Proceedings of the Estonian Academy of Sciences,
2015, **64**, 3S, 325–330
doi: 10.3176/proc.2015.3S.01
Available online at www.eap.ee/proceedings

MECHANICS

Wave propagation in pantographic 2D lattices with internal discontinuities

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Received 11 December 2014, accepted 7 May 2015, available online 28 August 2015

Abstract. In the present paper we consider a 2D pantographic structure composed of two orthogonal families of Euler beams. Pantographic rectangular 'long' waveguides are considered in which imposed boundary displacements can induce the onset of travelling (possibly non-linear) waves. We performed numerical simulations concerning a set of dynamically interesting cases. The system undergoes large rotations, which may involve geometrical non-linearities, possibly opening a path to appealing phenomena such as the propagation of solitary waves. Boundary conditions dramatically influence the transmission of the considered waves at discontinuity surfaces. The theoretical study of this kind of objects looks critical, as the concept of pantographic 2D sheets seems to have promising possible applications in a number of fields, e.g. acoustic filters, vascular prostheses, and aeronautic/aerospace panels.

Key words: pantographic structures, wave propagation, homogenization, solitons.

1. INTRODUCTION

In the present study, we employed the concept, suggested by dell'Isola [17], of pantographic lattices, whose technological importance is rapidly increasing and, especially in nano-technology, could be very relevant. Pantographic structures are mechanical systems in which arrays of beams or rods are connected by internal kinematic pivots. Actually, a very wide class of objects can be effectively described and studied by means of suitably chosen pantographic models (see e.g. [15]). In this work, a numerical analysis of wave propagation is performed on the basis of the discrete mechanical model presented in [17].

Let us briefly recall the model considered therein. In Fig. 1a, the reference configuration C^* is shown. Lines indicate beams, which are divided in two families of parallel and equally spaced beams (with distance d), reciprocally orthogonal in C^*. The beams are arranged in a rectangle (sized $\sqrt{2}\mathrm{L}d \times \sqrt{2}\mathrm{W}d$ in C^*, where L and W are integers representing the number of intervals between the height and the width, respectively) whose sides are crossed by the beams at 45 degrees in C^*. Each beam has a standard linearized Euler strain energy given by

$$\mathscr{E} = \int_\Lambda \frac{k_M(u'')^2 + k_N(w')^2}{2}. \tag{1}$$

Here u and w are respectively the values of transverse and axial displacements $\mathbf{u}$ and $\mathbf{w}$ with respect to C^*, and k_M and k_N are respectively bending and axial stiffness coefficients, which in the real object depend of course on the diameter of the beams, while the integral is extended over the entire length Λ of the beams.

Dots in Fig. 1a represent hinges that allow free rotations and do not interrupt the continuity of the beams. The configuration at a given time t can be considered as characterized by a 'large' displacement (with respect to C^*) due to the contribution of the rigid rotations, and

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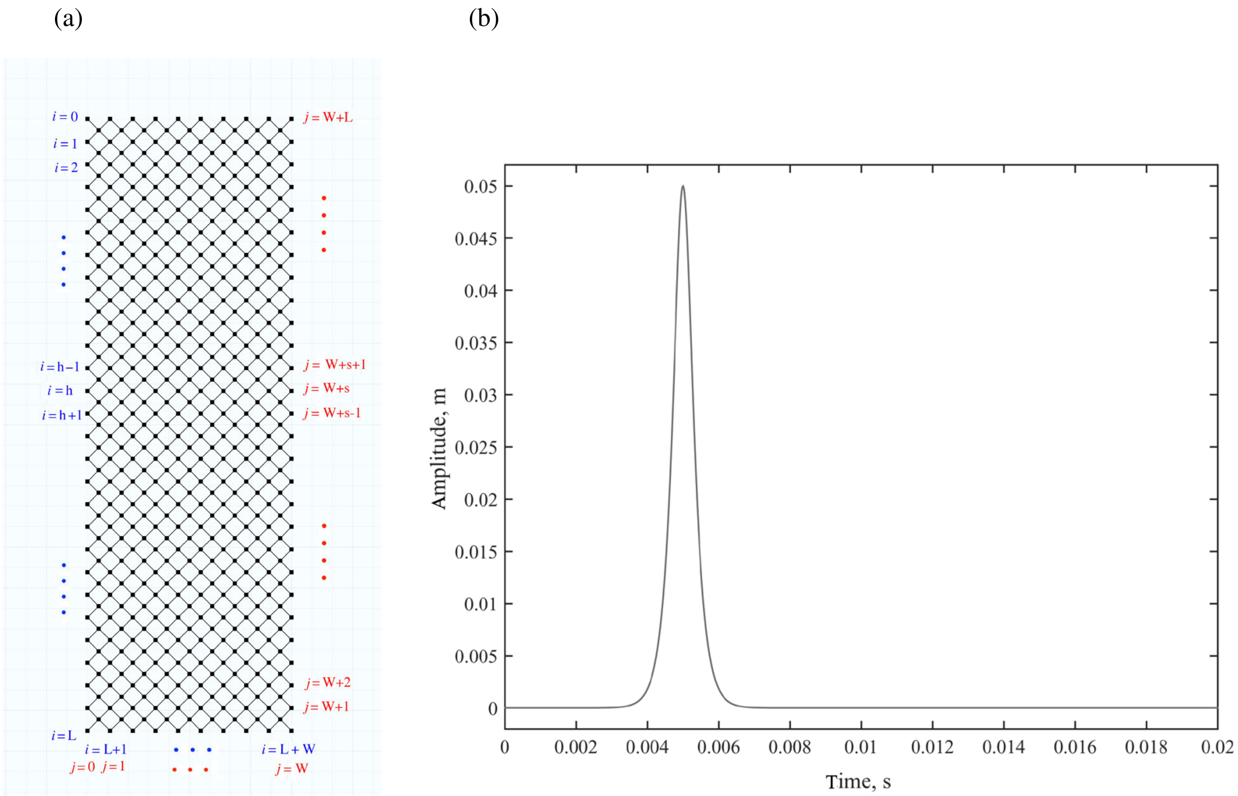

Fig. 1. Reference configuration (a) and time history of the impulse (b).

a ‘small’ displacement due to axial and bending elastic deformations.

2. NUMERICAL RESULTS

Wave propagation in non-trivial structures is of course widely studied in the literature (see e.g. [34,35,40]), and in particular several numerical and experimental results on woven fabrics can be found (see e.g. [7]). For our numerical analysis we used a length l of 0.1 m for the lower and upper sides, while the height h of the rectangle is either 2 m or 2.5 m in the simulations.

We chose the values 1.96×10^{-2} N m^2 and 7.85×10^4 N for k_M and k_N respectively, which can be thought as relative to a beam with an elliptic section of semiaxes $a = 0.001$ m and $b = 0.00025$ m (area $A = 7.85 \times 10^{-7}$ m^2, inertia moment of the cross-section around its minor axis $J = 1.96 \times 10^{-13}$ m^4) rotating around the minor one. We set $d = 0.0(1)$ m or $d = 0.00(5)$ m for the simulations.

As for mechanical parameters, we chose the mass density $\rho = 1450$ kg/m^3, $Y = 100$ GPa for Young’s modulus, and $\nu = 0.2$ for Poisson’s ratio. The material was assumed to be linearly elastic.

For all our simulations, we imposed a displacement on the points of the system belonging to the upper side of C^*, oriented along the height of the rectangle. The displacement is analytically represented by an impulse function $\Im(t) = u_0 * \text{sech}[\tau(t - t_0)]$, where $u_0 = 0.05$ m and $t_0 = 0.005$ s, while τ is a parameter affecting the duration of the impulse; in Fig. 1b the impulse is plotted with $\tau = 4000$ s^{-1}. In all the figures, times (in s) are given on the horizontal axis. If not otherwise specified, the lower side is built in, and the absolute values of the rotations of the cross-sections are represented by a colour map.

The numerical problems that may arise when considering structures of this kind with peculiar geometric characteristics can be very complex, moreover, the behaviour of the system can very easily display instabilities of the type of those discussed in e.g. [4,25–27,31]. A set of numerical tools has been elaborated to take care of such problems (the reader is referred e.g. to [9,10,12,21,30]), also in the direction of extending the model to the case of inextensible fibres, which can be numerically addressed by means of Lagrangian multipliers methods, as performed in [12,13].

Of course, our numerical results are intended as a first step towards a general homogenized theory of this kind of systems. Homogenization problems of this kind can in fact be very difficult, but a series of related problems have already been attacked in the literature (see e.g. [3,24,39,42,43]). These homogenization methods are of course at the basis of many nowadays active lines of investigations, such as mechanical phase transition [18,19,32,44], dissipation in particular structures [8], and anisotropy problems [33].

Considering our pantographic system, it is reasonable to describe its homogenized limit as metamaterial (see e.g. [2,5,14,22,29]), and because the system responds to solicitations like double forces (as we will numerically show), higher-gradient theories are also called for (see e.g. [1,16,23,36]).

All numerical simulations were performed with *COMSOL Multiphysics*®.

2.1. Basic wave propagation and double impulse

In Fig. 2 a basic case of wave propagation after an impulse of the type depicted in Fig. 1b is shown. This results in quite ordinary (continuous-like) wave propagation. Dispersion is clearly observable, as the length of the perturbed zone is unquestionably increasing in time, and a reflection on the lower side is also visible in the last snapshots.

In Fig. 3 the effect of a double impulse applied in the middle height of the structure is shown. By double impulse we mean a displacement oriented in the beam direction and imposed on two points belonging to the opposite ends of two adjacent beams, i.e. the ends are almost in line with one of the two families of beams. Both the upper and lower sides are built in. The idea is to show that such a structure can respond to a stimulus that, in the continuous homogenized limit case, is a double force (i.e. a pair of forces with null resultant and moment). Indeed, the onset of standard travelling waves is visible.

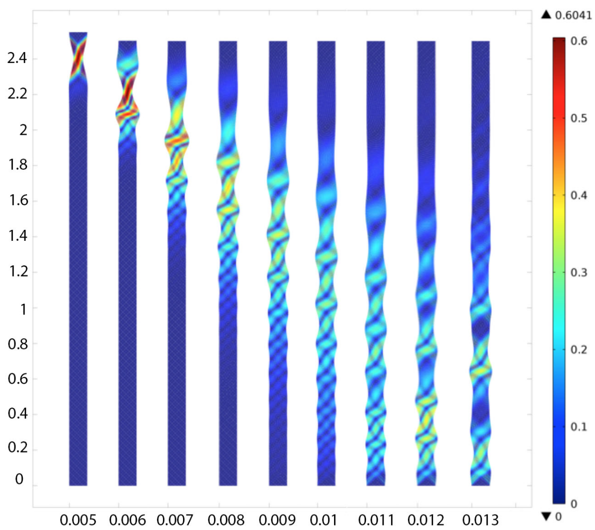

Fig. 2. Wave propagation after an imposed vertical displacement on the upper side ($\tau = 4000\ \mathrm{s}^{-1}$, $h = 2.5$ m, $d = 0.00(5)$ m).

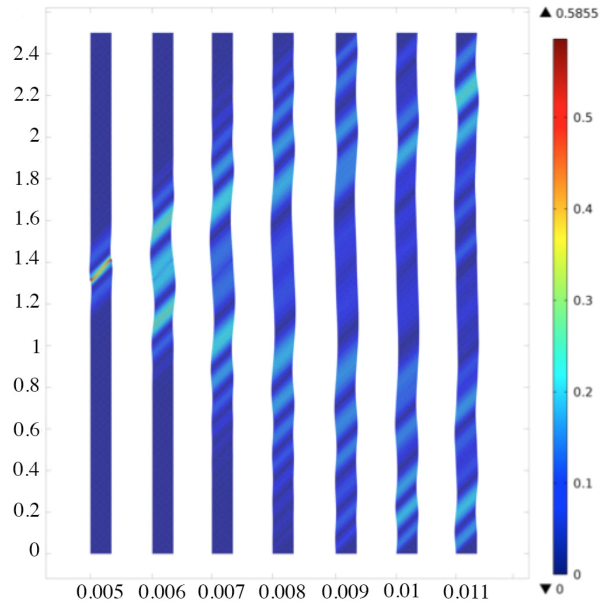

Fig. 3. Wave propagation after a double impulse ($h = 2.5$ m, $d = 0.00(1)$ m).

2.2. Internal discontinuities

In Fig. 4 the structure is provided with a horizontal set of hinges at the middle height. In the plot the local bending moment is represented by means of a colour map (in N m). The hinges, in this case, do interrupt the continuity of the beams, allowing energy-free angular displacements between the upper and the lower part of each beam. However, as it can be observed, due to the kind of internal connections between the whole system of beams, this does not change the overall character of wave propagation.

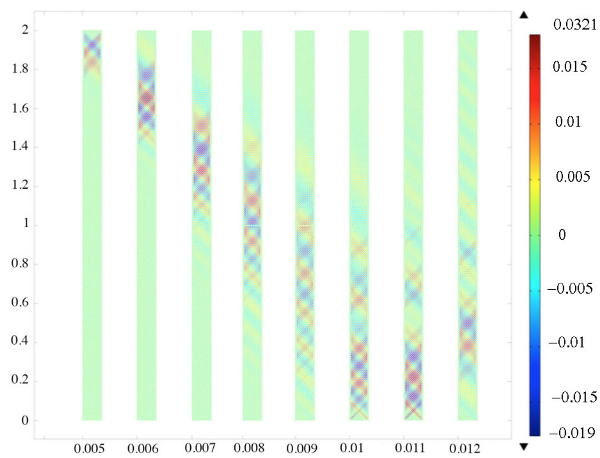

Fig. 4. Bending moment in two lattices connected by hinges ($h = 2$ m, $d = 0.00(1)$ m).

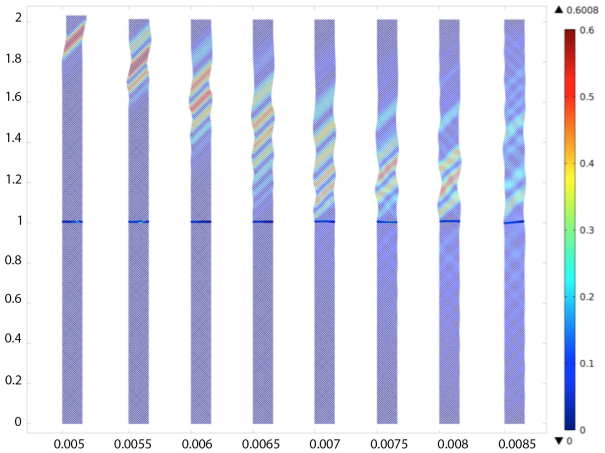

Fig. 5. Wave propagation in two lattices connected by an array of vertical beams ($h = 2$ m, $d = 0.00(5)$ m).

In Fig. 5 the upper and lower half of the system are connected by an array of vertical beams ($k_M = 1.96 \times 10^{-2}$ N m^2 and $k_N = 7.85 \times 10^4$ N). In this case, the imposed displacement is parallel to one of the two families of beams. It is interesting that the energy of the system remains more or less confined in the upper half, and waves practically do not propagate beyond the discontinuity, which therefore results in a simplified but potentially useful model for damping filters in the considered kind of structures.

2.3. Waves travelling in opposite directions

Finally, in Fig. 6 the initial impulse (parallel to one of the two families of beams) is imposed on both the upper and lower sides. Two waves travelling in opposite directions appear, and their interaction is shown. As one can see, the velocity of both wave fronts remains more or less unchanged after the crossing over. Moreover, in every snapshot a rather well-circumscribed travelling region displaying maximum perturbation is observable. These characteristics are shared by the well-known self-supporting, localized travelling perturbations, called 'solitons', originating in particular non-linear systems.

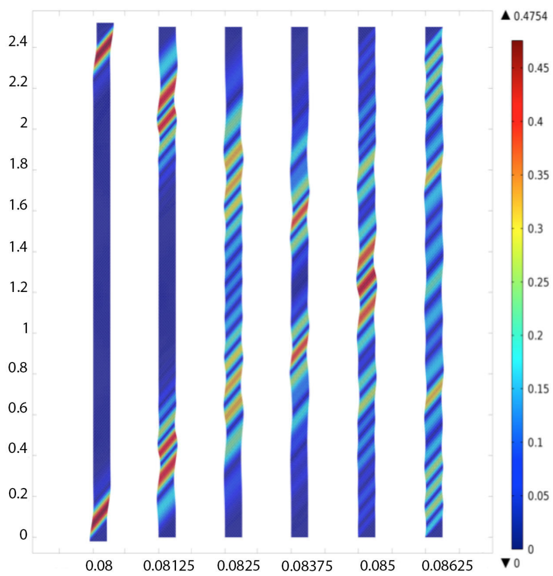

Fig. 6. Propagation of two waves travelling in opposite directions ($h = 2.5$ m, $d = 0.00(5)$ m).

3. CONCLUSIONS

A conjecture that can be validly proposed in this context is the possibility of the emerging of solitons, i.e. solitary waves propagating without changing their shape and speed due to the balance of all physical effects (e.g. between dispersion and nonlinearity [38]). Solitons were first observed in numerical simulations while studying the well-known Korteweg–de Vries (KdV) equation,

$$u_t = Kuu_x - u_{xxx}, \tag{2}$$

first applied in problems of hydrodynamics. The solutions of equations of this kind can be decomposed in localized perturbations with a well-defined shape that propagate at different velocities and preserve their shape and velocity when interacting with other waves. A short but very clear historical review on solitons in elastic solids is presented by Maugin [28]. Zabusky and Kruskal [45] demonstrated the emergence of a train of solitons from a harmonic initial condition for a given dispersion constant in the case of the KdV equation. The KdV equation is solved numerically using the pseudospectral method (see [20,37] for details). Soliton solutions are also appearing in nonlinear Cosserat models with a special coupling between rotations and deformations [6].

In fact, the study of the propagation of a pulse along the fabric can be made using a discrete model consisting of two linear orders of one-dimensional beams of identical geometrical and mechanical characteristics, interacting with each other through constraints, which impose the continuity of the displacement in a finite number of pivot points. The intensity of the coupling can be adjusted by varying the value of the elastic constants and the mass density of the beams. In a one-dimensional context, a beam discretized by means of concentrated masses and linear springs is studied e.g. in [41]. A case with dispersion is obtained by suitably adjusting the parameters that characterize the system. If we reduce the pulse duration and increase the value of the coupling constant, the width of the wave packet (train) will begin to be comparable with the pitch of the two rows of beams. The dispersion effects, as observed, are already visible in the simulations. Furthermore, the structure exhibits large rotations and hence the problem can be modelled taking into account the deformation of the structure when formulating the dynamic equations. On the whole, the structure and the model appears to be rich enough to allow the onset of true solitons if suitable non-linearities are considered. The soliton-like character of the perturbations shown in Fig. 6 suggests that further investigation in this direction would indeed be very interesting.

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Lainelevi kahemõõtmelises sisemiste katkevustega pantograafilises võres

Angela Madeo, Alessandro Della Corte, Leopoldo Greco ja Patrizio Neff

On vaadeldud kahest ortogonaalsest Euleri talade perest koosnevat kahemõõtmelist pantograafilist võret. Võret moodustavatele lainejuhtidele rakendatud rajasiirded võivad tekitada levivaid laineid, mis oma omadustelt võivad olla ka mittelineaarsed. Autorid on teostanud numbrilisi simulatsioone ja vaadelnud siin mõningaid dünaamika seisukohast huvitavaid juhte. Juhul kui võres tekivad suured pöörded, võib see kaasa tuua geomeetrilise mittelineaarsuse ja avada tee solitonide tekkimiseks.