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Decision Support

Pairwise comparison tables within the deck of cards method in multiple criteria decision aiding

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ABSTRACT

This paper deals with an improved version of the deck of cards method to render the construction of ratio and interval scales more “accurate” compared to the ones built in the original version. The improvement comes from the fact that we can account for a richer and finer preference information provided by the decision-maker, which permits a more accurate modeling of the strength of preference between different levels of a scale. Instead of considering only the number of blank cards between consecutive positions in the ranking of objects, such as criteria and scale levels, we consider also the number of blank cards between not consecutive positions in the ranking. This information is collected in a pairwise comparison table that is not necessarily built with precise values. We can consider imprecise information provided in the form of intervals and missing values. Since the provided information is not necessarily consistent, we propose also some procedures to help the decision-maker to make consistent her evaluations in a co-constructive way interacting with an analyst and reflecting and revising her judgments. A didactic example will illustrate the application of the method.

1. Introduction

Multiple Criteria Decision Aiding (MCDA) (Greco, Ehrgott, & Figueira, 2016) provides tools supporting the Decision-Maker (DM) to reflect, conjecture, discuss, and argue about decisions in which a plurality of points of view are taken into consideration (Roy, 1993). MCDA procedures are based on an exchange of information between the DM, expressing relevant aspects of her preferences, and the analyst that uses such elements to build a decision model in a co-constructive approach involving the DM’s feedback in each stage. To construct the decision model, it is necessary to assign a value to several preference parameters depending on the nature of the adopted formal approach. Some examples of these parameters are the following:

1. relative importance of criteria in outranking methods (Figueira, Greco, Roy, & Słowiński, 2013; Govindan & Jepsen, 2016);
2. weights and marginal value functions in Multiple Attribute Value Theory (MAVT) methods (Keeney & Raiffa, 1976);
3. capacities, that is non-additive weights, permitting to represent interaction between criteria for the Choquet integral preference model (Choquet, 1953; Grabisch, 1996).

An approach that has gained more and more attention for helping in eliciting parameters is the Deck of Cards Method (DCM) (Figueira & Roy, 2002). In the orginal version of the DCM for outranking methods, at first, the DM is asked to rank all the considered elements from the least important to the most important; then, to express the strength of preference between consecutive levels by adding blank cards between them and, finally, to define the ratio between the weight of the most important criterion and the weight of the least important one. In this paper, we want to apply this method (both in the context of outranking methods and MAVT methods) by using a richer information related to the difference of attractiveness between pairs of elements that are not necessarily consecutive. These differences in attractiveness are collected in a pairwise comparison table in which the value in line p and column q represents the number of blank cards corresponding to the difference in attractiveness between elements p and q.

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Let us remember that the idea of pairwise comparison table has been largely used in MCDA in very well-known methods such as AHP (Saaty, 1977) and MACBETH (Bana e Costa & Vansnick, 1994). To the best of our knowledge, pairwise comparison tables have never been coupled with the DCM. This paper proposes a methodology to fill this gap, using the pairwise comparison table to collect information that is not necessarily complete, and which allows for some imprecision or inconsistency. Moreover, since in case of imprecise and missing information, more than one comparison table can be compatible with the preference information provided by the DM, we propose to apply the Stochastic Multicriteria Acceptability Analysis (SMAA) (Lahdelma, Hokkanen, & Salminen, 1998; Pelissari, Oliveira, Ben Amor, Kandakoglu, & Helleno, 2019; Tervonen & Figueira, 2008). The application of SMAA in this case will permit to take into account all the comparison tables compatible with the information provided by the DM giving robust recommendations with respect to the problem at hand in probabilistic terms.

The paper is organized as follows. Section 2 introduces the basics of the DCM. The application of the DCM to assess interval scales is described in Section 3. Section 4 contains the description of the main concepts and definitions in our proposal. In Section 5, we explain how to restore consistency of the provided comparison table in case of inconsistent judgments. In Section 6, some extensions in the preference information provided by the DM in terms of imprecise and missing information are described. Section 7 contains a didactic example to which the proposed methodology is applied. Section 8 contains a comparison with other MCDA methods (SMART, SWING, SMARTS, AHP, MACBETH) and a discussion on the concept of weights of criteria. Finally, some conclusions and further directions of research are gathered in the last section.

2. The basic DCM

In this section we describe the DCM used in Figueira and Roy (2002) for the assessment of the weights of criteria for Electre methods. In these methods the weights represent the intrinsic importance of each criterion in terms of number of votes in a voting procedure in which, for each pair of alternatives a and b, each criterion is in favor or against the statement "a is at least as good as b". Remember that Figueira and Roy (2002) proposed a modified version of Simos' approach (see Maystre, Pictet, & Simos, 1994) for determining the weights of criteria in Electre methods and more generally for outranking based methods. This modified version is also known as SRF[1] method since it led to the implementation on SRF Software. For a list of applications of the DCM see Siskos and Tsotsolas (2015).

In the remaining of this paper we will use the following basic notation. Let $A=\{a_1,\ldots,a_i,\ldots,a_m\}$ denote the set of alternatives to be assessed, and $G=\{g_1,\ldots,g_j,\ldots,g_n\}$ denote the set of criteria.

Criterion $g_j\in G$ is a generic criterion to be maximized, while a_i represents a generic alternative to be assessed on g_j; consequently, $g_j(a_i)$ is the performance of a_i on g_j; moreover, E_j is the scale of criterion g_j, that is, E_j is the set of all possible evaluations that criterion g_j can take.

In addition, let $U=\{u_1,\ldots,u_j,\ldots,u_n,\ u_j:E_j\rightarrow[0,1]\}$, denote the set of non-decreasing utility/value functions (one per criterion); u_j denotes the generic utility/value function and $u_j(g_j(a_i))$ denotes the utility/value of the performance $g_j(a_i)$.

We show how the dialog between the analyst(s) and the decision-maker(s) or their representatives(s) must be conducted to gather the necessary preference information. We will use an example with six criteria, $G=\{g_1,g_2,g_3,g_4,g_5,g_6\}$. The process of preference elicitation should be performed in a co-constructive interactive way between the analyst and the DM or a representative of the DM:

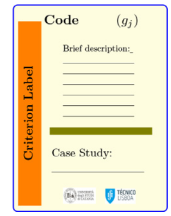

Fig. 1. Example of a criterion card.

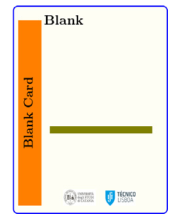

Fig. 2. Example of a blank card.

1. In the first step, the analyst must prepare a set of cards representing considered criteria (normally through a set of projects) with their names on the corresponding cards and, if necessary, some additional information (brief description, case study, criterion label, notation,...). An example of a criterion card is provided in Fig. 1. Then, the analyst should provide the DM with a first *set of cards* and explain her the contents of each one;
2. In the second step, the analyst must also prepare a set of blank cards (as that one shown in Fig. 2) and provide them to the DM. They will be used in the fourth step to define the difference of importance between consecutive levels;
3. In the third step, the analyst must start to gather preference information from the DM. The first task is to ask the DM to rank the criteria cards from the criteria with the lowest weight to the criteria with the highest weight. Whenever the DM feels that some criteria have the same weight she should put them in the same position in the ranking. Please, observe that the meaning of the term *weight* depends on the considered methodology. It has to be considered as "relative importance" in outranking methods, "substitution rates" in MAVT or "priorities ratio" in AHP.
Let us assume that in our case, the DM provided the following ranking:
$\{g_3\}\prec\{g_4,g_5\}\prec\{g_1\}\prec\{g_2\}\prec\{g_6\}$
(in MAVT like methods, this ranking can be obtained by constructing dummy projects as in Bottero, Ferretti, Figueira, Greco, & Roy 2018). The criterion with lowest weight is g_3. Then, there are two criteria having the same weight $\{g_4,g_5\}$ and being ranked higher than g_3. g_1 has higher weight than g_4 and g_5 but lower weight than g_2. Finally, g_6 has a higher weight than g_2 and, in particular, it is the criterion with the highest weight. In the example, we therefore have six criteria but only five ranking positions since g_4 and g_5 are in the same position having the same weight;
4. In the fourth step, an important piece of information should be presented to the DM by the analyst, i.e., related to the fact that two consecutive positions in the ranking may be more or less close. In order to model this *closeness* the DM can use blank cards and insert them in the consecutive intervals in the ranking. No blank card does not mean

[1] Simos-Roy-Figueira.

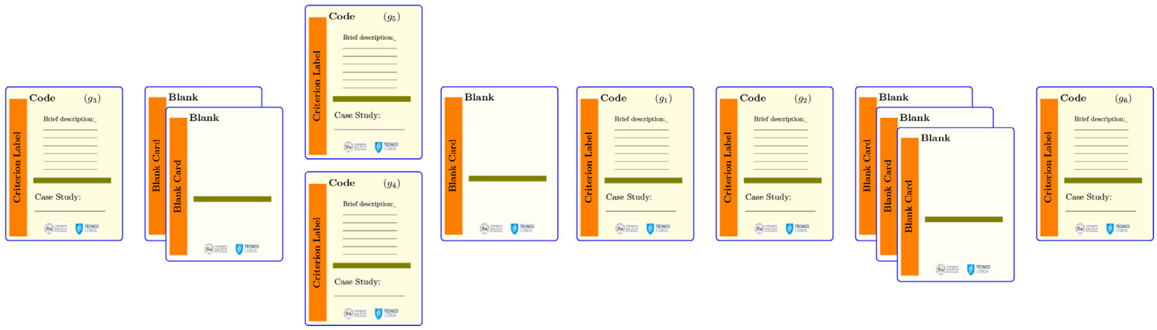

Fig. 3. Ranking of criteria with blank cards.

that the criteria of the two consecutive positions have the same weight, but that this difference will be minimal (technically will represent the unit α); one blank card means that the difference of weights is twice the minimal[2], and so on. Let us observe that the concept of minimal difference represented by no blank card can be related to the idea of just noticeable difference at the basis of psychophysics (Gescheider, 2013).
Let us assume that the DM provided the following blank cards (in square brackets) between the consecutive positions in the ranking;

$\{g_3\}$ [2] $\{g_4, g_5\}$ [1] $\{g_1\}$ [0] $\{g_2\}$ [3] $\{g_6\}$.

Fig. 3 provides a more visual information about this step;

5. Finally, in the fifth step, the analyst should gather from the DM crucial information for making possible the determination of the weights of criteria. Here, the DM must tell the analyst how many times the weight of criterion(a) ranking first (the one(s) in the top position) is higher than the weight of the criterion(a) ranking in the lowest position. In our example, this means how many times the weight of g_6, w_6, is higher than the weight of g_3, w_3. It is, in general, a difficult question; we do not have to work with only a precise value, sometimes two or three values or even a range can be provided and allow to have more sets of weights (Corrente, Figueira, Greco, & Słowiński, 2017). Let us denote by z (also called ratio z) this number.

Note that the information obtained in the fifth step is important in order to build a ratio scale. Indeed the nature of ratio scale characterizes both the weights of outranking methods, and the weights of MAVT based methods. In this last case the DCM can still be used with the adaptation made in the steps before; however, the blank cards have the meaning of difference in attractiveness between specific fictitious projects or alternatives.

3. The DCM for assessing utilities in interval scales

The utility values of the MAVT methods are the levels of a common interval scale, in general, within the range [0,1]. The translation from the original scales of the criteria to a single common interval scale requires the use of a procedure that should account for the intensity of preference between consecutive levels of the scale. In this section, we recall a procedure presented in Bottero et al. (2018) for defining an interval scale based on concepts of the DCM. The procedure allows scales not necessarily within the range [0,1] to be constructed.

In a first moment, let us suppose that criterion $g_j \in G$ has a discrete scale E_j, that is, $E_j = \{l_1, \ldots, l_k, \ldots, l_t\}$ where scale levels are totally ordered, i.e., $l_1 \prec \cdots \prec l_k \prec \cdots \prec l_t$ ($\prec$ means "strictly less preferred than"). In order to build an interval scale, we need to define at least two reference levels (instead of the definition of z, as in the case of ratio scales), to anchor the values assigned to levels l_k in the scale, with $k = 1, \ldots, t$. If more than two reference levels are defined, we can replicate the procedure for every two consecutive reference levels:

1. Consider the scale $E_j = \{l_1, \ldots, l_k, \ldots, l_t\}$;
2. Define two reference levels, l_p and l_q (not necessarily the extreme ones, that is l_1 and l_t) and define their utilities. It is frequent to use $u(l_p) = 0$ and $u(l_q) = 1$;
3. Insert the blank cards between the successive levels of the ranking (e_k is the number of blank cards to be included between levels l_k and l_{k+1}):

   $$l_1\ e_1\ l_2\ \cdots\ l_p\ e_p\ l_{p+1}\ e_{p+1}\ \cdots\ l_k$$
   $$e_k\ l_{k+1}\ \cdots l_{q-1}\ e_{q-1}\ l_q\ \cdots\ l_{t-1}\ e_{t-1}\ l_t;$$

4. Consider only the levels in between l_p and l_q and determine the value of the unit:

   $$\alpha = \frac{u(l_q) - u(l_p)}{h}, \tag{1}$$

   where

   $$h = \sum_{r=p}^{q-1} (e_r + 1), \tag{2}$$

   which is the number of units between levels l_p and l_q;
5. Compute the utility value, $u(l_k)$, for each level, $k = 1, \ldots, t$, as follows:

$$u(l_k) = \begin{cases} u(l_p) - \alpha\left(\displaystyle\sum_{r=k}^{p-1}(e_r+1)\right) & \text{for} \quad k = 1, \ldots, p-1, \\ u(l_p) + \alpha\left(\displaystyle\sum_{r=p}^{k-1}(e_r+1)\right) & \text{for} \quad k = p+1, \ldots, t. \end{cases}$$

Observe that the scale E_j of criterion g_j can be also continuous. In this case the levels $l_1, \ldots, l_t$ can be considered as reference levels whose utility values can be assessed with the DCM we have described. Once utility values $u(l_1), \ldots, u(l_t)$ have been assessed, all other levels in E_j can be assigned a utility value by linear interpolation (for an analogous approach in the context of the AHP method see Abastante, Corrente, Greco, Ishizaka, & Lami 2019).

[2] Denoting by g_j and g_{j+1} two criteria in two consecutive levels and by w_j and w_{j+1} their weight, if there is not any blank card between them, then $w_{j+1} - w_j = \alpha$. If, instead, between the two levels one blank card is included, then $w_{j+1} - w_j = \alpha + \alpha = 2\alpha$, therefore their difference (2α) is twice the minimal difference (α) corresponding to the case in which no blank card is included between the two consecutive levels.

4. Pairwise comparison tables based on the DCM

The DCM, as described in the literature and reviewed in the previous section, is based on the comparisons between the elements in consecutive levels. Such elements could be, for example, (i) criteria/projects to which should be associated a weight; (ii) alternatives' performances expressed on a qualitative scale and that need to be expressed on a cardinal scale or (iii) projects to which has to be assigned a single evaluation. Indeed, the DM is asked to rank order the cards corresponding to the considered elements and, after, to add blank cards between two consecutive cards to express the difference in attractiveness between the elements in one level and the elements in the following level. However, we can imagine that the DM could enrich the information she supplies by expressing the difference in attractiveness not only between consecutive levels of elements, but also between non consecutive ones. This information can be collected in a pairwise comparison table[3] that has a nature similar to the pairwise comparison tables considered in two well known MCDA methods, AHP (Saaty, 1977) and MACBETH (Bana e Costa & Vansnick, 1994). The pairwise comparison table of the DCM has the advantage of a visual support represented by the cards, that can aid the DM in defining and expressing her judgments and preferences. This seems an important point in the perspective of an MCDA methodology that has the primary scope of supporting the DM in discussing and arguing in order to construct a conviction on the decision to be taken. In the following we shall introduce and discuss in depth the theory and the practice of the pairwise comparison table based on the DCM.

We define scale levels as in the previous section, $l_1, \ldots, l_t$. In the classic DCM, the DM introduces cards between consecutive levels only. Now, we will enrich the preference information asking the DM to fill in a comparison table, C, where each entry, e_{pq}, denotes the number of blank cards that should be inserted between levels l_p and l_q. Of course, as in the classical DCM, the greater the number of blank cards between l_p and l_q, the greater the difference between the attractiveness of the two considered levels. The comparison table has to respect an important consistency condition:

Condition 1 (Consistency)**.** Given the comparison table, C,

	l_1	$\ldots$	l_p	$\ldots$	l_k	$\ldots$	l_q	$\ldots$	l_t
l_1	■								
$\vdots$		■							
l_p			■		e_{pk}	$\ldots$	e_{pq}		
$\vdots$				■		$\ddots$	$\vdots$		
l_k					■		e_{kq}		
$\vdots$						■			
l_q							■		
$\vdots$								■	
l_t									■

the following consistency condition

$$e_{pk} + e_{kq} + 1 = e_{pq}$$

must hold, for all $p, k, q = 1, \ldots, t$ such that $p < k < q$. Consequently, the pairwise comparison table C is consistent if all the $\frac{t(t-1)(t-2)}{6}$ equalities above are satisfied.

Proof. A formal proof of the consistency condition 1 is given in Appendix A. □

[3] In the following, we shall use "pairwise comparison table" and "pairwise table" interchangeably.

Once again, let us observe that the application of the classical DCM involves the knowledge of the number of blank cards to be inserted between two successive levels, that are the values e_{pq} with $p = 1, \ldots, t-1$ and $q = p+1$ in the comparison table above. From these values, considering the consistency condition 1, the whole comparison table can be filled.

For example, let us suppose we have five different levels $l_1, \ldots, l_5$ and that the DM applied the classical DCM specifying the values shown in the comparison table below

	l_1	l_2	l_3	l_4	l_5
l_1	■	1			
l_2		■	0		
l_3			■	3	
l_4				■	2
l_5					■

where 1 is the number of blank cards that should be included between l_1 and l_2, while 2 is the number of blank cards that should be included between l_4 and l_5. Consequently, taking into account the consistency condition 1, we can get all the other values in the following table.

	l_1	l_2	l_3	l_4	l_5
l_1	■	1	2	6	9
l_2		■	0	4	7
l_3			■	3	6
l_4				■	2
l_5					■

For example, $e_{13} = e_{12} + e_{23} + 1 = 1 + 0 + 1 = 2,$ while $e_{25} = e_{23} + e_{35} + 1 = 0 + 6 + 1 = 7$.

5. Detecting and correcting inconsistency in pairwise comparison tables based on the DCM

In the behavioral aspects of decision making it is important to develop tools for dealing with inconsistent judgments and ways of interacting with the DM to overcome, when possible, such inconsistency. In case of inconsistent judgments, the pairwise comparison table provided by the DM does not satisfy the consistency condition 1. Therefore, in this section we present some tools to detect and correct such inconsistency. These are mainly Mixed-Integer Linear Programming (MILP) based methods, which are used to check for the consistency of the preference and value information provided by the DM. If this information is not consistent, the solution of the MILP problem will suggest the minimum number of modifications necessary to restore the consistency.

To check if the pairwise comparison table C is consistent, that is, consistency condition 1 is satisfied, we shall proceed in the following way. For each ordered pair of levels (l_p, l_q) such that $p < q$, we define three different variables δ^+_{pq}, δ^-_{pq} and $\overline{e}_{pq}$ such that:

$$\overline{e}_{pq} = e_{pq} + \delta^+_{pq} - \delta^-_{pq}.$$

Their meaning is as follows:

1. δ^-_{pq} is a non-negative integer number, representing how many blank cards should be subtracted from e_{pq}, that is, how much e_{pq} has to be reduced to make the judgments consistent;
2. δ^+_{pq} is a non-negative integer number, representing how many blank cards should be added to e_{pq}, that is, how much e_{pq} has to be increased to make the judgments consistent;
3. $\overline{e}_{pq}$ is the *new* number of blank cards that should be included between levels l_p and l_q to make consistent the comparison table.

Moreover, for each pair of levels (l_p, l_q), a binary variable y_{pq} is defined to check if the starting evaluation e_{pq} has to be modified or not. Let P denote the set of all feasible ordered pairs (p,q) in the comparison table, that is $P=\{(p,q): p,q=1,\ldots,t \text{ and } p<q\}$.

To check if the comparison table provided by the DM is consistent, one has to solve the following MILP problem that, in the following, will be denoted by $MILP-P$:

$$y^{*} = argmin\; z(y) = \sum_{(p,q)\,\in\,P} y_{pq}$$

subject to:

$$\bar{e}_{pk}+\bar{e}_{kq}+1=\bar{e}_{pq}, \qquad (p,k),\ (k,q),\ (p,q)\in P, \tag{3a}$$

$$e_{pq}+\delta^{+}_{pq}-\delta^{-}_{pq}=\bar{e}_{pq}, \qquad (p,q)\in P, \tag{3b}$$

$$\delta^{+}_{pq}+\delta^{-}_{pq}\leqslant My_{pq}, \qquad (p,q)\in P, \tag{3c}$$

$$\delta^{-}_{pq},\ \delta^{+}_{pq}\in\mathbb{N}_0, \qquad (p,q)\in P, \tag{3d}$$

$$y_{pq}\in\{0,1\}, \qquad (p,q)\in P. \tag{3e}$$

In this model M is a big positive number. The objective function $z(y)$, that has to be minimized, counts the number of modifications necessary to make consistent the comparison table C. Constraint (3a) is the consistency condition of the new comparison table that should be fulfilled by all triples of levels (p,k,q) such that $p<k<q$. Constraint (3b) is used to link the starting evaluations (e_{pq}) to the new ones $(\bar{e}_{pq})$[4]. Constraint (3c) is used to check if the comparison e_{pq} has to be modified or not. If $y_{pq}=0$, then $\delta^{+}_{pq}=\delta^{-}_{pq}=0$ and, consequently, e_{pq} has not to be modified, while if $y_{pq}=1$, then constraint (3c) is always satisfied and, therefore, e_{pq} has to be modified. Constraints (3d) and (3e) express the nature of the used variables.

If $z^{*}=0$, with z^{*} being the optimal value of the objective function z, then the comparison table is consistent and, therefore, no evaluation e_{pq} needs to be modified. In the opposite case, the comparison table is not consistent and the evaluations for which $y^{*}_{pq}=1$ need to be modified adding δ^{+}_{pq} units to e_{pq} or reducing of δ^{-}_{pq} units the same evaluation. Let us observe that z^{*}, the optimal value of our objective function, is the number of 1s in the matrix y^{*}.

If the comparison table is not consistent, i.e., $z^{*}>0$, it is reasonable to find all the possible sets of modifications of cardinality z^{*} which, once done, can restore its consistency. Considering $[\bar{e}^{1*}_{pq}]_{(p,q)\in P}$ the matrix of $\bar{e}_{pq}$ obtained as solution of $MILP-P$ and denoting by $P_1=\{(p,q)\in P:\bar{e}^{1*}_{pq}\neq e_{pq}\}\subseteq P$ the set of pairs of levels needing to be modified to restore the consistency of the comparison table provided by the DM, to check for another possible solution, we need to solve $MILP-P$ with the addition of the following constraints:

$$z(y)=z^{*} \tag{4a}$$

$$\bar{e}_{pq}\leqslant\left(\bar{e}^{1*}_{pq}-1\right)+My^{1,1}_{pq}, \qquad (p,q)\in P_1, \tag{4b}$$

$$\bar{e}_{pq}\geqslant\left(\bar{e}^{1*}_{pq}+1\right)-My^{1,2}_{pq}, \qquad (p,q)\in P_1, \tag{4c}$$

$$\sum_{(p,q)\in P_1}\left(y^{1,1}_{pq}+y^{1,2}_{pq}\right)\leqslant 2|P_1|-1, \tag{4d}$$

$$y^{1,1}_{pq},y^{1,2}_{pq}\in\{0,1\}, \qquad (p,q)\in P_1. \tag{4e}$$

If $y^{1,1}_{pq}=1$ or $y^{1,2}_{pq}=1$, the corresponding constraints (4b) and (4c) are always satisfied and, consequently, $\bar{e}_{pq}=\bar{e}^{1*}_{pq}$. Constraint (4d) ensures, therefore, that at least one between $\bar{e}_{pq}\geqslant\bar{e}^{1*}_{pq}+1$ and $\bar{e}_{pq}\leqslant\bar{e}^{1*}_{pq}-1$ for at least one pair $(p,q)\in P_1$ hold. To simplify the notation, we shall denote by EP_1 the set composed of constraints (4b)–(4e). If $MILP-P$ with the addition of constraints in $\{z(y)=z^{*}\}\cup EP_1$ is infeasible, then the solution found solving $MILP-P$ with a number of modified pairwise judgments e_{pq} equal to z^{*} is the unique one. In the opposite case, there is another solution. Proceeding in an iterative way, all the consistent comparison tables obtained with z^{*} pairwise comparison modifications can be found. They are obtained modifying z^{*} entries of the inconsistent comparison table provided by the DM making it consistent. After k different solutions have already been found, the $(k+1)-th$ can be obtained solving $MILP-P$ with the addition of the constraints in $\{z(y)=z^{*}\}\cup EP_1\cup EP_2\cup\ldots\cup EP_k$ where EP_t, $t=1,\ldots,k$, is the set composed of the following constraints:

$$\begin{aligned}
&\bar{e}_{pq}\leqslant\left(\bar{e}^{t*}_{pq}-1\right)+My^{t,1}_{pq}, \qquad (p,q)\in P_t,\\
&\bar{e}_{pq}\geqslant\left(\bar{e}^{t*}_{pq}+1\right)-My^{t,2}_{pq}, \qquad (p,q)\in P_t,\\
&\sum_{(p,q)\in P_t}\left(y^{t,1}_{pq}+y^{t,2}_{pq}\right)\leqslant 2|P_t|-1,\\
&\qquad y^{t,1}_{pq},y^{t,2}_{pq}\in\{0,1\}, \qquad (p,q)\in P_t,
\end{aligned}$$

$[\bar{e}^{t*}_{pq}]_{(p,q)\in P}$ is the matrix of $\bar{e}_{pq}$ obtained at the t-th iteration and $P_t=\{(p,q)\in P:\bar{e}^{t*}_{pq}\neq e_{pq}\}$ is the set composed of the pairs of levels for which the information provided by the DM needs to be modified to restore the consistency.

In the following section we present an example to illustrate these concepts.

5.1. An illustrative example

Let us consider five different levels, l_1, l_2, l_3, l_4, l_5 ordered from the lowest to the highest. Let us assume that the DM is able to provide the comparison table below.

	l_1	l_2	l_3	l_4	l_5
l_1	■	2	4	5	9
l_2		■	1	2	6
l_3			■	0	4
l_4				■	3
l_5					■

It is easy to observe that the table is perfectly consistent since $e_{pq}=e_{pk}+e_{kq}+1$ for all (p,k), (k,q), $(p,q)\in P$.

Now, let us consider another example, in which the DM is able to provide the information in the following comparison table.

	l_1	l_2	l_3	l_4	l_5
l_1	■	2	4	5	8
l_2		■	1	2	6
l_3			■	0	4
l_4				■	3
l_5					■

Solving $MILP-P$, we obtain $z^{*}>0$ and, consequently, the comparison table provided by the DM is not consistent. In this case, we can observe that the inconsistency is present since $e_{14}+e_{45}+1=5+3+1=9\neq e_{15}=8$. Therefore, to restore the consistency of the provided comparison table we need to put $e_{15}=e^{*}_{15}=9$.

To check for the existence of another possible set of modifications of cardinality 1 (since only one modification was enough to

[4] Note that constraints (3a) and (3b) can be converted into a single one, that is, $\bar{e}_{pk}+\bar{e}_{kq}+1=e_{pq}+\delta^{+}_{pq}-\delta^{-}_{pq}$, but, in this way, the model looses readability.

make consistent the pairwise comparison table above), we have to add the following constraints to $MILP-P$:

$$z(y) = 1,$$
$$\overline{e}_{15} \leqslant (9-1) + My_{15}^{1,1},$$
$$\overline{e}_{15} \geqslant (9+1) - My_{15}^{1,2},$$
$$y_{15}^{1,1} + y_{15}^{1,2} \leqslant 1,$$
$$y_{15}^{1,1}, y_{15}^{1,2} \in \{0,1\}.$$

In this case the MILP problem is infeasible. This means that there is not any other single comparison that can be modified so that the pairwise comparison table provided by the DM is consistent.

6. Some further developments: Imprecise judgments and missing values

The section is devoted to some further developments of the method presented in the previous section. It comprises the case in which information is provided in an imprecise way through intervals and the missing data case. We also present some approaches to deal with these cases.

6.1. The imprecise judgments in the form of intervals

In this section we shall develop the whole methodology corresponding to the case in which the DM is not able to give a precise value to the number of blank cards that should be included between the levels l_p and l_q but she prefers to express an imprecise evaluation represented by an interval of possible values. For this reason, the evaluation e_{pq} taken into account in the previous section is now replaced by the interval $[e_{pq}^L, e_{pq}^R]$ with $e_{pq}^L \leqslant e_{pq}^R$ and $e_{pq}^L = e_{pq}^R = e_{pq}$ if the information regarding the pair of levels $(p, q) \in P$ is precise.

Of course, the starting point will be the definition of the *interval consistency condition* taking into account the interval evaluations provided by the DM.

Condition 2 (Interval Consistency Condition)**.** An interval pairwise comparison table is said consistent iff there exists a pairwise comparison table $C = [e_{pq}]_{(p,q)\in P}$, with $e_{pq} \in [e_{pq}^L, e_{pq}^R]$ for all $(p,q) \in P$, that satisfies the consistency condition 1.

For each interval evaluation $[e_{pq}^L, e_{pq}^R]$ we consider the new interval evaluation $[\overline{e}_{pq}^L, \overline{e}_{pq}^R]$, where:

1. $e_{pq}^L + \delta_{pq}^{L+} - \delta_{pq}^{L-} = \overline{e}_{pq}^L$, for all $(p,q) \in P$, is the new left bound of the interval evaluation,
2. $e_{pq}^R + \delta_{pq}^{R+} - \delta_{pq}^{R-} = \overline{e}_{pq}^R$, for all $(p,q) \in P$, is the new right bound of the interval evaluation,
3. $\delta_{pq}^{L+}, \delta_{pq}^{L-}, \delta_{pq}^{R+}, \delta_{pq}^{R-} \in \mathbb{N}_0$ are auxiliary variables used to quantify how much the left or right bounds of the interval evaluation provided by the DM have to be modified.

To check if the comparisons provided by the DM are consistent, we have to solve the following $MILP-I$ problem

y_{pq}

subject to:

$$\overline{e}_{pk} + \overline{e}_{kq} + 1 = \overline{e}_{pq}, \qquad (p,k), (k,q), (p,q) \in P,$$
$$e_{pq}^L + \delta_{pq}^{L+} - \delta_{pq}^{L-} = \overline{e}_{pq}^L, \qquad (p,q) \in P,$$
$$e_{pq}^R + \delta_{pq}^{R+} - \delta_{pq}^{R-} = \overline{e}_{pq}^R, \qquad (p,q) \in P,$$
$$\overline{e}_{pq}^L \leqslant \overline{e}_{pq} \leqslant \overline{e}_{pq}^R, \qquad (p,q) \in P,$$
$$\delta_{pq}^{L+} + \delta_{pq}^{L-} + \delta_{pq}^{R+} + \delta_{pq}^{R-} \leqslant My_{pq}, \qquad (p,q) \in P,$$
$$\delta_{pq}^{L-}, \delta_{pq}^{L+}, \delta_{pq}^{R-}, \delta_{pq}^{R+} \in \mathbb{N}_0, \qquad (p,q) \in P,$$
$$y_{pq} \in \{0,1\}, \qquad (p,q) \in P,$$

where M is a big positive number, while y_{pq} is a binary variable used to check if some bound of the interval $[e_{pq}^L, e_{pq}^R]$ has to be modified or not.

Two cases can be observed:

1. $z^* > 0$: in this case, the imprecise information provided by the DM is not consistent and, therefore, some interval evaluations need to be modified. Let $[[\overline{e}_{pq}^{1L*}, \overline{e}_{pq}^{1R*}]]_{(p,q)\in P}$ be the matrix composed of the intervals found as solution of $MILP-I$ and let us define $P_1^L = \{(p,q) \in P : \overline{e}_{pq}^{1L*} \neq e_{pq}^L\}$ and $P_1^R = \{(p,q) \in P : \overline{e}_{pq}^{1R*} \neq e_{pq}^R\}$, being the subsets of pairs of levels (p, q) such that the left or the right bound of $[e_{pq}^L, e_{pq}^R]$ need to be modified. To check if there exists another possible set of modifications of cardinality z^* restoring the consistency of the comparison table provided by the DM, one has to solve $MILP-I$ with the addition of the following constraints:

$$z(y) = z^*, \tag{8a}$$

$$\overline{e}_{pq}^L \leqslant \left(\overline{e}_{pq}^{1L*} - 1\right) + My_{pq}^{1,1L}, \qquad (p,q) \in P_1^L, \tag{8b}$$

$$\overline{e}_{pq}^L \geqslant \left(\overline{e}_{pq}^{1L*} + 1\right) - My_{pq}^{1,2L}, \qquad (p,q) \in P_1^L, \tag{8c}$$

$$\overline{e}_{pq}^R \leqslant \left(\overline{e}_{pq}^{1R*} - 1\right) + My_{pq}^{1,1R}, \qquad (p,q) \in P_1^R, \tag{8d}$$

$$\overline{e}_{pq}^R \geqslant \left(\overline{e}_{pq}^{1R*} + 1\right) - My_{pq}^{1,2R}, \qquad (p,q) \in P_1^R, \tag{8e}$$

$$\sum_{(p,q)\in P_1^L} \left(y_{pq}^{1,1L} + y_{pq}^{1,2L}\right) + \sum_{(p,q)\in P_1^R} \left(y_{pq}^{1,1R} + y_{pq}^{1,2R}\right) \qquad \text{(8f)}$$
$$\leqslant 2|P_1^L| + 2|P_1^R| - 1, \qquad \text{(8f)}$$

$$y_{pq}^{1,1L}, y_{pq}^{1,2L} \in \{0,1\}, \qquad (p,q) \in P_1^L, \tag{8g}$$

$$y_{pq}^{1,1R}, y_{pq}^{1,2R} \in \{0,1\}, \qquad (p,q) \in P_1^R. \tag{8h}$$

Constraints (8b)–(8f) are used to avoid that the MILP problem gives back the previous solution, while constraints (8g)–(8h) express the nature of the y variables. Finally, constraint (8a) is imposing that the objective value obtained by solving $MILP-I$ is not deteriorated. If $MILP-I$ with the addition of the previous constraints is infeasible, then, the solution found by solving $MILP-I$ above is the unique one. In the opposite case, another set of modifications of cardinality z^* is able to restore the consistency of the comparison table and, proceeding in an iterative way, other sets can eventually be discovered. Let us use the following notation:

(a) $[[\overline{e}_{pq}^{tL*}, \overline{e}_{pq}^{tR*}]]_{(p,q)\in P}$ is the matrix composed of the intervals found as solution at the iteration t,

(b) $P_t^L = \{(p,q) \in P : \overline{e}_{pq}^{tL*} \neq e_{pq}^L\}$ and $P_t^R = \{(p,q) \in P : \overline{e}_{pq}^{tR*} \neq e_{pq}^R\}$ are the sets composed of pairs of levels for which the corresponding left or right bound of the interval evaluation provided by the DM has to be modified with respect to the solution obtained at the iteration t,

(c) EP_t^I the set of constraints

$$\begin{aligned}
&\bar{e}_{pq}^{L} \leqslant \left(\bar{e}_{pq}^{tL*}-1\right)+My_{pq}^{t,1L}, && (p,q)\in P_t^L,\\
&\bar{e}_{pq}^{L} \geqslant \left(\bar{e}_{pq}^{tL*}+1\right)-My_{pq}^{t,2L}, && (p,q)\in P_t^L,\\
&\bar{e}_{pq}^{R} \leqslant \left(\bar{e}_{pq}^{tR*}-1\right)+My_{pq}^{t,1R}, && (p,q)\in P_t^R,\\
&\bar{e}_{pq}^{R} \geqslant \left(\bar{e}_{pq}^{tR*}+1\right)-My_{pq}^{t,2R}, && (p,q)\in P_t^R,\\
&\sum_{(p,q)\in P_t^L}\left(y_{pq}^{t,1L}+y_{pq}^{t,2L}\right)+\sum_{(p,q)\in P_t^R}\left(y_{pq}^{t,1R}+y_{pq}^{t,2R}\right)\\
&\qquad \leqslant 2|P_t^L|+2|P_t^R|-1,\\
&y_{pq}^{t,1L},y_{pq}^{t,2L}\in\{0,1\}, && (p,q)\in P_t^L,\\
&y_{pq}^{t,1R},y_{pq}^{t,2R}\in\{0,1\}, && (p,q)\in P_t^R.
\end{aligned}$$

Then, the $(k+1)-th$ solution can be found solving $MILP-I$ with the addition of constraints in $\{z(y)=z^*\}\cup EP_1^I\cup\ldots\cup EP_k^I$.

2. $z^*=0$: in this case, the imprecise comparison table provided by the DM is consistent and, therefore, the values of $\bar{e}_{pq}^{1*}$ obtained as solutions of $MILP-I$, where $\bar{e}_{pq}^{1*}\in[\bar{e}_{pq}^{1L*},\bar{e}_{pq}^{1R*}]=[e_{pq}^L,e_{pq}^R]$, will form a precise consistent comparison table. In this case, it is reasonable asking if there is another precise comparison table or if the found comparison table is unique. To answer to this question, we have to solve $MILP-I$ with the addition of the following constraints:

$$z(y)=z^*=0, \tag{10a}$$

$$\bar{e}_{pq} \leqslant \left(\bar{e}_{pq}^{1*}-1\right)+My_{pq}^{1,1}, \quad (p,q)\in P, \tag{10b}$$

$$\bar{e}_{pq} \geqslant \left(\bar{e}_{pq}^{1*}+1\right)-My_{pq}^{1,2}, \quad (p,q)\in P, \tag{10c}$$

$$\sum_{(p,q)\in P}\left(y_{pq}^{1,1}+y_{pq}^{1,2}\right)\leqslant 2|P|-1, \tag{10d}$$

$$y_{pq}^{1,1},y_{pq}^{1,2}\in\{0,1\}, \quad (p,q)\in P, \tag{10e}$$

where constraints (10b)–(10d) avoid that the MILP problem gives back the same $\bar{e}_{pq}^{1*}$ value; if the problem is infeasible, then the precise comparison table found by solving $MILP-I$ is the unique one. In the opposite case, there is another precise comparison table compatible with the information provided by the DM and, proceeding in an iterative way, all of them can be found. Denoting by $[\bar{e}_{pq}^{t*}]_{(p,q)\in P}$ the matrix solution obtained at the iteration t, and by EP_t^{IF} the set composed of the following constraints

$$\begin{aligned}
&\bar{e}_{pq} \leqslant \left(\bar{e}_{pq}^{t*}-1\right)+My_{pq}^{t,1}, && (p,q)\in P,\\
&\bar{e}_{pq} \geqslant \left(\bar{e}_{pq}^{t*}+1\right)-My_{pq}^{t,2}, && (p,q)\in P,\\
&\sum_{(p,q)\in P}\left(y_{pq}^{t,1}+y_{pq}^{t,2}\right)\leqslant 2|P|-1,\\
&y_{pq}^{t,1},y_{pq}^{t,2}\in\{0,1\}, && (p,q)\in P,
\end{aligned}$$

the solution at the $(k+1)-th$ iteration can be found by solving $MILP-I$ with the addition of the constraints in $\{z(y)=z^*=0\}\cup EP_1^{IF}\cup\ldots\cup EP_k^{IF}$.

Let us observe that, however, the number of precise comparison tables that can be extracted in this case is equal to $\prod_{(p,q)\in P}\left(e_{pq}^R-e_{pq}^L+1\right)$ and, maybe, not all of them are consistent with respect to the consistency condition 1. If there exists more than one precise comparison table compatible with the information provided by the DM, then we can compute multicriteria preferences for each of them and the results can be discussed with the DM. In this perspective, taking into account robustness concerns, one could compute the frequency with which an alternative attains a certain rank position and the frequency with which an alternative is preferred to another considering all extracted consistent comparison tables. This probabilistic information can be seen as an application of the Stochastic Multicriteria Acceptability Analysis (SMAA) (Lahdelma et al., 1998; Pelissari et al., 2019; Tervonen & Figueira, 2008) to our approach.

Let us observe that if the DM provides imprecise information regarding all pairs of consecutive levels, this information is for sure consistent[5]. Therefore, if we assume that there is a not-integer card distance between two levels (representing the case of some hesitation of the DM between the precise number of blank cards, so that, for example, if 2 blank cards seem too few and 3 too many, [2, 3] can represent the perception of the DM), an infinite number of consistent comparison tables can be built from this preference information taking in a random way one value e_{pq} in each interval $[e_{pq}^L,e_{pq}^R]$ for all $(p,q)\in P$. The sampling procedure to be used in this case, has been already introduced in Corrente et al. (2017).

6.2. An illustrative example on the imprecise preference information

Let us suppose that the DM is able to provide the imprecise preference information shown in the comparison table below.

	l_1	l_2	l_3	l_4	l_5
l_1	■	[0, 1]	[5, 6]	[5, 8]	[7, 11]
l_2		■	[1, 2]	[4, 6]	[8, 11]
l_3			■	[2, 3]	[6, 8]
l_4				■	[3, 4]
l_5					■

Solving $MILP-I$, we obtain $z^*=1$ and $y_{13}^*=1$. In particular, we get $\delta_{13}^{L-*}=1$ meaning that, to restore the interval consistency, the interval [5, 6] should be replaced by the interval [4, 6] in which the old left bound has been decreased of one unit. Indeed, looking at the interval evaluations regarding pairs of levels (l_1,l_2) and (l_2,l_3), that is [0, 1] and [1, 2], one can easily observe that the maximum value for $\bar{e}_{13}$ should be equal to 4 (it is obtained when $\bar{e}_{12}=1$ and $\bar{e}_{23}=2$) but the minimum value for the comparison between levels l_1 and l_3 is, instead, $e_{13}^L=5$. Consequently, an interval pairwise comparison table consistent with the information provided by the DM is the following:

	l_1	l_2	l_3	l_4	l_5
l_1	■	[0, 1]	[4, 6]	[5, 8]	[7, 11]
l_2		■	[1, 2]	[4, 6]	[8, 11]
l_3			■	[2, 3]	[6, 8]
l_4				■	[3, 4]
l_5					■

To check for other possible modifications of minimum cardinality making consistent the imprecise comparison table provided by the DM we use the procedure described in the previous section

[5] We underlined in the end of Section 4 that if the DM provides information on the number of blank cards to be included between consecutive levels (as in the DCM), then the whole comparison table can be built following the consistency condition 1. Therefore, if the DM provides imprecise preference information on consecutive levels in terms of intervals, picking randomly one value in each interval $[e_{pq}^L,e_{pq}^R]$, with $q=p+1$, representing the number of blank cards that can be included between levels l_p and l_q, a consistent comparison table can be rebuilt.

obtaining the following consistent imprecise comparison tables:

	l_1	l_2	l_3	l_4	l_5
l_1	■	[0, 1]	[2, 6]	[5, 8]	[7, 11]
l_2		■	[1, 2]	[4, 6]	[8, 11]
l_3			■	[2, 3]	[6, 8]
l_4				■	[3, 4]
l_5					■

	l_1	l_2	l_3	l_4	l_5
l_1	■	[0, 1]	[3, 6]	[5, 8]	[7, 11]
l_2		■	[1, 2]	[4, 6]	[8, 11]
l_3			■	[2, 3]	[6, 8]
l_4				■	[3, 4]
l_5					■

	l_1	l_2	l_3	l_4	l_5
l_1	■	[0, 1]	[1, 6]	[5, 8]	[7, 11]
l_2		■	[1, 2]	[4, 6]	[8, 11]
l_3			■	[2, 3]	[6, 8]
l_4				■	[3, 4]
l_5					■

	l_1	l_2	l_3	l_4	l_5
l_1	■	[0, 1]	[0, 6]	[5, 8]	[7, 11]
l_2		■	[1, 2]	[4, 6]	[8, 11]
l_3			■	[2, 3]	[6, 8]
l_4				■	[3, 4]
l_5					■

At last, imposing that the previously found solutions are not obtained anymore, we obtain that the $MILP-I$ with the addition of constraints in $\{z(y)=z^*=1\}\cup EP_1^I\cup\ldots\cup EP_5^I$ is infeasible and, therefore, the five imprecise comparison tables above are the only tables that can be obtained with a minimal number of corrections from the imprecise and inconsistent comparison table provided by the DM.

Now, we check how many precise and consistent comparison tables can be extracted from each of the five imprecise comparison tables shown above by using the procedure described in the previous section. We obtain that, of course, the number of precise comparison tables depends on the interval $[\overline{e}_{13}^L,\overline{e}_{13}^R]$ being the only one that needs to be modified to make consistent the inconsistent comparison table provided by the DM. We obtain the results presented in the following table:

$[\overline{e}_{13}^L,\overline{e}_{13}^R]$	Precise tables extracted
[4, 6]	1
[2, 6]	11
[3, 6]	7
[1, 6]	11
[0, 6]	11

In particular, in this case, the number of precise comparison tables that can be extracted from the imprecise one, is non-decreasing with respect to inclusion of the intervals $[\overline{e}_{13}^L,\overline{e}_{13}^R]$ and, since the number of precise comparison tables that can be extracted from the imprecise one is the same for the intervals [2, 6], [1, 6] and [0, 6], then the narrowest should be taken into account, that is, [2, 6]. Of course, if the DM is only willing to build an imprecise and consistent comparison table from which the minimum number of precise and consistent comparison tables can be extracted, then, the interval [4,6] has to be preferred among the discovered ones. Indeed, in this case, the imprecise comparison table is consistent and only one precise comparison table can be extracted from it.

All precise comparison tables that can be extracted from the corresponding imprecise ones are provided in the e-Appendix.

6.3. Missing values

Let us consider the case in which some of the comparisons in the table are missing. This means that the DM was not able or did not want to provide any value for the number of blank cards to be inserted between two different levels. Using the notation introduced in Section 5, we will describe a procedure that will be composed of the following main steps:

1. Check if there exists at least one set of values to be assigned to the missing comparisons so that the consistency condition 1 is satisfied. If this is not the case, the inconsistent judgments will be highlighted to be shown to the DM; moreover, all sets of possible modifications of minimum cardinality restoring the consistency of the provided pairwise comparisons will be found;
2. In case the information provided by the DM is consistent or once the consistency has been restored in the previous step, show all the perfectly consistent precise comparison table that can be extracted from the imprecise and inconsistent table supplied by the DM.

To check if the few comparisons provided by the DM are consistent, we have to solve the following MILP problem denoted by $MILP-M$,

$$y^* = argmin\ z(y) = \sum_{(p,q)\,\in\,P^{DM}} y_{pq}$$

subject to:

$$\begin{array}{ll}
\overline{e}_{pk}+\overline{e}_{kq}+1=\overline{e}_{pq}, & (p,k),\ (k,q),\ (p,q)\in P,\\
e_{pq}+\delta^+_{pq}-\delta^-_{pq}=\overline{e}_{pq}, & (p,q)\in P^{DM},\\
\delta^+_{pq}+\delta^-_{pq}\leqslant My_{pq}, & (p,q)\in P^{DM},\\
\delta^-_{pq},\ \delta^+_{pq}\in\mathbb{N}_0, & (p,q)\in P^{DM},\\
y_{pq}\in\{0,1\}, & (p,q)\in P^{DM},\\
\overline{e}_{pq}\in\mathbb{N}_0, & (p,q)\in P,
\end{array}$$

where $P^{DM}\subseteq P$ is the set of pairs of levels for which the DM was able to provide some preference information. Let us underline that, differently from $MILP-P$ introduced in Section 5, the auxiliary variables δ^+_{pq}, δ^-_{pq} and y_{pq} are defined for the pairs $(p,q)\in P^{DM}$ only, i.e., the pairs of levels for which the DM expressed a preference information.

Two cases can be considered:

1. $z^*>0$: in this case, the pairwise comparisons provided by the DM are not consistent. Therefore, some of them need to be modified. Denoting by $[\overline{e}^{1*}_{pq}]_{(p,q)\in P}$ the matrix composed of the values obtained by solving $MILP-M$ and by $P_1^{DM}=\left\{(p,q)\in P^{DM}:\overline{e}^{1*}_{pq}\neq e_{pq}\right\}$ the set of pairs of levels for which the corresponding comparison provided by the DM needs to be modified, to check for another possible set of modifications of cardinality z^* making consistent the judgments provided by the DM, one has to solve $MILP-M$ with the addition of the following constraints:

$$\begin{array}{ll}
z(y)=z^*, & \\
\overline{e}_{pq}\leqslant\left(\overline{e}^{1*}_{pq}-1\right)+My^{1,1}_{pq}, & (p,q)\in P_1^{DM},\\
\overline{e}_{pq}\geqslant\left(\overline{e}^{1*}_{pq}+1\right)-My^{1,2}_{pq}, & (p,q)\in P_1^{DM},\\
\displaystyle\sum_{(p,q)\in P_1^{DM}}\left(y^{1,1}_{pq}+y^{1,2}_{pq}\right)\leqslant 2|P_1^{DM}|-1, & \\
y^{1,1}_{pq},\ y^{1,2}_{pq}\in\{0,1\}, & (p,q)\in P_1^{DM}.
\end{array}$$

If the new MILP problem is infeasible, then the previous solution is the unique one. In the opposite case, there is another possible solution and proceeding in an iterative way, one can check for other possible solutions. Indeed, after k solutions have been found, the $(k+1)-th$ can be obtained by solving $MILP-M$ with the addition of the constraints in $\{z(y)=z^*\}\cup EP_1^{DM}\cup\ldots\cup EP_k^{DM}$, where, EP_t^{DM} is the set composed of the following constraints:

$$\bar{e}_{pq}\leqslant\left(\bar{e}_{pq}^{t*}-1\right)+My_{pq}^{t,1},\qquad (p,q)\in P_t^{DM},$$
$$\bar{e}_{pq}\geqslant\left(\bar{e}_{pq}^{t*}+1\right)-My_{pq}^{t,2},\qquad (p,q)\in P_t^{DM},$$
$$\sum_{(p,q)\in P_t^{DM}}\left(y_{pq}^{t,1}+y_{pq}^{t,2}\right)\leqslant 2|P_t^{DM}|-1,$$
$$y_{pq}^{t,1},y_{pq}^{t,2}\in\{0,1\},\qquad (p,q)\in P_t^{DM},$$

$[\bar{e}_{pq}^{t*}]_{(p,q)\in P}$ is the matrix of $\bar{e}_{pq}$ obtained at the t-th iteration and $P_t^{DM}=\left\{(p,q)\in P^{DM}:\bar{e}_{pq}^{t*}\neq e_{pq}\right\}$;

2. $z^*=0$: in this case, the few comparisons provided by the DM are consistent and, consequently, $e_{pq}=\bar{e}_{pq}^{1*}$ for all $(p,q)\in P^{DM}$, while the values $\bar{e}_{pq}^{1*}$ with $(p,q)\in P\backslash P^{DM}$ will be the values to be assigned to the missing information. To check if the found solution, that is the matrix of values $\bar{e}_{pq}^{1*}$ with $(p,q)\in P\backslash P^{DM}$ obtained solving $MILP-M$, is the unique one, one can proceed by solving $MILP-M$ with the addition of the following constraints:

$$z(y)=z^*=0,$$
$$\bar{e}_{pq}\leqslant\left(\bar{e}_{pq}^{1*}-1\right)+My_{pq}^{1,1},\qquad (p,q)\in P\setminus P^{DM},$$
$$\bar{e}_{pq}\geqslant\left(\bar{e}_{pq}^{1*}+1\right)-My_{pq}^{1,2},\qquad (p,q)\in P\setminus P^{DM},$$
$$\sum_{(p,q)\in P\setminus P^{DM}}\left(y_{pq}^{1,1}+y_{pq}^{1,2}\right)\leqslant 2|P\setminus P^{DM}|-1,$$
$$y_{pq}^{1,1},y_{pq}^{1,2}\in\{0,1\},\qquad (p,q)\in P\setminus P^{DM}.$$

If the new problem is infeasible, then the previously found solution is the unique one. Otherwise, considering $[\bar{e}_{pq}^{t*}]_{(p,q)\in P}$ the solution obtained at the t-th iteration and $\overline{EP}_t^{DM}$ the set composed of constraints

$$\bar{e}_{pq}\leqslant\left(\bar{e}_{pq}^{t*}-1\right)+My_{pq}^{t,1},\qquad (p,q)\in P\setminus P^{DM},$$
$$\bar{e}_{pq}\geqslant\left(\bar{e}_{pq}^{t*}+1\right)-My_{pq}^{t,2},\qquad (p,q)\in P\setminus P^{DM},$$
$$\sum_{(p,q)\in P\setminus P^{DM}}\left(y_{pq}^{t,1}+y_{pq}^{t,2}\right)\leqslant 2|P\setminus P^{DM}|-1,$$
$$y_{pq}^{t,1},y_{pq}^{t,2}\in\{0,1\},\qquad (p,q)\in P\setminus P^{DM},$$

the $(k+1)$-th solution can be obtained by solving $MILP-M$ with the addition of the constraints in $\{z(y)=z^*=0\}\cup\overline{EP}_1^{DM}\cup\ldots\cup\overline{EP}_k^{DM}$.

Let us conclude this section observing that both the precise value case presented in Section 5 and the missing value case presented in this section can be considered as particular cases of the imprecise one introduced in Section 6.1 and, therefore, the imprecise case is the most general one. Indeed, if the DM is able to provide a precise information on the pairwise comparison between levels l_p and l_q, then the value e_{pq} could be replaced by the interval $[e_{pq}^L,e_{pq}^R]=[e_{pq},e_{pq}]$; if, instead, the DM is not able to provide any information on the pairwise comparison between levels l_p and l_q, then the missing information $e_{pq}=$ "?" could be replaced with the largest possible interval pairwise comparison, being $[e_{pq}^L,e_{pq}^R]=[0,M]$, where M is the big positive value defined in all MILP problems above.

6.3.1. An illustrative example

Let us assume that the DM was able to provide the preference information shown in the following table where the positions containing "?" denote some missing values since the DM was not able or was not sure about that comparison.

	l_1	l_2	l_3	l_4	l_5
l_1	■	?	?	?	?
l_2		■	?	?	6
l_3			■	0	5
l_4				■	3
l_5					■

Solving $MILP-M$, we obtain $z^*=1$ and, in particular, $y_{35}^*=1$ meaning that the value corresponding to the comparison between l_3 and l_5 is not consistent with the other information provided by the DM. Indeed, considering the consistency condition 1, we should have $e_{34}+e_{45}+1=e_{35}$, while this is not true since $0+3+1\neq 5$. Solving the problem, we get $\delta_{35}^-=1$ meaning that if the evaluation $e_{35}=5$ would be replaced by $e_{35}=4$, then the consistency is again restored. Doing the suggested replacement we get the new comparison table

	l_1	l_2	l_3	l_4	l_5
l_1	■	?	?	?	?
l_2		■	?	?	6
l_3			■	0	4
l_4				■	3
l_5					■

At this point, to check if there exists another possible single modification rendering consistent the provided comparison table, we add the following constraints to $MILP-M$:

$$z(y)=1,$$
$$\bar{e}_{35}\leqslant(4-1)+My_{35}^{1,1},$$
$$\bar{e}_{35}\geqslant(4+1)-My_{35}^{1,2},$$
$$y_{35}^{1,1}+y_{35}^{1,2}\leqslant 1,$$
$$y_{35}^{1,1},y_{35}^{1,2}\in\{0,1\}.$$

The MILP problem is again feasible and the new comparison table is as follows,

	l_1	l_2	l_3	l_4	l_5
l_1	■	?	?	?	?
l_2		■	?	?	6
l_3			■	1	5
l_4				■	3
l_5					■

where the modified comparison is (3,4). Avoiding to obtain $\bar{e}_{34}=1$, the last consistent comparison table is as follows,

	l_1	l_2	l_3	l_4	l_5
l_1	■	?	?	?	?
l_2		■	?	?	6
l_3			■	0	5
l_4				■	4
l_5					■

where the pairwise comparison needing to be modified is (4,5).

Now, let us assume that the preference information provided by the DM is contained in the comparison table below:

	l_1	l_2	l_3	l_4	l_5
l_1	■	?	4	?	9
l_2		■	?	?	6
l_3			■	?	?
l_4				■	3
l_5					■

Solving $MILP-M$, we obtain $z^*=0$ meaning that the preference information provided by the DM is consistent and, consequently, there is at least one comparison table that can be built on the basis of this information:

	l_1	l_2	l_3	l_4	l_5
l_1	■	2	4	5	9
l_2		■	1	2	6
l_3			■	0	4
l_4				■	3
l_5					■

To check for the existence of another comparison table, we have to solve $MILP-M$ with the addition of the following constraints:

$$\begin{array}{lll} & & z(y)=0, \\ \bar{e}_{12}\leqslant(2-1)+My_{12}^{1,1}, & \bar{e}_{12}\geqslant(2+1)-My_{12}^{1,2}, & \bar{e}_{14}\leqslant(5-1)+My_{14}^{1,1}, \\ \bar{e}_{14}\geqslant(5+1)-My_{14}^{1,2}, & \bar{e}_{23}\leqslant(1-1)+My_{23}^{1,1}, & \bar{e}_{23}\geqslant(1+1)-My_{23}^{1,2}, \\ \bar{e}_{24}\leqslant(2-1)+My_{24}^{1,1}, & \bar{e}_{24}\geqslant(2+1)-My_{24}^{1,2}, & \bar{e}_{34}\leqslant(0-1)+My_{34}^{1,1}, \\ \bar{e}_{34}\geqslant(0+1)-My_{34}^{1,2}, & \bar{e}_{35}\leqslant(4-1)+My_{35}^{1,1}, & \bar{e}_{35}\geqslant(4+1)-My_{35}^{1,2}, \end{array}$$

$$\sum_{(p,q)\in\{(1,2),(1,4),(2,3),(2,4),(3,4),(3,5)\}}\left(y_{pq}^{1,1}+y_{pq}^{1,2}\right)\leqslant 11,$$

$$y_{pq}^{1,1},y_{pq}^{1,2}\in\{0,1\},\ (p,q)\in\{(1,2),(1,4),(2,3),(2,4),(3,4),(3,5)\}.$$

In this case, the problem becomes infeasible, meaning that we cannot obtain another consistent comparison table. Therefore, the found comparison table is unique.

7. The approach of the comparison table based on the DCM: An illustrative example

We consider the example in Bottero et al. (2018) regarding the selection of the best strategy for the requalification of an abandoned quarry.

The set of alternatives are the following:

1. *Basic reclamation* (alternative a_1): This alternative consists of filling the quarry and implementing security measures favouring the evolution of natural vegetation and the growth of native trees.
2. *Valuable forest* (alternative a_2): This alternative also consists of filling the quarry and implementing security measures, but instead of native trees, an oak horn-beam wood should be established.
3. *Wetland* (alternative a_3): This alternative also consists of the partial filling the quarry and implementing security measures, the construction of a lake along with the plantation of wetland vegetation, and the natural evolution of the surrounding native wood.
4. *Ecological network* (alternative a_4): This alternative also consists of the partial filling the quarry and implementing security measures, the construction of small lakes, the predisposition of information and educational and the natural evolution of the surrounding native wood.
5. *Multifunctional area* (alternative a_5): This alternative also consists of the partial filling the quarry and implementing security measures, the construction of self-sufficient (energy and water disposal) sports and residential structures.

The set of criteria is as follows:

1. *Investment costs* (Scale unit: *K*€ ; Code: `COSTS`; notation: g_1; preference direction: minimization). This criterion comprises the requalification costs of the quarry.
2. *Profitability* (Scale unit: verbal levels (seven); Code: `PROFI`; notation: g_2; preference direction: maximization). This criterion comprises the future income the project is expected to produce for the local population.
3. *New services for the population* (Scale unit: verbal levels (seven); Code: `SERVI`; notation: g_3; preference direction: maximization). This criterion models the possibility of recruiting workers.
4. *Naturalized surface* (Scale unit: hectares; Code: `SURFA`; notation: g_4; preference direction: maximization). This criterion comprises the impacts of a project in the landscape quality and bio-diversity conservation.
5. *Environmental effects* (Scale unit: verbal levels (seven); Code: `ENVIR`; notation: g_5; preference direction: maximization). This criterion comprises the impacts of a project in the environmental system.
6. *Consistency with local planning requirements* (Scale unit: two levels (Yes-No); Code: `CONSI`; notation: g_6; preference direction: maximization). This criterion is related to the administrative feasibility of the project with respect to some urban constraints (if it is feasible, the answer is "yes" [1]; otherwise, the answer is "no" [0].

The verbal scale used for the criteria g_2, g_3, and g_5 comprises the following seven levels (in between parenthesis we used a numerical code for each level): `very bad`[1]; `bad`[2]; `rather bad`[3]; `average`[4]; `rather good`[5]; `good`[6]; `very good`[7].

Table 1 contains the performances of the 5 alternatives on the 6 considered criteria.

7.1. The multicriteria evaluation of the alternatives

In this section, we shall apply the extension of the DCM described in the previous sections to each criterion, to build a unique common scale [0,100] and, then, we shall aggregate the obtained values by means of an additive value function

$$U(a_i)=\sum_{j=1}^{6}w_ju_j\big(g_j(a_i)\big) \tag{18}$$

where $w_j>0$ for all $j=1,\ldots,6$ and, in general, $\sum_{j=1}^{6}w_j=1$. This means that, in this paper, for the sake of simplicity, we do not take into account interaction between criteria considered in Bottero et al. (2018).

To save space, in the following, we shall describe in detail the application of our proposal for two criteria only (`COSTS` and `SERVI`) on which the DM provided different types of information. Readers who are interested into the computations related to the other criteria are deferred to Appendix B.

1. Criterion g_1 `COSTS` (Investment costs). For this criterion, the DM considered as the lowest cost 0 and the highest 1,000,000 providing the following table which values represent the number of blank cards that should be included

Table 1
Performance table.

a	COSTS ($g_1(a)$)	PROFI ($g_2(a)$)	SERVI ($g_3(a)$)	SURFA ($g_4(a)$)	ENVIR ($g_5(a)$)	CONSI ($g_6(a)$)
a_1	30, 000	3	1	2.0	4	1
a_2	45, 000	3	5	5.0	5	1
a_3	90, 000	1	6	3.2	7	1
a_4	120,000	1	7	3.5	6	1
a_5	900,000	7	7	1.0	3	0

between pairs of levels.

	$l_1(1,000,000)$	$l_2(750,000)$	$l_3(500,000)$	$l_4(250,000)$	$l_5(0)$
$l_1(1,000,000)$	■	3	6	8	9
$l_2(750,000)$		■	2	4	5
$l_3(500,000)$			■	1	2
$l_4(250,000)$				■	0
$l_5(0)$					■

As one can easily check, the table satisfies the precise consistency condition 1. This means that there is not the necessity to revise the information provided by the DM and, consequently, we can assign a single value to the considered levels. Let us proceed by computing the number of units between the lowest level and the highest one, $h = e_{15} + 1 = 10$ and, therefore, $u(l_5) = u(l_1) + 10 \cdot \alpha$. Fixing the reference levels $u_1(l_1 = 1,000,000) = 0$ and $u_1(l_5 = 0) = 100$ the value of the unit is $\alpha = \frac{u(l_5)-u(l_1)}{10} = \frac{100-0}{10} = 10$. The remaining values are computed as follows:

- $u_1(l_2 = 750,000) = u(l_1) + (e_{12} + 1) \cdot \alpha = 0 + (3 + 1) \cdot 10 = 40$.
- $u_1(l_3 = 500,000) = u(l_1) + (e_{13} + 1) \cdot \alpha = 0 + (6 + 1) \cdot 10 = 70$.
- $u_1(l_4 = 250,000) = u(l_1) + (e_{14} + 1) \cdot \alpha = 0 + (8 + 1) \cdot 10 = 90$.

In order to compute the values of the performances we need to proceed with a linear interpolation:

- $u_1(30,000) = 100 + ((30,000 - 0)/(250,000 - 0))(90 - 100) = 98.8$.
- $u_1(45,000) = 100 + ((45,000 - 0)/(250,000 - 0))(90 - 100) = 98.2$.
- $u_1(90,000) = 100 + ((90,000 - 0)/(250,000 - 0))(90 - 100) = 96.4$.
- $u_1(120,000) = 100 + ((120,000 - 0)/(250,000 - 0))(90 - 100) = 95.2$.
- $u_1(900,000) = 40 + (900,000 - 750,000)/(1,000,000 - 750,000))(0 - 40) = 16$.

2. Criterion g_2 `PROFI` (Profitability). See Appendix B.
3. Criterion g_3 `SERVI` (New services for the population). In this case, the DM provided imprecise preference information as shown in the following comparison table:

	$l_1(vb)$	$l_2(b)$	$l_3(rb)$	$l_4(a)$	$l_5(rg)$	$l_6(g)$	$l_7(vg)$
$l_1(vb)$	■	[1, 2]	[3, 4]	[6, 7]	[9, 10]	[13, 14]	[18, 19]
$l_2(b)$		■	[1, 2]	[4, 5]	[7, 8]	[11, 12]	[16, 17]
$l_3(rb)$			■	[2, 3]	[5, 6]	[9, 10]	[14, 15]
$l_4(a)$				■	[2, 3]	[6, 7]	[11, 12]
$l_5(rg)$					■	[3, 4]	[8, 9]
$l_6(g)$						■	[4, 5]
$l_7(vg)$							■

Solving the $MILP-I$ problem, we find $z^* = 0$. This means that the information provided by the DM is intervally consistent and there exists at least one precise comparison table that can be extracted from the imprecise one. The precise comparison table obtained by solving the $MILP-I$ problem is shown below:

	$l_1(vb)$	$l_2(b)$	$l_3(rb)$	$l_4(a)$	$l_5(rg)$	$l_6(g)$	$l_7(vg)$
$l_1(vb)$	■	1	3	6	9	13	18
$l_2(b)$		■	1	4	7	11	16
$l_3(rb)$			■	2	5	9	14
$l_4(a)$				■	2	6	11
$l_5(rg)$					■	3	8
$l_6(g)$						■	4
$l_7(vg)$							■

This time, checking for other possible precise comparison tables compatible with the imprecise information provided by the DM we find 6 more tables. For space reasons, we do not report all of them in the manuscript but we shall take into account all these tables to obtain the final recommendation on the considered problem. These precise tables are shown in the e-Appendix. To obtain the utilities of the 7 levels on g_3, we consider the precise table above. We fix $u_3(vb = 1) = 0$ and $u_3(vg = 7) = 100$. Then, compute $h = e_{17} + 1 = 19$ and $u_3(vg = 7) = u_3(vb = 1) + 19 \cdot \alpha$. Consequently, the value of the unit is $\alpha = \frac{u_3(vg=7)-u_3(vb=1)}{19} = (100 - 0)/19 = 5.263$. The other five levels are computed as follows:

- $u_3(b = 2) = u_3(vb = 1) + (e_{12} + 1) \cdot \alpha = 0 + 2 \cdot 5.263 = 10.526$,
- $u_3(rb = 3) = u_3(vb = 1) + (e_{13} + 1) \cdot \alpha = 0 + 4 \cdot 5.263 = 21.052$,
- $u_3(a = 4) = u_3(vb = 1) + (e_{14} + 1) \cdot \alpha = 0 + 7 \cdot 5.263 = 36.841$,
- $u_3(rg = 5) = u_3(vb = 1) + (e_{15} + 1) \cdot \alpha = 0 + 10 \cdot 5.263 = 52.630$,
- $u_3(g = 6) = u_3(vb = 1) + (e_{16} + 1) \cdot \alpha = 0 + 14 \cdot 5.263 = 73.682$.

4. Criterion g_4 `SURFA` (Naturalized surface). See Appendix B.
5. Criterion g_5 `ENVIR` (Environmental effects). See Appendix B.
6. Criterion g_6 `CONSI` (Consistency with local requirements). For this criterion we simply consider that “no” has a zero value and “yes” a 100 value.

The table containing the utilities obtained by the previous steps is shown below. Let us observe that for criteria g_3 and g_5 we reported the utilities obtained solving the corresponding $MILP-I$ problems.

7.2. Getting the weighs w_j

In this section, we shall now use our approach based on the DCM to get the weights w_j to be used in eq. (18) to aggregate the normalized utilities computed as explained above. For

Table 2
Score table.

a	$u_1(g_1(a))$	$u_2(g_2(a))$	$u_3(g_3(a))$	$u_4(g_4(a))$	$u_5(g_5(a))$	$u_6(g_6(a))$
a_1	98.8	20.00	0.0000	11.111	36.364	100.00
a_2	98.2	20.00	52.630	100.00	54.546	100.00
a_3	96.4	0.000	73.682	22.222	100.00	100.00
a_4	95.2	0.000	100.00	66.666	72.728	100.00
a_5	16.0	100.0	100.00	0.000	18.182	0.0000

such a reason, at first, each criterion g_j, $j = 1, \ldots, 6$, is associated with a fictitious project p_j having the highest evaluation on g_j and the lowest on the remaining ones. Considering the normalized scores in Table 2, the highest and lowest evaluations in all criteria are 100 and 0, respectively (we can also use performances instead of evaluations). Therefore, $p_1 = (100, 0, 0, 0, 0, 0)$, while $p_6 = (0, 0, 0, 0, 0, 100)$. Then, the DM is asked to rank the p_j in a non-decreasing way, with respect to the satisfaction attached to each of them. The DM ranks the projects as follows

$$p_6 \prec p_1 \prec p_5 \prec \{p_2, p_3\} \prec p_4$$

meaning that p_6 is the least satisfying project; p_1 is more satisfying than p_6 but less satisfying than p_5 that, in turn, is less satisfying than p_2 and p_3 being equally satisfying. Finally, both of them are less satisfying than p_4, being the most satisfying project.

At this point, the DM is asked to include a certain number of blank cards between these projects (not only consecutive as in the classical DCM) representing the difference of satisfaction between them. In this way, the DM provides the values of the blank cards shown in the table below:

	p_6	p_1	p_5	$\{p_2, p_3\}$	p_4
p_6	■	1	3	4	6
p_1		■	1	2	4
p_5			■	0	2
$\{p_2, p_3\}$				■	1
p_4					■

Let us observe that the comparisons provided by the DM are precise and the consistency condition 1 is satisfied. Therefore, it is not necessary revising the given information.

Finally, the DM is asked to provide the ratio z between the satisfaction level attached to the best ranked project (p_4) and the satisfaction level attached to the worst ranked project (p_6). The DM provides a value of 8 for such a ratio meaning that the satisfaction attached to project p_4 is 8 times greater than the satisfaction attached to p_6.

The provided information can, therefore, be summarized in the following system of equalities[6]

$$\begin{cases} w_1 = w_6 + (e_{16}+1)\cdot\alpha = w_6 + 2\cdot\alpha, \\ w_2 = w_6 + (e_{26}+1)\cdot\alpha = w_6 + 5\cdot\alpha, \\ w_3 = w_6 + (e_{36}+1)\cdot\alpha = w_6 + 5\cdot\alpha, \\ w_4 = w_6 + (e_{46}+1)\cdot\alpha = w_6 + 7\cdot\alpha, \\ w_5 = w_6 + (e_{46}+1)\cdot\alpha = w_6 + 4\cdot\alpha \\ w_4 = z\cdot w_6 = 8\cdot w_6 \end{cases}$$

so that, considering $w_6 = 1$ we obtain

$$\alpha = \frac{z-1}{e_{46}+1} = \frac{8-1}{7} = 1,$$

and, consequently, the non-normalized (NNW) and normalized (NW) weights shown in the table below:

	NNW	NW
g_1	3	$\frac{3}{29} = 0.103$
g_2	6	$\frac{6}{29} = 0.207$
g_3	6	$\frac{6}{29} = 0.207$
g_4	8	$\frac{8}{29} = 0.276$
g_5	5	$\frac{5}{29} = 0.172$
g_6	1	$\frac{1}{29} = 0.035$

7.3. On the use of SMAA for obtaining more robust conclusions

As already observed in the previous section and in Appendix B, regarding criteria g_3 and g_5, there exists more than one comparison table compatible with the imprecise or missing information provided by the DM. Comprehensively, we have 7 comparison tables for g_3, 8 comparison tables for g_5 and only one comparison table for the remaining criteria. Since using each comparison table we can obtain one utility for the considered levels, we need to take into account 56 different cases. All the considered tables are shown in the e-Appendix. To get more robust recommendations on the problem at hand, we therefore take into account the 56 multicriteria evaluation combinations using the SMAA methodology and providing the rank acceptability indices and the pairwise winning indices shown in the tables below. Let us remind that the rank acceptability index $b_k(a)$ gives the frequency with which the alternative a gets the k-th position, with $k = 1, \ldots, 5$, while the pairwise winning index (Leskinen, Viitanen, Kangas, & Kangas, 2006), $p(a_i, a_r)$, gives the frequency with which the project a_i is preferred to the project a_r, with $i, r = 1, \ldots, 5$.

	$b_1(\cdot)$	$b_2(\cdot)$	$b_3(\cdot)$	$b_4(\cdot)$	$b_5(\cdot)$
a_1	0	0	0	0	100
a_2	76.786	23.214	0	0	0
a_3	0	0	100	0	0
a_4	23.214	76.786	0	0	0
a_5	0	0	0	100	0

$p(\cdot,\cdot)$	a_1	a_2	a_3	a_4	a_5
a_1	0	0	0	0	0
a_2	100	0	100	76.786	100
a_3	100	0	0	0	100
a_4	100	23.214	100	0	100
a_5	100	0	0	0	0

Looking at these tables, one can observe that the best project has to be chosen between a_2 and a_4 being the only that can take the first ranking position. In particular, a_2 is a bit in advantage since it has a first rank acceptability index equal to 76.786% against the 23.214% of a_2. Looking at the worst among the considered projects there is not any doubt about a_1 since it is always in the last position. In this way, the DM can choose a_2 or she can wish to investigate a bit more on both projects neglecting all the others.

To conclude this section, let us show how the results change if we admit that there is a not-integer card distance between two levels as assumed in Corrente et al. (2017). Let us consider the preference information provided by the DM on criterion g_3 and represented in the following comparison table:

	$l_1(vb)$	$l_2(b)$	$l_3(rb)$	$l_4(a)$	$l_5(rg)$	$l_6(g)$	$l_7(vg)$
$l_1(vb)$	■	$[1,2]$	$[3,4]$	$[6,7]$	$[9,10]$	$[13,14]$	$[18,19]$
$l_2(b)$		■	$[1,2]$	$[4,5]$	$[7,8]$	$[11,12]$	$[16,17]$
$l_3(rb)$			■	$[2,3]$	$[5,6]$	$[9,10]$	$[14,15]$
$l_4(a)$				■	$[2,3]$	$[6,7]$	$[11,12]$
$l_5(rg)$					■	$[3,4]$	$[8,9]$
$l_6(g)$						■	$[4,5]$
$l_7(vg)$							■

As we already observed in Section 7.1, solving the $MILP-I$ problem, we find $z^* = 0$. Therefore, there exists at least one precise

[6] Let us observe that, following the description provided in the previous sections, e_{pq} represents the number of blank cards that should be included between levels p and q. In the considered system of equalities e_{pq} denotes, instead, the number of blank cards that should be included between project p and project q.

comparison table, which elements are integer, compatible with the preferences provided by the DM. Consequently, there exists at least one comparison table which elements are not-integer, compatible with the same preferences. For example, the following comparison table

	$l_1(vb)$	$l_2(b)$	$l_3(rb)$	$l_4(a)$	$l_5(rg)$	$l_6(g)$	$l_7(vg)$
$l_1(vb)$	■	1.169	3.319	6.3557	9.7306	13.7817	18.9972
$l_2(b)$		■	1.15	4.1866	7.5616	11.6127	16.8282
$l_3(rb)$			■	2.0367	5.4146	9.4627	14.6782
$l_4(a)$				■	2.3749	6.426	11.6415
$l_5(rg)$					■	3.0511	8.2666
$l_6(g)$						■	4.2155
$l_7(vg)$							■

satisfies the consistency condition 1, it is concordant with the imprecise preferences provided by the DM and its entries are not-integer. Differently from the case in which there is an integer card distance between two levels and, therefore, the number of comparison tables compatible with the preferences provided by the DM is finite, considering a not-integer card distance between two levels, there is an infinite number of compatible comparison tables. In particular, defining $P=\{(p,q): p=1,\ldots,6; q=(p+1),\ldots,7\}$, all comparison tables $[e_{pq}]_{(p,q)\in P}$, which elements satisfy these constraints

$$e_{pk}+e_{kq}+1=e_{pq}, \qquad (p,k),(k,q),(p,q)\in P,$$
$$e^L_{pq}\leqslant e_{pq}\leqslant e^R_{pq}, \qquad (p,q)\in P,$$
$$e_{pq}\in\mathbb{R}^+_0, \qquad (p,q)\in P,$$

are compatible with the preferences provided by the DM. Therefore, we applied the Hit-And-Run method (Smith, 1984; Tervonen, Van Valkenhoef, Bastürk, & Postmus, 2013; Van Valkenhoef, Tervonen, & Postmus, 2014)[7] to sample from the space defined by the constraints above 10,000 precise compatible comparison tables. For each of these comparison tables, we can compute the evaluations for all levels. For example, considering the precise values in the comparison table above and assuming that $u_3(vg=7)=100$ and $u_3(vb=1)=0$ we get $h=e_{17}+1=19.9972$ and, consequently, $\alpha=\frac{u_3(vg=7)-u_3(vb=1)}{h}=\frac{100-0}{19.9972}=5.0007$. Therefore,

- $u_3(b=2)=u_3(vb=1)+(e_{12}+1)\cdot\alpha=0+2.169\cdot 5.0007=10.8467$,
- $u_3(rb=3)=u_3(vb=1)+(e_{13}+1)\cdot\alpha=0+4.319\cdot 5.0007=21.5982$,
- $u_3(a=4)=u_3(vb=1)+(e_{14}+1)\cdot\alpha=0+7.3557\cdot 5.0007=36.7835$,
- $u_3(rg=5)=u_3(vb=1)+(e_{15}+1)\cdot\alpha=0+10.7306\cdot 5.0007=53.6605$,
- $u_3(g=6)=u_3(vb=1)+(e_{16}+1)\cdot\alpha=0+14.7817\cdot 5.0007=73.9188$.

Repeating this procedure for the 10,000 sampled precise comparison tables and considering that on g_5 there were 8 different comparison tables, while, for all the other criteria there was just one precise comparison table, we computed the rank acceptability indices and the pairwise winning indices shown in the tables below taking into account all 80,000 multicriteria evaluation combinations:

	$b_1(\cdot)$	$b_2(\cdot)$	$b_3(\cdot)$	$b_4(\cdot)$	$b_5(\cdot)$
a_1	0	0	0	0	100
a_2	87.5	12.5	0	0	0
a_3	0	0	100	0	0
a_4	12.5	87.5	0	0	0
a_5	0	0	0	100	0

[7] See Corrente, Greco, and Słowiński (2019) for a detailed description of the application of the HAR method in MCDA.

$p(\cdot,\cdot)$	a_1	a_2	a_3	a_4	a_5
a_1	0	0	0	0	0
a_2	100	0	100	87.5	100
a_3	100	0	0	0	100
a_4	100	12.5	100	0	100
a_5	100	0	0	0	0

It is easy to observe that the obtained results are not very far from those previously computed.

8. Comparison with other MCDA methods

In this section we compare our extension of the DCM with three other very well-known multiple criteria elicitation procedures (SMART, AHP and MACBETH) pointing out the advantages of the methodology we are proposing as well as its drawbacks. Finally we briefly discuss the concept of importance and weights of criteria with respect to all these assessment procedures. Before comparing our approach with other methods, let us remark that it collapses to the basic SRF method (Figueira & Roy, 2002) in case the preference information supplied by the DM is given only by the number e_{pq} of blank cards between the level l_p and the subsequent level l_q, with $p=1,\ldots,t-1$ and $q=p+1$.

8.1. The SMART approach

The Simple Multi-Attribute Rating Technique, or SMART approach (Edwards, 1977; Edwards & Barron, 1994) is based on a direct rating technique. It requires the DM to assign to each item a numerical value according to its relative attractiveness, from 0, which is the value assigned to the least attractive item, to 100, which is the value assigned to the most attractive one. Any other item would be directly assigned to an intermediate value, according to the perceived difference in attractiveness from the items already scored. To aggregate all these partial values into a single score, the weights of the criteria are also acquired by assigning the least important criterion a value of 10 and assigning to other criteria values that reflect the ratio between their weights, so that if a criterion weight i is considered k times higher than criterion j, then, denoting by w_i and w_j the weights assigned to i and j, respectively, $w_i=kw_j$.

In the SWING method (Von Winterfeldt & Edwards, 1986), which is coupled with the SMART approach in the SMARTS method (Edwards & Barron, 1994), the DM is asked at the beginning to consider a fictitious alternative having the worst performance on all considered criteria. For the sake of simplicity, let us assume that three criteria are taken into account and the worst evaluation is 0 in all of them, so that this fictitious alternative is (0,0,0). Now, the DM is asked for which criterion g_i, the *swing* from the worst evaluation, let us say x_{i^*}, to the best evaluation, let us say x^*_i, gives the greatest increase to the overall value (Belton & Stewart, 2002). Assuming that 100 is the best evaluation on all criteria, that is, $x^*_i=100, i=1,2,3,$ if the DM states that the swing on the second criterion provides the greatest value to the initial fictitious alternative, then this criterion has the highest weight. The DM is then asked to sequentially repeat the same procedure with all the remaining criteria. If she states that the swings are as follows (0,100,0), (0,100,100) and (100,100,100), then g_2 has the highest weight, followed by g_3 and, finally, by g_1. At this point, the DM is asked some questions to infer numerical values for the criteria weights. Assuming that g_j results the criterion having the highest swing value and, consequently, the highest weight, for all other criteria g_i, $i\neq j$, she is asked to provide a value $x_{j|i}$ such that the alternative having the evaluation $x_{j|i}$ on the criterion j and the worst one on the other criteria is indifferent to the alternative having the best evaluation x^*_i on criterion i and the worst one on the remaining criteria. The ratio $\frac{x_{j|i}}{x_i^*}$ provides, therefore, the ratio between the weight of criterion i and the weight of criterion j.

Going back to the previous example, let us assume the DM specifies the values $x_{2|3} = 60$ and $x_{2|1} = 40$ such that, on the one hand, (0,0,100) is indifferent to (0, 60, 0) and, on the other hand, (100,0,0) is indifferent to (0, 40, 0). Consequently, $\frac{60}{100} = \frac{w_3}{w_2}$ and $\frac{40}{100} = \frac{w_1}{w_2}$.

With respect to SMART and SMARTS, the approach we are proposing presents the following relevant differences:

1. While in SMART and SMARTS the DM gives a direct rating of attractiveness of single alternatives with respect to considered criteria, in our approach, the DM can express pairwise comparisons of alternatives with respect to considered criteria in terms of number of blank cards representing difference of attractiveness. Analogously, in SMART and SMARTS the DM supplies evaluations of weights of single criteria or single fictitious alternatives related to considered criteria, while in our approach the DM gives number of cards expressing difference in evaluation. This permits to our approach to check for the consistency of the information supplied by the DM, while this is not possible in the SMART and SMARTS methods;
2. The expression of the difference in attractiveness by means of number of blank cards of our approach has an intermediate nature between the numerical value required by SMART and SMARTS and the ordinal qualitative evaluation used in AHP and MACBETH (see Sections 8.2 and 8.3 below). This intermediate nature makes the methodology we are proposing quite appealing in view of the results of the experiment illustrated in Fasolo and Bana e Costa (2014) pointing out that “more numerate DMs” express values more easily when assisted by SMART, while “more fluent DMs” find value elicitation easier with MACBETH. While Fasolo and Bana e Costa (2014) suggests to select more numerical or more qualitative elicitation techniques on the basis of the DM’s numeracy and fluency attitude, we advocacy our approach requiring an information having both numerical and qualitative nature.

8.2. The Analytic Hierarchy Process

The Analytic Hierarchy Process (AHP) (Saaty, 1977) requires that the DM supplies pairwise comparisons between items that can be criteria or performances of alternatives with respect to specific criteria. In case of comparison of criteria, AHP supplies priorities representing their priority level, while, in case of comparison of alternatives with respect to a given criterion, AHP supplies priorities representing a preference value of the corresponding performances. The pairwise comparisons given by the DM are expressed on a nine point ordinal qualitative scale with the following semantic: 1-indifference, 3-moderate preference, 5-strong preference, 7-very strong preference, and 9-extreme preference, and 2, 4, 6 and 8 represent intermediate degrees of preference between the two adjacent judgments. Observe that the ordinal qualitative judgments supplied by the DM are “automatically” coded in numerical terms, so that, for example, the judgment “item i is strongly preferred to item j” is interpreted as “the priority p_i of i should be 5 times greater than the priority p_j of item j”. Consequently, the priorities of considered items are supposed to be expressed on a ratio scale, so that, denoting by a_{ij} the numerical codification of the qualitative judgment related to items i and j, if the DM would be perfectly coherent for all pair of items i and j

$$a_{ij} = \frac{p_i}{p_j} \tag{20}$$

and the following consistency condition should be satisfied:

$$a_{ij}a_{jh} = a_{ih} \tag{21}$$

for all triples of items i, j and h. If consistency condition (21) would hold, the priorities p_i could be easily assessed by assigning a unitary value to an item h and by putting $p_i = a_{ih}$ for all items i. Unsurprisingly, consistency condition (21) is rarely satisfied so that it is not possible to fix priorities p_i satisfying condition (20). The priorities p_i are therefore assessed as elements of the eigenvector $\mathbf{p} = [p_i]$ of the matrix $\mathbf{A} = [a_{ij}]$, that is

$$\mathbf{Ap} = \lambda_{max}\mathbf{p}$$

with λ_{max} being the maximum eigenvalue of $\mathbf{A}$. The consistency of the DM is measured through the consistency index CI and the consistency ratio CR

$$CI = \frac{\lambda_{max} - n}{n-1}, \quad CR = \frac{CI}{RI},$$

where n denotes the number of items and RI is the mean value of CI corresponding to a random sample of comparison matrices $\mathbf{A}$. A value of RI not greater than 10% is considered acceptable, otherwise it is suggested to revise the judgments a_{ij} to increase the consistency. With respect to AHP, the approach we are proposing presents the following main differences:

1. In our approach there is not a prefixed evaluation scale with a well defined number of levels. Instead the DM can put any number of blank cards between two levels. This permits the DM to express more precisely and more freely her judgments;
2. The consistency condition of our approach is more understandable for the DM, because it amounts to count the number of blank cards between three levels. It has also a visual interpretation. Instead, the consistency condition of AHP remains relatively abstract for the DM, especially considering that the conversion of qualitative judgments to numerical values is “conventional” and very often perceived as a “black box” by the DM;
3. As pointed out by Bana e Costa and Vansnick (2008), the eingevalue approach adopted in AHP violates the Condition of Order Preservation (COP) for which if item i is preferred to item j more than item i' is preferred to item j', then the ratio of the priorities of i and j should be greater than the ratio of priorities of i' and j', that is, if $a_{ij} > a_{i'j'}$ then $\frac{p_i}{p_j} > \frac{p_{i'}}{p_{j'}}$. In the context of the extension of the DCM we are proposing, the COP should be reformulated as follows: “if the number of blank cards between levels l_p and l_q is greater than the number of blank cards between levels $l_{p'}$ and $l_{q'}$, then the difference between the values assigned to the levels l_p and l_q should be greater than the difference between the values assigned to levels $l_{p'}$ and $l_{q'}$”. Let us observe that the approach we are proposing aims to obtain pairwise comparisons that are consistent, by supporting the DM in correcting eventual inconsistencies when they are detected. Therefore, our approach satisfies a condition even stronger than the COP being the following: “the difference between the values assigned to items in levels l_p and l_q is proportional to the number of blank cards between them plus one, that is, $e_{pq} + 1$”;
4. AHP requires that the DM supplies pairwise comparison judgments a_{ij} for all pairs of considered items, which is a quite demanding procedure from the cognitive point of view. In our approach the DM is not required to express the number of blank cards to be included between any pair of levels l_p and l_q and, instead, she is encouraged to give only the information on which she is convinced enough. We have to

observe that also for AHP, several methodologies have been proposed to handle incomplete pairwise comparison matrices permitting the DM to provide only a part of the pairwise comparison judgments a_{ij} (see, for example, Bozóki, Fülöp, & Rónyai 2010; Harker 1987a; 1987b; Takeda & Yu 1995).

8.3. MACBETH

MACBETH (Measuring Attractiveness by a Categorical Based Evaluation TecHnique) (Bana e Costa & Vansnick, 1994; Bana e Costa, De Corte, & Vansnick, 2016) is a method for weighting criteria and for value assessment of alternatives with respect to considered criteria. Also MACBETH requires from the DM qualitative judgments on the difference of attractiveness of alternatives or criteria on a seven point scale: null, very weak, weak, moderate, strong, very strong, and extreme. The main idea of MACBETH is to build an interval scale compatible with the difference of attractiveness provided by the DM. With this aim, using some specific linear programming model, values are assigned to the criteria or alternatives so that if the difference of attractiveness between i and j is greater than the difference of attractiveness between i' and j', then $v_i - v_j > v_{i'} - v_{j'}$, where $v_i, v_j, v_{i'}$ and $v_{j'}$ are the values assigned to i, j, i' and j', respectively. In case of inconsistent judgments, MACBETH provides the DM with information permitting to eliminate such inconsistency.

With respect to MACBETH, the approach we are proposing presents the following relevant differences:

1. Our approach requires the DM to provide the number of blank cards between different levels, while the MACBETH requires a qualitative information in terms of difference of attractiveness expressed on its seven point ordinal scale. Similarly to the comparison with AHP provided above, we observe that even if the information required by our approach is not purely qualitative, it has a clear visual interpretation for the DM;
2. MACBETH requires that the difference of attractiveness is expressed on a fixed scale, while this is not the case for our approach, that, without this constraint, permits the DM to define with precision and accuracy her judgments;
3. Our approach requires that the difference between the values assigned to elements from l_p and l_q must be proportional to the number of blank cards between these levels plus one. MACBETH requires the quite weaker above mentioned COP property for which the difference of the values assigned to considered elements should maintain the order of the difference of attractiveness supplied by the DM. Consequently our approach is requiring a more restrictive coherence condition than MACBETH, which could be seen as a weak point for the methodology we are proposing. However, observe that this is related to the nature of the preference information required to the DM. Indeed, our approach permits the DM to supply a finer and more precise preference information, that is, not only qualitative judgments, such as “weak” or “moderate”, but more quantitative judgments related to the number of cards between different levels. Moreover, if the DM wants, she can use a qualitative scale defined by the procedure that will be explained in Section 8.4.

8.4. Drawbacks of the DCM

After comparing our approach with AHP and MACBETH, let us now present and discuss some possible drawbacks. We believe that they are the following:

1. *supplying the preference information related to a complete comparison table can be quite requiring from a cognitive point of view*: indeed, when the DM compares n items, she is requested to supply $\frac{n\cdot(n-1)}{2}$ pairwise comparisons, so that, if there are n alternatives and m criteria, the DM has to supply $\frac{m\cdot(m-1)}{2}$ pairwise comparison of criteria to define their weights and $\frac{n\cdot(n-1)}{2}$ pairwise comparisons of alternatives for each one of the m criteria, to define the value of each alternative with respect to each criterion. In fact, this is a weak point common also to AHP and MACBETH and we have already largely discussed on how our approach can process an incomplete pairwise comparison table. Therefore, when applying our approach, one has to be aware of this issue, but, one has also to take into account that there is a methodology to handle this drawback. Analogous approaches to deal with incomplete pairwise comparison matrices have been proposed for AHP and MACBETH (see, e.g., Bozóki et al. 2010; Bana e Costa et al. 2016; Harker 1987a; 1987b; Takeda & Yu 1995);
2. *providing the ratio z can be a difficult task for the DM:* indeed, differently from other preference information required by our method, the z value has not an intuitive and “physical” interpretation in terms of number of cards. Instead, z represents the ratio between the evaluations of two reference items (criteria, reference projects,...). Therefore the content of the information related to the ratio z is heterogeneous with respect to other preference information required by the method and, overall, it is much more abstract. We have to say that our method inherits this weak point from the basic SRF method proposed in Figueira and Roy (2002). Let us observe that, despite this weak point, SRF method has been successfully adopted in hundreds of real world applications. Therefore we can argue that this drawback is counterbalanced by the relative easy and understandable nature of the prevalent preference information in terms of number of cards required by the method. Observe also that, recently, Abastante, Corrente, Greco, Lami, and Mecca (2020) presented an extension of the SRF method in which the preference information captured by the ratio z is also expressed in terms of number of cards;
3. *the absence of a qualitative scale:* differently from the nine point AHP scale and the six point MACBETH scale, the absence of a qualitative scale can be a restrictive constraint for the preference information processed by our approach. In fact, a great share of the success of AHP and MACBETH is related to their plain and clear qualitative scales that make comfortable to express pairwise comparisons for the DM. Instead, our approach requires the DM to express pairwise comparisons in terms of number of blank cards and this can require a certain cognitive effort. Let us observe that this disadvantage is counterbalanced by the possibility to control the quantitative codification of the pairwise comparison judgments. Indeed, taking for example into account the nine point scale of AHP, why qualitative judgments “moderate”, “strong”, “very strong” and “extreme” have a value of 3, 5, 7 and 9, respectively? In fact, on the scale of AHP there has been a quite articulated and critical discussion (see e.g. Ishizaka, Balkenborg, & Kaplan 2011) and several other alternative scales have been proposed in literature (see, e.g., Harker & Vargas 1987; Lootsma 1989; Salo & Hämäläinen 1997).

Taking into account the last point, we propose a specific procedure to assign quantitative values to levels of qualitative scales by means of our approach. The basic idea is the following. Consider a qualitative scale such as, for example, “indifferent”, “moderately preferred”, “strongly preferred”, “very strongly preferred” and “extremely preferred”. Each level in the scale should be assigned an integer number representing the number of blank cards between the items for which the corresponding judgment is expressed. The level "indifferent", representing a null preference judgment, means that two items are put in the same equivalence class and consequently receive the same evaluation. Therefore, for the moment, "indifferent" is not associated to any number of blank cards. The other levels will be given a non-negative integer number, increasing with respect to the intensity of preference expressed by the judgment.

We have already seen that zero blank cards between two items represent the minimal difference among them. The DM can assign directly the number of blank cards to each level in the scale. For example, the DM can give zero blank cards to "moderately preferred", 3 blank cards to "strongly preferred", 6 blank cards to "very strongly preferred" and 8 blank cards to "extremely preferred". These values can be considered as number of blank cards between the considered level in the scale and the null preference level "indifferent". In addition or alternatively, the DM can express judgments in terms of number of blank cards to be included between other pairs of levels in the scale. For example, the DM can assign 2 blank cards between "moderately preferred" and "strongly preferred", and 4 blank cards between "strongly preferred" and "extremely preferred". This information can be used to obtain a compatible comparison table if the information is not complete and to verify that the consistency condition 1 holds. If the consistency condition 1 is not satisfied, the pairwise comparison table has to be revised together with the DM to modify some judgments in order to restore the consistency. If more than one complete consistent comparison table is compatible with the preference information supplied by the DM, the whole set of compatible comparison matrices can be computed to take into account robustness concerns.

The DM can compare reference levels of the qualitative scale for which other intermediate levels are present. For instance, if we add an intermediate level between each contiguous pair of levels, the scale we are considering in our example becomes the nine point scale of AHP. In this case, the intermediate level l_{r+1} should be assigned a number of blank cards intermediate between the number of blank cards $\nu(l_r)$ and $\nu(l_{r+2})$ assigned to the two reference levels l_r and l_{r+2} between which l_{r+1} lies. This means that there must exist a non-negative integer number d such that the number of blank cards between l_r and l_{r+2} is equal to $2d+1$. If this condition holds, then the intermediate level l_{r+1} is assigned a number of blank cards $\nu(l_{r+1}) = \nu(l_r) + d + 1$ so that there are d blank cards between l_r and l_{r+1} and d blank cards between l_{r+1} and l_{r+2}.

Suppose, for example, that the nine point scale $l_1, \ldots, l_9$ of AHP is considered and that the DM gives 1 blank card to l_3 ="moderately preferred", 5 blank cards to l_5 ="strongly preferred", 9 blank cards to l_7 ="very strongly preferred" and 11 blank cards to l_9 ="extremely preferred". In this case, 0 blank cards are assigned to l_2, 3 blank cards to l_4, 7 blank cards to l_6 and, finally, 10 blank cards to l_8. One can verify that, therefore, there are 0 blank cards between l_2 and l_3, 1 blank card between l_3 and l_4, between l_4 and l_5, between l_5 and l_6 and between l_6 and l_7; finally, 0 blank cards are included between l_7 and l_8 and between l_8 and l_9.

Observe that if between two contiguous reference levels l_r and l_{r+k+1} there are k intermediate levels, then the number of blank cards between l_r and l_{r+k+1} must be $(k+1)d + k$ with d being a non-negative integer number. In this way, there are d blank cards between l_r and l_{r+1}, as well as between l_{r+1} and l_{r+2}, and so on, until between l_{r+k} and l_{r+k+1}. Consequently, if the number of blank cards assigned to l_r is $\nu(l_r)$, the number of blank cards assigned to $l_{r+1}, l_{r+2}, \ldots, l_{r+k+1}$ are $\nu(l_r) + d + 1, \nu(l_r) + 2d + 2, \ldots, \nu(l_r) + (k+1)d + (k+1)$, respectively. This means that to check consistency and to assess the desired qualitative scale, for all contiguous reference levels l_p and l_q the following constraints have to be added to $MILP - P$ problem:

$$\bar{e}_{pq} = (k_{pq} + 1)d_{pq} + k_{pq},$$

$$d_{pq} \in \mathbb{N}_0,$$

where k_{pq} is the number of intermediate levels between l_p and l_q (so that $k_{pq} = q - p + 1$) and d_{pq} is the number of blank cards between any two consecutive levels between l_p and l_q.

Let us show with an example, how the procedure we are proposing is applied. Suppose that a DM wants to express qualitative pairwise judgments using the nine point scale of AHP. The procedure starts by collecting the preference information that permits to assign the number of blank cards to the nine levels of the AHP qualitative scale. With this aim the DM provides the pairwise comparison table based on the DCM considering the reference levels 1, 3, 5, 7 and 9 (in fact, "indifferent", "moderately preferred", "strongly preferred", "very strongly preferred" and "extremely preferred") shown in Table (3,left).

To check the consistency of the pairwise comparison Table (3,left), we have to solve $MILP - P$ with the addition of the following constraints permitting to define also the number of cards for the intermediate levels 2,4,6,8:

$$\bar{e}_{pq} = 2d_{pq} + 1, d_{pq} \in \mathbb{N}_0, \ (p,q) \in \{(1,3), (3,5), (5,7), (7,9)\}.$$

We get that $z^* = 2$ meaning that the information provided by the DM is inconsistent. However, the same program suggests the consistent comparison table shown in (3,right) being also the unique that can be obtained by the inconsistent information provided by the DM with only two corrections.

Considering the values in Table (3,right) we obtain also the following values of d: $d_{13} = 1, \ d_{35} = 2, \ d_{57} = 1, \ d_{79} = 4$. To apply the approach we are proposing, as already explained in the previous sections, the ratio z is necessary. However, in this case, we can use another implicit information being related to the value of the level 1, that is $\nu(1)$, representing the number of blank cards that should be included between two indifferent levels. Indeed, if levels l_p and l_q are indifferent, the value that should be assigned them is the same, that is, $u(l_p) = u(l_q)$. However, at the same time, $u(l_p) = u(l_q) + (\nu(1) + 1) \cdot \alpha$, where $\alpha > 0$ is the value assigned to a blank card. The previous equality reduces to $(\nu(1) + 1) \cdot \alpha = 0$ and, consequently, $\nu(1) = -1$.

Starting from this, taking $\alpha = 1$ as the unit is the single blank card, we therefore get the other values assigned to the qualitative scale, that is:

$$\begin{aligned}
\nu(2) &= \nu(1) + d_{13} + 1 = -1 + 1 + 1 = 1, \\
\nu(3) &= \nu(2) + d_{13} + 1 = 1 + 1 + 1 = 3, \\
\nu(4) &= \nu(3) + d_{35} + 1 = 3 + 2 + 1 = 6, \\
\nu(5) &= \nu(4) + d_{35} + 1 = 6 + 2 + 1 = 9, \\
\nu(6) &= \nu(5) + d_{57} + 1 = 9 + 1 + 1 = 11, \\
\nu(7) &= \nu(6) + d_{57} + 1 = 11 + 1 + 1 = 13, \\
\nu(8) &= \nu(7) + d_{79} + 1 = 13 + 4 + 1 = 18, \\
\nu(9) &= \nu(8) + d_{79} + 1 = 18 + 4 + 1 = 23.
\end{aligned}$$

The previous equalities permit, therefore, to convert the 1–9 qualitative scale of the AHP in number of blank cards to be included between two items in our approach. However, to fully compare our method with the AHP, we need a way to translate the values $\frac{1}{l}$, with $l = 1, \ldots, 9$, used in AHP. To this aim, we therefore need to define the number of blank cards e_{ab} that should be included between two items a and b, when a has a better evaluation than b.

We know that, when a has a better evaluation than b

$$u(a) = u(b) + (e_{ba} + 1) \cdot \alpha \tag{22}$$

with α being the value of a card. From Eq. (22) we get

$$u(b) = u(a) - e_{ba} \cdot \alpha - \alpha \tag{23}$$

Table 3
Pairwise comparisons of reference levels in the nine point scale (left) and consistent pairwise comparisons of reference levels found by solving $MILP-P$ (right).

	1	3	5	7	9
1	■	3	9	13	23
3		■	5	10	19
5			■	3	12
7				■	9
9					■

	1	3	5	7	9
1	■	3	9	13	23
3		■	5	9	19
5			■	3	13
7				■	9
9					■

Table 4
Pairwise comparisons of three schools with respect to vocational training applying AHP (left) and our approach translation explained above (right).

	S_1	S_2	S_3
S_1	1	7	9
S_2	$\frac{1}{7}$	1	5
S_3	$\frac{1}{9}$	$\frac{1}{5}$	1

	S_3	S_2	S_1
S_3	■	9	23
S_2		■	13
S_1			■

so that, taking $e_{ab}=-e_{ba}$ we obtain

$$u(b)=u(a)+e_{ab}\cdot\alpha-\alpha. \tag{24}$$

This means that if a has a better evaluation than b the number e_{ab} of cards between a and b is the same of the number e_{ba} of cards between b and a, but with a negative sign, and, moreover, to obtain the value $u(b)$, Eq. (24) has to be used instead of the usual

$$u(b)=u(a)+e_{ab}\cdot\alpha+\alpha \tag{25}$$

which holds when b has a better evaluation than a.

Summarizing,

$$u(b)=\begin{cases}u(a)+e_{ab}\cdot\alpha+\alpha & \text{if } b \text{ has an evaluation not worse than } a,\\ u(a)-e_{ab}\cdot\alpha-\alpha & \text{if } a \text{ has a better evaluation than } b.\end{cases} \tag{26}$$

In both cases, e_{ab} denotes the number of blank cards that should be included between items a and b in the DM's opinion.

Reminding that $v(l)$, $l=1,\ldots,9$, found above represent the number of blank cards corresponding to the nine point qualitative AHP scale, since $e_{ab}=-e_{ba}$, we find the corresponding values of $\frac{1}{l}$, with $l=1,\ldots,9$, used in AHP:

$$v\left(\frac{1}{2}\right)=-v(1)=-1,\quad v\left(\frac{1}{3}\right)=-v(3)=-3,$$
$$v\left(\frac{1}{4}\right)=-v(4)=-6,\ v\left(\frac{1}{5}\right)=-v(5)=-9,$$

$$v\left(\frac{1}{6}\right)=-v(6)=-11,\quad v\left(\frac{1}{7}\right)=-v(7)=-13,$$
$$v\left(\frac{1}{8}\right)=-v(8)=-18,\ v\left(\frac{1}{9}\right)=-v(9)=-23.$$

Using the number of blank cards $v(p)$, $p=1,\ldots,9$, so obtained, the pairwise comparison table supplied by the DM in terms of the nine point qualitative AHP scale can be translated in the pairwise comparison table of our approach. For example, considering the comparisons in Tables 4, representing pairwise judgments of three schools with respect to the criterion vocational training in the celebrated selection school example by Saaty (1977), using the above number of blank cards $v(p)$, $p=1,\ldots,9$, we get the corresponding comparisons in terms of number of blank cards shown in Table (4,right).

Observe that to obtain the value of the schools with respect to vocational training using our approach, in addition to determining a consistent comparison table solving $MILP-P$ considering the data in Table (4,right), it is also necessary that the DM orders all items (alternatives or criteria) from the least important to the most important and that she provides the ratio z. Therefore, the procedure we are proposing to assign number of blank cards to levels of a qualitative scale needs some more information than AHP or MACBETH that use their scales without requiring further information to the DM. Nevertheless, the procedure we are proposing permits the DM to actively participate in the definition of the quantitative values in the qualitative scale.

Observe also that the procedure we are proposing to assign a number of blank cards associated to the levels in a qualitative scale, permits to compare AHP (but also MACBETH) and our approach on the same preference information. In this perspective, in Appendix C we propose a comparison of the AHP method with our approach on the well known selection of vacation plan problem presented in Saaty (1977).

8.5. Weights of criteria

The concept of weights of criteria has been largely discussed in literature (see e.g. Choo, Schoner, and Wedley 1999 for a an extensive survey on the subject). It is important to point out that there is a great difference between the weights of criteria used in the additive value functions considered in Multiple Attribute Utility Theory (MAUT, Keeney & Raiffa 1976) and the weights used in the outranking methods such as ELECTRE (Figueira et al., 2013; Figueira, Mousseau, & Roy, 2016) and PROMETHEE (Brans & De Smet, 2016; Brans, Vincke, & Mareschal, 1986) methods (see Roy 2005). Indeed, the weights used in MAUT value functions are substitution rates permitting to aggregate value scoring $v_j(g_j(a))$ of alternatives a on different criteria g_j expressed on different scales. For example, a value function expressing an overall evaluation of a car aggregates maximum speed (expressed in km/h), acceleration (expressed in seconds to pass from 0 to 100 km/h), fuel consumption (expressed in km/l) and so on. In this context, the weight w_j is not dimensionless and it has to change if the value scoring function v_j changes. This is not the case of weights used in outranking methods which have the meaning of number of votes assigned to each criterion in a voting procedure where each criterion expresses its favor or disfavor on the preference of an alternative over another (Roy, 1991). In this context, weights are directly related to the concept of relative importance of a criterion and are dimensionless. Indeed, changing the unit or the scale range of a criterion does not change its relative importance and, consequently, the weight representing it.

This different meaning attached to weights in MAUT and outranking methods can be handled with our procedure by means of the type of questions asked to the DM, which must be adapted to the type of method. In case of outranking methods, the DM is asked to rank the criteria and to put between each pair of levels of criteria a number of blank cards representing the difference between their relative importance, as in the original DCM. Instead, in case of MAUT value functions, the weights are more correctly fixed in terms of values assigned to fictitious projects or alternatives having the worst evaluation in all criteria with the exception of a reference criterion where they have the best evaluation. These fictitious alternatives are the same considered by the SWING method and to assign a value to them the DM is asked to put between each pair of these alternatives a number of blank cards representing the difference of their attractiveness.

Since it is natural for many DMs to use the concept of weights, several researchers investigated relationships between the weights of MAUT value functions and the importance of criteria even if, as explained above, these are quite heterogeneous concepts. For example, Schenkerman (1991) observed that there is a widespread misconception to identify weights of MAUT value functions with importance of criteria. On the basis of some researches that detected a sensitivity of MAUT weights to the range of criteria evaluations smaller than what normative theory suggests (see e.g. Beattie & Baron 1991; Fischer 1995; Von Nitzsch & Weber 1993), Goldstein (1990) proposed a concept of impact of a criterion as the product of the weight of a criterion for the range of variation of the same criterion. Pajala, Korhonen, and Wallenius (2019) investigated different definitions of impact taking into account:

1. Weights supplied by AHP and most discriminant weights (priority levels) with respect to preferences of the DMs estimated with a linear programming model;
2. Different indices of variability for the evaluations of alternatives on considered criteria such as, among others, range of the criterion measured as difference between the maximal and the minimal evaluation, and coefficient of variation corresponding to the ratio between mean and standard deviation of the evaluations of alternatives with respect to the considered criterion.

The most promising definition of impact obtained with this experiment was the product of the AHP weights and the coefficient of variation, so that, on the basis of these results, the authors propose to use AHP weights as good approximation of weights for value functions. The advantage is that AHP requires a small number of pairwise comparison for the DM.

A method that requires a small number of comparison has some advantages, especially for the consideration of the cognitive burden for the DM. However, we also believe that using our approach one can maintain under a reasonable level the involved mental effort continuing to make questions homogeneous to the nature of weights to be elicited. This can be obtained taking advantage of the possibility to elicit weights on the basis of incomplete pairwise comparison tables that contain differences on attractiveness not necessarily between all pairs of items, but, instead, with respect to pairs of items on which the DM is more confident. In this way, we can continue to ask the DM to provide comparisons in terms of difference in attractiveness of “SWING type fictitious projects” in case of a value function is adopted or pairwise comparison in terms of importance of criteria in case an outranking approach is used.

9. Conclusions

We proposed a new methodology to elicit preference information from a DM to assign values to parameters of preference models in MCDA. These parameters can be the relative importance of criteria in outranking methods or values representing evaluations of alternatives on considered criteria and weights of criteria in a SWING based methodology. The methodology we presented puts together the simple interpretation and the visual support of the Deck of Cards Method and the richer and finer information supplied by comparative judgments representing the difference of appreciation between pairs of elements of the comparison tables approaches. This permits to improve the reliability of the values elicited with our methodology. Taking into account the limited human cognitive capacity we provide also procedures permitting to handle inconsistency of information supplied by the DM as well as incompleteness or imprecise or approximate definition of pairwise comparison tables. We believe that the methodology we are proposing has a great potential because it combines several basic aspects of very successful approaches in multiple criteria decision aiding, that are:

1. Deck of cards method (Figueira & Roy, 2002),
2. Pairwise comparison tables from AHP (Saaty, 1977) and MACBETH (Bana e Costa & Vansnick, 1994)
3. SWING method (Von Winterfeldt & Edwards, 1986) to define the weights of criteria in a multiattribute value theory approach.

The methodology we are proposing is strongly based on the interaction with the DM that can supply the information about values and preferences on which she is convinced enough. Indeed, the pairwise comparison tables we are considering have not to be necessarily complete, which means that the DM is not forced to make comparisons on which she is not sure. Moreover, the DM can supply also imprecise information in terms of intervals for the number of cards expressing differences of appreciation of considered elements. The robustness concerns related to the intervals of number of blank cards are taken into account in the methodology through the Stochastic Multicriteria Acceptability Analysis (Lahdelma et al., 1998). The methodology we are proposing permits to test the consistency of the information supplied by the DM and, in case of inconsistency, it supplies the DM with explanations of such inconsistency as well as with suggestions to correct it. These are clearly important elements for a discussion between the DM and the analyst that permits her to mature a more comprehensive understanding of the decision problem.

Due to the above mentioned interesting “good” properties, we expect, therefore, that our approach could be applied to several MCDA decision models, under certain adaptations. As future perspective, following and enriching the approach proposed in Bottero et al. (2018), we plan to apply our proposal to the Choquet integral decision model permitting to represent interaction between criteria. Another class of decision models to which very naturally our approach can be applied is the family of outranking methods for which pairwise comparison tables can be used to assess weights of criteria. The methodology we are proposing could also been coupled with the ordinal regression (Jacquet-Lagreze & Siskos, 1982) and the robust ordinal regression (Greco, Mousseau, & Słowiński, 2008) approaches. Indeed, the information contained in the pairwise comparison table can be used as a form of preference and value information supplied by the DM in order to assess one value function or a family of compatible value functions.

From the behavioral point of view, it would be interesting to study the cognitive burden implied in this method especially in comparison with the one involved in other methods using similar preference information from the DM such as deck of cards, AHP, and MACBETH.

Together and beyond the methodological development we have just outlined, we expect several applications of our approach in real world decision problems. Indeed, it can support the search of good compromise solutions in situations in which several highly conflicting criteria have to be taken into account such as, for example, in territorial and urban planning, energy system management and sustainable development.

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Supplementary material

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