$\|u\|_{r(z)} \le c \big\| |Du| \big\| \quad \text{for some } c > 0, \text{ all } u \in W_0^{1,r(z)}(\Omega).$$

In the sequel, we write

$$\rho_r(Du) = \rho_r(|Du|) \quad \text{and} \quad \|Du\|_{r(z)} = \big\| |Du| \big\|_{r(z)}.$$

We have that

$$W_0^{1,r(z)}(\Omega)^* = W^{-1,r'(z)}(\Omega).$$

A comprehensive analysis of variable exponent Lebesgue and Sobolev spaces can be found in the book of Diening-Harjulehto-Hästo-Ruzicka [7].

Let $A_{r(z)} : W_0^{1,r(z)}(\Omega) \to W^{-1,r'(z)}(\Omega)$ be the nonlinear operator defined by

$$\langle A_{r(z)}(u), h \rangle = \int_\Omega |Du|^{r(z)-2} (Du, Dh)_{\mathbb{R}^N} \, \mathrm{d}z \quad \text{for all } u, h \in W_0^{1,r(z)}(\Omega).$$

The following proposition summarizes the main properties of this operator (see Gasiński-Papageorgiou [12, Proposition 2.5] and Rădulescu-Repovš [38, p. 40]).

Proposition 2.2. *The operator $A_{r(z)}(\cdot)$ is continuous and strictly monotone (hence it is maximal monotone too) and of type $(S)_+$, that is*

$$\text{“}u_n \xrightarrow{w} u \textit{ in } W_0^{1,r(z)}(\Omega) \textit{ and } \limsup_{n\to\infty} \langle A_{r(z)}(u_n), u_n - u \rangle \le 0$$

$$\textit{imply that}$$

$$u_n \to u \textit{ in } W_0^{1,r(z)}(\Omega) \textit{ as } n \to \infty\text{''}.$$

In addition to the variable exponent spaces, we will also use the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. This is an ordered Banach space with positive cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} < 0 \right\}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

If $x \in \mathbb{R}$, then we set $x^{\pm} = \max\{\pm x, 0\}$. For $u \in W_0^{1,r(z)}(\Omega)$, we define $u^{\pm}(z) = u(z)^{\pm}$ for all $z \in \Omega$. We have

$$u^{\pm} \in W_0^{1,r(z)}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

If $u, v \in W^{1,r(z)}(\Omega)$ with $u \leq v$, then we define

$$\begin{aligned} &[u,v] = \left\{ h \in W_0^{1,r(z)}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega \right\}, \\ &\operatorname{int}_{C_0^1(\overline{\Omega})}[u,v] = \text{ the interior in } C_0^1(\overline{\Omega}) \text{ of } [u,v] \cap C_0^1(\overline{\Omega}), \\ &[u) = \left\{ h \in W_0^{1,r(z)}(\Omega) : u(z) \leq h(z) \text{ for a.a. } z \in \Omega \right\}. \end{aligned}$$

When X is a Banach space and $\varphi \in C^1(X, \mathbb{R})$, we set

$$K_{\varphi} = \{u \in X : \varphi'(u) = 0\} \quad \text{(the critical set of } \varphi).$$

Also, we say that $\varphi(\cdot)$ satisfies the *C-condition*, if the following property holds:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ *such that* $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ *is bounded and* $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ *in* X^* *as* $n \to \infty$, *admits a strongly convergent subsequence".*

This is a compactness-type condition on the functional $\varphi(\cdot)$. In most cases of interest, the ambient space X is infinite dimensional and so it is not locally compact. So, the burden of compactness is passed on the functional $\varphi(\cdot)$. Using the C-condition, one can prove a deformation theorem from which follow the minimax theorems of critical point theory (see Papageorgiou-Rădulescu-Repovš [30, Chapter 5]).

Finally, by $\widehat{u}_1(p_-) \in W_0^{1,p_-}(\Omega)$ we denote the positive, L^{p_-}-normalized (that is, $\|\widehat{u}_1(p_-)\|_{p_-} = 1$) eigenfunction corresponding to the principal eigenvalue $\widehat{\lambda}_1(p_-) > 0$ of $(-\Delta_{p_-}, W_0^{1,p_-}(\Omega))$. We know (see, for example, Gasiński-Papageorgiou [11, p. 739]) that $\widehat{u}_1(p_-) \in \operatorname{int} C_+$. Also, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Now we introduce our hypotheses on the data of (1.1).

H_0: $p, q, \eta \in C^{0,1}(\overline{\Omega})$, $0 < \eta(z) < 1$ and $1 < q(z) < p(z)$ for all $z \in \overline{\Omega}$, $p_- < N$.

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $0 \leq f(z,x) \leq a(z)[1 + |x|^{r(z)-1}]$ for a.a. $z \in \Omega$, all $x \geq 0$, with $a \in L^\infty(\Omega)$, $r \in C(\overline{\Omega})$, $p(z) < r(z) < p_-^*$ for all $z \in \overline{\Omega}$;

(ii) if $F(z,x) = \int_0^x f(z,s)\,\mathrm{d}s$, then $\lim_{x\to\pm\infty} \frac{F(z,x)}{x^{p_+}} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) there exists $\mu \in C(\overline{\Omega})$ such that

$$\begin{aligned} &\mu(z) \in \left((r_+ - p_-)\frac{N}{p_-}, p_+^* \right) \\ &0 < \gamma_0 \leq \liminf_{x\to+\infty} \frac{f(z,x)x - p_+ F(z,x)}{x^{\mu(z)}} \text{ uniformly for a.a. } z \in \Omega; \end{aligned}$$

(iv) there exist $\tau \in C(\overline{\Omega})$, $\delta > 0$ and $\vartheta > 0$ such that

$$\begin{aligned} &1 < \tau(z) < q_-,\\ &c_0 x^{\tau(z)-1} \leq f(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \in [0,\delta_0], \text{ some } c_0 > 0,\\ &\vartheta^{-\eta(z)} + f(z,\vartheta) \leq -\widehat{c}_\vartheta < 0, \text{ for a.a. } z \in \Omega; \end{aligned}$$

(v) there exists $\widehat{\xi}_\vartheta > 0$ such that for a.a. $z \in \Omega$, the function

$$x \mapsto f(z,x) + \widehat{\xi}_\vartheta x^{p(z)-1}$$

is nondecreasing on $[0,\vartheta]$.

Remarks. Since we aim to find positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality, we may assume that

$$f(z,x) = 0 \text{ for a.a } z \in \Omega, \text{ all } x \leq 0. \tag{2.1}$$

Hypotheses H_1(ii),(iii) imply that for a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p_+ - 1)$-superlinear. However, this superlinearity is not expressed using the well-known *Ambrosetti-Rabinowitz condition* (the *AR-condition* for short, see Ambrosetti-Rabinowitz [1]). Instead, we employ hypothesis H_1(iii) which is less restrictive and incorporates in our framework $(p_+ - 1)$-superlinear nonlinearities with "slower" growth near $+\infty$. For example, consider the following function

$$f(z,x) = \begin{cases} x^{\tau(z)-1} - 2x^{\vartheta(z)-1} & \text{if } 0 \leq x \leq 1\\ x^{p_+-1}\ln x + x^{s(z)-1} - 2x^{\lambda(z)-1} & \text{if } 1 < x \end{cases} \quad (\text{see } (2.1))$$

with $\vartheta, s, \lambda \in C(\overline{\Omega})$, $\tau(z) < \vartheta(z)$, $1 < s(z)$, $\lambda(z) < p(z)$ for all $z \in \overline{\Omega}$. Then, this function satisfies hypotheses H_1, but fails to satisfy the AR-condition.

On account of hypotheses H_1(i),(iv), we have

$$f(z,x) \geq c_0 x^{\tau(z)-1} - c_1 x^{r(z)-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0, \text{ with } c_1 > 0. \tag{2.2}$$

We introduce the following truncation of the right-hand side of (2.2):

$$k(z,x) = \begin{cases} c_0(x^+)^{\tau(z)-1} - c_1(x^+)^{r(z)-1} & \text{if } x \leq \vartheta\\ c_0\vartheta^{\tau(z)-1} - c_1\vartheta^{r(z)-1} & \text{if } \vartheta < x \end{cases}, \tag{2.3}$$

with $\vartheta > 0$ as in hypothesis H_1(iv). Evidently, this is a Carathéodory function. Using $k(\cdot,\cdot)$ as the source term, we consider the following auxiliary Dirichlet problem:

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) = k(z,u(z)) & \text{in } \Omega\\ u\Big|_{\partial\Omega} = 0, \ u > 0 \end{cases}. \tag{2.4}$$

Proposition 2.3. *Problem* (2.4) *admits a unique positive solution* $\overline{u} \in \mathrm{int}_{C_0^1(\overline{\Omega})}[0,\vartheta]$.

Proof. First we prove the existence of a positive solution. So, let $K(z,x) = \int\limits_0^x k(z,s)\,\mathrm{d}s$ and consider the C^1-functional $\sigma : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\sigma(u) = \int\limits_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,\mathrm{d}z + \int\limits_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,\mathrm{d}z - \int\limits_\Omega K(z,u^+)\,\mathrm{d}z \quad \text{for all } u \in W_0^{1,p(z)}(\Omega).$$

From (2.4) and Proposition 2.1, we see that $\sigma(\cdot)$ is coercive. Also by the anisotropic Sobolev embedding theorem and the convexity of the map $u \mapsto \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,\mathrm{d}z + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,\mathrm{d}z$, we see that $\sigma(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem we can find $\overline{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\sigma(\overline{u}) = \min\left\{\sigma(u) : u \in W_0^{1,p(z)}(\Omega)\right\}. \tag{2.5}$$

Let $u \in \operatorname{int} C_+$ and choose $t \in (0,1)$ small such that $tu(z) \leq \vartheta$ for all $z \in \overline{\Omega}$. Then, using (2.3) we have

$$\sigma(tu) \geq \frac{t^{p_+}}{p_+}\rho_p(Du) + \frac{t^{q_+}}{q_+}\rho_q(Du) + \frac{t^{r_+}}{r_+}\rho_r(u) - \frac{t^{\tau_-}}{\tau_-}\rho_\tau(u).$$

Since $1 < \tau_- < q_+ < p_+ < r_+$, by choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} & \sigma(tu) < 0, \\ \Rightarrow\quad & \sigma(\overline{u}) < 0 = \sigma(0) \quad (\text{see } (2.5)), \\ \Rightarrow\quad & \overline{u} \neq 0. \end{aligned}$$

From (2.5), we have

$$\begin{aligned} & \sigma'(\overline{u}) = 0, \\ \Rightarrow\quad & \langle A_{p(z)}(\overline{u}), h\rangle + \langle A_{q(z)}(\overline{u}), h\rangle = \int_\Omega k(z,\overline{u})h\,\mathrm{d}z \quad \text{for all } h \in W_0^{1,p(z)}(\Omega) \end{aligned} \tag{2.6}$$

In (2.6) first we choose $h = -\overline{u}^- \in W_0^{1,p(z)}(\Omega)$. We obtain

$$\begin{aligned} & \rho_p(D\overline{u}^-) + \rho_q(D\overline{u}^-) = 0 \quad (\text{see } (2.1)), \\ \Rightarrow\quad & \overline{u} \geq 0,\ \overline{u} \neq 0 \quad (\text{see Proposition } 2.1). \end{aligned}$$

Next in (2.6), we choose $h = (\overline{u} - \vartheta)^+ \in W_0^{1,p(z)}(\Omega)$. Then,

$$\begin{aligned} & \langle A_{p(z)}(\overline{u}), (\overline{u}-\vartheta)^+\rangle + \langle A_{q(z)}(\overline{u}), (\overline{u}-\vartheta)^+\rangle \\ & = \int_\Omega [c_0\vartheta^{\tau(z)-1} - c_1\vartheta^{r(z)-1}](\overline{u}-\vartheta)^+\,\mathrm{d}z \quad (\text{see } (2.3)) \\ & \leq \int_\Omega f(z,\vartheta)(\overline{u}-\vartheta)^+\,\mathrm{d}z \quad (\text{see } (2.2)) \\ & \leq 0 \quad (\text{see hypothesis } \mathrm{H}_1(\text{iv})), \end{aligned}$$

$$\Rightarrow \overline{u} \leq \vartheta.$$

We have proved that

$$\overline{u} \in [0,\vartheta],\ \overline{u} \neq 0. \tag{2.7}$$

From (2.7), (2.3) and (2.6), it follows that

$$-\Delta_{p(z)}\overline{u}(z) - \Delta_{q(z)}\overline{u}(z) = c_0\overline{u}(z)^{\tau(z)-1} - c_1\overline{u}(z)^{r(z)-1} \text{ in } \Omega, \quad \overline{u}|_{\partial\Omega} = 0.$$

From Fan-Zhao [9, Theorem 4.1] (see also Gasiński-Papageorgiou [12, Proposition 3.1]), we have that

$$\overline{u} \in L^\infty(\Omega).$$

Applying Lemma 3.3 of Fukagai-Narukawa [10] (see also Lieberman [24]), we have that

$$\overline{u} \in C_+\setminus\{0\}.$$

Moreover, Lemma 3.5 of [10] implies that

$$\overline{u} \in \operatorname{int} C_+.$$

Let $\widehat{\xi}_\vartheta > 0$ be as postulated by hypothesis H_1(v). We have

$$\begin{aligned}
&-\Delta_{p(z)}\overline{u} - \Delta_{q(z)}\overline{u} + \widehat{\xi}_\vartheta \overline{u}^{p(z)-1} \\
&\quad = c_0\overline{u}^{\tau(z)-1} - c_1\overline{u}^{r(z)-1} + \widehat{\xi}_\vartheta \overline{u}^{p(z)-1} \\
&\quad \le f(z,\overline{u}) + \widehat{\xi}_\vartheta \overline{u}^{p(z)-1} \quad \text{(see (2.2))} \\
&\quad \le f(z,\vartheta) + \widehat{\xi}_\vartheta \vartheta^{p(z)-1} \quad \text{(see (2.7) and hypothesis } H_1(\text{v})) \\
&\quad \le -\vartheta^{-\eta(z)} + \widehat{\xi}_\vartheta \vartheta^{p(z)-1} \quad \text{(see hypothesis } H_1(\text{iv})) \\
&\quad \le -\Delta_{p(z)}\vartheta - \Delta_{q(z)}\vartheta + \widehat{\xi}_\vartheta \vartheta^{p(z)-1} \text{ in } \Omega,
\end{aligned}$$

$$\Rightarrow \quad \overline{u}(z) < \vartheta \text{ for all } z \in \overline{\Omega}$$

(see Proposition 2.5 of Papageorgiou-Rădulescu-Repovš [29]).

We conclude that

$$\overline{u} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[0,\vartheta].$$

Next we show the uniqueness of this positive solution.

To this end, we consider the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega \frac{1}{p(z)}|Du^{\frac{1}{q_-}}|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du^{\frac{1}{q_-}}|^{q(z)}\,dz & \text{if } u \ge 0,\ u^{\frac{1}{q_-}} \in W_0^{1,p(z)}(\Omega) \\ +\infty & \text{otherwise} \end{cases}.$$

From Theorem 2.2 of Takač-Giacomoni [41], we have that the functional $j(\cdot)$ is convex.

Suppose that $\overline{v} \in W_0^{1,p(z)}(\Omega)$ is another positive solution of the auxiliary problem (2.4). As above, we show that $\overline{v} \in \operatorname{int} C_+$. Then, from Proposition 4.1.22, p. 274, of Papageorgiou-Rădulescu-Repovš [30], we have

$$\frac{\overline{u}}{\overline{v}} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\overline{v}}{\overline{u}} \in L^\infty(\Omega).$$

Hence, by Theorem 2.5 of Takač-Giacomoni [41] and the convexity of $j(\cdot)$, we have

$$\begin{aligned}
0 &\le \frac{1}{q_-}\left[\int_\Omega \frac{-\Delta_{p(z)}\overline{u} - \Delta_{q(z)}\overline{u}}{\overline{u}^{q_- -1}}(\overline{u}^{q_-} - \overline{v}^{q_-})\,dz + \int_\Omega \frac{-\Delta_{p(z)}\overline{v} - \Delta_{q(z)}\overline{v}}{\overline{v}^{q_- -1}}(\overline{u}^{q_-} - \overline{v}^{q_-})\,dz\right] \\
&= \frac{1}{q_-}\left[\int_\Omega c_0\left(\frac{1}{\overline{u}^{q_- -\tau(z)}} - \frac{1}{\overline{v}^{q_- -\tau(z)}}\right)(\overline{u}^{q_-} - \overline{v}^{q_-})\,dz\right. \\
&\quad \left.+ \int_\Omega c_1\left(\overline{v}^{r(z)-q_-} - \overline{u}^{r(z)-q_-}\right)(\overline{u}^{q_-} - \overline{v}^{q_-})\,dz\right] \le 0 \quad (\text{see hypotheses } H_0,\ H_1(iv)),
\end{aligned}$$

$$\Rightarrow \quad \overline{u} = \overline{v}.$$

This proves the uniqueness of the positive solution of problem (2.4). □

We consider the Banach space $C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. This is an ordered Banach space with positive cone $K_+ = \{u \in C_0(\overline{\Omega}) : u(z) \geq 0$ for all $z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} K_+ = \left\{ u \in K_+ : c_u \widehat{d} \leq u \text{ for } c_u > 0 \right\},$$

where $\widehat{d}(z) = d(z, \partial\Omega)$ for all $z \in \overline{\Omega}$. Lemma 14.16, p. 335, of Gilbarg-Trudinger [20], says that we can find $\delta_0 > 0$ such that $\widehat{d} \in C^2(\Omega_{\delta_0})$ with $\Omega_{\delta_0} = \{z \in \overline{\Omega} : d(z, \partial\Omega) < \delta_0\}$. Hence, $\widehat{d} \in \operatorname{int} C_+$ and so we can use Proposition 4.1.22, p. 274, of Papageorgiou-Rădulescu-Repovš [30] and find $0 < c_2 < c_3$ such that

$$\begin{aligned} & c_2 \widehat{d} \leq \overline{u} \leq c_3 \widehat{d} \\ \Rightarrow \quad & \overline{u} \in \operatorname{int} K_+. \end{aligned} \tag{2.8}$$

Let $s > N$. We have $\widehat{u}_1(p_-)^{\frac{1}{s}} \in K_+$ and so on account of (2.8), we can find $c_4 > 0$ such that

$$\begin{aligned} & 0 \leq \widehat{u}_1(p_-)^{\frac{1}{s}} \leq c_4 \overline{u} \\ \Rightarrow \quad & 0 \leq \overline{u}^{-\eta(z)} \leq c_5 \widehat{u}_1(p_-)^{-\frac{\eta(z)}{s}} \quad \text{for some } c_5 > 0. \end{aligned}$$

Note that

$$\begin{aligned} & \int_\Omega \left[\widehat{u}_1(p_-)^{-\frac{\eta(z)}{s}} \right]^s \mathrm{d}z \\ = \; & \int_\Omega \widehat{u}_1(p_-)^{-\eta(z)} \, \mathrm{d}z \\ = \; & \int_{\{\widehat{u}_1(p_-) \leq 1\}} \widehat{u}_1(p_-)^{-\eta(z)} \, \mathrm{d}z + \int_{\{\widehat{u}_1(p_-) > 1\}} \widehat{u}_1(p_-)^{-\eta(z)} \, \mathrm{d}z \\ \leq \; & \int_\Omega \widehat{u}_1(p_-)^{-\eta_+} \, \mathrm{d}z + |\Omega|_N, \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & \widehat{u}_1(p_-)(\cdot)^{-\eta(\cdot)} \in L^s(\Omega) \\ & \text{(see the Lemma of Lazer-McKenna [23] and recall that } \eta_+ < 1), \\ \Rightarrow \quad & \overline{u}(\cdot)^{-\eta(\cdot)} \in L^s(\Omega), \ s > N. \end{aligned} \tag{2.9}$$

3. Positive solutions

In this section, we prove a multiplicity theorem for the positive solutions of problem (1.1).

To produce the first positive solution of (1.1), we use (2.7) and (2.9) to define the following truncation of the reaction in problem (1.1):

$$e(z, x) = \begin{cases} \overline{u}(z)^{-\eta(z)} + f(z, \overline{u}(z)) & \text{if } x < \overline{u}(z) \\ x^{-\eta(z)} + f(z, x) & \text{if } \overline{u}(z) \leq x \leq \vartheta \\ \vartheta^{-\eta(z)} + f(z, \vartheta) & \text{if } \vartheta < x \end{cases}. \tag{3.1}$$

This is a Carathéodory function. We set $E(z, x) = \int_0^x e(z, s) \, \mathrm{d}s$ and introduce the functional $\psi : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} \, \mathrm{d}z + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} \, \mathrm{d}z - \int_\Omega E(z, u) \, \mathrm{d}z \quad \text{for all } u \in W_0^{1,p(z)}(\Omega).$$

From (2.9) it follows that $\psi \in C^1(W_0^{1,p(z)}(\Omega))$ (see also Papageorgiou-Smyrlis [32, Proposition 3]). Using this functional, we can now produce the first positive solution of (1.1).

Proposition 3.1. *If hypotheses* H_0*,* H_1*(i),(iv),(v) hold, then problem* (1.1) *has a positive solution* $u_0 \in [\overline{u},\vartheta] \cap \operatorname{int} C_+$*,* $u_0(z) < \vartheta$ *for all* $z \in \overline{\Omega}$*.*

Proof. From (3.1) and Proposition 2.1, we see that $\psi(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$\begin{aligned}
&\psi(u_0) = \min\left[\psi(u) : u \in W_0^{1,p(z)}(\Omega)\right],\\
\Rightarrow\quad &\psi'(u_0) = 0,\\
\Rightarrow\quad &\langle A_{p(z)}(u_0), h\rangle + \langle A_{q(z)}(u_0), h\rangle = \int_\Omega e(z,u_0) h \,\mathrm{d}z \quad \text{for all } h \in W_0^{1,p(z)}(\Omega).
\end{aligned} \tag{3.2}$$

In (3.2) first we choose $h = (\overline{u} - u_0)^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}
&\langle A_{p(z)}(u_0), (\overline{u}-u_0)^+\rangle + \langle A_{q(z)}(u_0), (\overline{u}-u_0)^+\rangle\\
&\quad= \int_\Omega \left[\overline{u}^{-\eta(z)} + f(z,\overline{u})\right](\overline{u}-u_0)^+ \,\mathrm{d}z \quad (\text{see } (3.1))\\
&\quad\geq \int_\Omega f(z,\overline{u})(\overline{u}-u_0)^+ \,\mathrm{d}z\\
&\quad\geq \int_\Omega \left[c_0\overline{u}^{\tau(z)-1} - c_1\overline{u}^{r(z)-1}\right](\overline{u}-u_0)^+ \,\mathrm{d}z \quad (\text{see } (2.2))\\
&\quad= \langle A_{p(z)}(\overline{u}), (\overline{u}-u_0)^+\rangle + \langle A_{q(z)}(\overline{u}), (\overline{u}-u_0)^+\rangle \quad (\text{see Proposition } 2.3),
\end{aligned}$$

$$\Rightarrow \overline{u} \leq u_0.$$

Next we test (3.2) with $(u_0 - \vartheta)^+ \in W_0^{1,p(z)}(\Omega)$. Then

$$\begin{aligned}
&\langle A_{p(z)}(u_0), (u_0-\vartheta)^+\rangle + \langle A_{q(z)}(u_0), (u_0-\vartheta)^+\rangle\\
&\quad= \int_\Omega \left[\vartheta^{-\eta(z)} + f(z,\vartheta)\right](u_0-\vartheta)^+ \,\mathrm{d}z \quad (\text{see } (3.1))\\
&\quad\leq 0 \quad (\text{see hypothesis } \mathrm{H}_1(\mathrm{iv})),
\end{aligned}$$

$$\Rightarrow u_0 \leq \vartheta.$$

So, we have proved that

$$u_0 \in [\overline{u},\vartheta]. \tag{3.3}$$

From (3.3), (3.1) and (3.2) it follows that

$$-\Delta_{p(z)} u_0 - \Delta_{q(z)} u_0 = u_0^{-\eta(z)} + f(z,u_0) \ \text{ in } \Omega, \quad u_0|_{\partial\Omega} = 0. \tag{3.4}$$

From (2.9), (3.3), (3.4) and Theorem 4.1 of Fan-Zhao [9] (see also Tan-Fang [42, Theorem 3.1]), we have that

$$u_0 \in L^\infty(\Omega) \quad (\text{recall that } s > N \text{ is arbitrary}).$$

Then (2.9) and hypothesis H_1(i) imply that

$$\beta(\cdot) = u_0(\cdot)^{-\eta(\cdot)} + f(\cdot, u_0(\cdot)) \in L^s(\Omega), \quad s > N. \tag{3.5}$$

We consider the following linear Dirichlet problem

$$-\Delta y(z) = \beta(z) \text{ in } \Omega, \quad y|_{\partial\Omega} = 0. \tag{3.6}$$

Then (3.5) and Theorem 9.15, p. 241, of Gilbarg-Trudinger [20], imply that problem (3.6) admits a unique solution $y \in W^{2,s}(\Omega)$, $s > N$, (in fact $y \geq 0$ since $\beta \geq 0$). From the Sobolev embedding theorem, we have

$$\begin{aligned} & W^{2,s}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega}) \quad \text{with } \alpha = 1 - \frac{N}{s} \in (0,1), \\ \Rightarrow \quad & y \in C_0^{1,\alpha}(\overline{\Omega}) = C^{1,\alpha}(\overline{\Omega}) \cap C_0^1(\overline{\Omega}), \\ \Rightarrow \quad & w = Dy \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N). \end{aligned}$$

We rewrite (3.4) as follows:

$$-\text{div}\left(|Du_0|^{p(z)-2}Du_0 + |Du_0|^{q(z)-2}Du_0 - w\right) = 0 \quad \text{in } \Omega.$$

As before, from Fukagai-Narukawa [10] (see also Lieberman [24]), we have that

$$u_0 \in \operatorname{int} C_+ \quad (\text{see } (3.3)).$$

Let $\widehat{\xi}_\vartheta > 0$ be as postulated by hypothesis H_1(v). We have

$$\begin{aligned} & -\Delta_{p(z)}u_0 - \Delta_{q(z)}u_0 + \widehat{\xi}_\vartheta u_0^{p(z)-1} - u_0^{-\eta(z)} \\ & = f(z, u_0) + \widehat{\xi}_\vartheta u_0^{p(z)-1} \\ & \leq f(z, \vartheta) + \widehat{\xi}_\vartheta \vartheta^{p(z)-1} \quad (\text{see } (3.3) \text{ and hypothesis } H_1(\text{v})) \\ & \leq -\widehat{c}_\vartheta - \vartheta^{-\eta(z)} + \widehat{\xi}_\vartheta \vartheta^{p(z)-1} \quad (\text{see hypothesis } H_1(\text{iv})) \\ & \leq -\Delta_{p(z)}\vartheta - \Delta_{q(z)}\vartheta + \widehat{\xi}_\vartheta \vartheta^{p(z)-1} - \vartheta^{-\eta(z)} \text{ in } \Omega. \end{aligned}$$

But then from the anisotropic strong comparison principle (see Proposition 2.5 of [29] and Proposition 6 of [28]), we have

$$u_0(z) < \vartheta \quad \text{for all } z \in \overline{\Omega}.$$

□

To produce a second positive solution for problem (1.1), we introduce the following truncation of the reaction:

$$l(z,x) = \begin{cases} \overline{u}(z)^{-\eta(z)} + f(z, \overline{u}(z)) & \text{if } x \leq \overline{u}(z) \\ x^{-\eta(z)} + f(z,x) & \text{if } \overline{u}(z) < x \end{cases}. \tag{3.7}$$

This is a Carathéodory function. We set $L(z,x) = \int\limits_0^x l(z,s)\,\mathrm{d}s$ and consider the functional $\varphi : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) = \int\limits_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,\mathrm{d}z + \int\limits_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,\mathrm{d}z - \int\limits_\Omega L(z,u)\,\mathrm{d}z \quad \text{for all } u \in W_0^{1,p(z)}(\Omega).$$

As before, on account of (2.9), we have that $\varphi \in C^1(W_0^{1,p(z)}(\Omega))$.

From (3.1) and (3.7), we see that

$$\varphi\Big|_{[0,\vartheta]} = \psi\Big|_{[0,\vartheta]} \quad \text{and} \quad \varphi'\Big|_{[0,\vartheta]} = \psi'\Big|_{[0,\vartheta]}. \tag{3.8}$$

Proposition 3.2. *If hypotheses* H_0, H_1 *hold, then* $u_0 \in \operatorname{int} C_+$ *is a local minimizer of* φ.

Proof. From the proof of Proposition 3.1, we know that

$$\begin{aligned} & u_0 \in \operatorname{int} C_+ \text{ is a minimizer of } \psi(\cdot) \text{ and } u_0(z) < \vartheta \text{ for all } z \in \overline{\Omega}, \\ \Rightarrow \quad & u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[0, \vartheta]. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), it follows that

$$\begin{aligned} & u_0 \text{ is a local } C_0^1(\overline{\Omega}) - \text{minimizer of } \varphi(\cdot), \\ \Rightarrow \quad & u_0 \text{ is a local } W_0^{1,p(z)}(\Omega) - \text{minimizer of } \varphi \\ & \text{(see Fan [8], Gasiński-Papageorgiou [12] and Tan-Fang [42]).} \end{aligned}$$

□

Proposition 3.3. *If hypotheses* H_0, H_1 *hold, then* $K_\varphi \subseteq [\overline{u}) \cap \operatorname{int} C_+$.

Proof. Let $u \in K_\varphi$. We have

$$\begin{aligned} & \varphi'(u) = 0 \\ \Rightarrow \quad & \langle A_{p(z)}(u), h\rangle + \langle A_{q(z)}(u), h\rangle = \int_\Omega l(z,u)h \, \mathrm{d}z \quad \text{for all } h \in W_0^{1,p(z)}(\Omega). \end{aligned} \tag{3.10}$$

In (3.10) we choose $h = (\overline{u} - u)^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned} & \langle A_{p(z)}(u), (\overline{u}-u)^+\rangle + \langle A_{q(z)}(u), (\overline{u}-u)^+\rangle \\ & = \int_\Omega \left[\overline{u}^{-\eta(z)} + f(z,\overline{u})\right] (\overline{u}-u)^+ \, \mathrm{d}z \quad \text{(see (3.7))} \\ & \geq \int_\Omega f(z,\overline{u})(\overline{u}-u)^+ \, \mathrm{d}z \\ & \geq \int_\Omega \left[c_0 \overline{u}^{\tau(z)-1} - c_1 \overline{u}^{r(z)-1}\right] (\overline{u}-u)^+ \, \mathrm{d}z \quad \text{(see (2.2))} \\ & = \langle A_{p(z)}(\overline{u}), (\overline{u}-u)^+\rangle + \langle A_{q(z)}(\overline{u}), (\overline{u}-u)^+\rangle \quad \text{(see Proposition 2.3)}, \end{aligned}$$

$$\Rightarrow \overline{u} \leq u.$$

From (3.7) and (3.10) it follows that

$$-\Delta_{p(z)} u - \Delta_{q(z)} u = u^{-\eta(z)} + f(z,u) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

As before, the anisotropic regularity theory (see [9,10]) implies that

$$\begin{aligned} & \overline{u} \in \operatorname{int} C_+, \\ \Rightarrow \quad & K_\varphi \subseteq [\overline{u}) \cap \operatorname{int} C_+. \end{aligned}$$

□

From Proposition 3.3 and (3.7), we see that we may assume

$$K_\varphi \text{ is finite.} \tag{3.11}$$

Otherwise, we already have an infinity of positive smooth solutions and so we are done.

From (3.11), Proposition 3.2 and Theorem 5.7.6, p. 449, of Papageorgiou-Rădulescu-Repovš [30], we know that we can find $\rho \in (0,1)$ small such that

$$\varphi(u_0) < \inf \left[\varphi(u) : u \in W_0^{1,p(z)}(\Omega) : \|u-u_0\| = \rho\right] = m_0. \tag{3.12}$$

On account of hypothesis H_1(ii) we, have:

Proposition 3.4. *If hypotheses* H_0, H_1 *hold and* $u \in \operatorname{int} C_+$, *then* $\varphi(tu) \to -\infty$ *as* $t \to +\infty$.

Proposition 3.5. *If hypotheses* H_0, H_1 *hold, then the functional* $\varphi(\cdot)$ *satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p(z)}(\Omega)$ such that

$$|\varphi(u_n)| \leq c_6 \quad \text{for some } c_6 > 0, \text{ all } n \in \mathbb{N}, \tag{3.13}$$

$$(1+\|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } W^{-1,p'(z)}(\Omega) \text{ as } n \to \infty. \tag{3.14}$$

From (3.14) we have

$$\left|\langle A_{p(z)}(u_n), h\rangle + \langle A_{q(z)}(u_n), h\rangle - \int_\Omega l(z,u_n)h\,\mathrm{d}z\right| \leq \frac{\epsilon_n\|h\|}{1+\|u_n\|}$$
$$\text{for all } h \in W_0^{1,p(z)}(\Omega), \text{ with } \epsilon_n \to 0^+. \tag{3.15}$$

In (3.15) we choose $h = -u_n^- \in W_0^{1,p(z)}(\Omega)$. Then using (3.7), we obtain

$$\rho_p(Du_n^-) + \rho_q(Du_n^-) \leq c_7\|u_n^-\| \quad \text{for some } c_7 > 0, \text{ all } n \in \mathbb{N},$$
$$\Rightarrow \quad \{u_n^-\}_{n\geq 1} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded.} \tag{3.16}$$

If in (3.15) we choose $h \in u_n^+ \in W_0^{1,p(z)}(\Omega)$, then

$$-\rho_p(Du_n^+) - \rho_q(Du_n^+) + \int_\Omega l(z,u_n^+)u_n^+\,\mathrm{d}z \leq \epsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.17}$$

On the other hand, from (3.13) and (3.16), we have

$$\left|\int_\Omega \frac{1}{p(z)}|Du_n^+|^{p(z)}\,\mathrm{d}z + \int_\Omega \frac{1}{q(z)}|Du_n^+|^{q(z)}\,\mathrm{d}z - \int_\Omega L(z,u_n^+)\,\mathrm{d}z\right| \leq c_8$$
$$\text{for some } c_8 > 0, \text{ all } n \in \mathbb{N},$$
$$\Rightarrow \quad \rho_p(Du_n^+) + \rho_q(Du_n^+) - \int_\Omega p_+ L(z,u_n^+)\,\mathrm{d}z \leq p_+c_8 \quad \text{for all } n \in \mathbb{N}. \tag{3.18}$$

We add (3.17) and (3.18) and obtain

$$\int_\Omega \left[l(z,u_n^+)u_n^+ - p_+L(z,u_n^+)\right]\,\mathrm{d}z \leq c_9 \quad \text{for some } c_9 > 0, \text{ all } n \in \mathbb{N},$$
$$\Rightarrow \quad \int_\Omega \left[f(z,u_n^+)u_n^+ - p_+F(z,u_n^+)\right]\,\mathrm{d}z \leq c_{10}\left[1 + \int_\Omega (u_n^+)^{1-\eta(z)}\,\mathrm{d}z\right]$$
$$\text{for some } c_{10} > 0, \text{ all } n \in \mathbb{N} \text{ (see (3.7))}. \tag{3.19}$$

From hypotheses H_1(i),(iiii), we see that we can find $\gamma_1 \in (0,\gamma_0)$ and $c_{11} = c_{11}(\gamma_1) > 0$ such that

$$\gamma_1 x^{\mu(z)} - c_1 1 \leq f(z,x)x - p_+F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.20}$$

We return to (3.19) and use (3.20). Then

$$\begin{aligned} &\rho_\mu(u_n^+) \le c_{12}\left[1 + \|u_n^+\|_{\mu(z)}\right] \quad \text{for some } c_{12} > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow\ &\{u_n^+\}_{n\ge1} \subseteq L^{\mu(z)}(\Omega) \text{ is bounded (see Proposition 2.1).} \end{aligned} \tag{3.21}$$

From hypothesis H_1(iii), we see that without any loss of generality, we may assume that $\mu(z) < r(z) < p_-^*$ for all $z \in \overline{\Omega}$ (see hypothesis H_1(i)). Hence,

$$\mu_- < r_+ < p_-^*.$$

We choose $t \in (0,1)$ such that

$$\frac{1}{r_+} = \frac{1-t}{\mu_-} + \frac{t}{p_-^*}. \tag{3.22}$$

From the interpolation inequality (see Papageorgiou-Winkert [36, Proposition 2.3.17, p. 116]), we have

$$\begin{aligned} &\|u_n^+\|_{r_+} \le \|u_n^+\|_{\mu_-}^{1-t}\|u_n\|_{p_-^*}^t, \\ \Rightarrow\ &\|u_n^+\|_{r_+}^{r_+} \le c_{13}\|u_n^+\|_{p_+^*}^{tr_+} \quad \text{for some } c_{13} > 0, \text{ all } n \in \mathbb{N} \\ &\text{(see (3.21) and recall that } L^{\mu(z)}(\Omega) \hookrightarrow L^{\mu_-}(\Omega) \text{ continuously)}, \\ \Rightarrow\ &\|u_n^+\|_{r_+}^{r_+} \le c_{14}\|u_n^+\|^{tr_+} \quad \text{for some } c_{14} > 0, \text{ all } n \in \mathbb{N} \\ &\text{(since } W_0^{1,p(z)}(\Omega) \hookrightarrow L^{p_-^*}(\Omega) \text{ continuously).} \end{aligned} \tag{3.23}$$

We test (3.15) with $h = u_n^+ \in W_0^{1,p(z)}(\Omega)$ and obtain

$$\begin{aligned} &\rho_p(Du_n^+) + \rho_q(Du_n^+) \le \epsilon_n + \int_\Omega l(z,u_n^+)u_n^+ \,\mathrm{d}z, \\ \Rightarrow\ &\rho_p(Du_n^+) + \rho_q(Du_n^+) \le c_{15}\left[1 + \int_\Omega f(z,u_n^+)u_n^+ \,\mathrm{d}z\right] \quad \text{for some } c_{15} > 0, \text{ all } n \in \mathbb{N} \text{ (see (3.7))} \\ &\le c_{16}\left[1 + \|u_n^+\|_{r_+}^{r_+}\right] \quad \text{for some } c_{16} > 0, \text{ all } n \in \mathbb{N} \text{ (see hypothesis } H_1\text{(i))} \\ &\le c_{17}\left[1 + \|u_n^+\|^{tr_+}\right] \quad \text{for some } c_{17} > 0, \text{ all } n \in \mathbb{N} \text{ (see (3.23))}. \end{aligned} \tag{3.24}$$

From (3.22), we have

$$tr_+ = \frac{p_-^*(r_+ - \mu_-)}{p_-^* - \mu_-} < p_- \quad \text{(see hypothesis } H_1\text{(iii))}.$$

Then from (3.24), it follows that

$$\begin{aligned} &\{u_n^+\}_{n\ge1} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded}, \\ \Rightarrow\ &\{u_n\}_{n\ge1} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded (see (3.16))}. \end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p(z)}(\Omega) \text{ and } u_n \to u \text{ in } L^{r(z)}(\Omega) \text{ as } n \to \infty. \tag{3.25}$$

In (3.15), we choose $h = u_n - u \in W_0^{1,p(z)}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.25). Then

$$\lim_{n\to+\infty} \left[\langle A_{p(z)}(u_n), u_n - u\rangle + \langle A_{q(z)}(u_n), u_n - u\rangle\right] = 0,$$

$$\Rightarrow \quad \limsup_{n\to\infty} \left[\langle A_{p(z)}(u_n), u_n - u\rangle + \langle A_{q(z)}(u), u_n - u\rangle\right] \le 0 \quad \text{(since } A_{q(z)}(\cdot) \text{ is monotone)},$$

$$\Rightarrow \quad \limsup_{n\to\infty} \langle A_{p(z)}(u_n), u_n - u\rangle \le 0 \quad \text{(see (3.25))},$$

$$\Rightarrow \quad u_n \to u \text{ in } W_0^{1,p(z)} \text{ as } n \to \infty \text{ (see Proposition 2.2)}.$$

This proves that the functional $\varphi(\cdot)$ satisfies the C-condition. □

Now we are ready for the multiplicity theorem.

Theorem 3.6. *If hypotheses* H_0, H_1 *hold, then problem* (1.1) *has at least two positive solutions*

$$u_0, \widehat{u} \in \operatorname{int} C_+, \quad u_0 \ne \widehat{u}, \quad u_0(z) < \vartheta \textit{ for all } z \in \overline{\Omega}.$$

Proof. From Proposition 3.1, we already have one positive solution

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[0, \vartheta]. \tag{3.26}$$

Propositions 3.4, 3.5 and (3.12) permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\widehat{u} \in K_\varphi \subseteq [\overline{u}) \cap \operatorname{int} C_+ \text{ (see Proposition 3.3) and } m_0 \le \varphi(\widehat{u}) \text{ (see (3.12))}. \tag{3.27}$$

From (3.27) and (3.7), it follows that

$$\widehat{u} \in \operatorname{int} C_+ \text{ is a positive solution of problem (1.1)},$$
$$\widehat{u} \ne u_0,$$

and $u_0(z) < \vartheta$ for all $z \in \overline{\Omega}$ (see (3.26)). □

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Nikolaos S. Papageorgiou
Department of Mathematics, Zografou Campus
National Technical University
15780 Athens
Greece
e-mail: npapg@math.ntua.gr

Andrea Scapellato
Dipartimento di Matematica e Informatica
Università degli Studi di Catania
Viale Andrea Doria 6
95125 Catania
Italy
e-mail: scapellato@dmi.unict.it

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