THE POLYNOMIAL DUNFORD-PETTIS PROPERTY

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Abstract. A Banach space E has the Dunford-Pettis property (DPP, for short) if every weakly compact (linear) operator on E is completely continuous. The $\mathcal{L}_1$ and the $\mathcal{L}_\infty$-spaces have the DPP. In 1979 R. A. Ryan proved that E has the DPP if and only if every weakly compact polynomial on E is completely continuous.

Every k-homogeneous (continuous) polynomial $P \in \mathcal{P}(^kE, F)$ between Banach spaces E and F admits an extension $\widetilde{P} \in \mathcal{P}(^kE^{**}, F^{**})$ called the Aron-Berner extension. The Aron-Berner extension of every weakly compact polynomial $P \in \mathcal{P}(^kE, F)$ is F-valued, that is, $\widetilde{P}(E^{**}) \subseteq F$, but there are nonweakly compact polynomials with F-valued Aron-Berner extension.

We strengthen Ryan's result by showing that E has the DPP if and only if every polynomial $P \in \mathcal{P}(^kE, F)$ with F-valued Aron-Berner extension is completely continuous. This answers a question raised in 2003 by I. Villanueva and the second named author. They proved the result for certain spaces E, for instance, the $\mathcal{L}_\infty$-spaces, but the question remained open for other spaces such as the $\mathcal{L}_1$-spaces.

1. Introduction

Throughout E, F, G, X, and Z denote Banach spaces, E^* is the dual of E, and B_E stands for its closed unit ball. The closed unit ball B_{E^*} will always be endowed with the weak-star topology. By $\mathbb{N}$ we represent the set of all natural numbers and by $\mathbb{K}$ the scalar field (real or complex). We use the symbol $\mathcal{L}(E, F)$ for the space of all (linear bounded) operators from E into F endowed with the operator norm. For $T \in \mathcal{L}(E, F)$ we denote its adjoint by $T^* \in \mathcal{L}(F^*, E^*)$.

Given $k \in \mathbb{N}$, we use $\mathcal{P}(^kE, F)$ for the space of all k-homogeneous (continuous) polynomials from E into F endowed with the supremum norm. When $F = \mathbb{K}$, we omit the range space, writing $\mathcal{P}(^kE) := \mathcal{P}(^kE, \mathbb{K})$. For the general theory of polynomials on Banach spaces, we refer the reader to [Di] and [Mu]. For unexplained notation and results in Banach space theory, the reader may see [Di, DJT, DU].

A polynomial $P \in \mathcal{P}(^kE, F)$ is *(weakly) compact* if $P(B_{E^*})$ is relatively (weakly) compact in F.

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Given a polynomial $P \in \mathcal{P}(^kE, F)$, its *adjoint* P^* is the operator

$$P^* : F^* \longrightarrow \mathcal{P}(^kE)$$

given by $P^*(\psi) := \psi \circ P$ for every $\psi \in F^*$. It is well-known that P is (weakly) compact if and only if P^* is (weakly) compact (see [AS, Proposition 3.2] for the compact case and [R2, Proposition 2.1] for the weakly compact case).

We say that a polynomial $P \in \mathcal{P}(^kE, F)$ is *completely continuous* if it takes weak Cauchy sequences into norm convergent sequences. We say that P is *unconditionally converging* if, for every weakly unconditionally Cauchy series $\sum x_n$ in E, the sequence $(P(s_m))_{m=1}^{\infty}$ is norm convergent, where $s_m := \sum_{n=1}^{m} x_n$.

Every polynomial $P \in \mathcal{P}(^kE, F)$ between Banach spaces admits an extension $\widetilde{P} \in \mathcal{P}(^kE^{**}, F^{**})$ called the Aron-Berner extension. We recall the construction of the Aron-Berner extension of a polynomial following [CGKM, §2]. Let A be the symmetric k-linear mapping associated with P. We can extend A to a k-linear mapping $\widetilde{A}$ from E^{**} into F^{**} in such a way that for each fixed j $(1 \leq j \leq k)$ and for each fixed $x_1, \ldots, x_{j-1} \in E$ and $z_{j+1}, \ldots, z_k \in E^{**}$, the linear mapping

$$z \longmapsto \widetilde{A}(x_1, \ldots, x_{j-1}, z, z_{j+1}, \ldots, z_k) \qquad (z \in E^{**}) \tag{1}$$

is weak*-to-weak* continuous. In other words, we define the image of the mapping in (1) to be the weak*-limit of the net $\left(\widetilde{A}(x_1, \ldots, x_{j-1}, x_\alpha, z_{j+1}, \ldots, z_k)\right)$ for a weak*-convergent net $(x_\alpha) \subset E$. By this weak*-to-weak* continuity, A can be extended to a k-linear mapping $\widetilde{A}$ from E^{**} into F^{**} beginning with the last variable and working backwards to the first. Then the restriction

$$\widetilde{P}(z) := \widetilde{A}(z, \ldots, z) \qquad (z \in E^{**})$$

is called the *Aron-Berner extension* of P. Given $z \in E^{**}$ and $w \in F^*$, we have

$$\widetilde{P}(z)(w) = \widetilde{w \circ P}(z)\,. \tag{2}$$

Actually this equality is often used as the definition of the vector-valued Aron-Berner extension based upon the scalar-valued Aron-Berner extension. Recall that $\widetilde{A}$ is not symmetric in general.

The Aron-Berner extension was introduced in [AB]. A survey of its properties may be seen in [Z]. It has been studied by many mathematicians. We only mention a few examples: [AB, ACG, Ca, CG, CL, CGKM, DG, DGG, GGMM, GV, PVWY].

Given a Banach space E, we denote by

$$\Theta_E : \mathcal{P}(^kE) \longrightarrow \mathcal{P}(^kE^{**})$$

the *Aron-Berner extension operator,* given by $\Theta_E(P) := \widetilde{P}$ for every $P \in \mathcal{P}(^kE)$. The operator Θ_E is an isometric embedding [DG, Theorem 3].

The Aron-Berner extension of every weakly compact polynomial is F-valued, that is, $\widetilde{P}(E^{**}) \subseteq F$ [Ca, Proposition 1.4], but there are polynomials with F-valued Aron-Berner extension which are not weakly compact. The most typical and basic example may be the polynomial $Q \in \mathcal{P}(^k\ell_2, \ell_1)$ given by $Q(x) := \left(x_n^k\right)_{n=1}^{\infty}$ for all $x = (x_n)_{n=1}^{\infty} \in \ell_2$. The polynomials with F-valued Aron-Berner extension are often more useful than the weakly compact polynomials when it comes to characterize isomorphic properties of Banach spaces: see for instance [GV]. In the polynomial setting they play somehow the role of the weakly compact operators in the linear setting.

The *bounded weak-star topology* bw* on a dual Banach space E^* is the finest topology that coincides with the weak-star topology on bounded subsets of E^*. The bw* topology is locally convex [Me, Theorem 2.7.2].

It should be noted that the statement of [GV, Lemma 3.3] (given without proof) is wrong. This lemma is used in several places of [GV]. A corrected version of the lemma and subsequent results of [GV] is given in [PVWY, §2].

A Banach space has the *Dunford-Pettis property* (*DPP,* for short) if every weakly compact operator on E is completely continuous. Ryan proved [R1] that E has the DPP if and only if every weakly compact polynomial on E is completely continuous. An attempt to strengthen this result was made in [GV] where the question was raised of knowing if the DPP of E implies the complete continuity of every polynomial from E into an arbitrary Banach space F so that the Aron-Berner extension of P is F-valued. A partial affirmative answer was given in [GV] for spaces E such that every operator from E into E^* is weakly compact, but the question remained open in general and was unknown for instance for $\mathcal{L}_1$-spaces.

In the present paper we prove that E has the DPP if and only if whenever P is a polynomial from E into an arbitrary Banach space F with F-valued Aron-Berner extension, then P is completely continuous. We achieve this result by a careful study of the composition of Dunford-Pettis operators (see definition below) with polynomials having F-valued Aron-Berner extension.

We summarize some characterizations of isomorphic properties of Banach spaces that can be obtained using polynomials with F-valued Aron-Berner extension:

(a) The DPP as mentioned above.

(b) Recall that E has the reciprocal Dunford-Pettis property (RDPP, for short) if every completely continuous operator on E is weakly compact. A space E has the RDPP if and only if every completely continuous polynomial from E into an arbitrary Banach space F has F-valued Aron-Berner extension [GV, Corollary 3.5].

(c) E is said to have property (V) if every unconditionally converging operator on E is weakly compact. A space E has property (V) if and only if every unconditionally converging polynomial from E into an arbitrary Banach space F has F-valued Aron-Berner extension [GV, Corollary 4.3].

A subset A of a Banach space E is a *Dunford-Pettis set* (*DP set,* for short) [An, Theorem 1] if, for every weakly null sequence $(x_n^*) \subset E^*$, we have

$$\lim_n \sup_{x\in A} |\langle x, x_n^*\rangle| = 0\,.$$

An operator $S \in \mathcal{L}(G,E)$ is a *Dunford-Pettis operator* if $S(B_G)$ is a DP set in E. We denote by $\mathcal{DP}$ the ideal of Dunford-Pettis operators which has been studied under a different notation in [GG1].

A subset A of a Banach space E is said to be a *Rosenthal set* if every sequence in A contains a weak Cauchy subsequence.

Given an operator $S \in \mathcal{L}(G, E)$, we denote by $S_1 \in \mathcal{L}(G, S(G))$ the operator given by $S_1(g) := S(g)$ for $g \in G$. Note that the normed space $S(G)$ is not necessarily complete.

2. The results

Given $k \in \mathbb{N}$ and an operator $S \in \mathcal{L}(G, E)$, we define the operator

$$S_k^* : \mathcal{P}(^kE) \longrightarrow \mathcal{P}(^kG) \qquad (\text{or} \quad S_k^* : \mathcal{P}(^kS(G)) \to \mathcal{P}(^kG))$$

by $S_k^*(P)(g) := P(S(g))$ for all $P \in \mathcal{P}(^kE)$ (or $\mathcal{P}(^kS(G))$) and $g \in G$. Similarly, we define

$$S_k^{***} : \mathcal{P}(^kE^{**}) \longrightarrow \mathcal{P}(^kG^{**}) \qquad (\text{or} \quad S_k^{***} : \mathcal{P}(^kS(G)^{**}) \to \mathcal{P}(^kG^{**}))$$

by $S_k^{***}(Q)(g^{**}) := Q(S^{**}(g^{**}))$ for all $Q \in \mathcal{P}(^kE^{**})$ (or $\mathcal{P}(^kS(G)^{**})$) and $g^{**} \in G^{**}$.

Given a polynomial $P \in \mathcal{P}(^kE, F)$ and an operator $S \in \mathcal{L}(G, E)$, we shall use the following diagram:

$$\begin{array}{ccccccc} F^* & \xrightarrow{P^*} & \mathcal{P}(^kE) & \xrightarrow{\rho_k} & \mathcal{P}(^kS(G)) & \xrightarrow{S_k^*} & \mathcal{P}(^kG) \\ & & \Theta_E \downarrow & & \Theta_{S(G)} \downarrow & & \Theta_G \downarrow \\ & & \mathcal{P}(^kE^{**}) & \xrightarrow{\rho_k^{\mathrm{b}}} & \mathcal{P}(^kS(G)^{**}) & \xrightarrow{S_k^{***}} & \mathcal{P}(^kG^{**}) \end{array} \tag{3}$$

where ρ_k and ρ_k^{b} are restriction operators. The superscript "b" stands for "bidual".

We show that the diagram commutes. Indeed, the only part which needs a proof is the right hand rectangle. For $R \in \mathcal{P}(^kS(G))$, we have

$$\begin{aligned} \Theta_G \circ S_k^*(R)(g^{**}) &= \Theta_G(R \circ S_1)(g^{**}) = \widetilde{R} \circ S_1^{**}(g^{**}) = \Theta_{S(G)}(R)(S_1^{**}(g^{**})) \\ &= S_k^{***}\left(\Theta_{S(G)}(R)\right)(g^{**}) . \end{aligned}$$

We say that a net of polynomials $(P_\alpha) \subset \mathcal{P}(^kE)$ is $\tau_{\mathrm{p}}^{\mathrm{b}}$-convergent to $P \in \mathcal{P}(^kE)$ if, for every $x^{**} \in E^{**}$, we have

$$\widetilde{P}_\alpha(x^{**}) \underset{\alpha}{\longrightarrow} \widetilde{P}(x^{**})$$

for every $x^{**} \in E^{**}$. The subscript "p" stands for "pointwise" and the superscript "b" for "bidual".

Proposition 2.1. *Let $S \in \mathcal{L}(G, E)$ and $P \in \mathcal{P}(^kE, F)$. The following assertions are equivalent:*
(a) $\widetilde{P} \circ S^{**}(G^{**}) \subseteq F$*;*
(b) $\Theta_G \circ S_k^* \circ \rho_k \circ P^*$ *is weak***-to-*$\tau_{\mathrm{p}}^{\mathrm{b}}$ *continuous;*
(c) $\Theta_G \circ S_k^* \circ \rho_k \circ P^*$ *is* bw**-to-*$\tau_{\mathrm{p}}^{\mathrm{b}}$ *continuous.*

Proof. (a) $\Rightarrow$ (b). Let $(f_\alpha^*) \subset F^*$ be a weak*-null net. Then, for all $g^{**} \in G^{**}$, we have

$$\begin{aligned} \Theta_G \circ S_k^* \circ \rho_k \circ P^*\left(f_\alpha^*\right)(g^{**}) &= S_k^{***} \circ \Theta_{S(G)} \circ \rho_k \circ P^*\left(f_\alpha^*\right)(g^{**}) \\ &= \Theta_{S(G)} \circ \rho_k \circ P^*\left(f_\alpha^*\right)(S^{**}(g^{**})) \end{aligned}$$

$$= \left(f_\alpha^* \circ \widetilde{P}\right)(S^{**}(g^{**}))$$
$$= \left\langle \widetilde{P}(S^{**}(g^{**})), f_\alpha^* \right\rangle \underset{\alpha}{\longrightarrow} 0\,,$$

since $\widetilde{P}(S^{**}(g^{**})) \in F$.

(b) $\Rightarrow$ (c) is obvious since bw* is finer than the weak-star topology.

(c) $\Rightarrow$ (a). Let $(f_\alpha^*) \subset F^*$ be a bw*-null net. By the above calculations, we have for all $g^{**} \in G^{**}$:

$$\left\langle \widetilde{P}(S^{**}(g^{**})), f_\alpha^* \right\rangle \underset{\alpha}{\longrightarrow} 0\,,$$

so $\widetilde{P}(S^{**}(g^{**})) \in (F^*, \mathrm{bw}^*)^* = F$ [Me, Theorem 2.7.8]. □

Proposition 2.2. *Let $Q \in \mathcal{P}(^kE)$ and $S \in \mathcal{DP}(G, E)$. Then $\widetilde{Q} \in \mathcal{P}(^kE^{**})$ is weak-star continuous on $S^{**}(B_{G^{**}})$.*

Proof. We modify the proof of [GG1, Proposition 3.1]. Let $A := S(B_G)$ which is an absolutely convex DP set in E. By [DFJP, Lemma 1], we can find a Banach space Z and an operator $j \in \mathcal{L}(Z, E)$ so that:

(a) $A \subseteq j(B_Z)$;

(b) $j^{**} : Z^{**} \to E^{**}$ is injective and $j^{**-1}(E) = Z$;

(c) $j(B_Z) \subseteq 2^n A + 2^{-n} B_E$ for every $n \in \mathbb{N}$.

Using Goldstine's theorem, it is easy to check that $S^{**}(B_{G^{**}}) \subseteq j^{**}(B_{Z^{**}})$. Let $(z_\alpha^{**}) \subset B_{Z^{**}}$ be a net such that weak*-$\lim j^{**}(z_\alpha^{**}) = 0$. Assume that (z_α^{**}) is not weak-star null. Then every subnet has a weak-star Cauchy subnet, so this subnet must be weak-star convergent and its limit has to be 0 by the injectivity of j^{**}. Therefore, the original net (z_α^{**}) is itself weak-star null. So $B_{Z^{**}}$ and $j^{**}(B_{Z^{**}})$ are weak-star homeomorphic.

Let $(g_\alpha^{**}) \subset B_{G^{**}}$ be a net such that

$$S^{**}(g_\alpha^{**}) \xrightarrow{\mathrm{w}^*} x^{**} \in E^{**}\,.$$

Since $S^{**}(B_{G^{**}})$ is weak-star compact, we can find $g^{**} \in B_{G^{**}}$ so that $x^{**} = S^{**}(g^{**})$. Let $z_\alpha^{**} \in B_{Z^{**}}$ with $S^{**}(g_\alpha^{**}) = j^{**}(z_\alpha^{**})$ and $z^{**} \in B_{Z^{**}}$ so that $S^{**}(g^{**}) = j^{**}(z^{**})$. By the above weak-star homeomorphism, we have weak*-$\lim z_\alpha^{**} = z^{**}$.

By (c), $j(B_Z)$ is a DP set (see the proof in [GG1, Proposition 3.1]). By [GG1, Proposition 3.6 and Theorem 3.5], the polynomial $Q \circ j$ is weakly continuous on bounded subsets of Z. By [ACG, Theorem 7.1], its Aron-Berner extension $\widetilde{Q} \circ j^{**}$ is weak-star continuous on bounded sets of Z^{**}, and

$$\widetilde{Q} \circ S^{**}(g_\alpha^{**}) = \widetilde{Q} \circ j^{**}(z_\alpha^{**}) \underset{\alpha}{\longrightarrow} \widetilde{Q} \circ j^{**}(z^{**}) = \widetilde{Q} \circ S^{**}(g^{**})\,.$$ □

Corollary 2.3. *Given a polynomial $P \in \mathcal{P}(^kE, F)$ and an operator $S \in \mathcal{DP}(G, E)$ so that $\widetilde{P} \circ S^{**}(G^{**}) \subseteq F$, we have that $\widetilde{P}$ is weak*-to-weak continuous on $S^{**}(B_{G^{**}})$ and so the polynomial $\widetilde{P} \circ S^{**}$ is weakly compact.*

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc} G & \xrightarrow{S} & E & \xrightarrow{P} & F \\ {\scriptstyle k_G}\downarrow & & {\scriptstyle k_E}\downarrow & \nearrow {\scriptstyle \widetilde{P}} & \\ G^{**} & \xrightarrow{S^{**}} & E^{**} & & \end{array}$$

where k_E and k_G are the canonical isometric embeddings. Let $(g_\alpha^{**}) \subset B_{G^{**}}$ be a net and $g^{**} \in B_{G^{**}}$ so that

$$S^{**}\left(g_\alpha^{**}\right) \xrightarrow{\mathrm{w}^*} S^{**}(g^{**}) .$$

By Proposition 2.2, we have for all $f^* \in B_{F^*}$,

$$\begin{aligned} \left(f^* \circ \widetilde{P}\right)\left(S^{**}\left(g_\alpha^{**}\right)\right) &= \widetilde{f^* \circ P}\left(S^{**}\left(g_\alpha^{**}\right)\right) \\ \underset{\alpha}{\longrightarrow} \widetilde{f^* \circ P}\left(S^{**}\left(g^{**}\right)\right) &= \left(f^* \circ \widetilde{P}\right)\left(S^{**}\left(g^{**}\right)\right) . \end{aligned}$$

Therefore,

$$\widetilde{P}\left(S^{**}\left(g_\alpha^{**}\right)\right) \xrightarrow{\text{weak}} \widetilde{P}\left(S^{**}\left(g^{**}\right)\right) \quad \text{in} \quad F . \qquad \square$$

Theorem 2.4. *Let $S \in \mathcal{DP}(G, E)$ and $(P_n) \subset \mathcal{P}(^kE)$ be a sequence of polynomials such that, for every $g^{**} \in G^{**}$, we have $\widetilde{P_n} \circ S^{**}(g^{**}) \to 0$. Then we have*

$$\lim_n \sup_{g \in B_G} |P_n \circ S(g)| = 0 .$$

Proof. Assume the result fails. Then, passing to a subsequence if necessary, we can find a sequence $(g_n) \subset B_G$ and $\delta > 0$ such that $|P_n \circ S(g_n)| > \delta$ for all $n \in \mathbb{N}$. Let $A_n : E \times \overset{(k)}{\cdots} \times E \to \mathbb{K}$ be the unique symmetric k-linear form associated with P_n. Then,

$$\left|A_n\left(S(g_n), \overset{(k)}{\ldots}, S(g_n)\right)\right| > \delta \qquad (n \in \mathbb{N}) .$$

Since $S(B_G)$ is a DP set in the Banach space $\overline{S(G)}$, the sequence

$$\left(A_n\left(S(g_n), \overset{(k-1)}{\ldots}, S(g_n), \cdot\right)\right)_{n=1}^\infty$$

is not weakly null in $S(G)^*$. So, passing again to a subsequence if necessary, there are $x_k^{**} \in S(G)^{**}$ and $\delta_1 > 0$ such that

$$\left|\widetilde{A_n}\left(S(g_n), \overset{(k-1)}{\ldots}, S(g_n), x_k^{**}\right)\right| > \delta_1 \qquad (n \in \mathbb{N}) . \tag{4}$$

Consider the operator $S_1 \in \mathcal{L}(G, S(G))$. By [Me, Theorem 3.1.17], the operator

$$S(G)^* \xrightarrow{S_1^*} G^*$$

is injective, and the operator

$$G^{**} \xrightarrow{S_1^{**}} S(G)^{**}$$

has weak-star dense range. Therefore, by the weak-star continuity of

$$A_n\left(S(g_n), \overset{(k-1)}{\ldots}, S(g_n), \cdot\right)$$

on $S(G)^{**}$ and using (4), we can find $g_k^{**} \in G^{**}$ so that

$$\left|\widetilde{A_n}\left(S(g_n), \overset{(k-1)}{\ldots}, S(g_n), S^{**}\left(g_k^{**}\right)\right)\right| > \delta_1 \qquad (n \in \mathbb{N}).$$

Iterating the argument, the sequence

$$\left(\widetilde{A_n}\left(S(g_n), \overset{(k-2)}{\ldots}, S(g_n), \ \cdot\ , S^{**}\left(g_k^{**}\right)\right)\right)_{n=1}^{\infty}$$

is not weakly null in $S(G)^*$ so, passing to a subsequence if necessary, there are $x_{k-1}^{**} \in S(G)^{**}$ and $\delta_2 > 0$ such that

$$\left|\widetilde{A_n}\left(S(g_n), \overset{(k-2)}{\ldots}, S(g_n), x_{k-1}^{**}, S^{**}\left(g_k^{**}\right)\right)\right| > \delta_2 \qquad (n \in \mathbb{N}).$$

As above, we can find $g_{k-1}^{**} \in G^{**}$ so that

$$\left|\widetilde{A_n}\left(S(g_n), \overset{(k-2)}{\ldots}, S(g_n), S^{**}\left(g_{k-1}^{**}\right), S^{**}\left(g_k^{**}\right)\right)\right| > \delta_2 \qquad (n \in \mathbb{N}).$$

Proceeding up to the first variable, we can find $g_1^{**} \in G^{**}$ and $\delta_k > 0$ so that

$$\left|\widetilde{A_n}\left(S^{**}\left(g_1^{**}\right), \ldots, S^{**}\left(g_k^{**}\right)\right)\right| > \delta_k \qquad (n \in \mathbb{N}).$$

By the polarization formula [Mu, Theorem 1.10], we obtain

$$\begin{aligned} k!2^k\delta_k &< \left|\sum_{\epsilon_j=\pm 1} \epsilon_1 \cdots \epsilon_k \widetilde{A_n} \circ S^{**}\left(\epsilon_1 g_1^{**} + \cdots + \epsilon_k g_k^{**}\right)^k\right| \\ &\leq \sum_{\epsilon_j=\pm 1} \left|\widetilde{A_n} \circ S^{**}\left(\epsilon_1 g_1^{**} + \cdots + \epsilon_k g_k^{**}\right)^k\right| \\ &= \sum_{\epsilon_j=\pm 1} \left|\widetilde{P_n} \circ S^{**}\left(\epsilon_1 g_1^{**} + \cdots + \epsilon_k g_k^{**}\right)\right| \qquad (n \in N), \end{aligned} \tag{5}$$

where we have used the notation

$$\widetilde{A_n} \circ S^{**}\left(g^{**}\right)^k := \widetilde{A_n}\left(S^{**}(g^{**}), \overset{(k)}{\ldots}, S^{**}(g^{**})\right)$$

for $g^{**} \in G^{**}$. Since each summand of (5) tends to zero as n goes to ∞, we reach a contradiction. □

Theorem 2.5. *Given $S \in \mathcal{DP}(G, E)$ and $P \in \mathcal{P}(^kE, F)$ with B_{F^*} weak-star sequentially compact, assume that $\widetilde{P} \circ S^{**}(G^{**}) \subseteq F$. Then the polynomial $P \circ S$ is compact.*

Proof. Suppose $P \circ S$ is not compact. Then its adjoint $S_k^* \circ P^*$ is not compact, so there is a sequence $(f_n^*) \subset B_{F^*}$ such that the sequence $(S_k^* \circ P^*(f_n^*))_{n=1}^{\infty}$ does not have any convergent subsequence. By the weak-star sequential compactness of B_{F^*}, we can assume that (f_n^*) is weak-star convergent. By linearity of $S_k^* \circ P^*$, we can assume that (f_n^*) is weak-star null.

By passing to a subsequence if necessary, we can find a sequence $(g_n) \subset B_G$ and $\delta > 0$ so that

$$\left|P^*\left(f_n^*\right)\left(S(g_n)\right)\right| = \left|S_k^* \circ P^*\left(f_n^*\right)\left(g_n\right)\right| > \delta \qquad (n \in \mathbb{N}). \tag{6}$$

Since $\widetilde{P} \circ S^{**}(G^{**}) \subseteq F$, Proposition 2.1 implies that $\Theta_G \circ S_k^* \circ \rho_k \circ P^*(f_n^*) \to 0$ pointwise on G^{**} so

$$\widetilde{f_n^* \circ P}(S^{**}(g^{**})) = \left(f_n^* \circ \widetilde{P}\right)(S^{**}(g^{**})) = \Theta_G \circ S_k^* \circ \rho_k \circ P^*(f_n^*)(g^{**}) \underset{n}{\longrightarrow} 0 .$$

By Theorem 2.4 we have

$$P^*(f_n^*)(S(g_n)) \underset{n}{\longrightarrow} 0$$

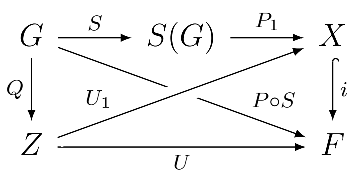
in contradiction with (6). □

Theorem 2.6. *Let $P \in \mathcal{P}(^kE, F)$ be a polynomial such that $\widetilde{P}(E^{**}) \subseteq F$. Then P is weakly continuous on DP sets of E.*

Proof. Let $S \in \mathcal{DP}(G, E)$. By Corollary 2.3, the polynomial $P \circ S$ is weakly compact. By [R2, Theorem 3.7], there are a reflexive Banach space Z, a polynomial $Q \in \mathcal{P}(^kG, Z)$ and an operator $U \in \mathcal{L}(Z, F)$ such that $P \circ S = U \circ Q$.

Let $X := \overline{U(Z)} \subseteq F$ with embedding $i : X \hookrightarrow F$. Then X is weakly compactly generated [FHHMZ, page 575]. By [Di, Chapter XIII, Theorem 4], B_{X^*} is weak-star sequentially compact. Denote by $U_1 \in \mathcal{L}(Z, X)$ the operator given by $U_1(z) := U(z)$ for $z \in Z$.

Let $P_1 \in \mathcal{P}(^kS(G), X)$ be the polynomial defined by $P_1(S(g)) := U_1 \circ Q(g)$ for all $g \in G$.



We have $i \circ P_1 \circ S = P \circ S$. Since Z is reflexive, we have $\widetilde{Q}(G^{**}) \subseteq Z$, so $\widetilde{P_1 \circ S} = \widetilde{P_1} \circ S^{**} = U_1 \circ \widetilde{Q}$ and $\widetilde{P_1} \circ S^{**}(G^{**}) \subseteq X$. By Theorem 2.5, the polynomial $P_1 \circ S$ is compact. Hence, $P \circ S = i \circ P_1 \circ S$ is compact. Since $S \in \mathcal{DP}(G, E)$ is arbitrary, [GG1, Proposition 3.6] implies that P is weakly continuous on DP sets of E. □

Corollary 2.7. *Given a Banach space E and $k \in \mathbb{N}$, the following assertions are equivalent:*

(a) *E has the DPP;*

(b) *for every Banach space F, every polynomial $P \in \mathcal{P}(^kE, F)$ with $\widetilde{P}(E^{**}) \subseteq F$ is completely continuous;*

(c) *every polynomial $P \in \mathcal{P}(^kE, c_0)$ with $\widetilde{P}(E^{**}) \subseteq c_0$ is completely continuous.*

Proof. (a) ⇒ (b). If $\widetilde{P}(E^{**}) \subseteq F$, Theorem 2.6 implies that P is weakly continuous on DP sets of E. Since E has the DPP, [GG1, Proposition 1.2] implies that P is weakly continuous on Rosenthal sets. By the comment preceding [GG1, Corollary 3.7], P is weakly *uniformly* continuous on Rosenthal sets and so P takes weak Cauchy sequences into norm convergent sequences.

(b) ⇒ (c) is obvious.

(c) ⇒ (a). Let $P \in \mathcal{P}(^kE, c_0)$ be a weakly compact polynomial. Then $\widetilde{P}(E^{**}) \subseteq c_0$ so P is completely continuous. By [GG1, Theorem 3.14], E has the DPP. □

3. The holomorphic DPP

Suppose now that E and F are complex Banach spaces. We denote by $\mathcal{H}_{\mathrm{b}}(E,F)$ the space of holomorphic mappings of bounded type from E into F, that is, every $f \in \mathcal{H}_{\mathrm{b}}(E,F)$ is bounded on bounded subsets of E. We refer the reader to Isidro's paper [I] for basic properties of this well-known space. Given $f \in \mathcal{H}_{\mathrm{b}}(E,F)$ with Taylor series expansion at the origin $f = \sum_{k=0}^{\infty} P_k$ where $P_k \in \mathcal{P}({}^kE,F)$, f has F-valued Aron-Berner extension $\widetilde{f} \in \mathcal{H}_{\mathrm{b}}(E^{**},F)$ if and only if $\widetilde{P_k}$ is F-valued for all $k \in \mathbb{N}$ [GGMM, Theorem 3.3]. Using the arguments of [GV, §5], we obtain the following results:

Theorem 3.1. *Given $f \in \mathcal{H}_{\mathrm{b}}(E,F)$ with F-valued Aron-Berner extension, f is weakly continuous on DP sets.*

Corollary 3.2. *A complex Banach space E has the DPP if and only if every $f \in \mathcal{H}_{\mathrm{b}}(E,F)$ with F-valued Aron-Berner extension is completely continuous.*

References

[An] K. T. Andrews, Dunford-Pettis sets in spaces of Bochner integrable functions, *Math. Ann.* **241** (1979), 35–41.

[AB] R. M. Aron and P. D. Berner, A Hahn-Banach extension theorem for analytic mappings, *Bull. Soc. Math. France* **106** (1978), 3–24.

[ACG] R. M. Aron, B. J. Cole and T. W. Gamelin, Weak-star continuous analytic functions, *Canad. J. Math.* **47** (1995), 673–683.

[AS] R. M. Aron and M. Schottenloher, Compact holomorphic mappings on Banach spaces and the approximation property, *J. Funct. Anal.* **21** (1976), 7–30.

[Ca] D. Carando, Extendible polynomials on Banach spaces, *J. Math. Anal. Appl.* **233** (1999), 359–372.

[CG] D. Carando and D. Galicer, Extending polynomials in maximal and minimal ideals, *Publ. Res. Inst. Math. Sci.* **46** (2010) 669-680.

[CL] D. Carando and S. Lassalle, Extension of vector-valued integral polynomials, *J. Math. Anal. Appl.* **307** (2005), 77–85.

[CGKM] Y. S. Choi, D. García, S. G. Kim, and M. Maestre, Composition, numerical range and Aron-Berner extension, *Math. Scand.* **103** (2008), 97–110.

[DG] A. M. Davie and T. W. Gamelin, A theorem on polynomial-star approximation, *Proc. Amer. Math. Soc.* **106** (1989), 351–356.

[DFJP] W. J. Davis, T. Figiel, W. B. Johnson, and A. Pełczyński, Factoring weakly compact operators, *J. Funct. Anal.* **17** (1974), 311–327.

[Di] J. Diestel, *Sequences and Series in Banach Spaces,* Graduate Texts in Math. **92,** Springer, Berlin 1984.

[DJT] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators,* Cambridge Stud. Adv. Math. **43,** Cambridge University Press, Cambridge 1995.

[DU] J. Diestel and J. J. Uhl, Jr., *Vector Measures,* Math. Surveys Monographs **15,** American Mathematical Society, Providence RI 1977.

[DGG] V. Dimant, D. Galicer, and R. García, Geometry of integral polynomials, M-ideals and unique norm preserving extensions, *J. Funct. Anal.* **262** (2012), 1987-2012.

[Di] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces,* Springer Monogr. Math., Springer, Berlin 1999.

[FHHMZ] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis,* CMS Books in Math., Springer, New York 2011.

[GGMM] P. Galindo, D. García, M. Maestre, and J. Mujica, Extension of multilinear mappings on Banach spaces, *Studia Math.* **108** (1994), 55–76.

[GG1] M. González and J. M. Gutiérrez, Weakly continuous mappings on Banach spaces with the Dunford-Pettis property, *J. Math. Anal. Appl.* **173** (1993), 470–482.

[GV] J. M. Gutiérrez and I. Villanueva, Extensions of multilinear operators and Banach space properties, *Proc. Roy. Soc. Edinburgh* **133A** (2003), 549–566. See Corrigendum in [PVWY, §2].

[I] J. M. Isidro, Topological duality on the functional space $(H_b(U,F),\tau_b)$, *Proc. Roy. Irish Acad. Sect. A* **79** (1979), 115–130.

[Me] R. E. Megginson, *An Introduction to Banach Space Theory,* Graduate Texts in Math. **183**, Springer, Berlin 1998.

[Mu] J. Mujica, *Complex Analysis in Banach Spaces,* Math. Studies **120**, North-Holland, Amsterdam 1986.

[PVWY] A. M. Peralta, I. Villanueva, J. D. Maitland Wright, and K. Ylinen, Quasi-completely continuous multilinear operators, *Proc. Roy. Soc. Edinburgh* **140A** (2010), 635–649.

[R1] R. A. Ryan, Dunford-Pettis properties, *Bull. Acad. Polon. Sci. Sér. Sci. Math.* **27** (1979), 373–379.

[R2] R. A. Ryan, Weakly compact holomorphic mappings on Banach spaces, *Pacific J. Math.* **131** (1988), 179–190.

[Z] I. Zalduendo, Extending polynomials on Banach spaces—A survey, *Rev. Un. Mat. Argentina* **46** (2005), 45–72.

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