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Solvability of Langevin equations with two Hadamard fractional derivatives via Mittag–Leffler functions

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ABSTRACT

In this paper we discuss the solvability of Langevin equations with two Hadamard fractional derivatives. The method of this discussion is to study the solutions of the equivalent Volterra integral equation in terms of Mittag–Leffler functions. The existence and uniqueness results are established by using Schauder’s fixed point theorem and Banach’s fixed point theorem, respectively. An example is given to illustrate the main results.

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1. Introduction

In recent years, The fractional differential equations have elicited as a rich area of research due to their applications in various fields of science and engineering such as aerodynamics, viscoelasticity, control theory, economics and blood flow phenomena, etc. For more details, we refer the reader to the text books [1–4].

Besides the well-known Riemann–Liouville and Caputo type fractional derivatives, there is an other fractional derivative introduced by Hadamard in 1892 [5], which is known as Hadamard derivative and differs from the preceding ones in the sense that the kernel of the integral in its definition contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in [1,6–8] and the references therein. For some recent results in Hadamard fractional differential equations, we refer the reader to [9–11] and the references therein.

The Langevin equation was able to attract the attention of researchers due to its importance in mathematical physics that is used in modeling the phenomena occurring in fluctuating environment such as Brownian motion [12].

In 1908, P. Langevin [13] derived the classical form of this equation in terms of ordinary derivatives. For the systems in complex media, classical Langevin equation does not provide the correct description of the dynamics. The generalized Langevin equation was introduced in 1966 by Kubo [14], where a frictional memory kernel was incorporated into the Langevin equation to describe the fractal and memory properties. In the 1990s, Mainardi and collaborators [15,16] introduced the fractional Langevin equation. Many interesting results regarding the existence, uniqueness and

stability results for fractional order Langevin equations have been studied by many researchers, see for example [17–23] and references cited therein.

As far as we know, there are no contributions associated with the solutions of the equivalent Volterra integral equations of the fractional Langevin equations in terms of Mittag–leffler functions.

The objective of this paper is to investigate the existence and uniqueness of solutions for the following Langevin equations with two Hadamard fractional derivatives:

$$\begin{aligned} {}^H D_{1,t}^\beta ({}^H D_{1,t}^\alpha - \lambda)x(t) &= f(t, x(t)), \quad t \in [1, e], \lambda > 0, 0 < \alpha, \beta \leq 1, \\ ({}^H D_{1,t}^\alpha - \lambda)x(e) &= 0, \quad {}^H I_{1+}^{1-\alpha} x(1) = c_0, c_0 \in \mathbb{R}, \end{aligned} \tag{1}$$

where ${}^H D_{1,t}^\alpha, {}^H D_{1,t}^\beta$ denote Hadamard fractional derivatives of orders $\alpha, \beta (0 < \alpha, \beta \leq 1)$, respectively, ${}^H I_{1+}^{1-\alpha}$ denotes the left-sided Hadamard fractional integral of order $1 - \alpha$ and $f : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ is given continuous function.

2. Preliminaries

Let $C([1, e], \mathbb{R})$ be the Banach space of all continuous functions from $[1, e]$ to \mathbb{R} endowed with the norm $\|x\|_C = \sup_{t \in [1, e]} |x(t)|$.

For $0 \leq \gamma < 1$, we define the weighted space of functions g on $[1, e]$ by

$$C_{\gamma, \ln}([1, e], \mathbb{R}) = \{g : [1, e] \rightarrow \mathbb{R} \mid (\ln t)^\gamma g(t) \in C([1, e], \mathbb{R})\}.$$

Clearly, $C_{\gamma, \ln}([1, e], \mathbb{R})$ is the Banach space with the norm

$$\|g\|_{\gamma, \ln} = \|(\ln t)^\gamma g(t)\|_C = \sup_{t \in [1, e]} |(\ln t)^\gamma g(t)|.$$

The following definitions are devoted to the basic concepts of Hadamard fractional integrals and fractional derivatives.

Definition 2.1 (see [1, p.110]): The left-sided Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of function $g(t)$ is defined by

$${}^H I_{1+}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} g(s) \frac{ds}{s}, \quad 0 < 1 < t \leq b < \infty,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 (see [1, p.111]): The left-sided Hadamard fractional derivative of order $\alpha \in [n - 1, n), n \in \mathbb{Z}^+$ of function $g(t)$ is defined by

$${}^H D_{1+}^\alpha g(t) = \frac{1}{\Gamma(n - \alpha)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} g(s) \frac{ds}{s}, \quad 0 < 1 < t \leq b < \infty,$$

where $\Gamma(\cdot)$ is the Gamma function.

Proposition 2.3 (see [1, p.114,115]): Let $\alpha > 0, \beta > 0$ and $0 < 1 < b < \infty$. Then for $f \in C_{\gamma, \ln}([1, e], \mathbb{R})$, the following properties hold:

$${}^H D_{1,t}^\alpha {}^H I_{1+}^\beta f = {}^H I_{1+}^{\beta-\alpha} f \quad \text{and} \quad {}^H D_{1,t}^\alpha {}^H I_{1+}^\alpha f = f.$$

Lemma 2.4 (see [1, Theorem 2.3, p.116]): Let $\beta > 0, 0 < 1 < b < \infty$ and $f \in C_{\gamma, \ln}([1, e], \mathbb{R})$. Then

$${}^H I_{1+}^{\beta} {}^H D_{1,t}^{\beta} u(t) = u(t) - \sum_{j=1}^n c_j (\ln t)^{\beta-j},$$

where $c_j \in \mathbb{R}$ and $n - 1 < \beta < n$.

2.1. Auxiliary lemma

Lemma 2.5: Given $\sigma \in C_{\gamma, \ln}([1, e], \mathbb{R})$. Then the linear problem

$$\begin{aligned} {}^H D_{1,t}^{\beta} ({}^H D_{1,t}^{\alpha} - \lambda)x(t) &= \sigma(t), & 1 < t < e, \\ ({}^H D_{1,t}^{\alpha} - \lambda)x(e) &= 0, \quad {}^H I_{1+}^{1-\alpha} x(1) = c_0, & c_0 \in \mathbb{R}, \end{aligned} \tag{2}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) &= c_0 (\ln t)^{\alpha-1} \mathbb{E}_{\alpha, \alpha}(\lambda (\ln t)^{\alpha}) + \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t}{s}\right)^{\alpha} \right) \\ &\quad \times \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau}\right)^{\beta-1} \sigma(\tau) \frac{d\tau}{\tau} - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} \sigma(\tau) \frac{d\tau}{\tau} \right] \frac{ds}{s} \end{aligned} \tag{3}$$

Proof: Firstly, we apply the Hadamard fractional integral of order β to both sides of (2) and using the result of Lemma 2.4, we get

$$({}^H D_{1,t}^{\alpha} - \lambda)x(t) = {}^H I_{1+}^{\beta} \sigma(t) + c_1 (\ln t)^{\beta-1}, \tag{4}$$

where $c_1 \in \mathbb{R}$. Using the boundary condition $({}^H D_{1,t}^{\alpha} - \lambda)x(e) = 0$, we get

$$c_1 = -\frac{1}{\Gamma(\beta)} \int_1^e (1 - \ln s)^{\beta-1} \sigma(s) \frac{ds}{s}.$$

Then we can rewrite Equation (4) in the form

$$({}^H D_{1,t}^{\alpha} - \lambda)x(t) = \frac{1}{\Gamma(\beta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\beta-1} \sigma(s) \frac{ds}{s} - \frac{(\ln t)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln s)^{\beta-1} \sigma(s) \frac{ds}{s}. \tag{5}$$

Now, in view of [1, p. 234, (4.1.89)–(4.1.95)], we conclude the desired Equation (5) with the boundary condition ${}^H I_{1+}^{1-\alpha} x(1) = c_0$ has a solution $x \in C_{\gamma, \ln}([1, e], \mathbb{R})$ that satisfies the volterra integral Equation (3), which yields the required solution. By direct computation we can prove the converse. This ends the proof. ■

Lemma 2.6 ([24]): Let $\alpha, \gamma \in (0, 1), 1 < t_1 < t_2 \leq e$ and $\alpha - 1 + \frac{1}{q} > 0, \gamma < \frac{1}{p}$. Then

$$\begin{aligned} &\int_1^{t_2} (\ln s)^{-\gamma} \left(\left(\ln \frac{t_2}{s}\right)^{\alpha-1} - \left(\ln \frac{t_1}{s}\right)^{\alpha-1} \right) \frac{ds}{s} \\ &\leq \left(\frac{\ln(t_2)^{1-p\gamma}}{1-p\gamma} \right)^{\frac{1}{p}} \left[\frac{(\ln t_2)^{q(\alpha-1)+1} - (\ln t_1)^{q(\alpha-1)+1} + \left(\ln \frac{t_1}{t_2}\right)^{q(\alpha-1)+1}}{q(\alpha-1)+1} \right], \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1, p, q > 1$.

It is worth noting that the Mittag–Leffler function $\mathbb{E}_{\alpha,\beta}(\cdot)$ in two variables is defined as

$$\mathbb{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + k\alpha)}, \quad \alpha > 0, \beta > 0, z \in \mathbb{C}.$$

Lemma 2.7 (see [25, 4.2.3 and 4.2.7]): *The Mittag–Leffler function $\mathbb{E}_{\alpha,\beta}(\cdot)$ satisfies the following identities:*

- (i) $\mathbb{E}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z\mathbb{E}_{\alpha,\beta+\alpha}(z),$
- (ii) $\mathbb{E}_{\frac{1}{2},1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\frac{k}{2}+1)} = e^{z^2} \operatorname{erfc}(-z),$

where *erfc* is complementary to the error function *erf*:

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-u^2} du = 1 - \operatorname{erf}(z), \quad z \in \mathbb{C}.$$

Lemma 2.8 (The mean value theorem for integrals): *If f is a continuous function on the closed, bounded interval $[a, b]$, then there is at least one number ξ in (a, b) for which*

$$f(\xi) = \frac{1}{b-a} \int_a^b f(t) dt.$$

In the following Lemma, we present some useful integrals which are used further in this paper.

Lemma 2.9: *Using the famous beta function $\mathbb{B}(m, n) = \int_0^1 (1-z)^{m-1} z^{n-1} dz$, we get*

- (i) $\int_1^t \left(\ln \frac{t}{s}\right)^{\beta-1} (\ln s)^{-\gamma} \frac{ds}{s} = (\ln t)^{\beta-\gamma} \mathbb{B}(\beta, 1-\gamma).$
- (ii) $\int_1^t \left(\ln \frac{t}{s}\right)^{\beta-1} \frac{ds}{s} = \frac{1}{\beta} (\ln t)^\beta.$
- (iii) $\int_1^e (1-\ln s)^{\beta-1} (\ln s)^{-\gamma} \frac{ds}{s} = \mathbb{B}(\beta, 1-\gamma).$
- (iv) $\int_1^e (1-\ln s)^{\beta-1} \frac{ds}{s} = \frac{1}{\beta}.$

To end this section, we present the Schauder’s fixed point theorem which plays a main tool in the existence of the solution of (1).

Lemma 2.10 (Schauder’s fixed point theorem [26]): *Let E be a Banach space and Q be a nonempty bounded convex and closed subset of E , and $N : Q \rightarrow Q$ is a compact and continuous map. Then N has at least one fixed point in Q .*

3. Main results

Let $B_r = \{x \in C_{\gamma, \ln}([1, e], \mathbb{R}) : \|x\|_{\gamma, \ln} \leq r\}$ with $r \geq \frac{\omega_1}{1-L_1\omega_2}$, where

$$\omega_1 = \left(|c_0| + \frac{L_2 \mathbb{B}(\alpha, 1 + \beta)}{\Gamma(\beta + 1)} + \frac{L_2 \mathbb{B}(\alpha, \beta)}{\Gamma(\beta + 1)} \right) \mathbb{E}_{\alpha, \alpha}(\lambda),$$

and

$$\omega_2 = \frac{\mathbb{B}(\beta, 1 - \gamma)}{\Gamma(\beta)} (\mathbb{B}(\alpha, 1 + \beta - \gamma) + \mathbb{B}(\alpha, \beta)) \mathbb{E}_{\alpha, \alpha}(\lambda).$$

We define the operator $(\mathcal{H}x)(t) : B_r \rightarrow C_{\gamma, \ln}([1, e], \mathbb{R})$ by

$$\begin{aligned} (\mathcal{H}x)(t) &= c_0 (\ln t)^{\alpha-1} \mathbb{E}_{\alpha, \alpha}(\lambda (\ln t)^\alpha) \\ &+ \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t}{s} \right)^\alpha \right) \\ &\times \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right] \frac{ds}{s} \end{aligned} \quad (6)$$

Theorem 3.1: Assume that:

(H1) $f : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $f(\cdot, x(\cdot)) \in C_{\gamma, \ln}([1, e], \mathbb{R})$ and there exist constants $0 \leq L_1 < \omega_2^{-1}$, $L_2 > 0$ such that

$$|f(t, x)| \leq L_1 |x| + L_2, \quad \forall (t, x) \in [1, e] \times \mathbb{R}.$$

Then (1) has at least one solution on $[1, e]$.

Proof: The proof will be through several steps.

Step 1. We show that $\mathcal{H}(B_r) \subset B_r$.

For any $x \in C_{\gamma, \ln}([1, e], \mathbb{R})$ and Lemma 2.9, we have

$$\begin{aligned} &|(\ln t)^\gamma (\mathcal{H}x)(t)| \\ &\leq |c_0 (\ln t)^{\gamma+\alpha-1} \mathbb{E}_{\alpha, \alpha}(\lambda (\ln t)^\alpha)| \\ &+ \left| (\ln t)^\gamma \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right. \right. \\ &\quad \left. \left. - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right] \frac{ds}{s} \right| \\ &\leq |c_0| (\ln t)^{\gamma+\alpha-1} \mathbb{E}_{\alpha, \alpha}(\lambda) + (\ln t)^\gamma \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t}{s} \right)^\alpha \right) \\ &\quad \times \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} |f(\tau, x(\tau))| \frac{d\tau}{\tau} + \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} |f(\tau, x(\tau))| \frac{d\tau}{\tau} \right] \frac{ds}{s} \\ &\leq |c_0| \mathbb{E}_{\alpha, \alpha}(\lambda) + \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} (L_1 |x(\tau)| + L_2) \frac{d\tau}{\tau} \right. \\ &\quad \left. + \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} (L_1 |x(\tau)| + L_2) \frac{d\tau}{\tau} \right] \frac{ds}{s} \end{aligned}$$

$$\begin{aligned}
&\leq |c_0| \mathbb{E}_{\alpha,\alpha}(\lambda) + \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t}{s} \right)^\alpha \right) \\
&\quad \times \left[\frac{L_1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} (\ln \tau)^{-\gamma} \|x\|_{\gamma,\ln} \frac{d\tau}{\tau} + \frac{L_2}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} \frac{d\tau}{\tau} \right. \\
&\quad \left. + \frac{L_1 (\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} (\ln \tau)^{-\gamma} \|x\|_{\gamma,\ln} \frac{d\tau}{\tau} + \frac{L_2 (\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
&= |c_0| \mathbb{E}_{\alpha,\alpha}(\lambda) + \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \\
&\quad \times \left(\lambda \left(\ln \frac{t}{s} \right)^\alpha \right) \left[\frac{L_1 \|x\|_{\gamma,\ln}}{\Gamma(\beta)} (\ln s)^{\beta-\gamma} \mathbb{B}(\beta, 1 - \gamma) + \frac{L_2}{\Gamma(\beta + 1)} (\ln s)^\beta \right. \\
&\quad \left. + \left(\frac{L_1 \|x\|_{\gamma,\ln}}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) + \frac{L_2}{\Gamma(\beta + 1)} \right) (\ln s)^{\beta-1} \right] \frac{ds}{s} \\
&\leq |c_0| \mathbb{E}_{\alpha,\alpha}(\lambda) + \frac{L_1 \|x\|_{\gamma,\ln}}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) \mathbb{E}_{\alpha,\alpha}(\lambda) \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} (\ln s)^{\beta-\gamma} \frac{ds}{s} \\
&\quad + \frac{L_2}{\Gamma(\beta + 1)} \mathbb{E}_{\alpha,\alpha}(\lambda) \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} (\ln s)^\beta \frac{ds}{s} \\
&\quad + \left(\frac{L_1 \|x\|_{\gamma,\ln}}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) + \frac{L_2}{\Gamma(\beta + 1)} \right) \mathbb{E}_{\alpha,\alpha}(\lambda) \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} (\ln s)^{\beta-1} \frac{ds}{s} \\
&= |c_0| \mathbb{E}_{\alpha,\alpha}(\lambda) + \frac{L_1 \|x\|_{\gamma,\ln}}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) \mathbb{E}_{\alpha,\alpha}(\lambda) (\ln t)^{\alpha+\beta-\gamma} \mathbb{B}(\alpha, 1 + \beta - \gamma) \\
&\quad + \frac{L_2}{\Gamma(\beta + 1)} \mathbb{E}_{\alpha,\alpha}(\lambda) (\ln t)^{\alpha+\beta} \mathbb{B}(\alpha, 1 + \beta) \\
&\quad + \left(\frac{L_1 \|x\|_{\gamma,\ln}}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) + \frac{L_2}{\Gamma(\beta + 1)} \right) \mathbb{E}_{\alpha,\alpha}(\lambda) (\ln t)^{\alpha+\beta-1} \mathbb{B}(\alpha, \beta) \\
&\leq \omega_1 + L_1 \omega_2 \|x\|_{\gamma,\ln} \\
&\leq \omega_1 + L_1 \omega_2 r \leq r,
\end{aligned}$$

which implies that $\|\mathcal{H}x\|_{\gamma,\ln} \leq r$, therefore $\mathcal{H}(B_r) \subset B_r$.

Step 2. We show that \mathcal{H} is continuous.

$$\begin{aligned}
&|(\ln t)^\gamma ((\mathcal{H}x_n)(t) - (\mathcal{H}x)(t))| \\
&\leq \left| (\ln t)^\gamma \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} (f(\tau, x_n(\tau)) - f(\tau, x(\tau))) \frac{d\tau}{\tau} \right. \right. \\
&\quad \left. \left. - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} (f(\tau, x_n(\tau)) - f(\tau, x(\tau))) \frac{d\tau}{\tau} \right] \frac{ds}{s} \right| \\
&\leq \mathbb{E}_{\alpha,\alpha}(\lambda) \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} |f(\tau, x_n(\tau)) - f(\tau, x(\tau))| \frac{d\tau}{\tau} \right. \\
&\quad \left. + \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} |f(\tau, x_n(\tau)) - f(\tau, x(\tau))| \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
&\leq \mathbb{E}_{\alpha,\alpha}(\lambda) \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} (\ln \tau)^{-\gamma} \|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|_{\gamma,\ln} \frac{d\tau}{\tau} \right.
\end{aligned}$$

$$\begin{aligned}
 & + \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} (\ln \tau)^{-\gamma} \|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|_{\gamma, \ln} \frac{d\tau}{\tau} \Big] \frac{ds}{s} \\
 = & \frac{\mathbb{B}(\beta, 1 - \gamma)}{\Gamma(\beta)} \mathbb{E}_{\alpha, \alpha}(\lambda) \left[\int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} (\ln s)^{\beta-\gamma} \frac{ds}{s} \right. \\
 & \left. + \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} (\ln s)^{\beta-1} \frac{ds}{s} \right] \|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|_{\gamma, \ln} \\
 = & \frac{\mathbb{B}(\beta, 1 - \gamma)}{\Gamma(\beta)} \mathbb{E}_{\alpha, \alpha}(\lambda) [(\ln t)^{\alpha+\beta-\gamma} \mathbb{B}(\alpha, 1 + \beta - \gamma) \\
 & + (\ln t)^{\alpha+\beta-1} \mathbb{B}(\alpha, \beta)] \|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|_{\gamma, \ln} \\
 \leq & \frac{\mathbb{B}(\beta, 1 - \gamma)}{\Gamma(\beta)} (\mathbb{B}(\alpha, 1 + \beta - \gamma) + \mathbb{B}(\alpha, \beta)) \mathbb{E}_{\alpha, \alpha}(\lambda) \|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|_{\gamma, \ln}.
 \end{aligned}$$

Hence, we get

$$\|\mathcal{H}x_n - \mathcal{H}x\|_{\gamma, \ln} \leq \omega_2 \|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|_{\gamma, \ln},$$

and the continuity of f implies that \mathcal{H} is continuous.

Step 3. We show \mathcal{H} is relatively compact on B_r .

According to *Step 1*, we showed that $\mathcal{H}(B_r) \subset B_r$. Thus $\mathcal{H}(B_r)$ is uniformly bounded. It remains to show that \mathcal{H} is equicontinuous.

For $1 < t_1 < t_2 \leq e$ and $x \in B_r$, we have

$$\begin{aligned}
 & |(\mathcal{H}x)(t_2) - (\mathcal{H}x)(t_1)| \\
 \leq & \left| c_0 (\ln t_2)^{\alpha-1} \mathbb{E}_{\alpha, \alpha}(\lambda (\ln t_2)^\alpha) - c_0 (\ln t_1)^{\alpha-1} \mathbb{E}_{\alpha, \alpha}(\lambda (\ln t_2)^\alpha) \right| \\
 & + \left| c_0 (\ln t_1)^{\alpha-1} \mathbb{E}_{\alpha, \alpha}(\lambda (\ln t_2)^\alpha) - c_0 (\ln t_1)^{\alpha-1} \mathbb{E}_{\alpha, \alpha}(\lambda (\ln t_1)^\alpha) \right| \\
 & + \left| \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right. \right. \\
 & \left. \left. - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right] \frac{ds}{s} \right. \\
 & \left. - \int_1^{t_1} \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right. \right. \\
 & \left. \left. - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right] \frac{ds}{s} \right| \\
 \leq & \left| c_0 \mathbb{E}_{\alpha, \alpha}(\lambda (\ln t_2)^\alpha) ((\ln t_2)^{\alpha-1} - (\ln t_1)^{\alpha-1}) \right| \\
 & + \left| c_0 (\ln t_1)^{\alpha-1} (\mathbb{E}_{\alpha, \alpha}(\lambda (\ln t_2)^\alpha) - \mathbb{E}_{\alpha, \alpha}(\lambda (\ln t_1)^\alpha)) \right| \\
 & + \left| \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) \right. \\
 & \times \left. \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right] \frac{ds}{s} \right|
 \end{aligned}$$

$$\begin{aligned}
& - \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right. \\
& \left. - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right] \frac{ds}{s} \Big| \\
& + \left| \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right. \right. \\
& \left. \left. - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right] \frac{ds}{s} \right. \\
& - \int_1^{t_2} \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right. \\
& \left. - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right] \frac{ds}{s} \Big| \\
& + \left| \int_{t_1}^{t_2} \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right. \right. \\
& \left. \left. - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right] \frac{ds}{s} \right| \\
& \leq \mathbb{E}_{\alpha,\alpha}(\lambda) \left| c_0 \left((\ln t_2)^{\alpha-1} - (\ln t_1)^{\alpha-1} \right) \right| \\
& + \left| c_0 (\ln t_1)^{\alpha-1} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda (\ln t_2)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda (\ln t_1)^\alpha \right) \right) \right| \\
& + \left| \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right) \right. \\
& \times \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right. \\
& \left. - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right] \frac{ds}{s} \Big| \\
& + \left| \int_1^{t_2} \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \left(\left(\ln \frac{t_2}{s} \right)^{\alpha-1} - \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right. \right. \\
& \left. \left. - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right] \frac{ds}{s} \right| \\
& + \left| \int_{t_1}^{t_2} \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right. \right. \\
& \left. \left. - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} f(\tau, x(\tau)) \frac{d\tau}{\tau} \right] \frac{ds}{s} \right| \\
& \leq |c_0| \mathbb{E}_{\alpha,\alpha}(\lambda) \left((\alpha-1) \frac{(\ln \zeta)^{\alpha-2}}{\zeta} |t_2 - t_1| \right) + |c_0| (\mathcal{O}) |t_2 - t_1| \\
& + \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right)
\end{aligned}$$

$$\begin{aligned}
 & \times \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} |f(\tau, x(\tau))| \frac{d\tau}{\tau} \right. \\
 & + \left. \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} |f(\tau, x(\tau))| \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
 & + \int_1^{t_2} \mathbb{E}_{\alpha, \alpha}(\lambda) \left(\ln \frac{t_1}{s} \right)^\alpha \left(\left(\ln \frac{t_2}{s} \right)^{\alpha-1} - \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} |f(\tau, x(\tau))| \frac{d\tau}{\tau} \right. \\
 & + \left. \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} |f(\tau, x(\tau))| \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
 & + \int_{t_1}^{t_2} \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} |f(\tau, x(\tau))| \frac{d\tau}{\tau} \right. \\
 & + \left. \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} |f(\tau, x(\tau))| \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
 & \leq |c_0| \mathbb{E}_{\alpha, \alpha}(\lambda) \left((\alpha - 1) \frac{(\ln \zeta)^{\alpha-2}}{\zeta} |t_2 - t_1| \right) + |c_0| (\mathcal{O}|t_2 - t_1|) \\
 & + \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} \left(\mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right) \\
 & \times \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} (L_1 |x(\tau)| + L_2) \frac{d\tau}{\tau} \right. \\
 & + \left. \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} (L_1 |x(\tau)| + L_2) \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
 & + \int_1^{t_2} \mathbb{E}_{\alpha, \alpha}(\lambda) \left(\ln \frac{t_1}{s} \right)^\alpha \left(\left(\ln \frac{t_2}{s} \right)^{\alpha-1} - \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \right) \\
 & \times \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} (L_1 |x(\tau)| + L_2) \frac{d\tau}{\tau} \right. \\
 & + \left. \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} (L_1 |x(\tau)| + L_2) \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
 & + \int_{t_1}^{t_2} \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} (L_1 |x(\tau)| + L_2) \frac{d\tau}{\tau} \right. \\
 & + \left. \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} (L_1 |x(\tau)| + L_2) \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
 & \leq |c_0| \mathbb{E}_{\alpha, \alpha}(\lambda) \left((\alpha - 1) \frac{(\ln \zeta)^{\alpha-2}}{\zeta} |t_2 - t_1| \right) + |c_0| (\mathcal{O}|t_2 - t_1|) \\
 & + \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} \left(\mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right) \\
 & \times \left[\frac{L_1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} (\ln \tau)^{-\gamma} \|x\|_{\gamma, \ln} \frac{d\tau}{\tau} + \frac{L_2}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} \frac{d\tau}{\tau} \right. \\
 & + \left. \frac{L_1 (\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} (\ln \tau)^{-\gamma} \|x\|_{\gamma, \ln} \frac{d\tau}{\tau} + \frac{L_2 (\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} \frac{d\tau}{\tau} \right] \frac{ds}{s}
 \end{aligned}$$

$$\begin{aligned}
& + \int_1^{t_2} \mathbb{E}_{\alpha,\alpha}(\lambda) \left(\ln \frac{t_1}{s} \right)^\alpha \left(\left(\ln \frac{t_2}{s} \right)^{\alpha-1} - \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \right) \left[\frac{L_1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} (\ln \tau)^{-\gamma} \|x\|_{\gamma, \ln} \frac{d\tau}{\tau} \right. \\
& + \frac{L_2}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} \frac{d\tau}{\tau} \\
& + \left. \frac{L_1(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} (\ln \tau)^{-\gamma} \|x\|_{\gamma, \ln} \frac{d\tau}{\tau} + \frac{L_2(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
& + \int_{t_1}^{t_2} \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \\
& \times \left[\frac{L_1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} (\ln \tau)^{-\gamma} \|x\|_{\gamma, \ln} \frac{d\tau}{\tau} + \frac{L_2}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} \frac{d\tau}{\tau} \right. \\
& + \left. \frac{L_1(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} (\ln \tau)^{-\gamma} \|x\|_{\gamma, \ln} \frac{d\tau}{\tau} + \frac{L_2(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
& = |c_0| \mathbb{E}_{\alpha,\alpha}(\lambda) \left((\alpha - 1) \frac{(\ln \xi)^{\alpha-2}}{\xi} |t_2 - t_1| \right) + |c_0| (\mathcal{O}|t_2 - t_1|) \\
& + \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right) \\
& \times \left[\frac{L_1 r}{\Gamma(\beta)} (\ln s)^{\beta-\gamma} \mathbb{B}(\beta, 1 - \gamma) + \frac{L_2}{\Gamma(\beta + 1)} (\ln s)^\beta \right. \\
& + \left. \left(\frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) + \frac{L_2}{\Gamma(\beta + 1)} \right) (\ln s)^{\beta-1} \right] \frac{ds}{s} \\
& + \int_1^{t_2} \mathbb{E}_{\alpha,\alpha}(\lambda) \left(\ln \frac{t_1}{s} \right)^\alpha \left(\left(\ln \frac{t_2}{s} \right)^{\alpha-1} - \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \right) \\
& \times \left[\frac{L_1 r}{\Gamma(\beta)} (\ln s)^{\beta-\gamma} \mathbb{B}(\beta, 1 - \gamma) + \frac{L_2}{\Gamma(\beta + 1)} (\ln s)^\beta \right. \\
& + \left. \left(\frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) + \frac{L_2}{\Gamma(\beta + 1)} \right) (\ln s)^{\beta-1} \right] \frac{ds}{s} \\
& + \int_{t_1}^{t_2} \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \left[\frac{L_1 r}{\Gamma(\beta)} (\ln s)^{\beta-\gamma} \mathbb{B}(\beta, 1 - \gamma) + \frac{L_2}{\Gamma(\beta + 1)} (\ln s)^\beta \right. \\
& + \left. \left(\frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) + \frac{L_2}{\Gamma(\beta + 1)} \right) (\ln s)^{\beta-1} \right] \frac{ds}{s} \\
& = |c_0| \mathbb{E}_{\alpha,\alpha}(\lambda) \left((\alpha - 1) \frac{(\ln \xi)^{\alpha-2}}{\xi} |t_2 - t_1| \right) + |c_0| (\mathcal{O}|t_2 - t_1|) + I_1 + I_2 + I_3,
\end{aligned}$$

where $\xi \in (t_1, t_2)$ and

$$\begin{aligned}
I_1 & = \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right) \left[\frac{L_1 r}{\Gamma(\beta)} (\ln s)^{\beta-\gamma} \mathbb{B}(\beta, 1 - \gamma) \right. \\
& + \left. \frac{L_2}{\Gamma(\beta + 1)} (\ln s)^\beta + \left(\frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) + \frac{L_2}{\Gamma(\beta + 1)} \right) (\ln s)^{\beta-1} \right] \frac{ds}{s},
\end{aligned}$$

$$\begin{aligned}
 I_2 &= \int_1^{t_2} \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \left(\left(\ln \frac{t_2}{s} \right)^{\alpha-1} - \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \right) \right. \\
 &\quad \times \left[\frac{L_1 r}{\Gamma(\beta)} (\ln s)^{\beta-\gamma} \mathbb{B}(\beta, 1-\gamma) + \frac{L_2}{\Gamma(\beta+1)} (\ln s)^\beta \right. \\
 &\quad \left. \left. + \left(\frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1-\gamma) + \frac{L_2}{\Gamma(\beta+1)} \right) (\ln s)^{\beta-1} \right] \frac{ds}{s}, \\
 I_3 &= \int_{t_1}^{t_2} \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \left[\frac{L_1 r}{\Gamma(\beta)} (\ln s)^{\beta-\gamma} \mathbb{B}(\beta, 1-\gamma) + \frac{L_2}{\Gamma(\beta+1)} (\ln s)^\beta \right. \\
 &\quad \left. + \left(\frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1-\gamma) + \frac{L_2}{\Gamma(\beta+1)} \right) (\ln s)^{\beta-1} \right] \frac{ds}{s}.
 \end{aligned}$$

For $\frac{1}{p} + \frac{1}{q} = 1, p, q > 1$, we have the following estimations.

$$\begin{aligned}
 I_1 &= \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right) \left[\frac{L_1 r}{\Gamma(\beta)} (\ln s)^{\beta-\gamma} \mathbb{B}(\beta, 1-\gamma) \right. \\
 &\quad \left. + \frac{L_2}{\Gamma(\beta+1)} (\ln s)^\beta + \left(\frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1-\gamma) + \frac{L_2}{\Gamma(\beta+1)} \right) (\ln s)^{\beta-1} \right] \frac{ds}{s} \\
 &= \frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1-\gamma) \left[\int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} (\ln s)^{\beta-\gamma} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right) \frac{ds}{s} \right. \\
 &\quad \left. + \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} (\ln s)^{\beta-1} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right) \frac{ds}{s} \right] \\
 &\quad + \frac{L_2}{\Gamma(\beta+1)} \left[\int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} (\ln s)^\beta \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right) \frac{ds}{s} \right. \\
 &\quad \left. + \int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{\alpha-1} (\ln s)^{\beta-1} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right) \frac{ds}{s} \right] \\
 &\leq \frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1-\gamma) \left[\left(\int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{p(\alpha-1)} (\ln s)^{p(\beta-\gamma)} \frac{ds}{s} \right)^{\frac{1}{p}} \right. \\
 &\quad \times \left(\int_1^{t_2} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right)^q \frac{ds}{s} \right)^{\frac{1}{q}} \\
 &\quad \left. + \left(\int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{p(\alpha-1)} (\ln s)^{p(\beta-1)} \frac{ds}{s} \right)^{\frac{1}{p}} \right. \\
 &\quad \left. \times \left(\int_1^{t_2} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right)^q \frac{ds}{s} \right)^{\frac{1}{q}} \right] \\
 &\quad + \frac{L_2}{\Gamma(\beta+1)} \left[\left(\int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{p(\alpha-1)} (\ln s)^{p\beta} \frac{ds}{s} \right)^{\frac{1}{p}} \right.
 \end{aligned}$$

$$\begin{aligned}
 & \times \left(\int_1^{t_2} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right)^q \frac{ds}{s} \right)^{\frac{1}{q}} \\
 & + \left(\int_1^{t_2} \left(\ln \frac{t_2}{s} \right)^{p(\alpha-1)} (\ln s)^{p(\beta-1)} \frac{ds}{s} \right)^{\frac{1}{p}} \\
 & \times \left[\int_1^{t_2} \left(\mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_2}{s} \right)^\alpha \right) - \mathbb{E}_{\alpha,\alpha} \left(\lambda \left(\ln \frac{t_1}{s} \right)^\alpha \right) \right)^q \frac{ds}{s} \right]^{\frac{1}{q}} \\
 & \leq \frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) \left[(\mathbb{B}(p(\alpha - 1) + 1, p(\beta - \gamma) + 1))^{\frac{1}{p}} \left(\int_1^{t_2} \mathcal{O}(|t_2 - t_1|) \frac{ds}{s} \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + (\mathbb{B}(p(\alpha - 1) + 1, p(\beta - 1) + 1))^{\frac{1}{p}} \left(\int_1^{t_2} \mathcal{O}(|t_2 - t_1|) \frac{ds}{s} \right)^{\frac{1}{q}} \right] \\
 & + \frac{L_2}{\Gamma(\beta + 1)} \left[(\mathbb{B}(p(\alpha - 1) + 1, p\beta + 1))^{\frac{1}{p}} \left(\int_1^{t_2} \mathcal{O}(|t_2 - t_1|) \frac{ds}{s} \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + (\mathbb{B}(p(\alpha - 1) + 1, p(\beta - 1) + 1))^{\frac{1}{p}} \left(\int_1^{t_2} \mathcal{O}(|t_2 - t_1|) \frac{ds}{s} \right)^{\frac{1}{q}} \right],
 \end{aligned}$$

which implies that $I_1 \rightarrow 0$ as $t_1 \rightarrow t_2$.

For I_2 , using Lemma 2.6, we have

$$\begin{aligned}
 I_2 & \leq \frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) \mathbb{E}_{\alpha,\alpha}(\lambda) \left[\left(\int_1^{t_2} (\ln s)^{p(\beta-\gamma)} \frac{ds}{s} \right)^{\frac{1}{p}} \left(\int_1^{t_2} \left(\left(\ln \frac{t_2}{s} \right)^{\alpha-1} - \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \right)^q \frac{ds}{s} \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left(\int_1^{t_2} (\ln s)^{p(\beta-1)} \frac{ds}{s} \right)^{\frac{1}{p}} \left(\int_1^{t_2} \left(\left(\ln \frac{t_2}{s} \right)^{\alpha-1} - \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \right)^q \frac{ds}{s} \right)^{\frac{1}{q}} \right] \\
 & + \frac{L_2}{\Gamma(\beta + 1)} \mathbb{E}_{\alpha,\alpha}(\lambda) \left[\left(\int_1^{t_2} (\ln s)^{p\beta} \frac{ds}{s} \right)^{\frac{1}{p}} \left(\int_1^{t_2} \left(\left(\ln \frac{t_2}{s} \right)^{\alpha-1} - \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \right)^q \frac{ds}{s} \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left(\int_1^{t_2} (\ln s)^{p(\beta-1)} \frac{ds}{s} \right)^{\frac{1}{p}} \left(\int_1^{t_2} \left(\left(\ln \frac{t_2}{s} \right)^{\alpha-1} - \left(\ln \frac{t_1}{s} \right)^{\alpha-1} \right)^q \frac{ds}{s} \right)^{\frac{1}{q}} \right] \\
 & \leq \left\{ \frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) \mathbb{E}_{\alpha,\alpha}(\lambda) \left[\left(\frac{(\ln t_2)^{p(\beta-\gamma)+1}}{p(\beta - \gamma) + 1} \right)^{\frac{1}{p}} + \left(\frac{(\ln t_2)^{p(\beta-1)+1}}{p(\beta - 1) + 1} \right)^{\frac{1}{p}} \right] \right. \\
 & \quad \left. + \frac{L_2}{\Gamma(\beta + 1)} \mathbb{E}_{\alpha,\alpha}(\lambda) \left[\left(\frac{(\ln t_2)^{p\beta+1}}{p\beta + 1} \right)^{\frac{1}{p}} + \left(\frac{(\ln t_2)^{p(\beta-1)+1}}{p(\beta - 1) + 1} \right)^{\frac{1}{p}} \right] \right\} \\
 & \times \left(\frac{1}{q(\alpha - 1) + 1} \left((\ln t_2)^{q(\alpha-1)+1} - (\ln t_1)^{q(\alpha-1)+1} + \left(\ln \frac{t_1}{t_2} \right)^{q(\alpha-1)+1} \right) \right),
 \end{aligned}$$

which implies that $I_2 \rightarrow 0$ as $t_1 \rightarrow t_2$.

For I_3 , using Lemma 2.8, we have

$$I_3 = \frac{L_1 r}{\Gamma(\beta)} \mathbb{B}(\beta, 1 - \gamma) \mathbb{E}_{\alpha, \alpha}(\lambda) \left[\left(\ln \frac{t_1}{\xi} \right)^{\alpha-1} \frac{(\ln \xi)^{\beta-\gamma}}{\xi} |t_2 - t_1| + \left(\ln \frac{t_1}{\xi} \right)^{\alpha-1} \frac{(\ln \xi)^{\beta-1}}{\xi} |t_2 - t_1| \right] \\ + \frac{L_2}{\Gamma(\beta + 1)} \mathbb{E}_{\alpha, \alpha}(\lambda) \left[\left(\ln \frac{t_1}{\xi} \right)^{\alpha-1} \frac{(\ln \xi)^\beta}{\xi} |t_2 - t_1| + \left(\ln \frac{t_1}{\xi} \right)^{\alpha-1} \frac{(\ln \xi)^{\beta-1}}{\xi} |t_2 - t_1| \right],$$

where $\xi \in (t_1, t_2)$. Then $I_3 \rightarrow 0$ as $t_1 \rightarrow t_2$.

Thus we get $|(\mathcal{H}x)(t_2) - (\mathcal{H}x)(t_1)| \rightarrow 0$ as $t_1 \rightarrow t_2$ which proves that the operator \mathcal{H} is equicontinuous operator. Hence, by Arzelà-Ascoli theorem, we conclude that \mathcal{H} is relatively compact on B_r . Therefore, the Schauder's fixed point theorem shows that the operator \mathcal{H} has a fixed point which corresponds to the solution of (1). This completes the proof. ■

Next, we shall show that the following existence and uniqueness result via Banach's fixed point theorem.

Theorem 3.2: *Assume that:*

(H2) *Let $f : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. There exists a constant $L > 0$ such that $|f(t, x) - f(t, y)| \leq L|x - y|$, for each $t \in [1, e]$ and $x, y \in \mathbb{R}$.*

Then (1) has a unique solution on $[1, e]$, provided that $L\omega_2 < 1$.

Proof: As previously proven in Step 1 of Theorem 3.1, the operator $\mathcal{H} : B_k \rightarrow B_k$ defined in (6) is uniformly bounded. Then it remains to show that \mathcal{H} is a contraction mapping.

For $x, y \in B_k$, where $B_k = \{x \in C_{\gamma, \ln}([1, e], \mathbb{R}) : \|x\|_{\gamma, \ln} \leq k\}$, we have

$$|(\ln t)^\gamma ((\mathcal{H}x)(t) - (\mathcal{H}y)(t))| \\ = \left| (\ln t)^\gamma \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} (f(\tau, x(\tau)) - f(\tau, y(\tau))) \frac{d\tau}{\tau} \right. \right. \\ \left. \left. - \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} (f(\tau, x(\tau)) - f(\tau, y(\tau))) \frac{d\tau}{\tau} \right] \frac{ds}{s} \right| \\ \leq \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t}{s} \right)^\alpha \right) \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} |f(\tau, x(\tau)) - f(\tau, y(\tau))| \frac{d\tau}{\tau} \right. \\ \left. + \frac{(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} |f(\tau, x(\tau)) - f(\tau, y(\tau))| \frac{d\tau}{\tau} \right] \frac{ds}{s} \\ \leq \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t}{s} \right)^\alpha \right) \left[\frac{L}{\Gamma(\beta)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} |x(\tau) - y(\tau)| \frac{d\tau}{\tau} \right. \\ \left. + \frac{L(\ln s)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \ln \tau)^{\beta-1} |x(\tau) - y(\tau)| \frac{d\tau}{\tau} \right] \frac{ds}{s} \\ \leq \frac{L}{\Gamma(\beta)} \left\{ \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \mathbb{E}_{\alpha, \alpha} \left(\lambda \left(\ln \frac{t}{s} \right)^\alpha \right) \left[\int_1^s \left(\ln \frac{s}{\tau} \right)^{\beta-1} (\ln \tau)^{-\gamma} \frac{d\tau}{\tau} \right. \right. \\ \left. \left. + (\ln s)^{\beta-1} \int_1^e (1 - \ln \tau)^{\beta-1} (\ln \tau)^{-\gamma} \frac{d\tau}{\tau} \right] \frac{ds}{s} \right\} \|x - y\|_{\gamma, \ln}$$

$$\begin{aligned} &\leq \frac{L\mathbb{B}(\beta, 1 - \gamma)\mathbb{E}_{\alpha,\alpha}(\lambda)}{\Gamma(\beta)} \left\{ \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} (\ln s)^{\beta-\gamma} \frac{ds}{s} + \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} (\ln s)^{\beta-1} \frac{ds}{s} \right\} \|x-y\|_{\gamma,\ln} \\ &\leq \frac{L\mathbb{B}(\beta, 1 - \gamma)\mathbb{E}_{\alpha,\alpha}(\lambda)}{\Gamma(\beta)} (\mathbb{B}(\alpha, 1 + \beta - \gamma) + \mathbb{B}(\alpha, \beta)) \|x - y\|_{\gamma,\ln}, \end{aligned}$$

which implies that $\|\mathcal{H}x - \mathcal{H}y\|_{\gamma,\ln} \leq L\omega_2 \|x - y\|_{\gamma,\ln}$. It follows that \mathcal{H} is a contraction. As a consequence of Banach’s fixed point theorem, the operator \mathcal{H} has a fixed point which corresponds to the unique solution of (1). This completes the proof. ■

4. An example

Consider the following Langevin equation with two Hadamard fractional derivatives:

Example 4.1:

$$\begin{aligned} {}^H D_{1,t}^{\frac{3}{4}} ({}^H D_{1,t}^{\frac{1}{2}} - 1)x(t) &= f(t, x(t)), \quad t \in [1, e], \\ ({}^H D_{1,t}^{\frac{1}{2}} - 1)x(e) &= 0, \quad {}^H I_{1+}^{\frac{1}{2}} x(1) = 1. \end{aligned} \tag{7}$$

Here, $\alpha = \frac{1}{2}, \beta = \frac{3}{4}, \lambda = 1, \gamma = \frac{1}{4}$ and $c_0 = 1$.

In order to illustrate Theorem 3.1, we take $f(t, x) = \sin \frac{1}{81}|x| + \frac{1}{(1+t)^2}$ for all $t \in [1, e]$.

Clearly, $|f(t, x)| \leq \frac{1}{81}|x| + \frac{1}{4}$. According to the assumption (H1), we get $L_1 = \frac{1}{81}$ and $L_2 = \frac{1}{4}$.

Thus,

$$L_1\omega_2 = \frac{\mathbb{B}\left(\frac{3}{4}, \frac{3}{4}\right)}{81\Gamma\left(\frac{3}{4}\right)} \left[\mathbb{B}\left(\frac{1}{2}, \frac{3}{2}\right) + \mathbb{B}\left(\frac{1}{2}, \frac{3}{4}\right) \right] \mathbb{E}_{\frac{1}{2}, \frac{1}{2}}(1) = 0.06772116862\mathbb{E}_{\frac{1}{2}, \frac{1}{2}}(1).$$

Using Lemma 2.7, we get

$$\mathbb{E}_{\frac{1}{2}, \frac{1}{2}}(1) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} + \mathbb{E}_{\frac{1}{2}, 1}(1) = \frac{1}{\sqrt{\pi}} + \operatorname{erfc}(-1) \times e = 5.573170227.$$

Therefore, $L_1\omega_2 = 0.3774216007 < 1$, and according to Theorem 3.1, we conclude that the Langevin equation (7) with $f(t, x) = \sin \frac{1}{81}|x| + \frac{1}{(1+t)^2}$ has at least one solution on $[1, e]$.

For the illustration of Theorem 3.2, let us take $f(t, x) = \frac{|x|}{(99+t^2)(1+|x|)}$ for all $t \in [1, e]$. Obviously, $|f(t, x) - f(t, y)| \leq \frac{1}{100}|x - y|$. Thus, the assumption (H2) implies that $L = \frac{1}{100}$.

The direct computations give

$$L\omega_2 = \frac{\mathbb{B}\left(\frac{3}{4}, \frac{3}{4}\right)}{100\Gamma\left(\frac{3}{4}\right)} \left[\mathbb{B}\left(\frac{1}{2}, \frac{3}{2}\right) + \mathbb{B}\left(\frac{1}{2}, \frac{3}{4}\right) \right] \mathbb{E}_{\frac{1}{2}, \frac{1}{2}}(1) = 0.05485414658\mathbb{E}_{\frac{1}{2}, \frac{1}{2}}(1).$$

Hence $L\omega_2 = 0.3057114966 < 1$. According to Theorem 3.2, the Langevin equation (7) with $f(t, x) = \frac{|x|}{(99+t^2)(1+|x|)}$ has a unique solution on $[1, e]$.

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