Research Article

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On a Singular Robin Problem with Convection Terms

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Abstract: In this paper, the existence of smooth positive solutions to a Robin boundary-value problem with non-homogeneous differential operator and reaction given by a nonlinear convection term plus a singular one is established. Proofs chiefly exploit sub-super-solution and truncation techniques, set-valued analysis, recursive methods, nonlinear regularity theory, as well as fixed point arguments. A uniqueness result is also presented.

Keywords: Robin Problem, Quasilinear Elliptic Equation, Gradient Dependence, Singular Term

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$, $g\colon \Omega \times (0, +\infty) \to [0, +\infty)$ be two Carathéodory functions. In this paper, we study existence and uniqueness of solutions to the following Robin problem:

$$\begin{cases} -\operatorname{div} a(\nabla u) = f(x, u, \nabla u) + g(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_a} + \beta u^{p-1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a\colon \mathbb{R}^N \to \mathbb{R}^N$ denotes a continuous strictly monotone map having suitable properties, which basically stem from Liebermann's nonlinear regularity theory [12] and Pucci–Serrin's maximum principle [19]; see Section 2 for details. Moreover, $\beta > 0$, $1 < p < +\infty$, while $\frac{\partial}{\partial \nu_a}$ denotes the co-normal derivative associated with a.

This problem gathers together several hopefully interesting technical features, namely:

- The involved differential operator appears in a general form that includes non-homogeneous cases.
- f depends on the solution and its gradient. So, the reaction exhibits nonlinear convection terms.
- g can be singular at zero, i.e., $\lim_{s\to 0^+} g(x, s) = +\infty$.
- Robin boundary conditions are imposed instead of (much more frequent) Dirichlet ones.

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All these things have been extensively investigated, although separately. For instance, both differential operator and Robin conditions already appear in [8], where, however, the problem has a fully variational structure, whilst [15] falls inside non-variational settings. The paper [4] addresses the presence of convection terms; see also [14, 15, 20], which exhibit more general contexts. Last but not least, singular problems were considered especially after the seminal works of Crandall, Rabinowitz and Tartar [2], and Lazer and McKenna [10]. Among recent contributions on this subject, we mention [7, 16]. Finally, [13] treats a p-Laplacian Dirichlet problem whose right-hand side has the same form as that in (1.1). It represented the starting point of our research.

Several issues arise when passing from Dirichlet to Robin boundary conditions. Accordingly, here, we try to develop some useful tools in this direction, including the location of solutions to an auxiliary variational problem inside an opportune sublevel of its energy functional, constructed for preserving some compactness and semicontinuity properties (cf. Section 3).

Our main result, Theorem 3.19, establishes the existence of a regular solution to (1.1) chiefly via sub-super-solution and truncation techniques, set-valued analysis, recursive methods, nonlinear regularity theory, as well as Schaefer's fixed point theorem. Uniqueness is also addressed, but only when $p = 2$ (vide Section 4).

Usually, linear problems possess only one solution, whereas multiplicity is encountered in nonlinear phenomena. Hence, it might be of interest to seek hypotheses on f and g that yield uniqueness even if $p \neq 2$. As far as we know, this is still an open problem.

Let us finally note that replacing the constant β with a nontrivial non-negative function $\beta \in L^\infty(\Omega)$ does not invalidate our results.

2 Preliminaries

Let X be a set and let $C \subseteq X$. We denote by χ_C the characteristic function of C. If $C \neq \emptyset$ and $\Gamma\colon C \to C$, then

$$\mathrm{Fix}(\Gamma) := \{x \in C : x = \Gamma(x)\}$$

is the fixed point set of Γ. The following result, usually called Schaefer's theorem [6, p. 827] or Leray–Schauder's alternative principle, will play a basic role in the sequel.

Theorem 2.1. *Let X be a Banach space, let $C \subseteq X$ be nonempty convex, and let $\Gamma\colon C \to C$ be continuous. Suppose Γ maps bounded sets into relatively compact sets. Then either $\{x \in C : x = t\Gamma(x)$ for some $t \in (0,1)\}$ turns out unbounded or $\mathrm{Fix}(\Gamma) \neq \emptyset$.*

Given a partially ordered set $(X, \leq)$, we say that X is downward directed when, for every $x_1, x_2 \in X$, there exists $x \in X$ such that $x \leq x_i$, $i = 1, 2$. The notion of upward directed set is analogous.

If Y is a real function space on a set $\Omega \subseteq \mathbb{R}^N$ and $u, v \in Y$, then $u \leq v$ means $u(x) \leq v(x)$ for almost every $x \in \Omega$. Moreover, $Y_+ := \{u \in Y : u \geq 0\}$, $\Omega(u \leq v) := \{x \in \Omega : u(x) \leq v(x)\}$, etc.

Let X, Y be two metric spaces and let $\mathscr{S}\colon X \to 2^Y$. The multifunction $\mathscr{S}$ is called lower semicontinuous when for every $x_n \to x$ in X, $y \in \mathscr{S}(x)$, there exists a sequence $\{y_n\} \subseteq Y$ having the following properties:

$$y_n \to y \quad \text{in } Y, \qquad y_n \in \mathscr{S}(x_n) \quad \text{for all } n \in \mathbb{N}.$$

Finally, if X is a Banach space and $J \in C^1(X)$, then

$$\mathrm{Crit}(J) := \{x \in X : J'(x) = 0\}$$

is the critical set of J.

The monograph [1] represents a general reference on these topics.

Given any $s > 1$, the symbol s' will indicate the conjugate exponent of s, namely, $s' := \frac{s}{s-1}$.

Henceforth, for $1 < p < +\infty$, $\beta > 0$, Ω as in the Introduction, and $u\colon \overline{\Omega} \to \mathbb{R}$ appropriate, the notation below will be adopted:

$$\|u\|_\infty := \operatorname*{ess\,sup}_{x\in\Omega}|u(x)|, \qquad \|u\|_{C^1(\overline{\Omega})} := \|u\|_\infty + \|\nabla u\|_\infty,$$
$$\|u\|_p := \Big(\int_\Omega |u|^p\,dx\Big)^{\frac{1}{p}}, \qquad \|u\|_{p,\partial\Omega} := \Big(\int_{\partial\Omega} |u|^p\,d\sigma\Big)^{\frac{1}{p}},$$
$$\|u\|_{1,p} := (\|u\|_p^p + \|\nabla u\|_p^p)^{\frac{1}{p}}, \quad \|u\|_{\beta,1,p} := (\beta\|u\|_{p,\partial\Omega}^p + \|\nabla u\|_p^p)^{\frac{1}{p}}.$$

Here, σ denotes the $(N-1)$-dimensional Hausdorff measure on $\partial\Omega$. If $\nu(x)$ is the outward unit normal vector to $\partial\Omega$ at its point x, then $\frac{\partial}{\partial\nu_a}$ stands for the co-normal derivative associated with a, defined by extending the map $u \mapsto \langle a(\nabla u), \nu\rangle$ from $C^1(\overline{\Omega})$ to $W^{1,p}(\Omega)$.

Remark 2.2. The trace inequality ensures that $\|u\|_{p,\partial\Omega}$ makes sense whenever $u \in W^{1,p}(\Omega)$; see, for instance, [3] or [9].

Remark 2.3. It is known that (see [5])

$$\operatorname{int}(C^1(\overline{\Omega})_+) = \{u \in C^1(\overline{\Omega}) : u(x) > 0 \text{ for all } x \in \overline{\Omega}\}.$$

Remark 2.4. $\|\cdot\|_{\beta,1,p}$ is a norm on $W^{1,p}(\Omega)$ equivalent to $\|\cdot\|_{1,p}$. In particular, there exists $c_1 = c_1(p,\beta,\Omega) \in (0,1)$ such that

$$c_1\|u\|_{1,p} \le \|u\|_{\beta,1,p} \le \frac{1}{c_1}\|u\|_{1,p} \quad \text{for all } u \in W^{1,p}(\Omega)\,. \tag{2.1}$$

For the proof we refer to [17].

Let $\omega \in C^1(0,+\infty)$ satisfy

$$C_1 \le \frac{t\omega'(t)}{\omega(t)} \le C_2, \quad C_3 t^{p-1} \le \omega(t) \le C_4(1+t^{p-1})$$

in $(0,+\infty)$, with C_i suitable positive constants.

Assumption 2.5. The operator $a\colon \mathbb{R}^N \to \mathbb{R}^N$ has the following properties:
(i) $a(\xi) = a_0(|\xi|)\xi$ for all $\xi \in \mathbb{R}^N$, where $a_0\colon (0,+\infty) \to (0,+\infty)$ is C^1, $t \mapsto ta_0(t)$ turns out strictly increasing, and

$$\lim_{t\to 0^+} ta_0(t) = 0, \quad \lim_{t\to 0^+} \frac{ta_0'(t)}{a_0(t)} > -1.$$

(ii) $|Da(\xi)| \le C_5 \frac{\omega(|\xi|)}{|\xi|}$ in $\mathbb{R}^N \setminus \{0\}$.
(iii) $\langle Da(\xi)y, y\rangle \ge \frac{\omega(|\xi|)}{|\xi|}|y|^2$ for every $y, \xi \in \mathbb{R}^N$, $\xi \ne 0$.

Example 2.6. Various differential operators comply with Assumption 2.5. Three classical examples are listed below.
- The so-called p-Laplacian: $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, which stems from $a_0(t) := t^{p-2}$.
- The (p,q)-Laplacian: $\Delta_p u + \Delta_q u$, where $1 < q < p < +\infty$. In this case, $a_0(t) := t^{p-2} + t^{q-2}$.
- The generalized p-mean curvature operator:

$$u \mapsto \operatorname{div}[(1+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u],$$

corresponding to $a_0(t) := (1+t^2)^{\frac{p-2}{2}}$.

Finally, define

$$G_0(t) := \int_0^t s a_0(s)\,\mathrm{d}s \quad \text{for all } t \in \mathbb{R}, \qquad G(\xi) := G_0(|\xi|) \quad \text{for all } \xi \in \mathbb{R}^N.$$

Proposition 2.7. *Under Assumption 2.5, there exists $c_2 \in (0,1)$ such that*

$$|a(\xi)| \le \frac{1}{c_2}(1+|\xi|^{p-1}) \quad \textit{and} \quad c_2|\xi|^p \le \langle a(\xi), \xi\rangle \le \frac{1}{c_2}(1+|\xi|^p)$$

for all $\xi \in \mathbb{R}^N$. In particular,

$$c_2|\xi|^p \le G(\xi) \le \frac{1}{c_2}(1+|\xi|^p), \quad \xi \in \mathbb{R}^N.$$

Proof. See [8, Lemmas 2.1–2.2] or [17, Lemma 2.2 and Corollary 2.3]. □

3 Existence

Throughout this section, the convection term f and the singularity g will fulfill the assumptions below, where, to avoid unnecessary technicalities, 'for all x' takes the place of 'for almost all x'.

Assumption 3.1. $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,+\infty)$ is a Carathéodory function. Moreover, to every $M > 0$, there correspond $c_M, d_M > 0$ such that

$$f(x,s,\xi) \le c_M + d_M|s|^{p-1} \quad \text{for all } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \text{ with } |\xi| \le M.$$

Assumption 3.2. $g\colon \Omega \times (0,+\infty) \to [0,+\infty)$ is a Carathéodory function having the following properties:
(i) $g(x,\cdot)$ turns out nonincreasing on $(0,1]$ whenever $x \in \Omega$, and $g(\cdot,1) \not\equiv 0$.
(ii) There exist $c, d > 0$ such that

$$g(x,s) \le c + ds^{p-1} \quad \text{for all } (x,s) \in \Omega \times (1,+\infty).$$

(iii) With appropriate $\theta \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and $\varepsilon_0 > 0$, the map $x \mapsto g(x, \varepsilon\theta(x))$ belongs to $L^{p'}(\Omega)$ for any $\varepsilon \in (0,\varepsilon_0)$.

The paper [13] contains meaningful examples of functions g that satisfy Assumption 3.2. A very simple case is $g(x,s) := s^{-\gamma}$ for all $(x,s) \in \Omega \times (0,+\infty)$, where $\gamma > 0$ and $\theta(\cdot) \equiv 1$.

Fix $w \in C^1(\overline{\Omega})$. We first focus on the singular problem (without convection terms)

$$\begin{cases} -\operatorname{div} a(\nabla u) = f(x,u,\nabla w) + g(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_a} + \beta u^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Definition 3.3. $u \in W^{1,p}(\Omega)$ is called a subsolution to (3.1) when

$$\int_\Omega \langle a(\nabla u), \nabla v\rangle \,dx + \beta \int_{\partial\Omega} |u|^{p-2}uv \,d\sigma \le \int_\Omega [f(\cdot,u,\nabla w) + g(\cdot,u)]v \,dx$$

for all $v \in W^{1,p}(\Omega)_+$. The set of subsolutions will be denoted by $\underline{U}_w$.

We say that $u \in W^{1,p}(\Omega)$ is a supersolution to (3.1) if

$$\int_\Omega \langle a(\nabla u), \nabla v\rangle \,dx + \beta \int_{\partial\Omega} |u|^{p-2}uv \,d\sigma \ge \int_\Omega [f(\cdot,u,\nabla w) + g(\cdot,u)]v \,dx \tag{3.2}$$

for every $v \in W^{1,p}(\Omega)_+$, and indicate with $\overline{U}_w$ the supersolution set.

Finally, $u \in W^{1,p}(\Omega)$ is called a solution of (3.1), provided

$$\int_\Omega \langle a(\nabla u), \nabla v\rangle \,dx + \beta \int_{\partial\Omega} |u|^{p-2}uv \,d\sigma = \int_\Omega [f(\cdot,u,\nabla w) + g(\cdot,u)]v \,dx$$

for all $v \in W^{1,p}(\Omega)_+$. The corresponding solution set will be denoted by U_w. Obviously, $U_w = \overline{U}_w \cap \underline{U}_w$.

Lemma 3.4. *If* $u_1, u_2 \in \overline{U}_w$ *(resp.* $u_1, u_2 \in \underline{U}_w$*), then* $\min\{u_1, u_2\} \in \overline{U}_w$ *(resp.* $\max\{u_1, u_2\} \in \underline{U}_w$*). In particular, the set* $\overline{U}_w$ *(resp.* $\underline{U}_w$*) is downward (resp. upward) directed.*

Proof. This proof is patterned after that of [13, Lemma 10] (see also [1]). Thus, we only sketch it. Pick $u_1, u_2 \in \overline{U}_w$, set $u := \min\{u_1, u_2\}$, and define, for every $t \in \mathbb{R}$,

$$\eta_\varepsilon(t) := \begin{cases} 0 & \text{if } t < 0, \\ \frac{t}{\varepsilon} & \text{if } 0 \le t \le \varepsilon, \\ 1 & \text{if } t > \varepsilon, \end{cases}$$

where $\varepsilon > 0$. Further, to shorten the notation, write $\bar\eta_\varepsilon(x) := \eta_\varepsilon(u_2(x) - u_1(x))$. Evidently, we have both $\bar\eta_\varepsilon \in W^{1,p}(\Omega)_+$ and

$$\nabla\bar\eta_\varepsilon = \eta'_\varepsilon(u_2 - u_1)\,\nabla(u_2 - u_1).$$

Let $\hat v \in C^1(\overline{\Omega})_+$. Since u_i fulfills (3.2), one has

$$\int_\Omega \langle a(\nabla u_i), \nabla v\rangle \,\mathrm{d}x + \beta \int_{\partial\Omega} |u_i|^{p-2} u_i v \,\mathrm{d}\sigma \ge \int_\Omega [f(\cdot\,, u_i, \nabla w) + g(\cdot\,, u_i)] v \,\mathrm{d}x$$

whenever $v \in W^{1,p}(\Omega)_+$. Choosing $v := \bar\eta_\varepsilon \hat v$ if $i = 1$, $v := (1 - \bar\eta_\varepsilon)\hat v$ if $i = 2$, and adding them term by term produces

$$\begin{aligned} &\int_\Omega \big\langle a(\nabla u_1) - a(\nabla u_2), \nabla(u_2 - u_1)\big\rangle \eta'_\varepsilon(u_2 - u_1)\hat v \,\mathrm{d}x + \int_\Omega \langle a(\nabla u_1), \nabla\hat v\rangle\, \bar\eta_\varepsilon \,\mathrm{d}x + \int_\Omega \langle a(\nabla u_2), \nabla\hat v\rangle(1 - \bar\eta_\varepsilon) \,\mathrm{d}x \\ &\quad + \beta\Big(\int_{\partial\Omega} |u_1|^{p-2} u_1 \bar\eta_\varepsilon \hat v \,\mathrm{d}\sigma + \int_{\partial\Omega} |u_2|^{p-2} u_2 (1 - \bar\eta_\varepsilon)\hat v \,\mathrm{d}\sigma \Big) \\ &\ge \int_\Omega [f(\cdot\,, u_1, \nabla w) + g(\cdot\,, u_1)]\bar\eta_\varepsilon \hat v \,\mathrm{d}x + \int_\Omega [f(\cdot\,, u_2, \nabla w) + g(\cdot\,, u_2)](1 - \bar\eta_\varepsilon)\hat v \,\mathrm{d}x. \end{aligned} \tag{3.3}$$

The strict monotonicity of a combined with $\eta'_\varepsilon(u_2 - u_1)\hat v \ge 0$ lead to

$$\int_\Omega \big\langle a(\nabla u_1) - a(\nabla u_2), \nabla(u_2 - u_1)\big\rangle \eta'_\varepsilon(u_2 - u_1)\hat v \,\mathrm{d}x \le 0.$$

For almost every $x \in \Omega$, we have

$$\nabla u(x) = \begin{cases} \nabla u_1(x) & \text{if } u_1(x) < u_2(x), \\ \nabla u_2(x) & \text{otherwise}, \end{cases}$$

as well as

$$\lim_{\varepsilon\to 0^+} \bar\eta_\varepsilon(x) = \chi_{\Omega(u_1<u_2)}(x).$$

Hence, letting $\varepsilon \to 0^+$ and using the dominated convergence theorem, inequality (3.3) becomes

$$\int_\Omega \langle a(\nabla u), \nabla\hat v\rangle \,\mathrm{d}x + \beta \int_{\partial\Omega} |u|^{p-2} u \hat v \,\mathrm{d}\sigma \ge \int_\Omega [f(\cdot\,, u, \nabla w) + g(\cdot\,, u)]\hat v \,\mathrm{d}x,$$

see [13, Lemma 10] for more details. Since $\hat v \in C^1(\overline{\Omega})_+$ was arbitrary, by density, one arrives at $u \in \overline{U}_w$. □

Lemma 3.5. *Let Assumptions 3.1–3.2 be satisfied. Then there exists a subsolution* $\underline{u} \in \operatorname{int}(C^1(\overline{\Omega})_+)$ *to* (3.1) *independent of* w *and such that* $\|\underline{u}\|_\infty \le 1$.

Proof. Given any $\delta > 0$, consider the problem

$$\begin{cases} -\operatorname{div} a(\nabla u) = \tilde g(x, u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_a} + \beta |u|^{p-2} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where

$$\tilde{g}(x,s) := \min\{g(x,s),\delta\}, \quad (x,s)\in\Omega\times(0,+\infty). \tag{3.5}$$

Standard arguments yield a nontrivial solution $\underline{u}\in W^{1,p}(\Omega)$ to (3.4), because $\tilde{g}$ is bounded. Testing with $-\underline{u}^-$, we get

$$-\int_\Omega \langle a(\nabla\underline{u}),\nabla\underline{u}^-\rangle\,dx - \beta\int_\Omega |\underline{u}|^{p-2}\underline{u}\underline{u}^-\,d\sigma = -\int_\Omega \tilde{g}(x,\underline{u})\underline{u}^-\,dx \le 0,$$

whence, by Proposition 2.7,

$$c_2\|\underline{u}^-\|^p_{\beta,1,p} \le \int_\Omega \langle a(\nabla\underline{u}^-),\nabla\underline{u}^-\rangle\,dx + \beta\int_\Omega (\underline{u}^-)^p\,d\sigma \le 0.$$

Therefore, $\underline{u}\ge 0$. Regularity up to the boundary [12] and the strong maximum principle [19] then force $\underline{u}\in\operatorname{int}(C^1(\overline{\Omega})_+)$. Now, if $u_\delta\in C^{1,\alpha}(\overline{\Omega})_+$ satisfies

$$\begin{cases} -\operatorname{div} a(\nabla u) = \delta & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu_a} + \beta|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

then, by compactness of the embedding $C^{1,\alpha}(\overline{\Omega})) \hookrightarrow C^1(\overline{\Omega})$, we can find $u\in C^1(\overline{\Omega})$ such that $\lim_{\delta\to 0^+} u_\delta = u$ in $C^1(\overline{\Omega})$ up to subsequences. One evidently has $u\equiv 0$, because u_δ solves (3.6). Thus, $0\le u_\delta\le 1$ once δ is small enough. Using (3.5), the comparison principle finally entails

$$\|\underline{u}\|_\infty \le \|u_\delta\|_\infty \le 1. \tag{3.7}$$

Let θ and ε_0 be as in Assumption 3.2 (iii). Since $\underline{u},\theta\in\operatorname{int}(C^1(\overline{\Omega})_+)$, there exists $\varepsilon\in(0,\varepsilon_0)$ such that $\underline{u}-\varepsilon\theta\in\operatorname{int}(C^1(\overline{\Omega})_+)$. From Assumption 3.2 (i), (iii), and (3.7), we thus infer

$$0\le g(\cdot,\underline{u})\le g(\cdot,\varepsilon\theta)\in L^{p'}(\Omega). \tag{3.8}$$

The conclusion is achieved by verifying that $\underline{u}\in \underline{U}_w$ for any $w\in C^1(\overline{\Omega})$. Pick such a w, test (3.4) with $v\in W^{1,p}(\Omega)_+$, and recall (3.5), to arrive at

$$\int_\Omega \langle a(\nabla\underline{u}),\nabla v\rangle\,dx + \beta\int_{\partial\Omega} \underline{u}^{p-1}v\,d\sigma = \int_\Omega \tilde{g}(\cdot,\underline{u})v\,dx \le \int_\Omega g(\cdot,\underline{u})v\,dx \le \int_\Omega [f(\cdot,u,\nabla w)+g(\cdot,\underline{u})]v\,dx,$$

as desired. □

Remark 3.6. This proof shows that the subsolution $\underline{u}$ constructed in Lemma 3.5 enjoys further the property:

$$\int_\Omega \langle a(\nabla\underline{u}),\nabla v\rangle\,dx + \beta\int_{\partial\Omega} |\underline{u}|^{p-2}\underline{u}v\,d\sigma \le \int_\Omega g(\cdot,\underline{u})v\,dx \quad \text{for all } v\in W^{1,p}(\Omega)_+. \tag{3.9}$$

Given $w\in C^1(\overline{\Omega})$, consider the truncated problem

$$\begin{cases} -\operatorname{div} a(\nabla u) = \hat{f}(x,u)+\hat{g}(x,u) & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu_a} + \beta u^{p-1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

where

$$\hat{f}(x,s) := \begin{cases} f(x,\underline{u}(x),\nabla w(x)) & \text{if } s\le \underline{u}(x),\\ f(x,s,\nabla w(x)) & \text{otherwise,} \end{cases} \tag{3.11}$$

$$\hat{g}(x,s) := \begin{cases} g(x,\underline{u}(x)) & \text{if } s\le \underline{u}(x),\\ g(x,s) & \text{otherwise.} \end{cases} \tag{3.12}$$

The energy functional corresponding to (3.10) reads

$$\mathscr{E}_w(u) := \frac{1}{p}\int_\Omega G(\nabla u)\,\mathrm{d}x + \frac{\beta}{p}\int_{\partial\Omega} |u|^p\,\mathrm{d}\sigma - \int_\Omega \hat{F}(\cdot\,,u)\,\mathrm{d}x - \int_\Omega \hat{G}(\cdot\,,u)\,\mathrm{d}x$$

for all $u \in W^{1,p}(\Omega)$, with

$$\hat{F}(x,s) := \int_0^s \hat{f}(x,t)\,\mathrm{d}t, \quad \hat{G}(x,s) := \int_0^s \hat{g}(x,t)\,\mathrm{d}t.$$

Assumptions 3.1–3.2 ensure that $\mathscr{E}_w$ is of class C^1 and weakly sequentially lower semicontinuous; see, e.g., [8, Lemma 3.1]. Under the additional condition

$$d_M + d < c_1^p c_2 \quad \text{for all } M > 0, \tag{3.13}$$

it turns out also coercive, as the next lemma shows.

Lemma 3.7. *Let $\mathscr{B}$ be a nonempty bounded set in $C^1(\overline{\Omega})$. If Assumptions 3.1–3.2 and condition* (3.13) *hold true, then there exist $\alpha_1 \in (0,1)$, $\alpha_2 > 0$ such that*

$$\mathscr{E}_w(u) \geq \frac{\alpha_1}{p}\|u\|_{1,p}^p - \alpha_2(1 + \|u\|_{1,p}) \quad \textit{for all } (u,w) \in W^{1,p}(\Omega)\times\mathscr{B}.$$

Proof. Put $\hat{M} := \sup_{w\in\mathscr{B}}\|w\|_{C^1(\overline{\Omega})}$. By (3.11)–(3.12), Proposition 2.7 entails

$$\begin{aligned}\mathscr{E}_w(u) \geq \frac{c_2}{p}\|\nabla u\|_p^p + \frac{\beta}{p}\|u\|_{p,\partial\Omega}^p - \int_\Omega [f(\cdot\,,\underline{u},\nabla w) + g(\cdot\,,\underline{u})]\underline{u}\,\mathrm{d}x \\ - \int_{\Omega(u>\underline{u})}\bigg(\int_{\underline{u}}^{u} f(\cdot\,,t,\nabla w)\,\mathrm{d}t\bigg)\mathrm{d}x - \int_{\Omega(u>\underline{u})}\bigg(\int_{\underline{u}}^{u} g(\cdot\,,t)\,\mathrm{d}t\bigg)\mathrm{d}x.\end{aligned}$$

Assumption 3.1 along with Hölder's inequality imply

$$\begin{aligned}\int_{\Omega(u>\underline{u})}\bigg(\int_{\underline{u}}^{u} f(\cdot\,,t,\nabla w)\,\mathrm{d}t\bigg)\mathrm{d}x &\leq \int_{\Omega(u>\underline{u})}\bigg(\int_0^u f(\cdot\,,t,\nabla w)\,\mathrm{d}t\bigg)\mathrm{d}x \\ &\leq c_{\hat{M}}|\Omega|^{\frac{1}{p'}}\|u\|_p + \frac{d_{\hat{M}}}{p}\|u\|_p^p \leq c_{\hat{M}}|\Omega|^{\frac{1}{p'}}\|u\|_{1,p} + \frac{d_{\hat{M}}}{p}\|u\|_{1,p}^p.\end{aligned}$$

Exploiting (3.7), Assumption 3.2 (i), (ii), and Hölder's inequality again, we have

$$\begin{aligned}\int_{\Omega(u>\underline{u})}\bigg(\int_{\underline{u}}^{u} g(\cdot\,,t)\,\mathrm{d}t\bigg)\mathrm{d}x &\leq \int_{\Omega(u>\underline{u})}\bigg(\int_{\underline{u}}^{1} g(\cdot\,,t)\,\mathrm{d}t\bigg)\mathrm{d}x + \int_{\Omega(u>1)}\bigg(\int_1^u g(\cdot\,,t)\,\mathrm{d}t\bigg)\mathrm{d}x \\ &\leq \int_{\Omega(u>\underline{u})} g(\cdot\,,\underline{u})\,\mathrm{d}x + \int_{\Omega(u>1)}\bigg(\int_1^u (c + dt^{p-1})\,\mathrm{d}t\bigg)\mathrm{d}x \\ &\leq \int_\Omega g(\cdot\,,\underline{u})\,\mathrm{d}x + c|\Omega|^{\frac{1}{p'}}\|u\|_p + \frac{d}{p}\|u\|_p^p \\ &\leq \int_\Omega g(\cdot\,,\underline{u})\,\mathrm{d}x + c|\Omega|^{\frac{1}{p'}}\|u\|_{1,p} + \frac{d}{p}\|u\|_{1,p}^p.\end{aligned}$$

Hence, through (2.1) we easily arrive at

$$\begin{aligned}\mathscr{E}_w(u) &\geq \frac{c_2}{p}\|u\|_{\beta,1,p}^p - \frac{d_{\hat{M}} + d}{p}\|u\|_{1,p}^p - (c_{\hat{M}} + c)|\Omega|^{\frac{1}{p'}}\|u\|_p - K \\ &\geq \frac{c_1^p c_2 - d_{\hat{M}} - d}{p}\|u\|_{1,p}^p - (c_{\hat{M}} + c)|\Omega|^{\frac{1}{p'}}\|u\|_{1,p} - K \\ &\geq \frac{c_1^p c_2 - d_{\hat{M}} - d}{p}\|u\|_{1,p}^p - \max\{(c_{\hat{M}} + c)|\Omega|^{\frac{1}{p'}}, K\}(1 + \|u\|_{1,p}),\end{aligned}$$

where

$$\begin{aligned}K &:= \int_\Omega [f(\cdot,\underline{u},\nabla w)+g(\cdot,\underline{u})]\underline{u}\,\mathrm{d}x+\int_\Omega g(\cdot,\underline{u})\,\mathrm{d}x\\ &\le \int_\Omega (c_{\hat M}+d_{\hat M})\,\mathrm{d}x+2\int_\Omega g(\cdot,\varepsilon\theta)\,\mathrm{d}x\\ &\le (c_{\hat M}+d_{\hat M})|\Omega|+2\|g(\cdot,\varepsilon\theta)\|_{p'}|\Omega|^{\frac{1}{p}},\end{aligned}$$

due to Assumption 3.1 and (3.7)–(3.8). Now, the conclusion follows from (3.13). □

Remark 3.8. A standard application of Moser's iteration technique [11] shows that any solution to (3.10) lies in $L^\infty(\Omega)$. By Liebermann's regularity theory [12], it actually is Hölder continuous up to the boundary.

Lemma 3.9. *Let Assumptions 3.1–3.2 and condition* (3.13) *be satisfied. Then*

$$\emptyset \neq \operatorname{Crit}(\mathscr{E}_w) \subseteq U_w \cap \{u \in C^1(\overline{\Omega}) : u \ge \underline{u}\}.$$

Proof. Since $\mathscr{E}_w$ is coercive (cf. Lemma 3.7), the Weierstrass–Tonelli theorem produces $\operatorname{Crit}(\mathscr{E}_w)\neq\emptyset$. Pick any $u \in \operatorname{Crit}(\mathscr{E}_w)$, test (3.10) with $(\underline{u}-u)^+$, and exploit (3.11)–(3.12), besides (3.9), to achieve

$$\begin{aligned}\int_\Omega \langle a(\nabla u),\nabla(\underline{u}-u)^+\rangle\,\mathrm{d}x+\beta\int_{\partial\Omega}|u|^{p-2}u(\underline{u}-u)^+\,\mathrm{d}\sigma &= \int_\Omega [\hat f(\cdot,u)+\hat g(\cdot,u)](\underline{u}-u)^+\,\mathrm{d}x\\ &\ge \int_\Omega \hat g(\cdot,u)(\underline{u}-u)^+\,\mathrm{d}x=\int_\Omega g(\cdot,\underline{u})(\underline{u}-u)^+\,\mathrm{d}x\\ &\ge \int_\Omega \langle a(\nabla\underline{u}),\nabla(\underline{u}-u)^+\rangle\,\mathrm{d}x+\beta\int_{\partial\Omega}|\underline{u}|^{p-2}\underline{u}(\underline{u}-u)^+\,\mathrm{d}\sigma.\end{aligned}$$

Rearranging terms, we get

$$\int_\Omega \langle a(\nabla\underline{u})-a(\nabla u),\nabla(\underline{u}-u)^+\rangle\,\mathrm{d}x+\beta\int_{\partial\Omega}(|\underline{u}|^{p-2}\underline{u}-|u|^{p-2}u)(\underline{u}-u)^+\,\mathrm{d}\sigma\le 0.$$

The strict monotonicity of a, combined with [18, Lemma A.0.5], entail

$$\nabla(\underline{u}-u)^+=0 \quad \text{in } \Omega, \qquad (\underline{u}-u)^+=0 \quad \text{on } \partial\Omega.$$

So, $\|(\underline{u}-u)^+\|_{\beta,1,p}=0$, which means $u\ge\underline{u}$. Finally, by (3.11)–(3.12), one has $u\in U_w$, while $u\in C^1(\overline{\Omega})$ according to Remark 3.8. □

For every $w\in C^1(\overline{\Omega})$, we define

$$\mathscr{S}(w) := \{u \in C^1(\overline{\Omega}) : u\in U_w,\ u\ge\underline{u},\ \mathscr{E}_w(u)<1\}.$$

Lemma 3.10. *Under Assumptions 3.1–3.2 and condition* (3.13)*, the multifunction* $\mathscr{S} : C^1(\overline{\Omega}) \to 2^{C^1(\overline{\Omega})}$ *takes nonempty values, and maps bounded sets into relatively compact sets.*

Proof. If $w\in C^1(\overline{\Omega})$, then there exists $\hat u_w \in \operatorname{Crit}(\mathscr{E}_w)$ such that

$$\hat u_w \in C^1(\overline{\Omega}), \quad \hat u_w \ge \underline{u}, \quad \mathscr{E}_w(\hat u_w)=\inf_{W^{1,p}(\Omega)}\mathscr{E}_w \le \mathscr{E}_w(0)=0<1,$$

cf. the proof of Lemma 3.9. Hence, $\mathscr{S}(w)\neq\emptyset$, because $\hat u_w\in\mathscr{S}(w)$. Let $\mathscr{B}\subseteq C^1(\overline{\Omega})$ be nonempty and bounded. From Lemma 3.7, it follows that

$$\frac{\alpha_1}{p}\|u\|_{1,p}^p-\alpha_2(1+\|u\|_{1,p})\le\mathscr{E}_w(u)<1 \quad \text{for all } u\in\mathscr{S}(w),\ w\in\mathscr{B},$$

whence $\mathscr{S}(\mathscr{B})$ turns out bounded in $W^{1,p}(\Omega)$. By nonlinear regularity theory [12], the same holds when $C^{1,\alpha}(\overline{\Omega})$, with suitable $\alpha\in(0,1)$, replaces $W^{1,p}(\Omega)$. Recalling that $C^{1,\alpha}(\overline{\Omega})\hookrightarrow C^1(\overline{\Omega})$ compactly yields the conclusion. □

To see that $\mathscr{S}$ is lower semicontinuous, we shall employ the next technical lemma.

Lemma 3.11. *Let $\alpha, \beta, \gamma > 0$, let $1 < p < +\infty$, and let $\{a_k\} \subseteq [0, +\infty)$ satisfy the recursive relation*

$$\alpha a_k^p \le \beta a_k + \gamma a_{k-1}^p \quad \text{for all } k \in \mathbb{N}. \tag{3.14}$$

If $\gamma < \alpha$, then the sequence $\{a_k\}$ is bounded.

Proof. Using the obvious inequality

$$a_k \le T + T^{1-p} a_k^p, \quad T > 0,$$

(3.14) becomes

$$(\alpha - \beta T^{1-p}) a_k^p \le \beta T + \gamma a_{k-1}^p \quad \text{for all } k \in \mathbb{N}.$$

Since $\sigma := 1/p < 1$, this entails

$$(\alpha - \beta T^{1-p})^\sigma a_k \le (\beta T + \gamma a_{k-1}^p)^\sigma \le (\beta T)^\sigma + \gamma^\sigma a_{k-1}$$

or, equivalently,

$$a_k \le \left(\frac{\beta T}{\alpha - \beta T^{1-p}}\right)^\sigma + \left(\frac{\gamma}{\alpha - \beta T^{1-p}}\right)^\sigma a_{k-1}, \quad k \in \mathbb{N}, \tag{3.15}$$

provided $T > 0$ is large enough. Choosing $T > (\frac{\beta}{\alpha-\gamma})^{\frac{1}{p-1}}$, the coefficient of a_{k-1} turns out strictly less than 1. A standard computation based on (3.15) completes the proof. □

Lemma 3.12. *Suppose Assumptions 3.1–3.2 hold and, moreover,*

$$d_M + d < \frac{c_1^p c_2}{p} \quad \text{for all } M > 0. \tag{3.16}$$

Then the multifunction $\mathscr{S} : C^1(\overline{\Omega}) \to 2^{C^1(\overline{\Omega})}$ is lower semicontinuous.

Proof. The proof is patterned after that of [13, Lemma 20]. So, some details will be omitted. Let

$$w_n \to w \quad \text{in } C^1(\overline{\Omega}). \tag{3.17}$$

We claim that to each $\tilde{u} \in \mathscr{S}(w)$, there corresponds a sequence $\{u_n\} \subseteq C^1(\overline{\Omega})$ enjoying the following properties:

$$u_n \in \mathscr{S}(w_n), \quad n \in \mathbb{N}, \qquad u_n \to \tilde{u} \quad \text{in } C^1(\overline{\Omega}).$$

Fix $\tilde{u} \in \mathscr{S}(w)$. For every $n \in \mathbb{N}$, consider the auxiliary problem

$$\mathrm{P}(\tilde{u}, w_n) = \begin{cases} -\operatorname{div} a(\nabla u) = f(x, \tilde{u}, \nabla w_n) + \hat{g}(x, \tilde{u}) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_a} + \beta u^{p-1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.18}$$

with $\hat{g}(x, s)$ given by (3.12). One has $\hat{g}(x, \tilde{u}) = g(x, \tilde{u})$, because $\tilde{u} \in \mathscr{S}(w)$, while the associated energy functional reads

$$\mathscr{E}_{\tilde{u}, w_n}(u) := \frac{1}{p} \int_\Omega G(\nabla u)\, dx + \beta \int_{\partial\Omega} |u|^p\, d\sigma - \int_\Omega f(x, \tilde{u}, \nabla w_n) u\, dx - \int_\Omega \hat{g}(x, \tilde{u}) u\, dx, \quad u \in W^{1,p}(\Omega).$$

Since $\mathscr{E}_{\tilde{u}, w_n}$ turns out strictly convex, the same argument exploited to show Lemma 3.9 yields here a unique solution $u_n^0 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of (3.18) such that

$$\mathscr{E}_{\tilde{u}, w_n}(u_n^0) \le 0. \tag{3.19}$$

Via (3.17)–(3.19), reasoning as in Lemmas 3.7 and 3.10 (but for $\mathscr{E}_{\tilde{u}, w}$ instead of $\mathscr{E}_w$ and $\mathscr{B} := \{w_n : n \in \mathbb{N}\}$), we deduce that $\{u_n^0\} \subseteq C^1(\overline{\Omega})$ is relatively compact. Consequently, $u_n^0 \to u^0$ in $C^1(\overline{\Omega})$, where a subsequence is considered when necessary. By (3.17) again and Lebesgue's dominated convergence theorem, u^0 solves problem $\mathrm{P}(\tilde{u}, w)$. Thus, a fortiori, $u^0 = \tilde{u}$, because $\mathrm{P}(\tilde{u}, w)$ possesses one solution at most. An induction procedure provides now a sequence $\{u_n^k\}$ such that u_n^k solves problem $\mathrm{P}(u_n^{k-1}, w_n)$, the inequality $\mathscr{E}_{u_n^{k-1}, w_n}(u_n^k) \le 0$ holds, and

$$\lim_{n \to +\infty} u_n^k = \tilde{u} \quad \text{in } C^1(\overline{\Omega}) \text{ for all } k \in \mathbb{N}. \tag{3.20}$$

Claim: $\{u_n^k\}_{k\in\mathbb{N}} \subseteq C^1(\overline{\Omega})$ is relatively compact. In fact, recalling (3.17), pick $M = \sup_{n\in\mathbb{N}}\|w_n\|_{C^1(\overline{\Omega})}$. Through Hölder's and Young's inequalities, besides (3.8), we obtain

$$\frac{1}{p}\int_\Omega G(\nabla u_n^k)\,dx + \frac{\beta}{p}\int_{\partial\Omega}|u_n^k|^p\,d\sigma \geq \frac{c_1^p c_2}{p}\|u_n^k\|_{1,p}^p, \tag{3.21}$$

$$\begin{aligned}\int_\Omega f(\cdot\,, u_n^{k-1}, \nabla w_n)u_n^k\,dx &\leq c_M|\Omega|^{\frac{1}{p'}}\|u_n^k\|_p + d_M\int_\Omega |u_n^{k-1}|^{p-1}|u_n^k|\,dx \\ &\leq c_M|\Omega|^{\frac{1}{p'}}\|u_n^k\|_p + d_M\Big(\frac{1}{p'}\|u_n^{k-1}\|_p^p + \frac{1}{p}\|u_n^k\|_p^p\Big),\end{aligned} \tag{3.22}$$

as well as

$$\begin{aligned}\int_\Omega \hat{g}(\cdot\,, u_n^{k-1})u_n^k\,dx &= \int_{\Omega(u_n^{k-1}\leq 1)} \hat{g}(\cdot\,, u_n^{k-1})u_n^k\,dx + \int_{\Omega(u_n^{k-1}>1)} \hat{g}(\cdot\,, u_n^{k-1})u_n^k\,dx \\ &\leq \int_{\Omega(u_n^{k-1}\leq 1)} g(\cdot\,, \underline{u})u_n^k\,dx + \int_{\Omega(u_n^{k-1}>1)} g(\cdot\,, u_n^{k-1})u_n^k\,dx \\ &\leq (\|g(\cdot\,, \underline{u})\|_{p'} + c|\Omega|^{\frac{1}{p'}})\|u_n^k\|_p + d\int_\Omega |u_n^{k-1}|^{p-1}|u_n^k|\,dx \\ &\leq (\|g(\cdot\,, \underline{u})\|_{p'} + c|\Omega|^{\frac{1}{p'}})\|u_n^k\|_p + d\Big(\frac{1}{p'}\|u_n^{k-1}\|_p^p + \frac{1}{p}\|u_n^k\|_p^p\Big).\end{aligned} \tag{3.23}$$

Since $\mathscr{E}_{u_n^{k-1},w_n}(u_n^k) \leq 0$, estimates (3.21)–(3.23) entail

$$\frac{c_1^p c_2 - d_M - d}{p}\|u_n^k\|_{1,p}^p \leq (\|g(\cdot\,, \underline{u})\|_{p'} + (c_M + c)|\Omega|^{\frac{1}{p'}})\|u_n^k\|_{1,p} + \frac{d_M + d}{p'}\|u_n^{k-1}\|_{1,p}^p$$

for all $k \in \mathbb{N}$. Thanks to (3.16), Lemma 3.11 applies, and the sequence $\{u_n^k\}_{k\in\mathbb{N}}$ turns out bounded in $W^{1,p}(\Omega)$. Standard arguments involving regularity up to the boundary (cf. the proof of Lemma 3.10) yield the claim.

We may thus assume there exists $\{u_n\} \subseteq C^1(\overline{\Omega})$ fulfilling

$$\lim_{k\to\infty} u_n^k = u_n \quad \text{in } C^1(\overline{\Omega}) \tag{3.24}$$

whenever $n \in \mathbb{N}$. By (3.24) and Lebesgue's dominated convergence theorem, one has $u_n \in U_{w_n}$. Moreover, as in the proof of Lemma 3.9, $u_n \geq \underline{u}$. Due to (3.20) and (3.24), the double limit lemma [6, Proposition A.2.35] gives

$$u_n \to \tilde{u} \quad \text{in } C^1(\overline{\Omega}). \tag{3.25}$$

Thus, it remains to show that $\mathscr{E}_{w_n}(u_n) < 1$. From (3.17), we easily infer $\mathscr{E}_{w_n}(\tilde{u}) \to \mathscr{E}_w(\tilde{u})$. Since $\mathscr{E}_{w_n}$ is of class C^1, via (3.17) and (3.25), one arrives at

$$\lim_{n\to+\infty}(\mathscr{E}_{w_n}(u_n) - \mathscr{E}_w(\tilde{u})) = 0,$$

namely $\mathscr{E}_{w_n}(u_n) \to \mathscr{E}_w(\tilde{u})$. This completes the proof, because $\tilde{u} \in \mathscr{S}(w)$, whence $\mathscr{E}_w(\tilde{u}) < 1$. □

Lemma 3.13. *Under Assumptions 3.1–3.2 and condition* (3.13)*, the set* $\mathscr{S}(w)$*,* $w \in C^1(\overline{\Omega})$*, is downward directed.*

Proof. Let $u_1, u_2 \in \mathscr{S}(w)$ and let $\hat{u} := \min\{u_1, u_2\}$. By Lemma 3.4, we have $\hat{u} \in \overline{U}_w$. Consider the problem

$$\begin{cases} -\operatorname{div} a(\nabla u) = h(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_a} + \beta u^{p-1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.26}$$

where

$$h(x,s)=\begin{cases} f(x,\underline{u}(x),\nabla w(x))+g(x,\underline{u}(x)) & \text{if } s\le \underline{u}(x),\\ f(x,s,\nabla w(x))+g(x,s) & \text{if } \underline{u}(x)<s<\hat{u}(x),\\ f(x,\hat{u}(x),\nabla w(x))+g(x,\hat{u}(x)) & \text{if } s\ge \hat{u}(x).\end{cases}$$

The associated energy functional reads

$$\tilde{\mathscr{E}}_w(u):=\frac{1}{p}\int_\Omega G(\nabla u)\,dx+\beta\int_{\partial\Omega}|u|^p\,dx-\int_\Omega dx\int_0^u h(\cdot,t)\,dt,\quad u\in W^{1,p}(\Omega).$$

Arguing as in Lemma 3.10 produces a solution $\tilde{u}\in C^1(\overline{\Omega})$ to (3.26) such that $\tilde{\mathscr{E}}_w(\tilde{u})\le 0$. Next, adapt the proof of Lemma 3.9 and exploit the fact that $\hat{u}$ is a supersolution of (3.26) to achieve $\underline{u}\le\tilde{u}\le\hat{u}$. Consequently, $\tilde{u}\in U_w$ and

$$\mathscr{E}_w(\tilde{u})=\tilde{\mathscr{E}}_w(\tilde{u})\le 0<1.$$

This forces $\tilde{u}\in\mathscr{S}(w)$, besides $\tilde{u}\le\min\{u_1,u_2\}$. □

Lemma 3.14. *If Assumptions 3.1–3.2 and condition* (3.13) *hold true, then for every $w\in C^1(\overline{\Omega})$, the set $\mathscr{S}(w)$ possesses absolute minimum.*

Proof. Fix $w\in C^1(\overline{\Omega})$. We already know (see Lemma 3.13) that $\mathscr{S}(w)$ turns out downward directed. If $\mathscr{C}\subseteq\mathscr{S}(w)$ is a chain in $\mathscr{S}(w)$, then there exists a sequence $\{u_n\}\subseteq\mathscr{S}(w)$ satisfying

$$\lim_{n\to\infty}u_n=\inf\mathscr{C}.$$

On account of Lemma 3.10 and up to subsequences, one has $u_n\to\hat{u}$ in $C^1(\overline{\Omega})$. Thus, $\hat{u}=\inf\mathscr{C}$. By Zorn's lemma, $\mathscr{S}(w)$ admits a minimal element u_w. It remains to show that $u_w=\min\mathscr{S}(w)$. Pick any $u\in\mathscr{S}(w)$. Through Lemma 3.13 we get $\tilde{u}\in\mathscr{S}(w)$ such that $\tilde{u}\le\min\{u_w,u\}$. The minimality of u_w entails $u_w=\tilde{u}$. Therefore, $u_w\le u$, as desired. □

Remark 3.15. This proof is patterned after the one in [13, Theorem 23].

Lemma 3.14 allows to consider the function $\Gamma\colon C^1(\overline{\Omega})\to C^1(\overline{\Omega})$ given by

$$\Gamma(w):=\min\mathscr{S}(w)\quad\text{for all } w\in C^1(\overline{\Omega}).$$

Lemma 3.16. *Under Assumptions 3.1–3.2 and condition* (3.16), *Γ is continuous and maps bounded sets into relatively compact sets.*

Proof. The proof is analogous to that of [13, Lemma 24]. So, we will omit details. Let $\mathscr{B}\subseteq C^1(\overline{\Omega})$ be bounded. Since $\Gamma(\mathscr{B})\subseteq\mathscr{S}(\mathscr{B})$ and $\mathscr{S}(\mathscr{B})$ turns out relatively compact (cf. Lemma 3.10), $\Gamma(\mathscr{B})$ enjoys the same property. Next, suppose $w_n\to w$ in $C^1(\overline{\Omega})$. Setting $u_n:=\Gamma(w_n)$, one evidently has $u_n\to u$ in $C^1(\overline{\Omega})$, where a subsequence is considered when necessary. The function u complies with $u\ge\underline{u}$ and $\mathscr{E}_w(u)<1$ (see the proof of Lemma 3.12). Via the Lebesgue dominated convergence theorem, from $u_n\in U_{w_n}$, it follows $u\in U_w$. Plugging them all together, we get $u\in\mathscr{S}(w)$. It remains to verify that $u=\Gamma(w)$. Lemma 3.12 provides a sequence $\{v_n\}\subseteq C^1(\overline{\Omega})$ fulfilling both $v_n\in\mathscr{S}(w_n)$ for all $n\in\mathbb{N}$ and $v_n\to\Gamma(w)$ in $C^1(\overline{\Omega})$. The choice of Γ entails $u_n=\Gamma(w_n)\le v_n$, besides $\Gamma(w)\le u$. Letting $n\to+\infty$, we thus arrive at

$$\Gamma(w)\le u=\lim_{n\to+\infty}u_n\le\lim_{n\to+\infty}v_n=\Gamma(w),$$

i.e., $u=\Gamma(w)$, which completes the proof. □

To establish our main result, the stronger version below of Assumption 3.1 will be employed.

Assumption 3.17. $f\colon\Omega\times\mathbb{R}\times\mathbb{R}^N\to[0,+\infty)$ is a Carathéodory function such that

$$f(x,s,\xi)\le c_3+c_4|s|^{p-1}+c_5|\xi|^{p-1}\quad\text{for all } (x,s,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^N,$$

with appropriate $c_3,c_4,c_5>0$.

Condition (3.13) is substituted by

$$c_4 + (2p-1)c_5 + d < c_1^p c_2\,. \tag{3.27}$$

Remark 3.18. Assumption 3.17 clearly implies Assumption 3.1, with $c_M := c_3 + c_5 M^{p-1}$ and $d_M := c_4$. Likewise, (3.27) forces (3.13) while (3.16) reads as

$$c_4 + d < \frac{c_1^p c_2}{p}. \tag{3.28}$$

Theorem 3.19. *Let Assumptions 3.17, 3.2 and conditions* (3.27)–(3.28) *be satisfied. Then problem* (1.1) *possesses a solution* $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. *The set of solutions to* (1.1) *is compact in* $C^1(\overline{\Omega})$.

Proof. Define

$$\Lambda(\Gamma) := \{u \in C^1(\overline{\Omega}) : u = \tau\,\Gamma(u) \text{ for some } \tau \in (0,1)\}.$$

Claim : $\Lambda(\Gamma)$ is bounded in $W^{1,p}(\Omega)$. To see this, pick any $u \in \Lambda(\Gamma)$. Since $\frac{u}{\tau} = \Gamma(u) \in \mathscr{S}(u)$, one has $\mathscr{E}_u(\frac{u}{\tau}) < 1$. Assumption 3.17, combined with Young's and Hölder's inequalities, produces

$$\begin{aligned}
\int\limits_{\Omega(\frac{u}{\tau}>\underline{u})} \Bigg(\int\limits_{\underline{u}}^{\frac{u}{\tau}} f(\cdot\,,t,\nabla u)\,\mathrm{d}t \Bigg)\mathrm{d}x &\le \int\limits_{\Omega}\Bigg(\int\limits_{0}^{\frac{u}{\tau}} (c_3 + c_4 t^{p-1} + c_5|\nabla u|^{p-1})\,\mathrm{d}t \Bigg)\mathrm{d}x \\
&\le c_3\Big\|\frac{u}{\tau}\Big\|_1 + \frac{c_4}{p}\Big\|\frac{u}{\tau}\Big\|_p^p + c_5\int\limits_{\Omega}|\nabla u|^{p-1}\Big|\frac{u}{\tau}\Big|\,\mathrm{d}x \\
&\le c_3|\Omega|^{\frac{1}{p'}}\Big\|\frac{u}{\tau}\Big\|_p + \frac{c_4}{p}\Big\|\frac{u}{\tau}\Big\|_p^p + c_5\Bigg(\frac{\|\frac{u}{\tau}\|_p^p}{p} + \frac{\|\nabla u\|_p^p}{p'}\Bigg) \\
&\le c_3|\Omega|^{\frac{1}{p'}}\Big\|\frac{u}{\tau}\Big\|_{1,p} + \frac{c_4 + c_5}{p}\Big\|\frac{u}{\tau}\Big\|_{1,p}^p + \frac{c_5}{p'}\|u\|_{1,p}^p.
\end{aligned}$$

Analogously, on account of (3.7),

$$\begin{aligned}
\int\limits_{\Omega} f(\cdot\,,\underline{u},\nabla u)\underline{u}\,\mathrm{d}x &\le \int\limits_{\Omega}(c_3 + c_4\underline{u}^{p-1} + c_5|\nabla u|^{p-1})\underline{u}\,\mathrm{d}x \\
&\le \Big(c_3 + c_4 + \frac{c_5}{p}\Big)|\Omega| + \frac{c_5}{p'}\|\nabla u\|_p^p \\
&\le \Big(c_3 + c_4 + \frac{c_5}{p}\Big)|\Omega| + \frac{c_5}{p'}\|u\|_{1,p}^p.
\end{aligned}$$

Reasoning as in Lemma 3.7 and recalling that $\tau \in (0,1)$, we thus achieve

$$1 > \mathscr{E}_u\Big(\frac{u}{\tau}\Big) \ge \frac{c_1^p c_2 - c_4 - (2p-1)c_5 - d}{p}\Big\|\frac{u}{\tau}\Big\|_{1,p}^p - (c_3 + c)|\Omega|^{\frac{1}{p'}}\Big\|\frac{u}{\tau}\Big\|_{1,p} - K',$$

where

$$K' := \Big(c_3 + c_4 + \frac{c_5}{p}\Big)|\Omega| + 2\|g(\cdot\,,\varepsilon\theta)\|_{p'}|\Omega|^{\frac{1}{p}}.$$

Thanks to (3.27), the above inequalities force

$$\|u\|_{1,p} \le \Big\|\frac{u}{\tau}\Big\|_{1,p} \le K^*,$$

with $K^* > 0$ independent of u and τ. Thus, the claim is proved.

By regularity [12], the set $\Lambda(\Gamma)$ turns out bounded in $C^1(\overline{\Omega})$. Hence, due to Lemma 3.16, Theorem 2.1 applies, which entails $\operatorname{Fix}(\Gamma) \neq \emptyset$. Let $u \in \operatorname{Fix}(\Gamma)$. From $u = \Gamma(u) \in \mathscr{S}(u)$, we deduce both $u \ge \underline{u}$ and $u \in U_u$. Accordingly,

$$\hat{f}(\cdot\,,u) = f(\cdot\,,u,\nabla u), \quad \hat{g}(\cdot\,,u) = g(\cdot\,,u),$$

namely, the function u solves problem (1.1). Further, $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$ because of the strong maximum principle.

Finally, arguing as in Lemma 3.8 ensures that each solution to (1.1) lies in $C^{1,\alpha}(\overline{\Omega})$. Since $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ compactly and the solution set of (1.1) is closed in $C^1(\overline{\Omega})$, the conclusion follows. $\square$

Remark 3.20. The same techniques can be applied for finding solutions to the Neumann problem

$$\begin{cases} -\operatorname{div} a(\nabla u) + |u|^{p-2}u = f(x,u,\nabla u) + g(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_a} = 0 & \text{on } \partial\Omega. \end{cases}$$

In fact, it is enough to replace the norm $\|\cdot\|_{\beta,1,p}$ with the standard one $\|\cdot\|_{1,p}$.

4 Uniqueness (for $p = 2$)

Throughout this section, $p = 2$, the operator a fulfills Assumption 2.5, while the nonlinearities f and g comply with Assumptions 3.1 and 3.2, respectively. The following further conditions will be posited:

(C1) There exists $c_6 \in (0,1]$ such that

$$\langle a(\xi) - a(\eta), \xi - \eta \rangle \ge c_6 |\xi - \eta|^2 \quad \text{for all } \xi, \eta \in \mathbb{R}^N.$$

(C2) With appropriate $c_7, c_8 > 0$, one has

$$[f(x,s,\xi) - f(x,t,\xi)](s-t) \le c_7 |s-t|^2, \tag{4.1}$$
$$|f(x,t,\xi) - f(x,t,\eta)| \le c_8 |\xi - \eta| \tag{4.2}$$

in $\Omega \times \mathbb{R} \times \mathbb{R}^N$.

(C3) There exists $c_9 > 0$ such that

$$[g(x,s) - g(x,t)](s-t) \le c_9 |s-t|^2 \quad \text{for all } x \in \Omega,\ s,t \in [1,+\infty). \tag{4.3}$$

Moreover,

$$g(x,s) \le g(x,1) \quad \text{in } \Omega \times (1,+\infty). \tag{4.4}$$

Example 4.1. The parametric $(2,q)$-Laplacian $\Delta + \mu\Delta_q$, where $1 < q < 2$, $\mu \ge 0$, satisfies Assumption 2.5 and (C1), cf. [18, Lemma A.0.5].

Theorem 4.2. *Under the above assumptions, problem* (1.1) *admits a unique solution, provided*

$$c_7 + c_1 c_8 + c_9 < c_1^2 c_6. \tag{4.5}$$

Proof. Suppose u, v solve (1.1). Test with $u - v$ and subtract to arrive at

$$\begin{aligned} &\int_\Omega \langle a(\nabla u) - a(\nabla v), \nabla(u-v) \rangle \,\mathrm{d}x + \beta \int_{\partial\Omega} |u-v|^2 \,\mathrm{d}\sigma \\ &\quad = \int_\Omega [f(\cdot,u,\nabla u) - f(\cdot,v,\nabla v)](u-v)\,\mathrm{d}x + \int_\Omega [g(\cdot,u) - g(\cdot,v)](u-v)\,\mathrm{d}x. \end{aligned} \tag{4.6}$$

The left-hand side of (4.6) can easily be estimated from below via (C1) as follows:

$$\int_\Omega \langle a(\nabla u) - a(\nabla v), \nabla(u-v) \rangle \,\mathrm{d}x + \beta \int_{\partial\Omega} |u-v|^2 \,\mathrm{d}\sigma \ge c_6 \|u-v\|_{\beta,1,2}^2. \tag{4.7}$$

Using (4.1)–(4.2) and Hölder's inequality, we get

$$\begin{aligned} &\int_\Omega [f(\cdot,u,\nabla u) - f(\cdot,v,\nabla v)](u-v)\,\mathrm{d}x \\ &\quad = \int_\Omega [f(\cdot,u,\nabla u) - f(\cdot,v,\nabla u)](u-v)\,\mathrm{d}x + \int_\Omega [f(\cdot,v,\nabla u) - f(\cdot,v,\nabla v)](u-v)\,\mathrm{d}x \\ &\quad \le c_7 \int_\Omega |u-v|^2 \,\mathrm{d}x + c_8 \int_\Omega |\nabla u - \nabla v||u-v|\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\leq c_7\|u-v\|_2^2 + c_8\|\nabla(u-v)\|_2\|u-v\|_2 \\
&\leq \frac{c_7}{c_1^2}\|u-v\|_{\beta,1,2}^2 + \frac{c_8}{c_1}\|u-v\|_{\beta,1,2}^2. \qquad (4.8)
\end{aligned}
$$

Observe now that

$$
\begin{aligned}
\int_\Omega [g(\cdot\,,u)-g(\cdot\,,v)](u-v)\,\mathrm{d}x = &\int_{\Omega(\max\{u,v\}\leq 1)} [g(\cdot\,,u)-g(\cdot\,,v)](u-v)\,\mathrm{d}x \\
&+ \int_{\Omega(\min\{u,v\}>1)} [g(\cdot\,,u)-g(\cdot\,,v)](u-v)\,\mathrm{d}x \\
&+ \int_{\Omega(u\leq 1<v)} [g(\cdot\,,u)-g(\cdot\,,v)](u-v)\,\mathrm{d}x \\
&+ \int_{\Omega(v\leq 1<u)} [g(\cdot\,,u)-g(\cdot\,,v)](u-v)\,\mathrm{d}x. \qquad (4.9)
\end{aligned}
$$

By Assumption 3.2 (i), one has

$$
\int_{\Omega(\max\{u,v\}\leq 1)} [g(\cdot\,,u)-g(\cdot\,,v)](u-v)\,\mathrm{d}x \leq 0. \qquad (4.10)
$$

Inequality (4.3) entails

$$
\int_{\Omega(\min\{u,v\}>1)} [g(\cdot\,,u)-g(\cdot\,,v)](u-v)\,\mathrm{d}x \leq c_9\|u-v\|_2^2 \leq \frac{c_9}{c_1^2}\|u-v\|_{\beta,1,2}^2. \qquad (4.11)
$$

Thanks to Assumption 3.2 (i) again and (4.4), we obtain

$$
\int_{\Omega(u\leq 1<v)} [g(\cdot\,,u)-g(\cdot\,,v)](u-v)\,\mathrm{d}x \leq \int_{\Omega(u\leq 1<v)} [g(\cdot\,,1)-g(\cdot\,,v)](u-v)\,\mathrm{d}x \leq 0. \qquad (4.12)
$$

Likewise,

$$
\int_{\Omega(v\leq 1<u)} [g(\cdot\,,u)-g(\cdot\,,v)](u-v)\,\mathrm{d}x \leq 0. \qquad (4.13)
$$

Plugging (4.10)–(4.13) into (4.9) and (4.7)–(4.9) into (4.6) yields

$$
c_6\|u-v\|_{\beta,1,2}^2 \leq \left(\frac{c_7}{c_1^2} + \frac{c_8}{c_1} + \frac{c_9}{c_1^2}\right)\|u-v\|_{\beta,1,2}^2.
$$

On account of (4.5), this directly leads to $u = v$, as desired. □

Remark 4.3. The conditions that guarantee existence or uniqueness, namely, (3.27), (3.28) and (4.5), represent a balance between data (growth or variation of reaction terms) and structure (driving operator and domain) of the problem .

Remark 4.4. The choice $p = 2$ directly stems from the technical approach adopted in proving Theorem 4.2. To treat the general case, a natural attempt is to replace both $|\xi-\eta|^2$ and $|s-t|^2$ by $|\xi-\eta|^p$ and $|s-t|^p$, respectively, in conditions (C1)–(C3). However, if $p > 2$, then (4.1)–(4.3) imply that $f(x,\cdot\,,\cdot)$ as well as $g(x,\cdot)$ are constants, whereas even the p-Laplacian would not meet (C1) for $1 < p < 2$.

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References

[1] S. Carl, V. K. Le and D. Motreanu, *Nonsmooth Variational Problems and Their Inequalities. Comparison Principles and Applications*, Springer Monogr. Math., Springer, New York, 2007.

[2] M. G. Crandall, P. H. Rabinowitz and L. Tartar, On a Dirichlet problem with a singular nonlinearity, *Comm. Partial Differential Equations* **2** (1977), no. 2, 193–222.

[3] L. C. Evans, *Partial Differential Equations*, Grad. Stud. Math. 19, American Mathematical Society, Providence, 1998.

[4] F. Faraci, D. Motreanu and D. Puglisi, Positive solutions of quasi-linear elliptic equations with dependence on the gradient, *Calc. Var. Partial Differential Equations* **54** (2015), no. 1, 525–538.

[5] G. Fragnelli, D. Mugnai and N. S. Papageorgiou, Positive and nodal solutions for parametric nonlinear Robin problems with indefinite potential, *Discrete Contin. Dyn. Syst.* **36** (2016), no. 11, 6133–6166.

[6] L. Gasiński and N. S. Papageorgiou, *Nonlinear Analysis*, Ser. Math. Anal. Appl. 9, Chapman & Hall/CRC, Boca Raton, 2006.

[7] L. Gasiński and N. S. Papageorgiou, Nonlinear elliptic equations with singular terms and combined nonlinearities, *Ann. Henri Poincaré* **13** (2012), no. 3, 481–512.

[8] U. Guarnotta, S. A. Marano and N. S. Papageorgiou, Multiple nodal solutions to a Robin problem with sign-changing potential and locally defined reaction, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **30** (2019), no. 2, 269–294.

[9] A. Kufner, O. John and S. Fučík, *Function Spaces*, Czechoslovak Academy of Sciences, Prague, 1977.

[10] A. C. Lazer and P. J. McKenna, On a singular nonlinear elliptic boundary-value problem, *Proc. Amer. Math. Soc.* **111** (1991), no. 3, 721–730.

[11] A. Lê, Eigenvalue problems for the p-Laplacian, *Nonlinear Anal.* **64** (2006), no. 5, 1057–1099.

[12] G. M. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations, *Comm. Partial Differential Equations* **16** (1991), no. 2–3, 311–361.

[13] Z. Liu, D. Motreanu and S. Zeng, Positive solutions for nonlinear singular elliptic equations of p-Laplacian type with dependence on the gradient, *Calc. Var. Partial Differential Equations* **58** (2019), no. 1, Paper No. 28.

[14] D. Motreanu, V. V. Motreanu and A. Moussaoui, Location of nodal solutions for quasilinear elliptic equations with gradient dependence, *Discrete Contin. Dyn. Syst. Ser. S* **11** (2018), no. 2, 293–307.

[15] D. Motreanu and P. Winkert, Existence and asymptotic properties for quasilinear elliptic equations with gradient dependence, *Appl. Math. Lett.* **95** (2019), 78–84.

[16] N. S. Papageorgiou and P. Winkert, Singular p-Laplacian equations with superlinear perturbation, *J. Differential Equations* **266** (2019), no. 2–3, 1462–1487.

[17] N. S. Papageorgiou and P. Winkert, Solutions with sign information for nonlinear nonhomogeneous problems, *Math. Nachr.* **292** (2019), no. 4, 871–891.

[18] I. Peral, Multiplicity of Solutions for the p-Laplacian, ICTP Lecture Notes of the Second School of Nonlinear Functional Analysis and Applications to Differential Equations, Trieste, 1997.

[19] P. Pucci and J. Serrin, *The Maximum Principle*, Progr. Nonlinear Differential Equations Appl. 73, Birkhäuser, Basel, 2007.

[20] S. Zeng, Z. Liu and S. a. Migórski, Positive solutions to nonlinear nonhomogeneous inclusion problems with dependence on the gradient, *J. Math. Anal. Appl.* **463** (2018), no. 1, 432–448.