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# A Navier–Stokes-Type Problem with High-Order Elliptic Operator and Applications

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**Abstract:** The existence, uniqueness and uniformly $L^p$ estimates for solutions of a high-order abstract Navier–Stokes problem on half space are derived. The equation involves an abstract operator in a Banach space $E$ and small parameters. Since the Banach space $E$ is arbitrary and $A$ is a possible linear operator, by choosing spaces $E$ and operators $A$, the existence, uniqueness and $L^p$ estimates of solutions for numerous classes of Navier–Stokes type problems are obtained. In application, the existence, uniqueness and uniformly $L^{\mathbf{p}}$ estimates for the solution of the Wentzell–Robin-type mixed problem for the Navier–Stokes equation and mixed problem for degenerate Navier–Stokes equations are established.

**Keywords:** stokes systems; Navier–Stokes equations; differential equations with small parameters; semigroups of operators; differential-operator equations; maximal $L^p$ regularity

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## 1. Introduction

We consider the mixed problem for Navier–Stokes (N–S)-type equation with small parameter

$$\frac{\partial u}{\partial t}+B_{\varepsilon}u+\left(u.B_{1\varepsilon}\right)u+B_{1\varepsilon}\varphi+Au=f\left(x,t\right),B_{2\varepsilon}u=0, \tag{1a}$$

$$L_{k}u=\sum_{i=0}^{\nu_{k}}\varepsilon_{n}^{\sigma_{i}}\alpha_{ki}\frac{\partial^{i}u}{\partial x_{n}^{i}}\left(x',0,t\right)=0,\nu_{k}\in\left\{0,1,\ldots 2m-1\right\},k=1,2,\ldots m, \tag{1b}$$

$$u\left(x,0\right)=a\left(x\right),x\in\mathbb{R}_{+}^{n},\ t\in\left(0,T\right), \tag{1c}$$

where

$$\mathbb{R}_{+}^{n}=\left\{x\in\mathbb{R}^{n},x_{n}>0,\ x=\left(x',x_{n}\right),\ x'=\left(x_{1},x_{2},\ldots,x_{n-1}\right)\right\},$$

$$\sigma_{i}=\frac{1}{2m}\left(i+\frac{1}{q}\right),q\in\left(1,\infty\right),B_{\varepsilon}u=\sum_{k=1}^{n}\left(-1\right)^{m}\varepsilon_{k}\frac{\partial^{2m}u}{\partial x_{k}^{2m}},$$

$$B_{1\varepsilon}\varphi=-\left(\varepsilon_{1}^{\frac{1}{2}}\frac{\partial^{m}\varphi}{\partial x_{1}^{m}},\varepsilon_{2}^{\frac{1}{2}}\frac{\partial^{m}\varphi}{\partial x_{2}^{m}},\ldots,\varepsilon_{n}^{\frac{1}{2}}\frac{\partial^{m}\varphi}{\partial x_{n}^{m}}\right),B_{2\varepsilon}u=-\sum_{k=1}^{n}\varepsilon_{k}^{\frac{1}{2}}\frac{\partial^{m}u_{k}}{\partial x_{k}^{m}}$$

$\alpha_{ki}$ are complex numbers, $\varepsilon=\left(\varepsilon_{1},\varepsilon_{2},\ldots,\varepsilon_{n}\right)$, $\varepsilon_{k}$ are small positive parameters, $m$ is a positive integer with $m\geq 1$ and $A$ is a linear operator in a Banach space $E$. Here, $u=u_{\varepsilon}\left(x,t\right)=$

$(u_1(x,t), u_2(x,t), \ldots, u_n(x,t))$, $u_k(x,t) = u_{k\varepsilon}(x,t)$ and $\varphi = \varphi(x,t)$ are represent the $E$-valued unknown velocity and pressure like functions, respectively, $f = (f_1(x,t), f_2(x,t), \ldots, f_n(x,t))$ and $a$ represent a given $E$-valued external force and the initial velocity.

**Remark 1.** *When we consider the N–S problem 1a–1c, it means that the solution u belongs to space* $X_q^{1,2m}(A)$. *Using Lions and Peetre result (see foe example [1], § 1.8.2) then the trace operator* $u \to \partial_t u(x,.)$ *is bounded from* $X_q^{1,2m}(A)$ *to* $\mathbb{Z}(\mathbb{E}_0, \mathbb{E})$. *We assume A is to be such that* $(u.\nabla)u \in E$ *for* $u \in X_q^{1,2m}(A)$, *where* $(\mathbb{E}_0, \mathbb{E})_{\theta,p}$ *denotes real interpolation spaces between* $\mathbb{E}_0$, $\mathbb{E}$ *(see e.g., [1], § 1.3.2),*

$$X_q^{1,2m}(A) = \left(W^{1,2m,p}(\not<_+; E(A), E)\right)^n, \mathbb{E} = (L^p(\mathbb{R}^n; E))^n, \theta = \frac{1}{2p},$$

$$\mathbb{E}_0 = \left(W^{2m,p}(\mathbb{R}^n; E(A), E)\right)^n, \mathbb{Z}(\mathbb{E}_0, \mathbb{E}) = \left(C\left(\mathbb{R}^n; (\mathbb{E}_0, \mathbb{E})_{\theta,p}\right)\right),$$

*here,* $L^p(\mathbb{R}^n_+; E)$, $W^{2m,p}(\mathbb{R}^n; E(A), E)$, $W^{1,2m,p}(\not<_+; E(A), E)$ *and* $(\mathbb{E}_0, \mathbb{E})_{\theta,p}$ *will be defined in the sequel.*

Boundary value problems (BVPs) for differential-operator equations (DOEs) in classes of functions such as Lebesgue ones have been object of interest of a lots of scientists (see, for example [1–3]). Presentations to differential-operator equations have been done by several authors [4–6]. Regularity results for differential-operator equations are contained in [7–9]. In the present note, authors study degenerate parameter-dependent Boundary Value Problems for arbitrary order differential-operator equations. These kinds of problems have been applied in several fields which are useful in lots of fluid mechanics models.

The focus of our work was to prove uniform existence and uniqueness of the stronger local and global solution of the Navier–Stokes problem with a small parameter (1a)–(1c). This problem is characterized by the presence of an abstract operator $A$ and a small parameter $\varepsilon_k$ that, respectively, corresponds to the inverse of a Reynolds number $Re$ that is very large for the N–S equations. Regularity results of N–S equations were obtained, for example, by the authors in [4–6,10–17]. The N–S equations with small viscosity when the boundary is either characteristic or non-characteristic have been well-studied; see for example in the papers [3,14,16]. In addition, regularity properties of abstract differential equation (ADE) were deeply studied in [2,7–9,18–22]. Here, the authors study abstract N–S equations with a high elliptic part in a Banach space $E$ with operator coefficient $A$. In [22], we derived the $L^p-$regularity properties of the abstract Stokes problem. For $E = \mathbb{C}$, $A = a > 0$, $m = 1$, $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = 1$ the problem (1a)–(1c) state to be usual N–S problem. In this paper, the authors prove that the corresponding Stokes type problem

$$\frac{\partial u}{\partial t} + B_\varepsilon u + Au + B_{1\varepsilon}\varphi = f(x,t), \; B_{2\varepsilon}u = 0, \; x \in \mathbb{R}^n_+, \; t \in (0,T),$$

$$\sum_{i=0}^{\nu_k} \varepsilon_n^{\sigma_i} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}(x',0,t) = 0, \nu_k \in \{0,1,\ldots m-1\}, k = 1,2,\ldots m, \tag{1d}$$

$$u(x,0) = a(x)$$

has a unique solution $(u, B_{1\varepsilon}\varphi)$ for $f \in L^p(0,T; X_q) = B(p,q)$, $p,q \in (1,\infty)$ and the following uniform estimate holds

$$\left\|\frac{\partial u}{\partial t}\right\|_{B(p,q)} + \sum_{k=1}^{n} \left\|\varepsilon_k \frac{\partial^{2m} u}{\partial x_k^{2m}}\right\|_{B(p,q)} + \|Au\|_{B(p,q)} + \|B_{1\varepsilon}\varphi\|_{B(p,q)} \leq$$

$$C\left(\|f\|_{B(p,q)} + \|a\|_{\left(X_q^{2m}(A), X_q\right)_{\frac{1}{p},p}}\right) \tag{1e}$$

with $C = C(T,p,q)$ independent of $f$ and $\varepsilon$, where $\left(X_q^{2m}(A), X_q\right)_{\frac{1}{p},p}$ denote the real interpolation space between $X_q^{2m}(A) = W^{2m,q}\left(\mathbb{R}^n_+; E(A), E\right)^n$ and $X_q = \left(L^q\left(\mathbb{R}^n_+; E\right)\right)^n$ defined by the $K$-method (see e.g., [1], §1.3.2). Then, by following Kato-Fujita [13] method, by using (1e) we derive a local a priori estimates for solutions of (1a)–(1c), i.e., we prove that for $\gamma < 1$ and $\delta \geq 0$ such that $\frac{n}{2q} - \frac{1}{2} \leq \gamma$, $-\gamma < \delta < 1 - |\gamma|$, $a \in D\left(Q_{\varepsilon q}^{\gamma}\right)$ there exists $T_* \in (0,T)$ independent of $\varepsilon_k \in (0\,,1]$ such that $\left\|Q_{\varepsilon q}^{-\delta} P_q f(t)\right\|$ is continuous on $(0,T)$ and satisfies $\left\|Q_{\varepsilon q}^{-\delta} P f(t)\right\| = o\left(t^{\gamma+\delta-1}\right)$ as $t \to 0$; there exists a local solution of (1a)–(1c) such that $u \in C\left([0,T_*]; D\left(Q_{\varepsilon q}^{\gamma}\right)\right)$, $u(0) = a$, $u \in C\left((0\ T_*]; D\left(Q_{\varepsilon q}^{\alpha}\right)\right)$ for some $T_* > 0$ and $\left\|Q_{\varepsilon q}^{\alpha} u(t)\right\| = o\left(t^{\gamma-\alpha}\right)$ as $t \to 0$ uniformly in $\varepsilon$ for $\alpha$ with $\gamma < \alpha < 1 - \delta$. Moreover, the solution of (1a)–(1c) is unique for some $\beta$ with $\beta > |\gamma|$. For sufficiently small date we show that there exists a global solution of (1a)–(1c). Particularly, we prove that there is a $\delta > 0$ such that if $\|a\|_{X_q} < \delta$, then there exists a global solution $u_\varepsilon$ of (1a)–(1c) so that

$$t^{\left(1-\frac{n}{q}\right)/2} u_\varepsilon,\ t^{\left(1-\frac{n}{2q}\right)} B_{1\varepsilon} u_\varepsilon \in C\left([0\,,\infty); X_q\right) \text{ for } n \leq q \leq \infty.$$

Moreover, the following uniform estimates hold

$$\sup_{t,\varepsilon_j} \left\| t^{\left(1-\frac{n}{q}\right)/2} u_\varepsilon \right\|_{X_q} \leq C,\ \sup_{t,\varepsilon_j} \left\| t^{\left(1-\frac{n}{2q}\right)} B_{1\varepsilon} u_\varepsilon \right\|_{X_q} \leq C,\ j = 1,2,\ldots,n,$$

where $Q_{\varepsilon q}$ denotes the corresponding Stokes operator and $P_q$ is a projection operator in $X_q$.

In application, we put $E = L^{p_1}(0,1)$ and $A$ to be differential operator in (1a) and (1b), with generalized Wentzell–Robin boundary condition defined by

$$D(A) = \left\{ u \in \left(W^2_{p_1}(0,1)\right)^n,\ B_j u = Au(j),\ j = 0,1 \right\},$$

$$Au = au^{(2)} + bu^{(1)},$$

where $a$, $b$ are complex-valued functions. Then, we obtain the existence, uniqueness and uniformly $L^{\mathbf{p}}(\bar{\Omega})$ estimates for solutions the following Wentzell–Robin type mixed problem for the N–S equation

$$\frac{\partial u}{\partial t} - B_\varepsilon u + (u.B_{1\varepsilon})\, u + B_{1\varepsilon}\varphi + a\frac{\partial^2 u}{\partial y^2} + b\frac{\partial u}{\partial y} = f(x,y,t), \tag{1f}$$

$$B_{2\varepsilon} u = 0,\ u = u(x,y,t),\ x \in \mathbb{R}^n_+,$$

$$\sum_{i=0}^{\nu_k} \varepsilon_n^{\sigma_i} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}(x',0,t) = 0, \nu_k \in \{0,1,\ldots 2m-1\}, k = 1,2,\ldots m, \tag{1g}$$

$$Au(x,j,t) = 0, j = 0,1,\ u(x,0) = a(x),\ x' \in \mathbb{R}^{n-1}, y \in (0,1). \tag{1h}$$

Note that the regularity properties of Wentzell–Robin-type boundary value problems (BVP) for elliptic equations were studied e.g., in [23,24] and the references therein. Here,

$$\tilde{\Omega} = \mathbb{R}^n_+ \times (0,1), \bar{\Omega} = \mathbb{R}^n_+ \times (0,b), \mathbf{p} = (p_1,p), p_1, p \in (1,\infty)$$

$L^{\mathbf{p}}\left(\tilde{\Omega}\right)$ denotes the space of all $\mathbf{p}$-summable complex-valued functions with mixed norm i.e., the space of all measurable functions $f$ defined on $\tilde{\Omega}$, for which

$$\|f\|_{L^{\mathbf{p}}(\tilde{\Omega})} = \left( \int\limits_{R_+^n} \left( \int\limits_0^1 |f(x,y)|^{p_1} dy \right)^{\frac{p}{p_1}} dx \right)^{\frac{1}{p}} < \infty.$$

By using the general abstract result above, the existence, uniqueness and uniformly $L^{\mathbf{p}}\left(\tilde{\Omega}\right)$ estimates for solution of the problem (1f)–(1h) is obtained.

Moreover, we choose $E = L^{p_1}(0,b)$ and $A$ to be degenerated differential operator in $L^{p_1}(0,b)$ defined by

$$D(A) = \left\{ u \in W_\gamma^{[2],p_1}(0,1),\ \sum_{i=0}^{\nu_k} \alpha_{ki} u^{[i]}(0) + \beta_{ki} u^{[i]}(b) = 0,\ k = 1,2 \right\},$$

$$A(x)u = b_1(x,y)u^{[2]} + b_2(x,y)u^{[1]},\ x \in \mathbb{R}_+^n,\ y \in (0,b),\ \nu_k \in \{0,1\}, \tag{1i}$$

where $u^{[i]} = \left(y^\gamma \frac{d}{dy}\right)^\gamma u$ for $0 \le \gamma < \frac{1}{2}$, $b_1 = b_1(x,y)$ is a contınous, $b_2 = b_2(x,y)$ is a bounded function on $y \in [0,1]$ for a.e. $x \in \mathbb{R}_+^n$, $\alpha_{ki}$, $\beta_{ki}$ are complex numbers and $W_\gamma^{[2],p_1}(0,b)$ is a weighted Sobolev space defined by

$$W_\gamma^{[2],p_1}(0,b) = \{ u : u \in L^{p_1}(0,b),\ u^{[2]} \in L^{p_1}(0,b),$$

$$\|u\|_{W_\gamma^{[2],p_1}} = \|u\|_{L^{p_1}} + \left\|u^{[2]}\right\|_{L^{p_1}} < \infty.$$

Then, we obtain the existence, uniqueness and uniformly $L^{\mathbf{p}}(\bar{\Omega})$ estimates for solutions of the following mixed problem for degenerate N–S equation

$$\frac{\partial u}{\partial t} - B_\varepsilon u + (u.B_{1\varepsilon})u + B_{1\varepsilon}\varphi + \left( b_1 \frac{\partial^{[2]} u}{\partial y^2} + b_2 \frac{\partial^{[1]} u}{\partial y} \right) = f(x,y,t), \tag{1j}$$

$$B_{2\varepsilon} u = 0,\ x \in \mathbb{R}_+^n,\ y \in (0,b),\ t \in (0,T),\ u = u(x,y,t),$$

$$\sum_{i=0}^{\nu_k} \varepsilon_n^{\sigma_i} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}(x',0,t) = 0, \nu_k \in \{0,1,\ldots 2m-1\}, k = 1,2,\ldots m,$$

$$L_k u = \sum_{i=0}^{\nu_k} \alpha_{ki} u^{[i]}(x,0,t) + \beta_{ki} u^{[i]}(x,b,t) = 0,\ k = 1,2, \tag{1k}$$

$$u(x,y,0) = a(x,y). \tag{1l}$$

Let $E$ be a Banach space and $L^p(\Omega;E)$ denotes the space of strongly measurable $E$-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_{L^p} = \|f\|_{L^p(\Omega;E)} = \left( \int\limits_\Omega \|f(x)\|_E^p dx \right)^{\frac{1}{p}},\ 1 \le p < \infty.$$

The Banach space $E$ is called an $UMD$-space if the Hilbert operator $(Hf)(x) = \lim\limits_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$ is bounded in $L^p(\mathbb{R},E)$, $p \in (1,\infty)$ (see. e.g., [19], § 4). $UMD$ spaces include e.g., $L^p$, $l^p$ spaces and Lorentz spaces $L_{pq}$, $p,q \in (1,\infty)$.

Let $E_1$ and $E_2$ be two Banach spaces. Let $B(E_1, E_2)$ denote the space of all bounded linear operators from $E_1$ to $E_2$. For $E_1 = E_2 = E$ it will be denoted by $B(E)$.

Here, $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{R}$ denotes the set of real numbers. Let $\mathbb{C}$ be the set of complex numbers and

$$S_\psi = \{\lambda \in \mathbb{C}, \ \ |\arg \lambda| \leq \psi\} \cup \{0\}, \ \ 0 \leq \psi < \pi.$$

A linear operator $A$ is said to be positive in a Banach space $E$ with bound $M > 0$ if $D(A)$ is dense on $E$ and $\left\|(A + \lambda I)^{-1}\right\|_{B(E)} \leq M(1 + |\lambda|)^{-1}$ for any $\lambda \in S_\psi$ where $I$ is the identity operator in $E$ (see e.g., [1], §1.15.1). The positive operator $A$ is said to be $R$-positive in a Banach space $E$ if the set $\left\{\lambda (A + \lambda)^{-1} : \lambda \in S_\psi\right\}$ is $R$-bounded (see [19], § 4). The operator $A(s)$ is said to be positive in $E$ uniformly with respect to parameter $s$ with bound $M > 0$ if $D(A(s))$ is independent on $s$, $D(A(s))$ is dense in $E$ and $\left\|(A(s) + \lambda)^{-1}\right\| \leq \frac{M}{1+|\lambda|}$ for all $\lambda \in S_\psi$, where the constant $M$ does not depend on $s$ and $\lambda$.

Assume $E_0$ and $E$ are two Banach spaces and $E_0$ is continuously and densely included into $E$. Here, $\Omega$ is a measurable set in $\mathbb{R}^n$ and $m$ is a positive integer. Let $W^{m,p}(\Omega; E_0, E)$ denote the space of all functions $u \in L^p(\Omega; E_0)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L^p(\Omega; E)$ with the norm

$$\|u\|_{W^{m,p}(\Omega;E_0,E)} = \|u\|_{L^p(\Omega;E_0)} + \sum_{k=1}^{n} \left\|\frac{\partial^m u}{\partial x_k^m}\right\|_{L^p(\Omega;E)} < \infty.$$

Let $H^{s,p}(\mathbb{R}^n; E)$, $-\infty < s < \infty$ denotes the $E-$valued fractional Sobolev space of order $s$ that is defined as:

$$H^{s,p}(\mathbb{R}^n; E) = (I - \Delta)^{-\frac{s}{2}} L^p(\mathbb{R}^n; E)$$

with the norm

$$\|u\|_{H^{s,p}} = \left\|(I - \Delta)^{\frac{s}{2}} u\right\|_{L^p(\mathbb{R}^n;E)}.$$

It clear that $H^{0,p}(\mathbb{R}^n; E) = L^p(\mathbb{R}^n; E)$. It is known that if $E$ is a UMD space, then $H^{m,p}(\mathbb{R}^n; E) = W^{m,p}(\mathbb{R}^n; E)$ for positive integer $m$ (see e.g., [25], § 15). $H^{s,p}(\mathbb{R}^n; E_0, E)$ denote the Fractional Sobolev-Lions type space i.e.,

$$H^{s,p}(\mathbb{R}^n; E_0, E) = \{u \in H^{s,p}(\mathbb{R}^n; E) \cap L^p(\mathbb{R}^n; E_0),$$

$$\|u\|_{H^{s,p}(\mathbb{R}^n;E_0,E)} = \|u\|_{L^p(\mathbb{R}^n;E_0)} + \|u\|_{H^{s,p}(\mathbb{R}^n;E)} < \infty\Big\}.$$

## 2. Regularity Properties of Solutions for System of ADEs with Parameters

In this section, we will derive the maximal regularity properties of the BVP for system of ADE with small parameters in half-space

$$B_\varepsilon u + (A + \lambda) u = f(x), \ x \in \mathbb{R}^n_+, \tag{2a}$$

$$L_k u = \sum_{i=0}^{\nu_k} \varepsilon_n^{\sigma_i} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}(x', 0) = \eta_k, \nu_k \in \{0, 1, \ldots 2m-1\}, k = 1, 2, \ldots, m, \tag{2b}$$

$$B_\varepsilon u = \sum_{k=1}^{n} (-1)^m \varepsilon_k \frac{\partial^{2m} u}{\partial x_k^{2m}}, \ \sigma_i = \frac{1}{2m}\left(i + \frac{1}{q}\right), q \in (1, \infty),$$

$\alpha_{ki}$ are complex numbers, $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, $\varepsilon_k$ are small positive parameters, $m$ is a positive integer with $m \geq 1$ and $A$ is a linear operator in a Banach space $E$. Here, $u = u_\varepsilon(x) =$

$(u_1(x), u_2(x), \ldots, u_n(x))$, $u_k(x) = u_{k\varepsilon}(x)$, $\varphi = \varphi(x,t)$ are represent the $E$-valued unknown velocity and pressure like functions, respectively and $f = (f_1(x), f_2(x), \ldots, f_n(x))$.

Let $X_q = X_q(E) = \left(L^q\left(\mathbb{R}^n_+; E\right)\right)^n$ denotes the class of $E-$valued system of function $f = (f_1(x), f_2(x), \ldots, f_n(x))$ with norm

$$\|f\|_{X_q} = \left(\sum_{i=1}^{n} \|f_i\|^q_{L^q\left(\mathbb{R}^n_+;E\right)}\right)^{\frac{1}{q}}, q \in (1,\infty).$$

and let

$$X^m_q(A) = \left(W^{m,q}\left(\mathbb{R}^n_+; E(A), E\right)\right)^n$$

with norm

$$\|u\|_{X^m_q(A)} = \sum_{i=1}^{n} \|u_i\|_{W^{m,q}\left(\mathbb{R}^n_+;E(A),E\right)}.$$

Let

$$\theta_k = \frac{\frac{1}{q} + \nu_k}{2m}, W^{k,,m,q}(E) = \left(W^{m\theta_k,q}\left(\mathbb{R}^{n-1}, (E(A), E)_{\theta_k,q}, E\right)\right)^n.$$

By reasoning as in ([9], Theorem 2) we have

**Theorem 1.** *Let $E$ be a UMD space and $A$ be an $R$-positive operator in $E$. Assume $m$ is a nonnegative number, $q \in (1,\infty)$, $\alpha_\nu \neq 0, 0 < \varepsilon_k \leq 1$. Then for all $f \in X$, $\lambda \in S_\psi$ with sufficiently large $|\lambda| > 0$ problem (2a) and (2b) has a unique solution $u$ that belongs to $X^m_q(A)$ and the following coercive uniform estimate holds*

$$\sum_{k=1}^{n}\sum_{i=0}^{2m} \varepsilon_k^{\frac{i}{2m}} |\lambda|^{1-\frac{i}{2m}} \left\|\frac{\partial^i u}{\partial x_k^i}\right\|_{X_q} + \|Au\|_{X_q} \leq \tag{2c}$$

$$C\left(\|f\|_{X_q} + \sum_{k=1}^{m} \|\eta_k\|_{W^{k,2m,q}(E)}.\right)$$

*Consider the operator $Q_\varepsilon$ generated by problem (2a) and (2b), i.e.,*

$$D(Q_\varepsilon) = \left\{u \in X^{2m}_q(A),\ L_k u = 0\right\},\ Q_\varepsilon u = B_\varepsilon u + Au. \tag{2d}$$

From Theorem 1 we obtain the following results:

**Result 1**. Suppose the all conditions of Theorem 1 are satisfied. Then, there exists a resolvent $(Q_\varepsilon + \lambda)^{-1}$ for $\lambda \in S_\psi$ satisfying the following uniform estimate

$$\sum_{k=1}^{n}\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} \varepsilon_k^{\frac{i}{2m}} \left\|\frac{\partial^i}{\partial x_k^i}(Q_\varepsilon + \lambda)^{-1}\right\|_{B\left(X_q\right)} \leq C.$$

It is clear that the solution (2a) and (2b) depend on parameters $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, i.e., $u = u_\varepsilon(x)$. In view of the Theorem 1, we derive the properties of solutions (2a) and (2b).

From Theorem 1 we obtain:

**Result 2**. For $\lambda \in S_\psi$ there exists a resolvent $(Q_\varepsilon + \lambda)^{-1}$ of the operator $Q_\varepsilon$ satisfying the following uniform estimate

$$\sum_{k=1}^{n}\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} \varepsilon_k^{\frac{i}{2m}} \left\|\frac{\partial^i}{\partial x_k^i}(Q_\varepsilon + \lambda)^{-1}\right\|_{B\left(X_q\right)} \leq C. \tag{2e}$$

## 3. The Stokes System with Small Parameters

In this section, we derive the maximal regularity properties of the stationary abstract Stokes problem

$$B_\varepsilon u + (u.B_{1\varepsilon})\, u + B_{1\varepsilon}\varphi + Au = f(x),\ B_{2\varepsilon}u = 0, \tag{3a}$$

$$L_k u = \sum_{i=0}^{\nu_k} \varepsilon_n^{\sigma_i} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}\left(x', 0\right) = 0, \nu_k \in \{0, 1, \ldots 2m-1\}, k = 1, 2, \ldots m,$$

where

$$B_\varepsilon u = \sum_{k=1}^{n} (-1)^m \varepsilon_k \frac{\partial^{2m} u}{\partial x_k^{2m}},\ \sigma_i = \frac{1}{2m}\left(i + \frac{1}{q}\right), q \in (1, \infty),$$

$$B_{1\varepsilon}\varphi = \left(\varepsilon_1^{\frac{1}{2}} \frac{\partial^m \varphi}{\partial x_1^m}, \varepsilon_2^{\frac{1}{2}} \frac{\partial^m \varphi}{\partial x_2^m}, \ldots, \varepsilon_n^{\frac{1}{2}} \frac{\partial^m \varphi}{\partial x_n^m}\right), B_{2\varepsilon}u = -\sum_{k=1}^{n} \varepsilon_k^{\frac{1}{2}} \frac{\partial^m u_k}{\partial x_k^m},$$

$\alpha_{ki}$ are complex numbers, $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, $\varepsilon_k$ are small positive parameters, $m$ is a positive integer with $m \geq 1$ and $A$ is a linear operator in a Banach space $E$. Here, $u = u_\varepsilon(x) = (u_1(x), u_2(x), \ldots, u_n(x))$, $u_k(x) = u_{k\varepsilon}(x)$, $\varphi = \varphi(x, t)$ are represent the $E$-valued unknown velocity and pressure like functions, respectively and $f = (f_1(x), f_2(x), \ldots, f_n(x))$.

Here and hereafter $E^*$ will denoted the conjugate of $E$ and $(,.)$ (resp. $<,.>$)) denotes the duality pairing of functions on $\mathbb{R}_+^n$ (resp. $\mathbb{R}^{n-1}$) and $X_{q'} = X_{q'}(E^*) = \left(L^{q'}\left(\mathbb{R}_+^n; E^*\right)\right)^n$ will denote the dual spase of $X_q$, where $q^{-1} + (q')^{-1} = 1$. Let $X_{\sigma q} = L_\sigma^q\left(\mathbb{R}_+^n; E\right)$ denote the $E$-valued solenoidal space. Let $A$ be a positive operator in $E$. Let $Y^{2,q}(A) = W^{m,q}\left(\mathbb{R}_+^n; E(A), E\right)$. The spaces $\left(W^{m,q}\left(\mathbb{R}_+^n; E\right)\right)^n$, $(Y^{m,q}(A))^n$ will be denoted by $X_q^m$ and $X_q^m(A)$, respectively. Let

$$Bu = \sum_{k=1}^{n} (-1)^m \frac{\partial^{2m} u}{\partial x_k^{2m}},\ B_1 u = -\left(\frac{\partial^m u_1}{\partial x_1^m}, \frac{\partial^m u_2}{\partial x_2^m}, \ldots, \frac{\partial^m u_n}{\partial x_n^m}\right),$$

$$B_2 u = -\sum_{k=1}^{n} \frac{\partial^m u_k}{\partial x_k^m}.$$

Consider the space

$$Y_q(A, B_1) = \left\{u \in X_q(E(A)),\ \operatorname{div} B_1 u \in L^q\left(\mathbb{R}_+^n; E\right),\right.$$

$$\left.\|u\|_{Y_q(A,B_1)} = \left(\|u\|_{X_q(E(A))}^q + \|\operatorname{div} B_1 u\|_{L^q\left(\mathbb{R}_+^n; E\right)}^q\right)^{\frac{1}{q}}\right\}.$$

$Y_q(A, B_1)$ becomes a Banach space with this norm. Consider the problem

$$Bu + (A + \lambda) u = f(x),\ x \in \mathbb{R}_+^n, \tag{3b}$$

$$L_k u = \sum_{i=0}^{\nu_k} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}\left(x', 0\right) = 0, \nu_k \in \{0, 1, \ldots m-1\}, k = 1, 2, \ldots m.$$

By using Theorem 1, we obtain the following

**Corollary 1.** *Let $E$ be a UMD space and $A$ be an R-positive operator in $E$. Assume $m$ is a nonnegative number, $q \in (1, \infty)$. Then for all $f \in X_q^{-m}$ problem (3b) has a unique solution $u \in X_q^m(A)$ and the following estimate holds*

$$\|u\|_{X_q^m(A)} \leq C \|f\|_{X_q^{-m}}.$$

It is known that (see e.g., [11,12]) vector field $u \in \left(L^q\left(\mathbb{R}^n_+\right)\right)^n$ has a Helmholtz decomposition. In the following theorem we generalize this result for $E-$ valued function space $X_q$.

By reasoning as in [11,12] and ([22], Theorem 3.1) we have decomposition result via operator $\tilde{B}$ generated by problem (3b).

**Theorem 2.** *Let $E$ be an UMD space and $q \in (1,\infty)$. Assume there exists a constant $C_0 > 0$ such that*

$$\|u\|_{(E(A),E)_{\frac{1}{q},q}} \geq C_0 \|\operatorname{div} B_1 u\|_E \text{ for } u \in X_q^m(A),\ t \in [0,T],\ x \in \mathbb{R}^n_+. \tag{3c}$$

*Then $u \in X_q$ has a Helmholtz decomposition i.e., there exists a bounded linear projection operator $P_q$ from $X_q$ onto $X_{\sigma q}$ with null space*

$$N(P_q) = \left\{B_1\varphi \in X_q : \varphi \in L^q_{loc}\left(\mathbb{R}^n_+;E\right)\right\}.$$

*In particular, all $u \in X_q$ has a unique decomposition $u = u_0 + B_1\varphi$ with $u_0 = P_q u \in X_{q\sigma}$ so that*

$$\|B_1\varphi\|_{X_q} + \|u_0\|_{X_q} \leq C\|u\|_{X_q}.$$

For proving the Theorem 2 we need the following lemma:

By reasoning as in ([12], Lemma 2) we get:

**Lemma 1.** *$C_0^\infty\left(\mathbb{R}^n_+;E\right)$ is dense in $Y_q(A,B_1)$.*

*Here, $<,>$ and $(,)$denotes the duality pairing of abstract functions defined in $\mathbb{R}^{n-1}$ and $\mathbb{R}^n_+$, respectively. From [22] we have*

**Proposition 1.** *There exists a unique bounded linear operator $u \to \frac{\partial^i u}{\partial x_n^i}(x',0)$, $i = 0,1,\ldots,m-1$ from $X_q^m(A)$, $q \in (1,\infty)$ onto*

$$\left(W^{i,m,q}(E)\right)^* = \left(W^{m,q}\left(\mathbb{R}^{n-1},\left(E(A^*),E^*\right)_{\theta_i,1/q'},E^*\right)\right)^n$$

*such that*

$$< u(x',0), v(x',0) >= (B_2 u, v) + (u, B_1 v),\ v \in W^{m,q'}\left(\mathbb{R}^n_+, E(A), E\right)$$

*and the following estimate holds*

$$\left\|\frac{\partial^i u}{\partial x_n^i}(x',0)\right\|_{W^{i,m,q}(E)} \leq C\left(\|u\|_{X_q} + \|B_2 u\|_{L^q\left(\mathbb{R}^n_+;E\right)}\right), \tag{3d}$$

*where*

$$\theta_i = \frac{i+\frac{1}{q}}{m},\ \frac{1}{q} + \frac{1}{q'} = 1.$$

**Proof.** For $u \in Y_q(A,B_1)$ consider the linear form

$$T_u(v_i) = (B_2 u, \Phi) + (u, B_1\Phi),\ \Phi \in X^{2m-1}_{q'}(A),\ \frac{\partial^i}{\partial x_n^i}\Phi(x',0) = v_i. \tag{3e}$$

□

By virtue of trace theorem in $W^{s,q}(0,a;E(A),E)$, the interpolation of intersection and dual spaces (see e.g., ([22], §1.8.2, 1.12.1, 1.11.2)) and by localization argument we obtain that the operator

$v \to \frac{\partial^i}{\partial x_n^i} v\left(x', 0\right)$ is a bounded linear and surjective from $X_{q'}^m\left(A\right)$ onto $W^{i,m,q}\left(E\right)$. Hence, we can find for each $v_i \in W^{i,m,q}\left(E\right)$ an element $\Phi \in X_{q'}^{2m-1}\left(A\right)$ so that

$$\frac{\partial^i}{\partial x_n^i} \Phi\left(x', 0\right) = v_i,\ \left\|\Phi\right\|_{X_{q'}^{2m-1}(A)} \leq C \left\|v_i\right\|_{W^{i,q}(E)}.$$

Therefore, from (3e) we get

$$\left|T_u\left(v_i\right)\right| \leq \left(\left\|u\right\|_{X_q} + \left\|B_2 u\right\|_{L^q\left(R_+^n;E\right)}\right) \left\|\Phi\right\|_{X_{q'}^{2m-1}(A)}$$

$$\leq C \left(\left\|u\right\|_{X_q} + \left\|B_2 u\right\|_{L^q\left(R_+^n;E\right)}\right) \left\|v_i\right\|_{W^{i,q}(E)}.$$

This implies the existence of an element

$$\frac{\partial^i}{\partial x_n^i} u\left(x', 0\right) \in \left(W^{i,m,q}\left(E\right)\right)^* = \left(W^{m_i,q}\left(\mathbb{R}^{n-1}, \left(E\left(A^*\right), E^*\right)_{\theta_i, 1/q'}, E^*\right)\right)^n$$

such that

$$< \frac{\partial^i}{\partial x_n^i} u\left(x', 0\right), v_i >= T_u\left(v_i\right) \text{ for } v_i \in W^{i,q}\left(E\right)$$

and

$$\left\| \frac{\partial^i}{\partial x_n^i} u\left(x', 0\right) \right\|_{\left(W^{i,q}(E)\right)^*} \leq C \left(\left\|u\right\|_{X_q} + \left\|B_2 u\right\|_{L^q\left(\mathbb{R}_+^n;E\right)}\right),$$

where

$$m_i = 1 - (2m-1)\,\theta_i.$$

Thus, we have proved the existence of the operators $u \to \frac{\partial^i}{\partial x_n^i} u\left(x', 0\right)$. The uniqueness follows from Lemma 1.

Now we are going to construct the projection operator $P_q$. Let $f \in X_q$ and $f = \left(f_1\left(x\right), f_2\left(x\right), \ldots, f_n\left(x\right)\right)$. Consider the boundary value problem

$$Bu + Au = B_2 f\left(x\right),\ x \in \mathbb{R}_+^n, \tag{3f}$$

$$L_k u = \sum_{i=0}^{\nu_k} \alpha_{ki} \frac{\partial^i u}{\partial x_n^i}\left(x', 0\right) = 0, k = 1, 2, \ldots m.$$

Since $B_2 f\left(x\right) \in X_q^{-m}$, in view of Corollary 1, then for all $f \in X_q$ problem (3f) has a unique solution $u_1 \in X_q^m\left(A\right)$ and the following estimate holds

$$\left\|u_1\right\|_{X_q^m(A)} \leq C \left\|B_2 f\right\|_{X_q^{-m}}.$$

Thus we have

$$\left\|u_1\right\|_{X_q^m(A)} \leq C \left\|f\right\|_{X_q}, f - B_1 u \in X_q. \tag{3g}$$

Now consider the problem

$$Bu + Au = 0,\ L_k u = v_k - L_k u_1,\ x \in \mathbb{R}_+^n, k = 1, 2, \ldots m. \tag{3h}$$

By Theorem 1, we obtain that for all $v_k \in W^{\theta_k,m,q}\left(E\right)$ problem (3h) has a unique solution $u_2 \in X_q^m\left(A\right)$ and the following estimate holds

$$\|u_2\|_{X_q^m(A)} \leq C \sum_{k=1}^{n} \|v_k - L_k u\|_{W^{\theta_k,m,q}(E)},$$

where $\theta_k = \frac{\nu_k + \frac{1}{q}}{m}$. For any $f \in X_q$, we take the solution of (3f), then that of (3h) and put $u = u_1 + u_2$. We define

$$P_q u = u - B_1 u.$$

Then by reasoning as in [12,16] we have

**Lemma 2.** *Let $E$ be an UDM space and $q \in (1, \infty)$. Then, $P_q X_q$ is a closed subspace of $X_q$.*

**Lemma 3.** *Let $E$ be an UMD space and $q \in (1, \infty)$. Then, the operator $P_q$ is a linear bounded operator in $X_q$ and $P_q f = f$ if $B_1 f(x) = 0$.*

**Lemma 4.** *Assume $E$ is an UMD space, $A$ is an R-positive operator in $E$ and $q \in (1, \infty)$. Then the conjugate of $P_q$ is defined as $P_q^* = P_{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$ and this operator is bounded linear in $\left(L^{q'}\left(\mathbb{R}_+^n; E^*\right)\right)^n$.*

*Let*

$$W_q = \left\{\nabla \varphi : \varphi \in W^{1,q}\left(\mathbb{R}_+^n; E\right)\right\}, \ \left(P_q X_q\right)^{\perp} =$$

$$\left\{f \in \left(L^{q'}\left(\mathbb{R}_+^n; E^*\right)\right)^n, \ \langle f, v\rangle = 0 \text{ for any } v \in P_q X_q\right\}.$$

*From Lemmas 3 and 4 we obtain*

**Lemma 5.** *Assume $E$ is an UMD space and $q \in (1, \infty)$. Then*

$$\left(P_q X_q\right)^{\perp} = W_{q'}, \frac{1}{q} + \frac{1}{q'} = 1.$$

**Lemma 6.** *Assume $E$ is an UMD space and $q \in (1, \in)$. Then*

$$X_{q\sigma}^{\perp} = W_{q'}, \frac{1}{q} + \frac{1}{q'} = 1.$$

Now we are ready to prove the Theorem 2.

**Proof of Theorem 2.** From Lemmas 5 and 6 we get that $X_{q\sigma} = \left(P_q X_q\right)^{\perp}$. Then, by construction of $P_q$ we have

$$X_q = X_{\sigma q} \oplus W_q.$$

By Lemmas 2 and 3, we obtain the estimate (3a). Moreover, by Lemma 5, $W_q$ is a close subspace of $X_q$. Then, it is known that the dual space of quotient space $X_q / W_q$ is $W_q^{\perp}$. By first assertion we have $X_q / W_q = X_{\sigma q}$. □

**Theorem 3.** *Let $E$ be an UMD space, $A$ is an R-positive operator in $E$, $q \in (1, \infty)$. Then, problem (3a) and (3b) has a unique solution $u \in X_q^{2m}(A)$ for $f \in X_q$, $\varphi \in W^{m,q}\left(\mathbb{R}_+^n; E\right)$, $\lambda \in S_\psi$ and the following coercive uniform estimate holds*

$$\sum_{k=1}^{n} \sum_{i=0}^{2m} \varepsilon_k^{\frac{i}{2m}} |\lambda|^{1 - \frac{i}{2m}} \left\| \frac{\partial^i u}{\partial x_k^i} \right\|_{X_q} + \|Au\|_{X_q} + \|B_{1\varepsilon}\varphi\|_{X_q} \leq C \|f\|_{X_q} \tag{3i}$$

*with $C = C(q, A)$ independent of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, $\lambda$ and $f$.*

**Proof.** By applying the operator $P_q$ to problem (2a) and (2b) we get the Stokes problem (3a) and (3b). It is clear to see that

$$D\left(O_{\varepsilon q}\right) = D\left(Q_\varepsilon\right) \cap X_{\sigma q},$$

where $O_{\varepsilon q}$ is the abstract Stokes operator generated by problem (3a) and (3b) and $Q_\varepsilon$ is an abstrat elliptic operator in $X_q$ defined by (2e). □

Then Theorem 2 we obtain the assertion.

**Result 3.** From the Theorem 3 we get that $Q_{\varepsilon q}$ is a positive operator in $X_q$ and also generates a bounded holomorphic semigroup $S_\varepsilon(t) = \exp\left(-Q_{\varepsilon q} t\right)$ for $t > 0$. In a similar way as in [11] we show

**Proposition 2.** *The following estimate holds*

$$\left\| Q_{\varepsilon q}^{\alpha} S_\varepsilon(t) \right\| \leq C t^{-\alpha},$$

*uniformly in* $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ *for* $\alpha \geq 0$ *and* $t > 0$.

**Proof.** From Theorem 3 we obtain that the operator $O_{\varepsilon q}$ is uniformly positive in $X_q$, i.e., for $\lambda \in S_{\psi,\varkappa}$, $0 < \psi < \pi$ the following estimate holds

$$\left\| \left(Q_{\varepsilon q} + \lambda\right)^{-1} \right\| \leq M |\lambda|^{-1},$$

where the constant $M$ is independent of $\lambda$ and $\varepsilon$. Then, by using Danford integral and operator calculus (see e.g., in [10]) we obtain the assertion.

Now we can prove the main result of this section □

**Theorem 4.** *Let* $0 < \varepsilon_k \leq 1$. *Then, for* $f \in L^p\left(0, T; X_q\right) = B(p, q)$ *and* $a \in \left(X_q^{2m}(A), X_q\right)_{\frac{1}{p}, p}$, $p, q \in (1, \infty)$ *there is a unique solution* $\left(u, B_{1\varepsilon}\varphi\right)$ *of the problem (1d) and the following uniform estimate holds*

$$\left\| \frac{\partial u}{\partial t} \right\|_{B(p,q)} + \sum_{k=1}^{n} \left\| \varepsilon_k \frac{\partial^{2m} u}{\partial x_k^{2m}} \right\|_{B(p,q)} + \|Au\|_{B(p,q)} + \|B_{1\varepsilon}\varphi\|_{L^{p,q}} \leq$$

$$C\left( \|f\|_{B(p,q)} + \|a\|_{\left(X_q^{2m}(A), X_q\right)_{\frac{1}{p}, p}} \right) \tag{3j}$$

*with* $C = C(T, p, q)$ *independent of* $f$ *and* $\varepsilon$.

**Proof.** The problem (1d) can be expressed as the following abstract parabolic problem

$$\frac{du}{dt} + Q_{\varepsilon q} u = f(t),\ u(0) = a. \tag{3k}$$

By Proposition 2, operator $O_{\varepsilon q}$ is uniform positive and generates holomorphic semigroup in $X_q$. Moreover, by using ([9], Theorem 3) we get that the operator $Q_{\varepsilon q}$ is $R$-positive in $X_q$ uniformly with respect to $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. Since $E$ is a UMD space, in a similar way as in ([20], Theorem 4.2) we obtain that for all $f \in L^p\left(0, T; X_q\right)$ and $a \in \left(X_q^{2m}(A), E\right)_{\frac{1}{p}, p}$ there is a unique solution $u \in W^{1,p}\left(0, T, D\left(O_{\varepsilon q}\right), E\right)$ of the problem (4b) so that the following uniform estimate holds

$$\left\| \frac{du}{dt} \right\|_{L^p\left(0,T;X_q\right)} + \left\| Q_{\varepsilon q} u \right\|_{L^p\left(0,T;X_q\right)} \leq \tag{3l}$$

$$C\left(\|f\|_{L^p(0,T;X_q)}+\|a\|_{\left(X_q^{2m}(A),E\right)_{\frac{1}{p},p}}\right).$$

From the estimates (3k) and (3l) we obtain the assertion. □

**Result 4.** It should be noted that if $\varepsilon_1=\varepsilon_2=\ldots\varepsilon_n=1$ we obtain maximal regularity properties of abstract Stokes problem without any parameters in principal part.

**Remark 2.** *There are a lot of positive operators in concrete Banach spaces. Therefore, putting in (1d) concrete Banach spaces instead of E and concrete positive differential, pseudo differential operators, or finite, infinite matrices, etc. instead of A, by virtue of Theorem 3 and 4 we can obtain the maximal regularity properties of different class of stationary and instationary Stokes problems, respectively, which occur in numerous physics and engineering problems.*

## 4. Existence and Uniqueness for N–S Equation with Parameters

In this section, we study the N–S problem (1a)–(1c) in $X_q$. The problem (1a)–(1c) can be expressed as

$$\frac{du}{dt}+Q_{\varepsilon q}u=Fu+P_qf,\ u(0)=0,\ t>0,\ Fu=-P_q(u,B_{1\varepsilon})u. \tag{4a}$$

We consider the Equation (4a) in integral form

$$u(t)=S_\varepsilon(t)a+\int_0^t S_\varepsilon(t-s)\left[Fu(s)+P_qf(s)\right]ds,\ t>0. \tag{4b}$$

For proving the main result we need the following lemma which is obtained from ([11], Theorem 2).

**Lemma 7.** *Let E be a UMD space, A an R-positive operator in E, $q\in(1,\infty)$ and $0<\varepsilon_k\leq 1$. For any $0\leq\alpha\leq 1$ the domain $D\left(Q_{\varepsilon q}^{\alpha}\right)$ is the complex interpolation space $\left[X_q,D\left(Q_{\varepsilon q}\right)\right]_\alpha$, [1], §1).*

**Lemma 8.** *Let E be a UMD space, A an R-positive operator in E, $q\in(1,\infty)$ and $0<\varepsilon_k\leq 1$. For each $k=1,2,\ldots,n$ the operator $u\to Q_{\varepsilon q}^{-\frac{1}{2}}P_q\left(\frac{\partial^m}{\partial x_k^m}\right)u$ extends uniquely to a uniformly bounded linear operator from $L^q\left(\mathbb{R}_+^n;E\right)$ to $X_q$.*

**Proof.** Since $Q_{\varepsilon q}$ is a positive operator, it has a fractional powers $O_{\varepsilon q}^{\alpha}$. From the Lemma 7, It follows that the domain $D\left(Q_{\varepsilon q}^{\alpha}\right)$ is continuously embedded in $\left(H_q^{2\alpha}\left(\mathbb{R}_+^n;E(A),E\right)\right)^n\cap X_{q\sigma}$ for any $\alpha>0$. Then by using the duality argument and due to uniform positivity of $O_{\varepsilon q}^{\frac{1}{2}}$ we obtain the following uniform estimate

$$\left\|Q_{\varepsilon q}^{-\frac{1}{2}}P_q\left(\frac{\partial^m}{\partial x_k^m}\right)u\right\|_{L^q\left(\mathbb{R}_+^n;E\right)}\leq C\|u\|_{X_q}. \tag{4c}$$

□

By reasoning as in [10] we obtain the following

**Lemma 9.** *Let E be a a UMD space, A an R-positive operator in E, $q\in(1,\infty)$ and $0<\varepsilon_k\leq 1$. Let $0\leq\delta<\frac{1}{2}+\frac{n}{2}\left(1-\frac{1}{q}\right)$. Then the following uniform estimate holds*

$$\left\|Q_{\varepsilon q}^{-\delta}P_q(u,B_{1\varepsilon})v\right\|_q\leq M\left\|O_{\varepsilon q}^{\theta}u\right\|_q\left\|O_{\varepsilon q}^{\sigma}u\right\|_q$$

*provided that* $\theta > 0$, $\sigma > 0$, $\sigma + \delta > \frac{1}{2}$ *and*

$$\theta + \sigma + \delta > \frac{n}{2q} + \frac{1}{2}.$$

**Proof.** Assume that $0 < \nu < \frac{n}{2}\left(1 - \frac{1}{q}\right)$. Since $D\left(Q_\varepsilon^\alpha\right)$ is continuously embedded in $\left(H_q^{2\alpha}\left(\mathbb{R}_+^n; E\left(A\right), E\right)\right)^n \cap X_{q\sigma}$ and $L^{q'}\left(\mathbb{R}_+^n; E\right) \cap X_{q'}$ is the same as $X_{s'}$, by Sobolev embedding theorem we obtain that the operators

$$Q_{\varepsilon,q'}^{-\nu} : X_{q'} \to D\left(Q_{\varepsilon,q'}^{\nu}\right) \to X_{s'}$$

is bounded, where

$$\frac{1}{s'} = \frac{1}{q'} - \frac{2\nu}{n},\ \frac{1}{q} + \frac{1}{q'} = 1.$$

By duality argument then, we get that the operator $u \to Q_{\varepsilon,q}^{-\nu}$ is bounded from $X_s$ to $X_q$, where

$$\frac{1}{s} = 1 - \frac{1}{s'} = \frac{1}{q} + \frac{2\nu}{n}.$$

Consider first the case $\delta > \frac{1}{2}$. Since $P(u, B_{1\varepsilon})v$ is bilinear in $u, v$, it suffices to prove the estimate on a dense subspace. Therefore assume that $u$ and $v$ are smooth. Since div $B_{1\varepsilon}u = 0$, we get

$$(u, B_{1\varepsilon})v = \sum_{k=1}^{n} \varepsilon_k^{\frac{1}{2}} \frac{\partial^m}{\partial x_k^m}\left(u_k v\right).$$

Taking $\nu = \delta - \frac{1}{2}$ and using the uniform boundedness of $Q_{\varepsilon,q}^{-\nu}$, from $X_s$ to $X_q$ and Lemma 8 for all $\varepsilon > 0$ we obtain the uniform estimate

$$\left\|Q_{\varepsilon q}^{-\delta} P\left(u, B_{1\varepsilon}\right) v\right\|_q = \left\|\varepsilon_k Q_{\varepsilon,q}^{\frac{1}{2}-\nu} \sum_{k=1}^{n} P_q B_{1\varepsilon}\left(u_k v\right)\right\|_q \leq \left\||u|\,|v|\right\|_s.$$

By assumption we can take $r$ and $\eta$ such that

$$\frac{1}{r} \geq \frac{1}{q} - \frac{2\theta}{n},\ \frac{1}{\eta} \geq \frac{1}{q} - \frac{2\sigma}{n},\ \frac{1}{r} + \frac{1}{\eta} = \frac{1}{s},\ r > 1,\ \eta < \infty.$$

Since $D\left(Q_{\varepsilon,q}^{\alpha}\right)$ is continuously embedded in $\left(H_q^{2\alpha}\left(\mathbb{R}_+^n; E\left(A\right), E\right)\right)^n \cap X_{q\sigma}$, then by Sobolev embedding we get

$$\left\||u|\,|v|\right\|_s \leq \|u\|_r \|v\|_\eta \leq M \left\|Q_{\varepsilon,q}^{\theta} u\right\|_r \left\|Q_{\varepsilon,q}^{\sigma} v\right\|_\eta,$$

i.e., we have the required result for $\delta > \frac{1}{2}$. In particular, we get

$$\left\|Q_{\varepsilon q}^{-\frac{1}{2}} P_q\left(u, B_{1\varepsilon}\right) v\right\|_q \leq M \left\|Q_{\varepsilon,q}^{\theta} u\right\|_r \left\|O_{\varepsilon,q}^{\sigma} v\right\|_\eta,\ \theta + \beta \geq \frac{n}{2q},\ \beta > 0.$$

Similarly we obtain

$$\left\|P_q\left(u, B_{1\varepsilon}\right) v\right\|_q \leq C \|u\|_r \|v\|_\eta \leq C \left\|Q_{\varepsilon,q}^{\theta} u\right\|_r \left\|Q_{\varepsilon,q}^{\beta+\frac{1}{2}} v\right\|_\eta$$

for $\frac{1}{r}+\frac{1}{\eta}=\frac{1}{q}$ and $\delta=0$. The above two estimates show that the map $v\to P_q\left(u,B_{1\varepsilon}\right)v$ is a uniform bounded operator from $D\left(Q_{\varepsilon q}^{\beta}\right)$ to $D\left(O_{\varepsilon q}^{-\frac{1}{2}}\right)$ and from $D\left(Q_{\varepsilon q}^{\beta+\frac{1}{2}}\right)$ to $X_q$. By using the Lemma 7 and the interpolation theory of Banach spaces for $0\leq\delta\leq\frac{1}{2}$ we obtain the uniform estimate

$$\left\|P_q\left(u,B_{1\varepsilon}\right)v\right\|_q\leq C\left\|Q_{\varepsilon,q}^{\theta}u\right\|_r\left\|Q_{\varepsilon,q}^{\sigma}v\right\|_{\eta}.$$

□

By using Lemma 9 and iteration argument, by reasoning as in Fujita and Kato [13] we obtain the following.

**Theorem 5.** *Let $E$ be a UMD space, $A$ an $R$-positive operator in $E$, $q\in(1,\infty)$ and $0<\varepsilon_k\leq 1$. Let $\gamma<1$ be a real number and $\delta\geq 0$ such that*

$$\frac{n}{2q}-\frac{1}{2}\leq\gamma,\ -\gamma<\delta<1-|\gamma|.$$

*Suppose that $a\in D\left(Q_{\varepsilon q}^{\gamma}\right)$, and that $\left\|Q_{\varepsilon q}^{-\delta}Pf\left(t\right)\right\|$ is continuous on $(0,T)$ and satisfies*

$$\left\|Q_{\varepsilon q}^{-\delta}Pf\left(t\right)\right\|=o\left(t^{\gamma+\delta-1}\right)\ as\ t\to 0.$$

*Then there is $T_*\in(0,T)$ independent of $\varepsilon$ and local solution of (4a) $u$ such that $u\in C\left(\left[0,T_*\right];D\left(Q_{\varepsilon q}^{\gamma}\right)\right)$, $u\left(0\right)=a$,*

*$u\in C\left(\left(0,\ T_*\right];D\left(Q_{\varepsilon q}^{\alpha}\right)\right)$ for some $T_*>0$, $\left\|Q_{\varepsilon q}^{\alpha}u\left(t\right)\right\|=o\left(t^{\gamma-\alpha}\right)$ uniformly in $\varepsilon$ as $t\to 0$ for all $\alpha$ with $\gamma<\alpha<1-\delta$. Moreover, the solution of (4a) is unique if $u\in C\left(\left(0,\ T_*\right];D\left(Q_{\varepsilon q}^{\beta}\right)\right)$ and $\left\|Q_{\varepsilon q}^{\alpha}u\left(t\right)\right\|=o\left(t^{\gamma-\beta}\right)$ uniformly in $\varepsilon=(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)$ as $t\to 0$ for $\beta>|\gamma|$.*

**Proof.** We introduce the following iteration scheme

$$u_0\left(t\right)=S_{\varepsilon}\left(t\right)a+\int_0^t S_{\varepsilon}\left(t-s\right)Pf\left(s\right)ds,$$

$$u_{m+1}\left(t\right)=u_0\left(t\right)+\int_0^t S_{\varepsilon}\left(t-s\right)Fu_m\left(s\right)ds,\ m\geq 0.$$

By estimating the term $u_0\left(t\right)$ in (4c) and by using the Lemma 9 for $\gamma\leq\alpha<1-\delta$ we get the uniform estimate

$$\left\|Q_{\varepsilon q}^{\alpha}u_0\left(t\right)\right\|\leq\left\|Q_{\varepsilon q}^{\alpha}S_{\varepsilon}\left(t\right)a\right\|+\int_0^t\left\|Q_{\varepsilon q}^{\alpha+\delta}S_{\varepsilon}\left(t-s\right)\right\|\left\|Q_{\varepsilon q}^{-\delta}Pf\left(s\right)\right\|ds\leq$$

$$\left\|Q_{\varepsilon q}^{\alpha}S_{\varepsilon}\left(t\right)a\right\|+C_{\alpha+\delta}\int_0^t\left|\left(t-s\right)\right|^{-(\alpha+\delta)}\left\|Q_{\varepsilon q}^{-\delta}Pf\left(s\right)\right\|ds\leq M_{\alpha}t^{\gamma-\alpha}$$

with

$$M_{\alpha}=\sup_{0<t\leq T,\,\varepsilon>0}t^{\alpha-\gamma}\left\|Q_{\varepsilon q}^{\alpha+\delta}S_{\varepsilon}\left(t\right)a\right\|+C_{\alpha+\delta}NB\left(1-\delta-\alpha,\gamma+\alpha\right),$$

where $N = \sup_{0<t\leq T} t^{1-\gamma-\delta} \left\| Q_{\varepsilon q}^{-\delta} Pf(t) \right\|$ and $B(\beta,\eta)$ is the beta function. Here we suppose $\gamma+\delta>0$. By induction assume that $u_m(t)$ satisfies the following

$$\left\| O_{\varepsilon q}^{\alpha} u_m(t) \right\| \leq M_{\alpha m} t^{\gamma-\alpha},\ \gamma \leq \alpha < 1-\delta. \tag{4d}$$

We shall estimate $Q_{\varepsilon q}^{\alpha} u_{m+1}(t)$ by using (4b). To estimate the term $\left\| Q_{\varepsilon q}^{-\delta} Fu_m(s) \right\|$ we suppose

$$\theta+\sigma+\delta = 1+\gamma,\ \gamma<\theta<1-\delta,\ \gamma<\sigma<1-\delta,$$

$$\theta>0,\ \sigma>0,\ \delta+\sigma>\frac{1}{2},$$

so that the numbers $\theta, \sigma, \delta$ satisfy the assumptions of Lemma 9. Using Lemma 9 and (4d), we get

$$\left\| Q_{\varepsilon q}^{-\delta} Fu_m(s) \right\| \leq CM_{\theta m} M_{\sigma m} s^{\gamma+\delta-1}.$$

Therefore, we obtain

$$\left\| Q_{\varepsilon q}^{\alpha} u_m(t) \right\| \leq M_{\alpha} t^{\gamma-\alpha} + M_{\alpha+\delta} \int_0^t \|(t-s)\|^{-(\alpha+\delta)} \left\| Q_{\varepsilon q}^{-\delta} Fu_m(s) \right\| ds \leq$$

$$M_{\alpha m+1} t^{\gamma-\alpha}$$

with

$$M_{\alpha m+1} = M_{\alpha} + M_{\alpha+\delta} MB(1-\delta-\alpha, \gamma+\delta) M_{\theta m} M_{\sigma m}.$$

We get the uniform estimate. So, the remaining part of proof is obtained the same as in ([10], Theorem 2.3). □

By reasoning as in [13] we obtain

**Lemma 10.** *Let the operator $A_\varepsilon$ be uniform positive in a Banach space $E$ and $\alpha$ be a positive number with $0<\alpha<1$. Then, the following uniform inequality holds*

$$\left\| A_{\varepsilon}^{\alpha} \left( e^{-A_\varepsilon t} - I \right) u \right\|_E \leq \frac{t^{\alpha}}{\alpha} \|A_{\varepsilon}^{\alpha} u\|_E$$

*for all $u \in E$.*

**Proposition 3.** *Let $E$ be a space satisfying a multiplier condition, $A$ an $R$-positive operator in $E$, $q \in (1,\infty)$ and $0<\varepsilon_k \leq 1$. Let $u$ be the solution given by Theorem 5 Then $Q_{\varepsilon q}^{\alpha} u$ for $\gamma<\alpha<1-\delta$ is uniform Hölder continuous on every interval $[\eta, T_*]$, $0<\eta<T_*$ for all parameters $\varepsilon_k>0$.*

**Proof.** It suffices to prove the Hölder continuity of $Q_{\varepsilon}^{\alpha} v$, where

$$v(t) = \int_0^t S_\varepsilon(t-s)\left[Fu(s) + Pf(s)\right] ds.$$

Using the Lemma 10 we get the uniform estimate

$$\left\| \left( e^{-hQ_{\varepsilon q}} - I \right) Q_{\varepsilon q}^{-\alpha} \right\|_{B(E)} \leq \frac{h^{\alpha}}{\alpha},\ h>0.$$

Then by reasoning as in ([10], Proposition 2.4) we obtain the assertion. □

**Theorem 6.** *Let $E$ be a UMD space, $A$ an $R$-positive operator in $E$, $q \in (1,\infty)$ and $0 < \varepsilon_k \leq 1$. Assume $P_q f : (0,\ T_*] \to X_q$ is Hölder continuous on each subinterval $[\eta, T_*]$. Then, the solution of (4b) given by Theorem 5 satisfies Equation (4a) for all $\varepsilon_k > 0$. Moreover, $u \in D\left(O_{\varepsilon q}\right)$ for $t \in (0\ T_*]$.*

**Proof.** It suffices to show Hölder continuity of $Fu\left(t\right)$ on each interval $[\eta, T_*]$. It is clear to see that $u\left(\eta\right) \in X_q$ and

$$u\left(t\right) = S_\varepsilon\left(t\right) u\left(\eta\right) + \int_0^t S_\varepsilon\left(t-s\right)\left[Fu\left(s\right) + P_q f\left(s\right)\right] ds,\ t \in [\eta, T_*].$$

Since $P_q f$ is continuous on $[\eta, T_*]$ we get

$$\left\|P_q f\left(t\right)\right\| = o\left(t-\eta\right)^{-\alpha},\ t \to \eta, \alpha > 0.$$

The uniqueness of $u\left(t\right)$ ensured by Theorem 5, implies the following uniform estimates

$$C\left([\eta, T_*]; D\left(Q_{\varepsilon q}^{\nu}\right)\right) \cap C\left((\eta\ , T_*]; D\left(Q_{\varepsilon q}^{\alpha}\right)\right),$$

$$Q_{\varepsilon q}^{\alpha}\left\|u\left(t\right)\right\| = o\left(t-\eta\right)^{\nu-\alpha},\ t \to \eta, \nu < \alpha < 1,$$

where $\nu = \max\left\{\gamma, 0\right\}$. So, by Proposition 4, $Q_{\varepsilon q}^{\alpha} u\left(t\right)$ is continuous on every subinterval $[\eta, T_*]$. Since we can choose $\theta, \sigma$ so that

$$\theta + \sigma = 1 + \nu, \nu < \theta < 1,\ \max\left\{\gamma, \frac{1}{2}\right\} < \sigma < 1.$$

□

Lemma 8 implies that $Fu\left(t\right)$ is Hölder continuous on every interval $[\eta, T_*]$.

## 5. Regularity Properties

The purposes of this section is to show that the solutions of (1a) are smooth if the data are smooth. For simplicity, we assume $P_q f = 0$. The proof when $P_q f \neq 0$ is the same. Consider first all of the Stokes problem (3d) and (3e).

By reasoning as in ([13], Lemma 2.14) we obtain

**Lemma 11.** *Let $E$ be a UMD space, $A$ an $R$-positive operator in $E$, $q \in (1,\infty)$ and $0 < \varepsilon_k \leq 1$. Let $f \in C^{\mu}\left([0,T]; X_q\right)$ for some $\mu \in (0,1)$. Then for every $\eta \in (0,\mu)$ we have*

$$v\left(t\right) = \int_0^t S_\varepsilon\left(t-s\right) f\left(s\right) ds\ \in C^{\eta}\left((0,\ T]; D\left(Q_{\varepsilon q}\right)\right) \cap C^{1+\eta}\left((0,\ T]; X_q\right).$$

*In a similar way as Lemmas 2, 5 and 6 in [10] we obtain, respectively:*

**Lemma 12.** *Let $E$ be a UMD space, $A$ an $R$-positive operator in $E$, $q \in (1,\infty)$ and $0 < \varepsilon_k \leq 1$. For $u$, $v \in W^{m,q}\left(\mathbb{R}_+^n; E\left(A\right), E\right)$, $q \in (1,\infty)$ the following hold:*

*(1)*

$$P_q u \in W^{m,q}\left(\mathbb{R}_+^n; E\left(A\right), E\right) \cap X_{q\sigma},\ \left\|P_q u\right\|_{W^{m,q}\left(\mathbb{R}_+^n; E\right)} \leq C_{m,q}\left\|u\right\|_{W^{m,q}\left(\mathbb{R}_+^n; E\right)};$$

*(2) for $m > \frac{n}{q}$ there exists a constant $C_{m,q}$ such that*

$$\left\|P_q\left(u, B_{1\varepsilon}\right) v\right\|_{W^{m,q}\left(\mathbb{R}_+^n; E\right)} \leq C_{m,q}\left\|u\right\|_{W^{m,q}\left(\mathbb{R}_+^n; E\right)}\left\|v\right\|_{W^{m,q}\left(\mathbb{R}_+^n; E\right)};$$

*(3) when $q > n$ we have*

$$\left\| P_q\left(u, B_{1\varepsilon}\right) v\right\|_{L^q\left(\mathbb{R}^n_+;E\right)} \leq C_q \left\|u\right\|_{W^{1,q}\left(\mathbb{R}^n_+;E\right)} \left\|v\right\|_{W^{1,q}\left(\mathbb{R}^n_+;E\right)}.$$

**Lemma 13.** *Let $E$ be a UMD space, $A$ an $R$-positive operator in $E$, $q \in (1,\infty)$ and $0 < \varepsilon_k \leq 1$. Let $u = u_\varepsilon(t)$ be solution of (4b) for $P_q f = 0$, then $u \in C^\mu\left((0,\ T]; D\left(Q_{\varepsilon q}\right)\right)$ and $\frac{du}{dt} \in C^\mu\left((0,\ T]; X_q\right)$ for $\mu \in \left(0, \frac{1}{2}\right)$. Moreover,*

$$Fu \in C^\mu\left((0,\ T]; W^{1,q}\left(\mathbb{R}^n_+; E(A), E\right)\right).$$

**Lemma 14.** *Let $E$ be a a UMD space, $A$ an $R$-positive operator in $E$, $q \in (1,\infty)$ and $0 < \varepsilon_k \leq 1$. Let $u = u_\varepsilon(t)$ be solution of (4b) for $P_q f = 0$, then $u \in C^\mu\left((0,\ T]; D\left(Q_{\varepsilon q}^{\frac{1}{2}}\right)\right)$ for $\mu \in \left(0, \frac{1}{2}\right)$.*

Now, by reasoning as in ([10], Proposition 3.5) we can state the following

**Proposition 4.** *Let $E$ be a UMD space, $A$ an $R$-positive operator in $E$, $q \in (1,\infty)$ and $0 < \varepsilon_k \leq 1$. Let $E$ be Banach algebra, $q > n$ and $a \in X_q$. Suppose that the solution $u = u_\varepsilon(t)$ of (4b) for $P_q f = 0$ given by Theorem 5 exists on $[0,T]$. Then $u \in C^\infty\left(\mathbb{R}^n_+ \times [0,T]; E\right)$.*

**Proof.** The solution $u = u_\varepsilon(t)$ of (4b) for $P_q f = 0$ given by Theorem 5 is expressed as

$$u(t) = S_\varepsilon(t)\, a + \int_0^t S_\varepsilon(t-s)\, Fu(s)\, ds,\ t > 0, \tag{5a}$$

where $Fu = -P_q(u, \nabla)\, u$. From (5a) we get

$$Q_{\varepsilon q}^{\frac{1}{2}} u(t) = S_\varepsilon(t-\eta)\, Q_{\varepsilon q}^{\frac{1}{2}} u(\eta) + \int_\eta^t Q_{\varepsilon q} S_\varepsilon(t-s)\, Q_{\varepsilon q}^{-\frac{1}{2}} Fu(s)\, ds,\ t > 0 =$$

$$S_\varepsilon(t-\eta)\, Q_{\varepsilon q}^{\frac{1}{2}} u(\eta) + v(t),\ v(t) = v_\varepsilon(t) = \int_\eta^t Q_{\varepsilon q} S_\varepsilon(t-s)\, Q_{\varepsilon q}^{-\frac{1}{2}} Fu(s)\, ds.$$

Since $S_\varepsilon(t-\eta)\, Q_{\varepsilon q}^{\frac{1}{2}} u(\eta) \in C^\infty\left((\delta,\ T]; X_q\right)$ and $0 < \eta < T$, we will examining only $v(t)$. Integrating by parts, we obtain

$$v(t) = \int_\eta^t \frac{d}{ds} S_\varepsilon(t-s)\, O_\varepsilon^{-\frac{1}{2}} Fu(s)\, ds = \varepsilon Q_{\varepsilon q}^{-\frac{1}{2}} Fu(t) - \tag{5b}$$

$$S_\varepsilon(t-\eta)\, Q_{\varepsilon q}^{\frac{1}{2}} Fu(\delta) - \int_\eta^t S_\varepsilon(t-s)\, Q_{\varepsilon q}^{-\frac{1}{2}} \frac{d}{ds}(Fu)(s)\, ds.$$

Moreover, since $u(s) \in D\left(Q_{\varepsilon q}\right)$ for all $\varepsilon_k > 0$, $0 < s \leq T$, we have

$$(Fu)(s) = -\sum_{k=1}^n P_q\left(\varepsilon_k^{\frac{1}{2}} \frac{\partial^m}{\partial x_k^m}\right)\left[u_k(s)\, u(s)\right],$$

where

$$u(s) = (u_1(s), u_2(s), \ldots, u_n(s)),\ u_k = u_{k\varepsilon}.$$

Hence, by Lemma 7 we get the following uniform estimate

$$\left\|Q_{\varepsilon q}^{-\frac{1}{2}}\frac{d}{ds}Fu\right\|_{X_q}=\left\|\sum_{k=1}^{n}Q_{\varepsilon q}^{-\frac{1}{2}}P_q\left(\varepsilon_k^{\frac{1}{2}}\frac{\partial^m}{\partial x_k^m}\right)\left[\frac{du_{k\varepsilon}}{ds}u_\varepsilon+u_{k\varepsilon}\frac{du_\varepsilon}{ds}\right]\right\|_{X_q}$$

$$\leq C\left\|u_\varepsilon\right\|_{L^\infty\left(R_+^n;E\right)}\left\|\frac{du_\varepsilon}{ds}\right\|_{X_q}\leq C\left\|Q_{\varepsilon q}^{\frac{1}{2}}u_\varepsilon\right\|_{X_q}\left\|\frac{du_\varepsilon}{ds}\right\|_{X_q}.$$

This estimates together with Lemma 13 shows that

$$O_{\varepsilon q}^{-\frac{1}{2}}\frac{d}{ds}Fu\in C^\mu\left((0,\ T];X_q\right).$$

Lemmas 11 and 12 now imply that

$$\frac{dv}{dt}\in C^\mu\left((0,\ T];X_q\right).$$

Since $D\left(Q_{\varepsilon q}^{\frac{1}{2}}\right)\subset\left(W^{1,q}\left(\mathbb{R}_+^n;E\left(A\right),E\right)\right)^n$, Corollary 5.1, Lemmas 5.3, 5.4 and the identity $u\left(t\right)=Q_{\varepsilon q}^{\frac{1}{2}}\left(Fu-\frac{du}{dt}\right)$ imply

$$u\in C^\mu\left((0,\ T];\left(W^{3,q}\left(R_+^n;E\left(A\right),E\right)\right)^n\right).$$

Then the proof will be completed as in ([10], Proposition 3.5) by using the induction. □

Now we can state the main result of this section

**Theorem 7.** *Let $E$ be a UMD space, $A$ an $R$-positive operator in $E$, $q\in(1,\infty)$ and $0<\varepsilon_k\leq 1$. Let $E$ be Banach algebra and $a\in X_q$. Suppose that the solution $u=u_\varepsilon\left(t\right)$ of (4b) for $P_qF=0$ given by Theorem 5 exists on $[0,T]$. Then $u\in C^\infty\left(\mathbb{R}_+^n\times[0,T_*];E\right)$.*

**Proof.** For $q>n$ the assertion is obtained from the Proposition 4. Let us show that the assertion is valid for $1<q\leq n$. Indeed, the solution $u=u_\varepsilon\left(t\right)$ of (5b) for $P_qF=0$ given by Theorem 5 satisfies the Equation (5a) on every subinterval $[\eta,T_*]$, $0<\eta<T$. Theorem 6 shows that $u_\varepsilon\left(\eta\right)\in D\left(Q_{\varepsilon q}\right)$. Since $0\leq\frac{n}{2q}-\frac{1}{2}\leq\gamma<1$, we have $D\left(Q_{\varepsilon q}^\gamma\right)\subset X_n$ so that $D\left(Q_{\varepsilon q}\right)\subset X_s$ for some $s>n$. By $(4b)$ this means that we may assume $q>n$ and $a\in X_q$. □

## 6. Existence of Global Solutions

In this section, we prove the existence and estimate of a global solution of the problem (1a)–(1c). The proofs of these theorems are based on the theory of holomorphic semigroups and fractional powers of generators. We assume for simplicity that $f=0$, although it is not difficult to include nonzero $f$ under appropriate conditions. The main result is the following

**Theorem 8.** *Let $E$ be a UMD space, $A$ an $R$-positive operator in $E$, $q\in(1,\infty)$ and $0<\varepsilon_k\leq 1$ and $a\in X_q$. There exists a $T>0$ and a unique solution $u=u_\varepsilon$ of (1a)–(1c) so that $t^{\left(1-\frac{n}{q}\right)/2}u\in C\left([0\ ,T);X_q\right)$ for $n\leq q\leq\infty$ and $t^{\left(1-\frac{n}{2q}\right)}B_{1\varepsilon}u\in C\left([0\ ,T);X_q\right)$ for $n\leq q<\infty$. Moreover, the following estimates hold*

$$\sup_{t\in[0,\ T),\varepsilon_k>0}\left\|t^{\left(1-\frac{n}{q}\right)/2}u_\varepsilon\right\|_{X_q}\leq C,\quad\sup_{t\in[0,\ T),\varepsilon_k>0}\left\|t^{\left(1-\frac{n}{2q}\right)}B_{1\varepsilon}u_\varepsilon\right\|_{X_q}\leq C.$$

**Proof.** The solution $u = u_\varepsilon(t)$ of (4b) for $Pf = 0$ given by Theorem 5 is expressed as

$$u(t) = u_0(t) + G_\varepsilon u(t), \tag{6a}$$

where,

$$u_0(t) = S_\varepsilon(t)\, a,\ G_\varepsilon u(t) = \int_0^t S_\varepsilon(t-s)\, Fu(s)\, ds,\ t > 0.$$

By applying the generalized Minkovskii inequality and by Proposition 1 we can see that

$$\|S_\varepsilon(t)\, u\|_{X_p} \le C\varepsilon_k^{\frac{n}{2}\left(1+\frac{1}{p}\right)} t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)} \|u\|_{X_p},\ k = 1, 2, \ldots, n.$$

By using the above estimate we get

$$\|S_\varepsilon(t)\, u\|_{X_q} \le C\varepsilon_k^{\frac{n}{2}\left(2+\frac{1}{q}-\frac{1}{p}\right)} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|_{X_p}, \tag{6b}$$

$$\|B_{1\varepsilon} S_\varepsilon(t)\, u\|_{X_q} \le C\varepsilon_k^{\frac{n}{2}\left(\frac{3}{2}-\left(\frac{1}{p}-\frac{1}{q}\right)\right)} t^{-\left(1+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\right)} \|u\|_{X_p} \text{ for } 1 < p \le q < \infty. \tag{6c}$$

Moreover, by using (6a), (6b) and by applying the Hölder inequality we get

$$\|F(u, v)\|_{X_q} \le C\|u\|_{L^r} \|\nabla v\|_{L^s},\ \frac{1}{q} = \frac{1}{r} + \frac{1}{s}. \tag{6d}$$

Then in view of (6a) and (6d) we obtain the following uniform estimate

$$\|G_\varepsilon u\|_{\left(L^{m/\gamma}\right)^n} \le C \int_0^t (t-s)^{-(\alpha+\beta-\gamma)/2} \|u(s)\|_{m/\alpha} \|B_{1\varepsilon} u(s)\|_{m/\beta}\, ds, \tag{6e}$$

$$\|B_{1\varepsilon} G_\varepsilon u\|_{\left(L^{m/\gamma}\right)^n} \le C \int_0^t (t-s)^{-(1+\alpha+\beta-\gamma)/2} \|u(s)\|_{m/\alpha} \|B_{1\varepsilon} u(s)\|_{m/\beta}\, ds, \tag{6f}$$

where

$$\alpha,\ \beta,\ \gamma > 0,\ \gamma \le \alpha + \beta < n.$$

Then solving the Equation (6a) by successive approximation, starting with $u_0 = S_\varepsilon(t)\, a$ we get

$$u_{k+1} = u_0 + G_\varepsilon u_k,\ u_k = u_{k\varepsilon}(t),\ k = 0, 1, 2, \ldots \tag{6g}$$

First by reasoning as in ([1], Theorem 1) and by using (6c)–(6e) we show by induction that $u_k = u_{\varepsilon k}$ exists, moreover,

$$t^{(1-\delta)/2} u_{\varepsilon k} \in C\left([0, \infty); \left(L^{n/\delta}\left(\mathbb{R}^n_+; E\right)\right)^n\right),$$

$$t^{1/2} B_{1\varepsilon} u_{\varepsilon k} \in C\left([0, \infty); \left(L^n\left(\mathbb{R}^n_+; E\right)\right)^n\right)$$

and for $\delta \in (0, 1)$ the following uniform estimates hold

$$\sup_{t, \varepsilon_k} \left\| t^{(1-\delta)/2} u_{\varepsilon k} \right\|_{L^{n/\delta}} \le M_k,\ \sup_{t, \varepsilon_k} \left\| t^{1/2} B_{1\varepsilon} u_{\varepsilon k} \right\|_{L^q} \le M_k'. \tag{6h}$$

By applying (6c)–(6e) for $q = n$ and $p = \frac{n}{\delta}$ we have

$$M_0 = M_0' = C\|a\|_{\left(L^n\left(R^n_+; E\right)\right)^n}, \tag{6i}$$

where $C$ is a positive constant. From (6e) and (6g) for $n \leq p < \infty$ we obtain

$$\|u_{\varepsilon k+1}\|_{X_p} \leq \|u_{\varepsilon 0}\|_{X_p} \leq$$

$$CM_k M_k^{'} \int_0^t (t-s)^{-(1+\delta-n/q)/2} s^{-(1-\delta/2)} ds \leq Mt^{-(1-n/q)/2}.$$

It follows that $u_{\varepsilon k}(t)$ converges to a limit function $u_\varepsilon$ uniformly with respect to $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, moreover, $u_\varepsilon \in C\left([0,\ T); L^n\left(\mathbb{R}^n_+; E\right)^n\right)$ for $p = n$ and $u_\varepsilon$ satisfies (6a) for $n < p < \infty$. □

**Theorem 9.** *Let $E$ be a a UMD space, $A$ an R-positive operator in $E$, $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. There is a $\mu > 0$ such that if $\|a\|_{X_q} < \mu$, then there exists a global solution $u_\varepsilon$ of $(1a) - (1c)$, so that $t^{\left(1-\frac{n}{q}\right)/2} u_\varepsilon \in C\left([0,\ \infty); X_q\right)$ for $n \leq q \leq \infty$, $t^{\left(1-\frac{n}{q}\right)/2} u_\varepsilon$ and $t^{\left(1-\frac{n}{2q}\right)} B_{1\varepsilon} u_\varepsilon \in C\left([0,\ \infty); X_q\right)$ for $n \leq q < \infty$. Moreover, the following uniform estimates hold*

$$\sup_{t,\varepsilon_k} \left\| t^{\left(1-\frac{n}{q}\right)/2} u_\varepsilon \right\|_{X_q} \leq C,\ \sup_{t,\varepsilon_k} \left\| t^{\left(1-\frac{n}{2q}\right)} B_{1\varepsilon} u_\varepsilon \right\|_{X_q} \leq C. \tag{6j}$$

**Proof.** It is clear to see from proof of Theorem 6.1 that $M_k$ and $M_k^{'}$ are bounded by a constant $M$ if $M_0 \leq \lambda$. By (7i) this is true if $\|a\|_{X_q}$ is sufficiently small. In this case, as in [15] we prove that the sequences $t^{(1-\delta)/2} u_{\varepsilon k}$, $t^{1/2} B_{1\varepsilon} u_{\varepsilon k}$ are bounded on $(0, \infty)$ uniformly in $k$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ i.e.,

$$\sup_{t,\varepsilon_k} \left\| t^{(1-\delta)/2} u_{\varepsilon k} \right\|_{\left(L^{n/\delta}\right)^n} \leq M_1,\ \sup_{t,\varepsilon_k 0} \left\| t^{1/2} B_{1\varepsilon} u_{\varepsilon k} \right\|_{X_q} \leq M_2. \tag{6k}$$

Then (6k) is obtained from (6j). □

**Remark 3.** *Let $E$ be a UMD space, $A$ an R-positive operator in $E$, $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Theorem 9 shows that $L^p$ norms of $u_\varepsilon(t)$ decay as $t \to \infty$ for $p > q$ uniformly in $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$.*

*For $p = q$ we obtain the following result.*

**Theorem 10.** *Let the all conditions of Theorem 9 hold. Then $\|u_\varepsilon(t)\|_{X_p} \to 0$ uniformly in $\varepsilon$ as $t \to \infty$. More precisely, we have*

$$\|u_\varepsilon(t) - u_{0\varepsilon}(t)\|_{X_p} = O\left(t^{-\frac{\delta}{2}}\right) \text{ as } t \to \infty,$$

*where, $u_{0\varepsilon}(t) = S_\varepsilon(t)\, a$ and $\delta < \min\left\{1, n - \frac{n}{q}, \frac{n}{q} - 1\right\}$.*

## 7. The Wentzell–Robin Type Mixed Problem for Navier–Stokes Equations

Consider the N–S problem (1f)–(1h). Let

$$\tilde{\Omega} = \mathbb{R}^n_+ \times (0,1),\ X_{\mathbf{p}} = \left(L^{\mathbf{p}}(\tilde{\Omega})\right)^n,\ X^{2m,2}_{\mathbf{p}} = W^{2m,2,\mathbf{p}}\left((\tilde{\Omega})\right)^n.$$

$$B(\mathbf{p}, q) = L^q\left(0, T; X_{\mathbf{p}}\right),\ \mathbf{p} = (p_1, p),\ p_1, p, q \in (1, \infty)$$

denotes the product of Lebesque spaces with corresponding mixed norm and $\left(X^{2m,2}_{\mathbf{p}}, X_{\mathbf{p}}\right)_{\frac{1}{p},p}$ denote real interpolation space between $Y^{2m,2}_{\mathbf{p}}$ and $X_{\mathbf{p}}$.

**Theorem 11.** *Suppose $a$ is positive, $b$ is a real-valued functions on $\tilde{\Omega}$. Moreover, $a(x,.) \in C[0,1]$ for all $x \in \mathbb{R}^n_+$, $a(.,y) \in C_b\left(\mathbb{R}^n_+\right)$ for all $y \in [0,1]$ and $b(x,y)$ is a bounded function on $\tilde{\Omega}$ with*

$$\exp\left(-\int_{\frac{1}{2}}^{x} b(x,y)\, a^{-1}(x,y)\, dy\right) \in L_1(0,1), \text{ for } x \in \mathbb{R}^n_+.$$

*Let $0 < \varepsilon_k \leq 1$ and $a \in \left(X^{2m,2}_{\mathbf{p}},\ X_{\mathbf{p}}\right)_{\frac{1}{p},p}$ for $p,\ p_1 \in (1,\infty)$. Then,there exists a $T > 0$ and a unique solution $u = u_\varepsilon$ of (1f)–(1h) so that $t^{\left(1-\frac{n}{q}\right)/2}u \in C([0\ ,\ T);B(\mathbf{p},q))$ for $n \leq p \leq \infty$ and $t^{\left(1-\frac{n}{2q}\right)}B_{1\varepsilon}u \in C([0\ ,\ T);B(\mathbf{p},q))$ for $n \leq p < \infty$. Moreover, the following uniform estimates hold*

$$\sup_{t\in[0,\ T),\varepsilon_k>0}\left\|t^{\left(1-\frac{n}{q}\right)/2}u_\varepsilon\right\|_{B(\mathbf{p},q)} \leq C, \quad \sup_{t\in[0,\ T),\varepsilon_k>0}\left\|t^{\left(1-\frac{n}{2q}\right)}B_{1\varepsilon}u_\varepsilon\right\|_{B(\mathbf{p},q)} \leq C.$$

**Proof.** Let $E = L^{p_1}(0,1)$. It is known [19] that $L^{p_1}(0,1)$ is an $UMD$ space for $p_1 \in (1,\infty)$. Consider the operator $A$ defined by

$$D(A) = W^{2,p_1}\left(\tilde{\Omega}; B_j u = 0\right),\ Au = a\frac{\partial^2 u}{\partial y^2} + b\frac{\partial u}{\partial y}.$$

Therefore, the problem (1f)–(1h) can be rewritten in the form of (1a)–(1c), where $u(x) = u(x,.)$, $f(x) = f(x,.)$ are functions with values in $E = L^{p_1}(0,1)$. From [23,24] we get that the operator $A$ generates analytic semigroup in $L^{p_1}(0,1)$. Moreover, we obtain that the operator $A$ is $R$-positive in $L^{p_1}$. Then from Theorem 8 we obtain the assertion. □

## 8. The Mixed Problem for Degenerate Navier–Stokes Equation

Consider the N–S problem (1j)–(1l). Let

$$\bar{\Omega} = \mathbb{R}^n_+ \times (0,b),\ X_{\mathbf{p}} = \left(L^{\mathbf{p}}(\bar{\Omega})\right)^n,\ X^{2m,2}_{\mathbf{p}} = W^{2m,[2],\mathbf{p}}\left((\bar{\Omega})\right)^n.$$

The main aim of this section is to prove the following result:

**Theorem 12.** *Let $0 < \varepsilon_k \leq 1$ and $a \in \left(X^{2m,[2]}_{\mathbf{p}}, X_{\mathbf{p}}\right)_{\frac{1}{p},p}$ for $p,\ p_1 \in (1,\infty)$. Suppose $b_1(.,y) \in C\left(\mathbb{R}^n_+\right)$ for all $y \in [0,b]$, $b_1(x,.) \in C([0,b])$ for all $x \in \mathbb{R}^n_+$, $b_1(.,) \in C[0,1]$ and $b_2(x,y)$ is a bounded function on $\bar{\Omega}$. Moreover, assume $0 \leq \gamma < \frac{1}{2}$, $\alpha_{10}\beta_{20} - \alpha_{20}\beta_{10} \neq 0$, $\alpha_{20}\beta_{11} + \alpha_{21}\beta_{10} - \alpha_{10}\beta_{21} - \alpha_{11}\beta_{20} \neq 0$, $\alpha_{11}\beta_{21} + \alpha_{21}\beta_{11} \neq 0, \alpha_{11}\beta_{21} - \alpha_{11}\alpha_{21} \neq 0, \alpha_{11} \neq \beta_{11}$ for $\nu_k = 1$ and $|\alpha_{k0}| + |\beta_{k0}| > 0, \alpha_{10}\alpha_{20} + \beta_{10}\beta_{20} \neq 0$ for $\nu_k = 0$. Then,there exists a $T > 0$ and a unique solution $u = u_\varepsilon$ of (1j)–(1l) so that $t^{\left(1-\frac{n}{q}\right)/2}u \in C([0,\ T);B(\mathbf{p},q))$ for $n \leq p \leq \infty$ and $t^{\left(1-\frac{n}{2q}\right)}B_{1\varepsilon}u \in C([0,\ T);B(\mathbf{p},q))$ for $n \leq p < \infty$. Moreover, the following uniform estimates hold*

$$\sup_{t\in[0,T),\varepsilon_k>0}\left\|t^{\left(1-\frac{n}{q}\right)/2}u_\varepsilon\right\|_{B(\mathbf{p},q)} \leq C, \quad \sup_{t\in[0,\ T),\varepsilon_k>0}\left\|t^{\left(1-\frac{n}{2q}\right)}B_{1\varepsilon}u_\varepsilon\right\|_{B(\mathbf{p},q)} \leq C.$$

**Proof.** Let $E = L^{p_1}(0,b)$. It is known [19] that $L^{p_1}(0,b)$ is an $UMD$ space for $p_1 \in (1,\infty)$. Consider the operator $A$ defined by

$$D(A) = W^{[2],p_1}(\Omega; L_k u = 0),\ Au = b_1\frac{\partial^{[2]}u}{\partial y^2} + b_2\frac{\partial^{[1]}u}{\partial y}.$$

Therefore, the problem (1j)–(1l) can be rewritten in the form of (1a)–(1c), where $u(x) = u(x,.)$, $f(x) = f(x,.)$ are functions with values in $E = L^{p_1}(0,b)$. From [7] we get that the operator $A$ generates analytic semigroup in $L^{p_1}(0,b)$. Moreover, we obtain that the operator $A$ is $R$-positive in $L^{p_1}$. Then from Theorem 8 we obtain the assertion. □

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